NONLINEAR VOLTERRA FUNCTIONAL EQUATIONS AND LINEAR PARABOLIC DIFFERENTIAL SYSTEMS[1]

BY

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1. Introduction. In his recent paper [1][2] Professor R. H. Cameron pointed out that there exists a certain duality between almost everywhere[3] type existence problems for solutions of nonlinear Volterra functional equations and minimality problems of positive solutions for certain linear parabolic differential systems. At the conclusion of that paper he stated that he expected to follow it with another paper where the duality principle is put to work and that he expected to prove at least one theorem of each type without using the duality principle, and then prove an equal number of corresponding theorems of opposite types by using the duality principle. The nonlinear Volterra functional equations that were studied were, for the two-dimensional case, functional equations of the type

$$y(t) = x(t) + \int_0^t F^2\left\{s, x(s), \int_0^s F^1[r, x(r)]dr\right\} ds.$$

The definition of the functional equations in the general n-dimensional case will be given in §2. The related linear parabolic differential systems were, in the general n-dimensional case,

$$\frac{1}{4}\frac{\partial^2 \theta}{\partial u^2} - \frac{\partial \theta}{\partial t} + J^1(t, u)\frac{\partial \theta}{\partial v_1} + J^2(t, u, v_1)\frac{\partial \theta}{\partial v_2} + \cdots$$

$$+ J^{n-1}(t, u, v_1, \cdots, v_{n-2})\frac{\partial \theta}{\partial v_{n-1}}$$

(1.1)

$$+ J^n(t, u, v_1, \cdots, v_{n-1})\theta = 0,$$

(1.2)
$$\theta(0, u, v_1, \cdots, v_{n-1}) = L(u, v_1, \cdots, v_{n-1}).$$

The task of proving theorems of the above mentioned nature was then handed over to the present author. Since then two everywhere-type existence

theorems, namely, Theorems I, II, and then by applying the duality principle on them, three corresponding minimality theorems, namely Theorems III, IV, V have been proved. On the other hand a uniqueness theorem for solutions of the differential system, namely Theorem VI, has been found.

The proofs of these theorems constitute this paper. Theorems I, II were derived by classical methods in the sense that no knowledge of Wiener measure was assumed in the proof. Theorems III, IV, V were derived from Theorems I, II by means of Theorems 1, 4, 6 of [1]. Theorem VI is based on Theorem 8 of [1].

2. **The general Volterra functional.** Throughout this paper the letter I denotes the interval $0 \leq t \leq 1$, I_0 the interval $0 \leq t \leq t_0$, R_n the n-dimensional Euclidean space. The symbol $C_w[a, b]$ denotes the Wiener space on the interval $a \leq t \leq b$, namely the space of continuous functions on $a \leq t \leq b$ which vanish at $t = a$. In particular $C_w[0, 1]$ is often abbreviated as C_w. In order to shorten the notation for functions and functionals of several variables, we adopt the following notational convention which was introduced in [1].

NOTATION. When a functional depends on a function and several numerical variables, the function will be written first, followed by a bar. The notation for functionals and also for functions may be abbreviated by dropping the function or some or all of the numerical variables, but we stipulate that if the function is dropped, the bar must also be dropped, and if a numerical variable is dropped, all those that follow it must also be dropped. Moreover, whenever a function or variable is dropped, it is understood that the suppressed arguments are the letters which originally stood in the function or functional when it was first introduced, with no substitutions. We also stipulate that if a numerical variable is replaced by some other letter, or by a number it will not be suppressed and none of the numerical variables preceding it will be suppressed.

Thus, for example, the functional $\Psi[x(\cdot) \mid t, u, v_1, \cdots, v_n]$ could be abbreviated as Ψ, $\Psi(x \mid \ \ )$, $\Psi(t)$, $\Psi(t, u)$, $\Psi(x \mid t, u, v_1)$ etc. but could not be abbreviated as $\Psi(x)$, $\Psi(u)$, $\Psi(u, v_1, v_3)$ etc. The symbol $\Psi(0)$ means that t has been replaced by zero while other arguments are left alone, $\Psi(u)$ would mean that t has been replaced by u and would be the same as $\Psi(u, u)$.

With the above notational convention, we now give our formal definition of the Volterra functional.

DEFINITION. Consider a finite sequence of real continuous functions $F^1(t, u)$, $F^2(t, u, v_1)$, $\cdots$, $F^n(t, u, v_1, \cdots, v_{n-1})$ defined for $t \in I$ and other variables unrestricted. From these construct inductively two sequences of functionals $\Phi^k(x \mid t)$, $\Lambda^k(x \mid t)$ depending on the function $x(\cdot)$ and the real variable t as follows:

$$(2.0) \qquad \Lambda^0(x \mid t) = x(t), \qquad\qquad\qquad \text{on } C_w \otimes I,$$

$$(2.1) \qquad \Phi^k(x \mid t) = F^k(t, \Lambda^0, \cdots, \Lambda^{k-1}), \qquad (k = 1, 2, \cdots, n) \text{ on } C_w \otimes I,$$

$$(2.2) \qquad \Lambda^k(x \mid t) = \int_0^t \Phi^k(\tau)d\tau, \qquad\qquad (k = 1, 2, \cdots, n) \text{ on } C_w \otimes I.$$

Then the functionals Φ^k and Λ^k will be called "Volterra functionals" and Φ^n and Λ^n will be called Volterra functionals of order n if no shorter string of functions could be used to define them.

Now to come to our problem, let $F^1, F^2, \cdots, F^n$ be defined and continuous on $I \otimes R_n$ and consider the Volterra functional equation

$$(2.3) \qquad\qquad y(t) = x(t) + \Lambda^n(x \mid t)$$

or with $f = F^n$ (4)

$$(2.4) \qquad\qquad y(t) = x(t) + \int_0^t f[s, \Lambda^0(x \mid s), \cdots, \Lambda^{n-1}(x \mid s)]ds.$$

Then for any given $x \in C_w$, y defined by (2.3) belongs to C_w. The question here is under what restrictions on $F^1, F^2, \cdots, F^n$ the equation (2.3) has a solution $x \in C_w$ for every or for almost every $y \in C_w$. Before we answer this question we quote a uniqueness theorem for solutions of (2.3) in the following remark. For the proof, the reader is referred to [1, p. 145].

REMARK 1. Let $F^1(t, u), F^2(t, u, v_1), \cdots, F^n(t, u, v_1, \cdots, v_{n-1})$ and their first derivatives with respect to $u, v_1, \cdots, v_{n-1}$ be continuous on $I_0 \otimes R_n$. Then if x_1 and x_2 both belong to $C_w[0, t_0]$ and satisfy (2.3) for a given $y \in C_w[0, t_0]$, $x_1 = x_2$.

3. **An everywhere-type existence theorem for solutions of Volterra functional equations.**

THEOREM I. *Let $F^1(t, u), F^2(t, u, v_1), \cdots, F^n(t, u, v_1, \cdots, v_{n-1})$ and their first derivatives with respect to $u, v_1, \cdots, v_{n-1}$ be continuous on $I \otimes R_k$, $(k = 1, 2, \cdots, n)$, and let $F^1, F^2, \cdots, F^n$ satisfy*

$$|F^k(t, u, v_1, \cdots, v_{k-1})| \leqq A \left\{1 + \sum_{j=0}^{k-1} |v_j|\right\} \log \left\{2 + \sum_{j=0}^{k-1} |v_j|\right\}$$
$$(3.1) \qquad\qquad\qquad an\ I \otimes R_k,\ (k = 1, 2, \cdots, n)(^5),$$

for some positive number A. Then for every $y \in C_w$ the Volterra functional equation (2.3) has a solution $x \in C_w$. Moreover the solution is unique in C_w.

The proof of Theorem I is based on Lemma 3 and Lemma 4. Lemma 3 is based on Lemma 1 and Lemma 2.

3.1. LEMMA 1. *Let $F^1, F^2, \cdots, F^n$ be continuous on $I \otimes R_k$ $(k = 1, 2, \cdots, n)$ and absolutely continuous with respect to each of $u, v_1, \cdots, v_{n-1}$. Let the first derivatives of $F^1, F^2, \cdots, F^n$ with respect to $u, v_1, \cdots, v_{n-1}$ exist everywhere*

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(4) Later, in §§5–6, as in [1], the notation F^n will mean a function other than the nth in the succession $F^1, F^2, \cdots$. For the latter, the notation f will be used.

(5) We use v_0 for u.

*except on a finite number of hyperplanes of the form $v_j = const.$ where the right
and the left hand derivatives exist, and satisfy*

(3.2)
$$\left| F_j^k(t, u, v_1, \cdots, v_{k-1}) \right| \leqq M_j^k, \qquad (j = 0, 1, 2, \cdots, k-1),$$
$$(k = 1, 2, \cdots n),$$

*where the M's are positive constants and F_j^k stands for $\partial F^k / \partial V_j$. Then there exist
positive constants $B_1, B_2, \cdots, B_n$ such that for any $x_1, x_2 \in C_w$*

(3.3)
$$\left| \Lambda^k(x_1 \mid t) - \Lambda^k(x_2 \mid t) \right| \leqq B_k \int_0^t \left| x_1(s) - x_2(s) \right| ds,$$
$$for \ t \in I, (k = 1, 2, \cdots, n).$$

Proof. We remark that if $F(v)$ is an absolutely continuous function of the
real variable v and dF/dv exists everywhere except at a finite number of points
v where the right and the left hand derivatives exist, and is bounded by M,
then

$$\left| F(v_2) - F(v_1) \right| \leqq M \left| v_2 - v_1 \right|.$$

From this remark, (3.3) holds by complete induction on k with $B_1 = M_0^1$
and in general $B_{N+1} = M_0^{N+1} + M_1^{N+1} B_1 + \cdots + M_N^{N+1} B_N$ for $N = 0, 1, \cdots,$
$n-1$.

At this point we make the following observation which will be needed
later.

REMARK 2. Suppose $F^1, F^2, \cdots, F^n$ satisfy the conditions in Lemma 1.
Then if for some $y \in C[0, 1]$, there exist $x_1, x_2 \in C[0, t_0]$ with $t_0 \leqq 1$ which
satisfy (2.3), it follows that $x_1(t) = x_2(t)$ for $t \in [0, t_0]$.

Proof. With (3.3), Remark 2 can be proved exactly in the same way as
Remark 1.

3.2. LEMMA 2. *If $F^1, F^2, \cdots, F^n$ are continuous on $I \otimes R_k$ ($k = 1, 2, \cdots, n$)
and $\{x_p(t)\}$ is a sequence of functions which are continuous on I and converges
to a function $x(t)$ uniformly on I, then*

(3.4) $$\lim_{p \to \infty} \Lambda^k(x_p \mid t) = \Lambda^k(x \mid t) \ uniformly \ in \ t \ on \ I, \qquad (k = 0, 1, 2, \cdots, n).$$

Proof. From the fact that F^k is uniformly continuous on $I \otimes S_k$ where S_k
is a closed rectangle in R_k, the conclusion of the lemma follows immediately
by complete induction on k.

3.3. LEMMA 3. *Let $F^1, F^2, \cdots, F^n$ satisfy the hypothesis of Lemma 1. Then
for any $y \in C_w$, there exists a unique element x of C_w which satisfies (2.3).*

Proof. Let $y \in C_w$ be given. Let us define a sequence of functions $\{x_p(t)\}$
on I by

(3.5) $x_{p+1}(t) = y(t) - \Lambda^n(x_p \,|\, t),$ $(p = 0, 1, 2, \cdots)$

with $x_0 = y$. Clearly $x_p \in C_w$ for $p = 0, 1, 2, \cdots$. By (3.5), (3.3) and an induction on p, we obtain

(3.6) $\big|\, x_{p+1}(t) - x_p(t)\,\big| \leqq \dfrac{b(B_n t)^p}{p!},$ for $t \in I,\ p = 0, 1, 2, \cdots,$

where $b = \max |F^n|$ on the bounded closed set $I \otimes I^{(0)} \otimes I^{(1)} \otimes \cdots \otimes I^{(n-1)}$ and

$$I^{(k)} = \big\{ v_k \in R_1 \,\big|\, \min_{t \in I} \Lambda^k(y \,|\, t) \leqq v_k \leqq \max_{t \in I} \Lambda^k(y \,|\, t) \big\}.$$

By (3.6) and by Cauchy's criterion, $\{x_p\}$ converges uniformly on I to an element x of C_w.

This x is a solution of (2.3), for

$$x(t) = \lim_{x \to \infty} x_p(t) = y(t) - \Lambda^n(x \,|\, t)$$

by (3.5), (3.4). The uniqueness of this solution is from Remark 2.

   3.4. LEMMA 4. *Let $F^1, F^2, \cdots, F^n$ be continuous on $I \otimes R_k$ $(k = 1, 2, \cdots, n)$ and satisfy (3.1). For a given $x \in C_w$, define $y \in C_w$ by (2.3). Then there exists a positive number B, independent of x, such that*[6]

(3.7) $\||\Lambda^k(x \,|\, t)\|| \leqq B\{1 + \|y\|^B\},$ $(k = 1, 2, \cdots, n),$

where $\|y\|$ is the Hilbert norm of y, i.e. $\|x\| = \{\int_0^1 [x(t)]^2 dt\}^{1/2}$.

   Proof. Let $x \in C_w$ be given and let $y \in C_w$ be defined by (2.3). Let

(3.8) $w_0(t) = y(t) - x(t) = \Lambda^n(x \,|\, t),$

(3.9) $w_k(t) = \Lambda^k(x \,|\, t),$ $(k = 1, 2, \cdots, n - 1).$

Then by (3.8), (2.2), (2.1), (3.1) we have the following estimate for $w_0(t)$;

(3.10) $\big|\, w_0(t)\,\big| \leqq A\left\{ Y(t) + \sum_{j=0}^{n-1} W_j(t) \right\}$

where

(3.11) $Y(t) = \displaystyle\int_0^t \big\{1 + (n+1)\,\big|\,y(s)\,\big|\,\big\} \log \big\{2 + (n+1)\,\big|\,y(s)\,\big|\,\big\}\,ds,$

(3.12) $W_k(t) = \displaystyle\int_0^t \big\{1 + (n+1)\,\big|\,w_k(s)\,\big|\big\} \log \big\{2 + (n+1)\,\big|\,w_k(s)\,\big|\,\big\}\,ds,$

                                        $(k = 0, 1, 2, \cdots, n - 1).$

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[6] We define $\||x\|| = \max_{t \in I} |x(t)|$ for any function $x(t)$ defined on $t \in I$.

By (3.11), (3.12), $Y(t)$, $W_k(t)$, $(k=0, 1, 2, \cdots, n-1)$ are positive and monotonically increasing functions of t on I. Similarly from (3.9), (2.2), (2.1), (3.1) we obtain

$$(3.13) \qquad \left| w_k(t) \right| \leq A \left\{ Y(t) + \sum_{j=0}^{k-1} W_j(t) \right\}, \qquad (k = 1, 2, \cdots, n-1).$$

We now proceed to eliminate $Y(t)$, $W_k(t)$ from (3.10), (3.13) by estimating first $W_k(t)$ by $Y(t)$ and then $Y(t)$ by $\|y\|$. Let $Y_0(t) \equiv \|\|Y\|\| = Y(1)$(⁶). Then by (3.12), (3.10), (3.13)

$$\frac{dW_k(t)}{dt} \leq \left\{ 1 + (n+1)A \left(Y_0 + \sum_{j=0}^{n-1} W_j(t) \right) \right\}$$

$$\cdot \log \left\{ 2 + (n+1)A \left(Y_0 + \sum_{j=0}^{n-1} W_j(t) \right) \right\}$$

for $k=0, 1, 2, \cdots, n-1$. From the above n inequalities

$$(n+1)A \frac{d}{dt} \left\{ \sum_{k=0}^{n-1} W_k(t) \right\} \left\{ 2 + (n-1)A \left(Y_0 + \sum_{k=0}^{n-1} W_k(t) \right) \right\}^{-1}$$

$$\cdot \log \left\{ 2 + (n+1)A \left(Y_0 + \sum_{k=0}^{n-1} W_k(t) \right) \right\}^{-1} \leq n(n+1)A$$

and upon integrating with respect to t

$$\log \log \left\{ 2 + (n+1)A \left(Y_0 + \sum_{k=0}^{n-1} W_k(s) \right) \right\} \int_0^t \leq n(n+1)At,$$

and then from the fact that $W_k(0) = 0$, $(k=0, 1, 2, \cdots, n-1)$

$$2 + (n+1)A \left(Y_0 + \sum_{k=0}^{n-1} W_k(t) \right) \leq \left\{ 2 + (n+1)A Y_0 \right\}^{B_1}$$

with $B_1 = \exp \left\{ n(n+1)A \right\}$ and finally

$$(3.14) \qquad W_k(t) \leq \frac{1}{(n+1)A} \left\{ 2 + (n+1)A Y_0 \right\}^{B_1}, \qquad (k = 0, 1, 2, \cdots, n-1).$$

By (3.11) and the fact that $\log (1+u) \leq u$ for $u \geq 0$ and $\left| y(t) \right| \leq 1 + \left| y(t) \right|^2$ on I,

$$(3.15) \qquad Y_0 \leq \int_0^1 \left\{ 1 + (n+1) \left| y(t) \right| \right\}^2 dt \leq 3(n+1)^2 (1 + \|y\|^2),$$

and from (3.14)

$$W_k(t) \leqq \frac{1}{(n+1)A} \{2 + 3(n+1)^3 A(1+\|y\|^2)\}^{B_1}, \quad (k = 0, 1, 2, \cdots, n-1).$$

We may assume without loss of generality that $A \geqq 1$. Then since $n \geqq 1$, it follows that $B_1 > 1$ and

$$(3.16) \quad W_k(t) \leqq \frac{1}{(n+1)A} \{4(n+1)^3 A(1+\|y\|^2)\}^{B_1} \quad (k = 0, 1, 2, \cdots, n-1).$$

Then from (3.10), (3.13), (3.15), (3.16)

$$|w_k(t)| \leqq 3(n+1)^2 A(1+\|y\|^2) + \frac{n}{n+1} \{4(n+1)^3 A\}^{B_1}(1+\|y\|^2)^{B_1},$$

$$(k = 0, 1, 2, \cdots, n-1)$$

and from the fact that $(n+1)^3 A > 1$ and $B_1 > 1$, it follows that $3(n+1)^2 A \leqq (n+1)^{-1}\{4(n+1)^3 A\}^{B_1}$ and hence

$$|w_k(t)| \leqq \{4(n+1)^3 A\}^{B_1}(1+\|y\|^2)^{B_1}, \quad (k = 0, 1, 2, \cdots, n-1).$$

Also from the fact that $B_1 > 1$, we have $(1+\|y\|^2)^{B_1} \leqq 2^{B_1}\{1+\|y\|^{2B_1}\}$ so that

$$|w_k(t)| \leqq \{8(n+1)^3 A\}^{B_1}\{1+\|y\|^{2B_1}\}, \quad (k = 0, 1, 2, \cdots, n-1).$$

If we let $B = \max\{\{8(n+1)^3 A\}^{B_1}, 2B_1\}$, then

$$|w_k(t)| \leqq B\{1+\|y\|\}^B, \quad (k = 0, 1, 2, \cdots, n-1).$$

3.5. **Proof of Theorem** I. Let $y \in C_w$ be given and let

$$(3.17) \qquad\qquad S = B(1 + \|y\|^B) + \|\|y\|\|$$

where B is a positive constant specified in Lemma 4. Let us define a sequence of functions $F^{1*}(t, u), F^{2*}(t, u, v_1), \cdots, F^{n*}(t, u, v_1, \cdots, v_{n-1})$ by stipulating that for $k = 1, 2, \cdots, n$, F^{k*} is defined to be equal to F^k if $|v_j| \leqq S$ for all $j = 0, 1, 2, \cdots, k-1$, and if $v_{j_0} > S$ or $v_{j_0} < -S, \cdots, v_{j_m} > S$ or $v_{j_m} < -S$ where m is any of $0, 1, 2, \cdots, k-1$, F^{k*} is defined to be equal to the value of $F^k(t, u, v_1, \cdots, v_{k-1})$ with $v_{j_0}, \cdots, v_{j_m}$ replaced by S or $-S$. Then $F^{k*}, (k = 1, 2, \cdots, n)$, satisfy the hypotheses of Lemma 3 and hence there exists $x \in C_w$ such that

$$y(t) = x(t) + \Lambda^{n*}(x \mid t)$$

where $\Lambda^{k*}(x \mid t)$ $(k = 0, 1, 2, \cdots, n)$ are defined by (2.0), (2.1), (2.2) with F^{k*} $(k = 0, 1, 2, \cdots, n)$. By Lemma 4 and (3.17)

$$\|\|x\|\| \leqq S, \qquad \|\|\Lambda^{k*}(x \mid t)\|\| \leqq S, \qquad (k = 1, 2, \cdots, n-1).$$

Now since F^k are identical with F^{k*} on $|v_j| \leqq S$ for $j = 0, 1, 2, \cdots, k-1$ and $k = 1, 2, \cdots, n$, $\Lambda^{k*}(x \mid t) = \Lambda^k(x \mid t)$ for $t \in I$ and $k = 0, 1, 2, \cdots$, for our x.

Thus our x satisfies (2.3). The uniqueness of this solution x follows by Remark 1.

4. Another everywhere-type existence theorem for solutions of Volterra functional equations.

THEOREM II. *Let* $F^1(t, u)$, $F^2(t, u, v_1)$, $\cdots$, $F^n(t, u, v_1, \cdots, v_{n-1})$ *and their first derivatives with respect to* u, v_1, $\cdots$, v_{n-1} *be continuous on* $I \otimes R_k$ $(k = 1, 2, \cdots, n)$ *and let* F^n *satisfy the condition*

$$(4.1) \qquad F^n(t, u, v_1, \cdots, v_{n-1}) \operatorname{sgn} u \geqq - A \qquad an\ I \otimes R_n$$

where A *is a constant. Then the Volterra functional equation* (2.3) *has a solution* $x \in C_w$ *for every* $y \in C_w$. *Moreover the solution* x *is unique in* C_w.

Theorem II is based on Lemma 7 which is based on Lemmas 5 and 6.

4.1. LEMMA 5. *Let* F^1, F^2, $\cdots$ *together with their first derivatives with respect to* $u, v_1, v_2, \cdots$ *be continuius on* $I \otimes R_k$ $(k = 1, 2, \cdots)$. *Let* a *and* b *satisfy* $0 \leqq a \leqq b \leqq 1$, *let* $\xi(t)$ *be an arbitrary continuous function defined on* $0 \leqq t \leqq a$ *and let* U *be an arbitrary constant satisfying* $U \geqq |\xi(a)|$. *Let* $\mathfrak{M}(a, b, \xi, U)$ *be the totality of functions* $x(t)$ *defined and continuous on* $0 \leqq t \leqq b$ *and satisfying*

$$(4.2) \qquad x(t) = \xi(t) \qquad on\ 0 \leqq t \leqq a,$$

$$(4.3) \qquad |x(t)| \leqq U \qquad on\ a \leqq t \leqq b.$$

Then there exists a sequence of constants B_1, B_2, $\cdots$ *such that*

$$(4.4) \qquad |\Lambda^k(x_2 | t) - \Lambda^k(x_1 | t)| \leqq B_k \int_a^t |x_2(s) - x_1(s)|\, ds, \qquad (k = 1, 2, \cdots)$$

hold for all t *in* $a \leqq t \leqq b$ *and any pair of elements* x_1, x_2 *of* $\mathfrak{M}$.

We remark that the conclusion of Lemma 5 follows by complete induction on k using boundedness of $\partial F^k / \partial v_j$.

REMARK 3. Consider $\mathfrak{M}(a, b_1, \xi, U)$ and $\mathfrak{M}(a, b_2, \xi, U)$ with $0 \leqq a \leqq b_1 \leqq b_2$ $\leqq 1$. If B_1, B_2, $\cdots$ is a sequence of constants with which (4.4) holds for all t in $a \leqq t \leqq b_2$ for any $x_1, x_2 \in \mathfrak{M}(a, b_2, \xi, U)$, then (4.4) holds with this sequence for all t in $a \leqq t \leqq b_1$ for any $x_1, x_2 \in \mathfrak{M}(a, b_1, \xi, U)$.

4.2. LEMMA 6. *Let* F^1, F^2, $\cdots$, F^n *and their first derivatives with respect to* $u, v_1, \cdots, v_{n-1}$ *be continuous on* $I \otimes R_k$ $(k = 1, 2, \cdots, n)$. *Let* a *satisfy* $0 \leqq a < 1$ *and let* $\xi(t)$ *be an arbitrary function defined and continuous on* $0 \leqq t \leqq a$. *Then for any function* $\eta(t)$ *defined and continuous on* $a \leqq t \leqq 1$, *there exists a function* $x(t)$ *which is continuous on* $0 \leqq t \leqq b$ *for some* b *such that* $a < b \leqq 1$ *and satisfies*

$$(4.5) \qquad x(t) = \xi(t) \qquad on\ 0 \leqq t \leqq a,$$

$$(4.6) \qquad \eta(t) - \eta(a) = x(t) - x(a) + [\Lambda^n(x | s)]_a^t \qquad for\ a \leqq t \leqq b,$$

where

$$[\Lambda^n(x\,|\,s)]_a^t = \Lambda^n(x\,|\,t) - \Lambda^n(x\,|\,a) = \int_a^t F^n[s,\,\Lambda^0(x\,|\,s),\,\cdots,\,\Lambda^{n-1}(x\,|\,s)]ds.$$

Proof. We define a sequence of continuous functions $\{x_p(t)\}$ on $0 \leq t \leq 1$ by

(4.7) $x_0(t) = \xi(t),$ on $0 \leq t \leq a,$ $x_0(t) = \xi(a)$ on $a \leq t \leq 1.$

(4.8)
$$x_p(t) = \xi(t) \qquad\qquad\qquad \text{on } 0 \leq t \leq a,\ (p = 1, 2, \cdots),$$
$$x_p(t) = x_p(a) + \eta(t) - \eta(a) - [\Lambda(x_{p-1}\,|\,s)]_a^t$$
$$\text{on } a \leq t \leq 1,\ (p = 1, 2, \cdots).$$

To prove that $\{x_p(t)\}$ converges uniformly on $a \leq t \leq b$ for some b on $a < b \leq 1$, we prove that for a suitably chosen b in the above interval

(4.9) $$|x_{p+1}(t) - x_p(t)| \leq \frac{B_n^p t^p M}{p!}, \qquad \text{for } a \leq t \leq b,\ (p = 0, 1, 2, \cdots),$$

where B_n, M are non-negative numbers.

To do this let U be an arbitrary positive number satisfying $U > |\xi(a)|$ and let b_1 be such that $a < b_1 \leq 1$. Then according to Lemma 5, there exists a constant B_n such that

(4.10) $$|\Lambda^n(x_l\,|\,t) - \Lambda^n(x_m\,|\,t)| \leq B_n \int_a^t |x_l(s) - x_m(s)|\,ds$$

for all t on $a \leq t \leq b_1$ and any pair $x_l,\ x_m \in \mathfrak{M}(a, b_1, \xi, U)$. Let β be a variable with the domain $a \leq \beta \leq b_1$ and consider the continuous function of β, $\phi(\beta)$ $= \max_{a \leq t \leq \beta} |x_1(t) - \xi(a)|$. From the continuity of $x_1(t)$ on I and from $x_1(a)$ $= \xi(a)$, we have $\lim_{\beta \to a+} \phi(\beta) = 0$. On the other hand since $U > |\xi(a)|$, we have $\{U - |\xi(a)|\}/\exp\{B_n\} > 0$. Therefore there is a number b such that $a < b \leq b_1$ and

(4.11) $$M \equiv \phi(b) \leq \{U - |\xi(a)|\}/\exp\{B_n\}.$$

With this number b consider $\mathfrak{M}(a, b, \xi, U)$. By Remark 3, the inequality (4.10) holds with same B_n for all t on $a \leq t \leq b$ and any pair $x_l, x_m \in \mathfrak{M}(a, b, \xi, U)$. Now we prove (4.9) by complete induction on p. When $p = 0$, (4.9) holds by (4.7) and the definition of M. Assume that (4.9) holds for $p \leq q$. Then for any $p \leq q$, by (4.11)

$$|x_{p+1}(t) - \xi(a)| \leq M \sum_{k=0}^p \frac{B_n^k t^k}{k!} \leq M \exp\{B_n\} \leq U - |\xi(a)|, \quad \text{for } a \leq t \leq b$$

so that $|x_{p+1}(t)| \leq U$ for $a \leq t \leq b$ and in particular for $p = q-1$, q, $|x_q(t)|$, $|x_{q+1}(t)| \leq U$ for $a \leq t \leq b$ and hence $x_q,\ x_{q+1} \in \mathfrak{M}(a, b, \xi, U)$. Then for $p = q+1$,

by (4.8), (4.10) and the induction hypothesis, (4.9) holds. By complete in-
duction on p, (4.9) is now established. Then by Cauchy's criterion, $x(t)$
$= \lim_{p \to \infty} x_p(t)$ exists uniformly on $a \leq t \leq b$. By the definition of $\{x_p(t)\}$, $x(t)$
is continuous on $0 \leq t \leq 1$ and $x(t) = \xi(t)$ on $0 \leq t \leq a$. Also by (4.8)

$$x(t) = \lim_{p \to \infty} x_p(t) = x(a) + \eta(t) - \eta(a) - [\Lambda^n(x \mid s)]_a^t \quad \text{on } a \leq t \leq b,$$

where the passing to the limit of p under the $n-1$ integral signs involved in
$\Lambda^n(x \mid t)$ is justified by the uniform convergence of $\{x_p(t)\}$ and the con-
tinuity of $F^1, F^2, \cdots, F^n$ according to Lemma 2.

4.3. LEMMA 7. *Let $F^1, F^2, \cdots, F^n$ and their first derivatives with respect
to $u, v_1, \cdots, v_{n-1}$ be continuous on $I \otimes R_k$ ($k = 1, 2, \cdots, n$). Then for every
$y \in C_w$, the Volterra functional equation (2.3) either has a solution $x \in C_w$ or
else it is satisfied by a function $x_\infty(t)$ which is defined and continuous on some
interval $0 \leq t < t_\infty$ with $0 < t_\infty \leq 1$, vanishes at $t = 0$, and becomes unbounded as t
approaches t_∞ from the left.*

Proof. Let $y \in C_w$ be given. Suppose there does not exist an element of C_w
that satisfies (2.3) on I for the given y. We prove the lemma by constructing
$x_\infty(t)$ by applying Lemma 6. We shall construct a solution on an interval
$[0, t_1]$ with $0 < t_1 \leq 1$ and then extend this solution on an interval $[0, t_2]$ with
$t_1 < t_2 \leq 1$ and so forth. Since the number b in Lemma 6 is not unique, the
choice of $t_1, t_2, \cdots$ at each step will not be unique. We shall use the super-
script α for a sequence of choices of $t_1, t_2. \cdots$.

Now let $a = 0$ and $\xi(t)$ be defined on the closed interval $t = 0$ by $\xi(0) = 0$
and let $\eta(t) = y(t)$ on $0 \leq t \leq 1$. By Lemma 6, there exists a function $x_1^\alpha(t)$
which is continuous on some interval $0 \leq t \leq t_1^\alpha$ with $0 < t_1^\alpha \leq 1$ and satisfies

$$x_1^\alpha(0) = 0, \qquad y(t) = x_1^\alpha(t) + [\Lambda^n(x_1^\alpha \mid s)]_0^t \qquad \text{on } 0 \leq t \leq t_1^\alpha.$$

We apply Lemma 6 again by setting $a = t_1^\alpha$, $\xi(t) = x_1^\alpha(t)$ on $0 \leq t \leq t_1^\alpha$ and $\eta(t)$
$= y(t)$ on $t_1^\alpha \leq t \leq 1$, and obtain a function $x_2^\alpha(t)$ which is continuous on some
interval $0 \leq t \leq t_2^\alpha$ with $t_1^\alpha < t_2^\alpha \leq 1$ and satisfies

$$x_2^\alpha(t) = x_1^\alpha(t) \qquad \text{on } 0 \leq t \leq t_1^\alpha,$$

and

$$y(t) - y(t_1^\alpha) = x_2^\alpha(t) - x_2^\alpha(t_1^\alpha) + [\Lambda^n(x_2^\alpha \mid s)]_{t_1^\alpha}^t \qquad \text{on } t_1^\alpha \leq t \leq t_2^\alpha.$$

We repeat the application of Lemma 6. This process will not be terminated
since $t = 1$ will never be reached as we shall see below. Now, in general
$x_m^\alpha(t)$ is continuous on $[0, t_m^\alpha]$ with $0 < t_1^\alpha < t_2^\alpha < \cdots < t_m^\alpha \leq 1$, vanishes at $x = 0$
and satisfies

(4.12) $x_1(0) = 0,$ $x_k(t) = x_{k-1}(t)$ on $[0, t_{k-1}]$ for $k = 2, 3, \cdots, m.$

$$y(t) = \overset{\alpha}{x_1}(t) + [\Lambda^n(\overset{\alpha}{x_1} \mid s)]_0^t \qquad \text{on} [0, \overset{\alpha}{t_1}],$$

(4.13) $y(t) - y(\overset{\alpha}{t_{k-1}}) = \overset{\alpha}{x_k}(t) - \overset{\alpha}{x_k}(\overset{\alpha}{t_{k-1}}) + [\Lambda^n(\overset{\alpha}{x_k} \mid s)]_{t_{k-1}}^{t_\alpha}$

on $[\overset{\alpha}{t_{k-1}}, \overset{\alpha}{t_k}]$ for $k = 2, 3, \cdots, m.$

To show that $x_m^\alpha(t)$ satisfies (2.3) on $[0, t_m^\alpha]$ we set, for any $k \leq m, t = t_1, t_2, \cdots,$ t_{k-1} respectively in the first $k-1$ equations of (4.13) and add these $k-1$ equations and the kth equation of (4.13) side by side. Thus we see that $x_k^\alpha(t)$ satisfies (2.3) on $[t_{k-1}^\alpha, t_k^\alpha]$ for $k \leq m$ and then, since $x_m(t) = x_k(t)$ on $0 \leq t \leq t_k$ for $k \leq m$, $x_m^\alpha(t)$ satisfies (2.3) on $[0, t_m^\alpha]$. From this it follows also that $t_m^\alpha \neq 1$ for any positive integer m and the sequence $\{x_m^\alpha(t)\}$ is an infinite sequence. Now since $0 < t_1^\alpha < t_2^\alpha < \cdots < 1$, let $t_\infty^\alpha = \lim_{m \to \infty} t_m^\alpha \leq 1$. We now construct a function $x_\infty^\alpha(t)$ which is continuous on $[0, t_\infty^\alpha)$, vanishes at $t = 0$ and satisfies (2.3) on $[0, t_\infty^\alpha)$. For any $\tau \in [0, t_\infty^\alpha)$, there is a positive integer M such that $t_{M-1}^\alpha < \tau \leq t_M^\alpha$. Then $x_M^\alpha(\tau) = x_{M+1}^\alpha(\tau) = x_{M+2}^\alpha(\tau) = \cdots$. Let $x_\infty^\alpha(\tau)$ be this value which is independent of m for $m \geq M$. Then a function $x_\infty^\alpha(t)$ is uniquely defined on $[0, t_\infty^\alpha)$. Since $x_\infty^\alpha(t) = x_m^\alpha(t)$ on $[0, t_m^\alpha]$ for all m, $x_\infty^\alpha(t)$ is continuous on $[0, t_\infty^\alpha)$, vanishes at $t = 0$, and satisfies (2.3) on $[0, t_\infty^\alpha)$.

Now consider the collection $\{x_\infty^\alpha(t)\}$ each of which is constructed in the manner described above. Since $t_\infty^\alpha \leq 1$ for all α, let $t_\infty \equiv \sup_\alpha t_\infty^\alpha \leq 1$. To construct a function $x_\infty(t)$ which is continuous on $[0, t_\infty)$, vanishes at $t = 0$ and satisfies (2.3) on $[0, t_\infty)$. Let $\tau \in [0, t_\infty)$. Consider all values of α for which $\tau < t_\infty^\alpha$. For these values of α, the functions $x_\infty^\alpha(t)$ are continuous on $[0, \tau]$, vanish at $t = 0$ and satisfy (2.3) on $[0, \tau]$. Then by uniqueness of solution of (2.3) on $[0, \tau]$ according to Remark 1, the value of $x_\infty^\alpha(\tau)$ is independent of α. Let $x_\infty(\tau)$ be $x_\infty^\alpha(\tau)$ which is independent of α for such α that $\tau < t_\infty^\alpha$. Then a function $x_\infty(t)$ is uniquely defined on $[0, t_\infty)$. It has the required properties.

To prove that $x_\infty(t)$ becomes unbounded as t approaches t_∞ from the left: Assume that $x_\infty(t) = \Lambda^0(x_\infty \mid t)$ is bounded on $[0, t_\infty)$. If $\Lambda^k(x_\infty \mid t)$ is bounded on $[0, t_\infty)$ for $k \leq N$, it follows from the continuity of F^{N+1} on $I \otimes R_{N+1}$ that $\Lambda^{N+1}(x_\infty \mid t)$ is also bounded on $[0, t_\infty)$. By complete induction on k, $\Lambda^k(x_\infty \mid t)$, $(k = 0, 1, 2, \cdots, n)$ are all bounded on $[0, t_\infty)$. Let V be a positive number such that

(4.14) $|\Lambda^k(x_\infty \mid t)| \leq V$ on $0 \leq t < t_\infty$ for $k = 0, 1, 2, \cdots, n.$

Let the sequence of functions $F^{k*}(t, u, v_1, \cdots, v_{k-1})$, $(k = 1, 2, \cdots, n)$ be so defined that $F^{k*} = F^k$ if $|v_j| \leq V$ for all $j = 0, 1, 2, \cdots, k-1$, for $k = 1, 2, \cdots, n$ and beyond the above k-dimensional rectangle F^{k*} is continued as constant in the way F^{k*} was defined in the proof of Theorem I. The sequence of functionals $\Lambda^{k*}(x \mid t)$, $(k = 0, 1, 2, \cdots, n)$ is constructed with F^{k*}, $(k = 1, 2, \cdots, n)$ by (2.0), (2.1), (2.2). Then $x_\infty(t)$ is a solution of $y(t) = x(t) + \Lambda^{n*}(x \mid t)$ on $0 \leq t < t_\infty$. On the other hand, this modified functional

equation has a solution $\bar{x} \in C_w$ by Lemma 3. According to Remark 2, $x_\infty(t)$ $= \bar{x}(t)$ on $[0, t_\infty)$. Let us now extend $x_\infty(t)$ to $t = t_\infty$ by defining $x_\infty(t_\infty) = \bar{x}(t_\infty)$. Then $x_\infty(t)$ satisfies the modified functional equation on $[0, t_\infty]$. Now since (4.14) holds not only for $[0, t_\infty)$ but also for $[0, t_\infty]$, $x_\infty(t)$ satisfies (2.3) on $[0, t_\infty]$. Thus if $t_\infty = 1$, we have a solution $x_\infty \in C_w$ for (2.3), which is a contradiction. But if $t_\infty < 1$, then by applying Lemma 6 we can extend $x_\infty(t)$ beyond t_∞, which contradicts the definition of t_∞. Therefore $x_\infty(t)$ must become unbounded as t approaches t_∞ from the left.

4.4. Proof of Theorem II. Suppose that for some $y \in C_w$ there is no $x \in C_w$ that satisfies (2.3). Then by Lemma 7, there is a function $x(t)$ which vanishes at $t = 0$, is continuous and satisfies (2.3) on some interval $0 \le t < t_\infty$ with $0 < t_\infty \le 1$ and becomes unbounded as t approaches t_∞ from the left.

If $x(t)$ is such that $x(t) > 0$ on some interval $t_0 < t < t_\infty$, then by (2.3), (2.2), (2.1), (2.0), (4.1), $y(t) - y(t_0) \ge x(t) - x(t_0) - A(t - t_0)$ for $t_0 < t < t_\infty$. Similarly, if $x(t) < 0$ on $t_0 < t < t_\infty$ then $y(t) - y(t_0) \le x(t) - x(t_0) + A(t - t_0)$ for $t_0 < t < t_\infty$. In any case $y(t)$ becomes unbounded as t approaches t_∞ from the left. This contradicts the continuity of $y(t)$ at t_∞.

Let us assume that for any t_0 satisfying $0 < t_0 < t_\infty$ there is always a point t such that $t_0 < t < t_\infty$ and $x(t) = 0$. Let $\{t_m\}$ be an increasing sequence of points such that $\lim_{m \to \infty} t_m = t_\infty$ and $x(t_m) = 0$. Let t'_m be so chosen that $t_m < t'_m < t_\infty$ and $|x(t'_m)| \ge m$. Let t''_m be the least upper bound of all points between t_m and t'_m at which $x(t)$ vanishes. Then $x(t) \ne 0$ on $t''_m < t \le t'_m$ and $x(t''_m) = 0$ by the continuity of $x(t)$. Now if $x(t) > 0$ on $t''_m < t \le t'_m$ then $y(t'_m) \ge m$ and by (2.3), (2.2), (2.1), (2.0), (4.1), $y(t'_m) - y(t''_m) \ge m - A(t'_m - t''_m)$. Similarly if $x(t) < 0$ on $t''_m < t \le t'_m$ then $y(t'_m) \le -m$ and $y(t'_m) - y(t''_m) \le -m + A(t'_m - t''_m)$. Since $\lim_{m \to \infty} t'_m = \lim_{m \to \infty} t''_m = t_\infty$, the two inequalities contradict the continuity of $y(t)$ at t_∞. This completes the proof of Theorem II.

5. Relationship between the functional equations and the differential systems. Let Δ_0 be the triangle $0 \le s \le t \le t_0$ and I_0 be the interval $0 \le t \le t_0$ and let $J^1(t, u), J^2(t, u, v_1), \cdots, J^n(t, u, v_1, \cdots, v_{n-1})$ be defined and continuous on $I_0 \otimes R_n$ and $y(t)$ be defined and continuous on I_0. Then let the functionals $\Omega^k, \Psi^k, \tilde{F}^k, \tilde{\Phi}^k, \tilde{\Lambda}^k$ be defined by

$$(5.1) \qquad \Omega^0(y \mid s, t) = y(t - s) \qquad \text{on } \Delta_0.$$

$$(5.2) \quad \Psi^k(y \mid s, t, u, v_1, \cdots, v_{k-1}) = J^k[s, v_0 + \Omega^0, \cdots, v_{k-1} + \Omega^{k-1}]$$
$$\text{on } \Delta_0 \otimes R_k, (k = 1, 2, \cdots, n).$$

$$(5.3) \quad \Omega^k(y \mid s, t, u, v_1, \cdots, v_{k-1}) \int_s^t \Psi^k(p) dp \quad \text{on } \Delta_0 \otimes R_k, (k = 1, 2, \cdots, n).$$

$$\tilde{F}^k(t', u', v'_1, \cdots, v'_{k-1}, t, u, v_1, \cdots, v_{k-1})$$
$$(5.4) \qquad = tJ^k[(1 - t')t, u + t^{1/2}u', v_1 + v'_1, \cdots, v_{k-1} + v'_{k-1}],$$
$$(k = 1, 2, \cdots, n).$$

$$\tilde{\Phi}^k(x \mid t', t, u, v_1, \cdots, v_{k-1}) = t\Psi^k\left\{t^{1/2}x\left[\frac{(\cdot)}{t}\right] \mid (1 - t')t\right\},$$
(5.5)

$$(k = 1, 2, \cdots, n).$$

$$\tilde{\Lambda}^k(x \mid t', t, u, v_1, \cdots, v_{k-1}) = \Omega^k\left\{t^{1/2}x\left[\frac{(\cdot)}{t}\right] \mid (1 - t')t\right\},$$
(5.6)

$$(k = 1, 2, \cdots, n).$$

(5.7)
$$\tilde{\Lambda}^0(x \mid t', t) = x(t').$$

We quote results of [1] in the following remarks.

REMARK 4. If we regard the variables t, u, $v_1, \cdots, v_{n-1}$ in $\tilde{F}^k$, $\tilde{\Phi}^k$, Λ^k as parameters, then the recurrence relationships (2.0), (2.1), (2.2) hold for $\tilde{F}^k$, $\tilde{\Phi}^k$, $\tilde{\Lambda}^k$ with the variables t', u', v_1', $\cdots$, v_{n-1}'.

REMARK 5. Let $J^1(t, u)$, $J^2(t, u, v_1)$, $\cdots$, $J^n(t, u, v_1, \cdots, v_{n-1})$, $L(u, v_1, \cdots, v_{n-1})$ and the first derivatives of $J^1, J^2, \cdots, J^{n-1}$ with respect to $u, v_1, \cdots, v_{n-2}$ be continuous on $I_0 \otimes R_n$. Let $\tilde{F}^k$, $\tilde{\Phi}^k$, $\tilde{\Lambda}^k$ be defined by (5.4), (5.5), (5.6), (5.7) with $J^1, J^2, \cdots, J^n$, let $\tilde{G}$ be defined by

(5.8)
$$\tilde{G}(t', u', v_1', \cdots, v_{n-1}', t, u, v_1, \cdots, v_{n-1})$$
$$= \theta[(1 - t')t, u + t^{1/2}u', v_1 + v_1', \cdots, v_{n-1} + v_{n-1}']$$

for $0 \leq t' \leq 1$ and v_j' $(j = 0, 1, 2, \cdots, n-1)$ unrestricted, and finally, let $\tilde{g}, \tilde{f}$ be defined by

(5.9) $\quad \tilde{g}(t', u', v_1', \cdots, v_{n-1}') = -\dfrac{1}{2}\log \tilde{G}(t', u', v_1', \cdots, v_{n-1}') \quad$ on $I \otimes R_n$.

(5.10) $\quad \tilde{f}(t', u', v_1', \cdots, v_{n-1}') = \tilde{g}_{u'} \quad\quad\quad\quad\quad$ on $I \otimes R_n$.

Then by Remark 4 and [1, Theorem 1], the transformation

(5.11)
$$y(t') = x(t') + \int_0^{t'} \tilde{f}[s', \tilde{\Lambda}^0(s'), \cdots, \tilde{\Lambda}^{n-1}(s')]ds', \quad\quad t' \in I,$$

maps C_w in a 1-1 manner into a Wiener measurable subset Γ of C_w. Furthermore if there exists a positive solution θ of (1.1), (1.2) such that all its first derivatives and the first derivatives of θ_u with respect to $v_1, v_2, \cdots, v_{n-1}$ are continuous on $I_0 \otimes R_n$, then

(5.12)
$$\theta(t, u, v_1, \cdots, v_{n-1})m_w(\Gamma)$$
$$= \int_{C_w} \exp\{\Omega^n(0)\}L[v_0 + \Omega^0(0), \cdots, v_{n-1} + \Omega^{n-1}(0)]d_w y$$

on $I_0 \otimes R_n$.

6. Minimality theorems. 6.1. A minimality theorem derived from Theorem I.

THEOREM III. *Let $J^1, J^2, \cdots, J^n, L$ and the first derivatives of $J^1, J^2, \cdots, J^{n-1}$ with respect to $u, v_1, \cdots, v_{n-2}$ be continuous on $I_0 \otimes R_n$. Let $J^1, J^2, \cdots, J^{n-1}$ satisfy the order of growth conditions*

$$
(6.1) \quad
\begin{aligned}
&\big| J^k(t, u, v_1, \cdots, v_{k-1}) \big| \\
&\quad \leqq A \Big\{ 1 + \sum_{j=0}^{k-1} |v_j| \Big\} \log \Big\{ 2 + \sum_{j=0}^{k-1} |v_j| \Big\} \qquad \text{on } I_0 \otimes R_k,
\end{aligned}
$$

$$(k = 1, 2, \cdots, n-1)$$

where A is a positive number. Suppose that there exists a positive solution θ of (1.1), (1.2) such that all its first derivatives and the first derivatives of θ_u with respect to $v_1, v_2, \cdots, v_{n-1}$ are continuous on $I_0 \otimes R_n$ and furthermore

$$
(6.2) \quad \frac{\big| \theta_u(t, u, v_1, \cdots, v_{n-1}) \big|}{\theta(t, u, v_1, \cdots, v_{n-1})} \leqq A \Big\{ 1 + \sum_{j=0}^{n-1} |v_j| \Big\} \log \Big\{ 2 + \sum_{j=0}^{n-1} |v_j| \Big\}
$$

$$\text{on } I_0 \otimes R_n,$$

then it follows that θ is the minimal positive solution with continuous first derivatives of (1.1), (1.2) on $I_0 \otimes R_n$.

Proof. We prove that for any given point $(t, u, v_1, \cdots, v_{n-1}) \in I_0 \otimes R_n$, $m_w(\Gamma) = 1$ in (5.12). Let $A' \geqq \max \{1, \log (1 + (t_0)^{1/2})\}$. Then from (5.4), (6.1)

$$
\begin{aligned}
&\big| \tilde{F}^k(t', u', v_1', \cdots, v_{k-1}', t, u, v_1, \cdots, v_{k-1}) \big| \\
&\quad \leqq t_0(1 + (t_0)^{1/2}) A \Big\{ 1 + \sum_{j=0}^{k-1} |v_j| + \sum_{j=0}^{k-1} |v_j'| \Big\} \\
&\qquad \cdot \log \Big\{ (1 + (t_0)^{1/2}) \Big(2 + \sum_{j=0}^{k-1} |v_j| + \sum_{j=0}^{k-1} |v_j'| \Big) \Big\} \\
&\quad \leqq 2t_0(1 + (t_0)^{1/2}) A A' \Big\{ 1 + \sum_{j=0}^{k-1} |v_j| + \sum_{j=0}^{k-1} |v_j'| \Big\} \\
&\qquad \cdot \log \Big\{ 2 + \sum_{j=0}^{k-1} |v_j| + \sum_{j=0}^{k-1} |v_j'| \Big\}, \qquad (k = 1, 2, \cdots, n-1).
\end{aligned}
$$

Let us remark that for any two real numbers a and x

$$
\begin{aligned}
(6.3) \quad &(1 + |a| + |x|) \log (2 + |a| + |x|) \\
&\quad < 4(1 + |x|) \log (2 + |x|) + (1 + 2|a|) \log (2 + 2|a|) \\
&\quad < 4[1 + (1 + 2|a|) \log (2 + 2|a|)](1 + |x|) \log (2 + |x|).
\end{aligned}
$$

Then

$$|\tilde{F}^k| \leqq 8t_0(1 + (t_0)^{1/2}) A A' A_k''(u, v_1, \cdots, v_{k-1})$$

(6.4)

$$\left\{1 + \sum_{j=0}^{k-1} |v_j'|\right\} \log \left\{2 + \sum_{j=0}^{k-1} |v_j'|\right\}$$

where

(6.5)

$$A_k''(u, v_1, \cdots, v_{k-1}) = 1 + \left(1 + 2\sum_{j=0}^{k-1} |v_j|\right) \log\left(2 + 2\sum_{j=0}^{k-1} |v_j|\right),$$

$$(k = 1, 2, \cdots, n - 1).$$

Also from (5.10), (5.9), (5.8)

(6.6) $$\tilde{f} = -\frac{1}{2} \frac{t^{1/2}\theta_u[(1 - t')t, u + t^{1/2}u', v_1 + v_1', \cdots, v_{n-1} + v_{n-1}']}{\theta[(1 - t')t, u + t^{1/2}u', v_1 + v_1', \cdots, v_{n-1} + v_{n-1}']}$$

and by (6.2), (6.3)

$$|\tilde{f}(t', u', v_1', \cdots, v_{n-1}', t, u, v_1, \cdots, v_{n-1})|$$

(6.7)

$$\leqq 4(t_0)^{1/2}(1 + (t_0)^{1/2}) A A' A_n''(u, v_1, \cdots, v_{n-1})$$

$$\cdot \left\{1 + \sum_{j=0}^{n-1} |v_j'|\right\} \log \left\{2 + \sum_{j=0}^{n-1} |v_j'|\right\}$$

where $A_n''(u, v_1, \cdots, v_{n-1})$ is defined by (6.5) with $k = n$.

By (6.4), (6.7), the functions $\tilde{F}^k$, $\tilde{f}^k$ of the variables t', u', v_1', $\cdots$, v_{n-1}' satisfy the conditions of Theorem I for any given parametric values t, u, v_1, $\cdots$, v_{n-1} and according to the conclusion of Theorem I, the transformation (5.11) maps C_w in a 1-1 manner onto itself so that $m_w(\Gamma) = 1$ in (5.12). Since t, u, v_1, $\cdots$, v_{n-1} are arbitrarily given, $m_w(\Gamma) = 1$ holds on $I_0 \otimes R_n$. By [1, Theorem 4], the proof is completed.

6.2. A minimality theorem derived from Theorem II.

LEMMA 8. *Under the hypothesis of Remark 5 on J^k, L and θ, assume further that there are numbers $t^* \in I_0$ and $u^* \in R_1$ such that for some constant A*

(6.8)

$$\frac{\theta_u[t, u^* + \Delta u, v_1, \cdots, v_{n-1}]}{\theta[t, u^* + \Delta u, v_1, \cdots, v_{n-1}]} \text{ sgn } [\Delta u] \leqq A$$

holds for $0 \leqq t \leqq t^*$, Δu and v_j $(j = 1, 2, \cdots, n-1)$ unrestricted, then $m_w(\Gamma) = 1$ in (5.12) on the set $[0, t^*] \otimes \{u^*\} \otimes R_{n-1}$.

Proof. For $0 \leqq t \leqq t^*$, Δu and v_j $(j = 1, 2, \cdots, n-1)$ unrestricted, we have from (6.6), (6.8)

$$\tilde{f}(t', u', v_1', \cdots, v_{n-1}', t, u^*, v_1, \cdots, v_{n-1}) \text{ sgn } u' \geqq -\frac{1}{2} (t^*)^{1/2} A$$

for $(t', u', v_1', \cdots, v_{n-1}') \in I \otimes R_n$. Then for the parametric values in the set

$[0, t^*] \otimes \{u^*\} \otimes R_{n-1}$, the functions $\tilde{F}^1, \tilde{F}^2, \cdots, \tilde{F}^{n-1}, \tilde{f}$ satisfy the conditions of Theorem II and according to its conclusion, the transformation (5.11) maps C_w in a 1-1 manner onto itself so that $m_w(\Gamma) = 1$ in (5.12) on the set $[0, t^*] \otimes \{u^*\} \otimes R_{n-1}$.

THEOREM IV. *Let $J^1, J^2, \cdots, J^n, L$ and the first derivatives of $J^1, J^2, \cdots, J^{n-1}$ with respect to $u, v_1, \cdots, v_{n-2}$ be continuous on $I_0 \otimes R_n$. If $n > 1$, let $J^1, J^2, \cdots, J^{n-1}$ be pervasive[7] in I_0. Suppose that there exists a positive solution θ of (1.1), (1.2) such that all its first derivatives and the first derivatives of θ_u with respect to $v_1, v_2, \cdots, v_{n-1}$ are continuous on $I_0 \otimes R_n$. If for some constant A,*

$$(6.9) \qquad \frac{\theta_u[t, u, v_1, \cdots, v_{n-1}]}{\theta[t, u, v_1, \cdots \quad v_{n-1}]} \, \mathrm{sgn}\, u \leqq A \qquad on\ I_0 \otimes R_n,$$

then θ is the minimal positive solution with continuous first derivatives of (1.1), (1.2).

Proof. If (6.9) holds then (6.8) holds with $t^* = t_0$, $u^* = 0$ and by the conclusion of Lemma 8, $m_w(\Gamma) = 1$ in (5.12) on the set $I_0 \otimes \{0\} \otimes R_{n-1}$. By [1, Theorem 6] the proof of the theorem is completed.

6.3. Another minimality theorem based on Theorem II.

LEMMA 9. *Under the hypothesis of Remark 5 on J^k, L and θ, assume further that there is a non-negative number A such that*

$$(6.10) \qquad \frac{|\theta_u[t, u, v_1, \cdots, v_{n-1}]|}{\theta[t, u, v_1, \cdots, v_{n-1}]} \leqq A \qquad on\ I_0 \otimes R_n,$$

then $m_w(\Gamma) = 1$ in (5.12) on $I_0 \otimes R_n$.

Proof. If (6.10) holds on $I_0 \otimes R_n$, then from (6.6)

$$\left| \tilde{f}(t', u', v_1', \cdots, v_{n-1}', t, u, v_1, \cdots, v_{n-1}) \right| \leqq \frac{1}{2} (t_0)^{1/2} A \quad on\ I \otimes R_n,$$

for $(t, u, v_1, \cdots, v_{n-1}) \in I_0 \otimes R_n$. Therefore $\tilde{F}^1, \tilde{F}^2, \cdots, \tilde{F}^{n-1}, \tilde{f}$ satisfy the hypothesis of Theorem II for any choice of the parametric values $t, u, v_1, \cdots, v_{n-1}$ in $I_0 \otimes R_n$ and by the conclusion of the theorem, the transformation (5.11) maps C_w in a 1-1 manner onto itself so that $m_w(\Gamma) = 1$ in (5.12) on $I_0 \otimes R_n$.

THEOREM V. *Let $J^1, J^2, \cdots, J^n, L$ and the first derivatives of $J^1, J^2, \cdots, J^{n-1}$ with respect to $u, v_1, \cdots, v_{n-2}$ be continuous in $I_0 \otimes R_n$. Suppose that there exists a positive solution θ of (1.1), (1.2) such that all its first derivatives and the first derivatives of θ_u with respect to $v_1, v_2, \cdots, v_{n-1}$ are continuous on $I_0 \otimes R_n$. Then if θ_u/θ is bounded on $I_0 \otimes R_n$, it follows that θ is the minimal positive solution with continuous first derivatives of (1.1), (1.2).*

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(7) See [1, p. 162].

Proof. By Lemma 9, $m_w(\Gamma) = 1$ in (5.12) on $I_0 \otimes R_n$. By [1, Theorem 4], the proof of the theorem is completed.

7. **A uniqueness theorem for solutions of the differential system.** Let the functions H^k and the functionals $\tilde{H}^k$ ($k = 1, 2, \cdots$) be defined by

$$(7.1) \quad H^k(A, B, M, \epsilon, u, v_1, \cdots, v_{k-1}) = A \exp\{B \, | \, u \, |^{2-\epsilon}\} \prod_{j=1}^{k-1} \{1 + |v_j|\}^M,$$

$$(7.2) \quad \tilde{H}^k(y \,|\, A, B, M, \epsilon, u, v_1, \cdots, v_{k-1}) = H^k \exp\left\{B \max_{\rho \in I_0} |y(\rho)|^{2-\epsilon}\right\}$$

where A, B, $M > 0$, $2 > \epsilon > 0$, $y(\rho)$ is continuous in $\rho \in I_0$ and it is understood that when $k = 1$, the product $\prod_{j=1}^{k-1}$ is equal to 1. The notational convention for order of growth conditions involving H^k is that $J^k(t, u, v_1, \cdots, v_{k-1}) \leqq H^k(A, B, M, \epsilon)$ would mean that J^k is bounded by a function H^k of the type (7.1) with the specific constants A, B, M, ϵ on $I_0 \otimes R_n$ whereas $J^k(t, u, v_1, \cdots, v_{k-1}) \leqq H^k$ would mean that J^k is bounded on $I_0 \otimes R_n$ by a function H^k of the type (7.1) with unspecified constants.

For a function $L(u, v_1, \cdots, v_{n-1})$ defined on R_n, let its "translation transform" $T_t^0 L$ be defined by

$$(7.3) \quad T_t^s L = \int_{C_w[0, t-s]} \exp\{\Omega^n\} L[v_0 + \Omega^0, \cdots, v_{n-1} + \Omega^{n-1}] d_w y \quad 0 \leqq s \leqq t \leqq t_0,$$

provided it exists, and for $t = s$

$$(7.4) \qquad\qquad\qquad\qquad T_s^s L = L, \qquad\qquad\qquad\qquad \text{for } s \in I_0.$$

$T_t^s L$ defined in this fashion with given $J^1, J^2, \cdots, J^n$ by (5.1), (5.2), (5.3) is a function defined on $I_0 \otimes R_n$.

We quote [1, Theorem 8] on which our uniqueness theorem, Theorem VI, will be based.

THEOREM 8 OF [1]. *Let $J^1, J^2, \cdots, J^n$, L and their first and second derivatives with respect to u, $v_1, \cdots, v_{n-1}$ (except possibly for J_{uu}^k) be continuous and satisfy in $I_0 \otimes R_n$ the order of growth conditions*

$$(7.5) \quad \begin{aligned} |J^k| &\leqq H^k, \quad |J_j^k| \leqq H^k, (k = 1, 2, \cdots, n), (j = 0, 1, \cdots, n-1), \\ |J_{ij}^k| &\leqq H^k, \qquad\qquad (k = 1, 2, \cdots, n), (i = 1, 2, \cdots, n-1), \\ &\qquad\qquad\qquad\qquad\qquad (j = 0, 1, \cdots, n-1). \end{aligned}$$

$$(7.6) \qquad\qquad\qquad\qquad J^n \leqq \log H^n.$$

$$(7.7) \quad |L| \leqq H^n, \quad |L_j| \leqq H^n, \quad |L_{ij}| \leqq H^n, \quad (i, j = 0, 1, \cdots, n-1).$$

Then the function $\theta(t, u, v_1, \cdots, v_{n-1})$ defined by $\theta = T_t^0 L$ is a solution of the differential system (1.1), (1.2) and has all its first derivatives continuous on

$I_0 \otimes R_n$. *Moreover if $L > 0$ throughout R_n, θ is the minimal positive solution with continuous first derivatives throughout $I_0 \otimes R_n$.*

REMARK 6. It is proved on $[1, \text{p. } 167]$ that if $J^1, J^2, \cdots, J^n$ and their first derivatives are continuous and satisfy (7.5), then

$$(7.8) \qquad \left| \Psi^k \right| \leq \tilde{H}^k, \qquad \left| \Omega^k \right| \leq \tilde{H}^k, \qquad (k = 1, 2, \cdots, n).$$

7.1. THEOREM VI. *Let $J^1, J^2, \cdots, J^n$ and their first and second derivatives with respect to $u, v_1, \cdots, v_{n-1}$ (except possibly for J^k_{uu}) be continuous and satisfy on $I_0 \otimes R_n$ the order of growth conditions (7.5) and let J^n further satisfy*

$$(7.9) \qquad \left| J^n \right| \leq \log H^n \qquad \qquad \text{on } I_0 \otimes R_n.$$

Then if θ is any solution of the differential system (1.1), (1.2) with continuous first derivatives on $I_0 \otimes R_n$ such that

$$(7.10) \qquad \left| \theta \right| \leq H^n \qquad \qquad \text{on } I_0 \otimes R_n,$$

it follows that θ is unique.

The proof of the theorem is based on the following lemma.

LEMMA 10. *Let $J^1, J^2, \cdots, J^n$ and their first and second derivatives with respect to $u, v_1, \cdots, v_{n-1}$ (except possibly for J^k_{uu}) be continuous and satisfy on $I_0 \otimes R_n$*

$$(7.11) \quad \begin{aligned} \left| J^k \right| &\leq H^k(A, B, M, \epsilon), & (k = 1, 2, \cdots, n), \\ \left| J^k_j \right| &\leq H^k(A, B, M, \epsilon), & (k = 1, 2, \cdots, n), (j = 0, 1, \cdots, n-1), \\ \left| J^k_{ij} \right| &\leq H^k(A, B, M, \epsilon), & (k = 1, 2, \cdots, n), (i = 1, 2, \cdots, n-1), \\ & & (j = 0, 1, \cdots, n-1). \end{aligned}$$

$$(7.12) \qquad \left| J^n \right| \leq \log H^n(A, B, M, \epsilon)$$

where A, B, M, ϵ are such that (7.8) holds with them and besides $A \geq 1$, and let

$$L^*(u, v_1, \cdots, v_{n-1}; M_1, M_2, \cdots, M_{n-1})$$

$$(7.13) \qquad \qquad = A \exp \left\{ B_0(1 + u^2)^{1-\epsilon/6} \right\} \prod_{j=1}^{n-1} \left\{ 2 + v_j^2 \right\}^{M_j},$$

where B_0 is a sufficiently large positive number that satisfies

$$(7.14) \qquad B_0(1 + u^2)^{1-\epsilon/6} \geq B(1 + \left| u \right|)^{2-\epsilon/2} \qquad \qquad \text{on } u \in R_1$$

and M_j ($j = 1, 2, \cdots, n-1$) are parameters with positive values. Then the function $\theta^* = T^0_t L^*$ defined by (7.3), (7.4) is the minimal positive solution with continuous first derivatives on $I_0 \otimes R_n$ of the differential system consisting of the differential equation (1.1) and the boundary condition

$$(7.15) \qquad \theta^*(0, u, v_1, \cdots, v_{n-1}) = L^*(u, v_1, \cdots, v_{n-1}) \qquad \qquad \text{on } R_n,$$

for whatever positive values of M_j ($j=1, 2, \cdots, n-1$). *Furthermore for some values of* M_j ($j=1, 2, \cdots, n-1$)

(7.16) $\qquad\qquad \theta^* > H^n(A_0, B, M, \epsilon) \ \text{ with } \ A_0 > 0 \qquad\qquad \text{on } I_0 \otimes R_n$

7.2. Proof of Lemma 10. $J^1, J^2, \cdots, J^n$ satisfy the conditions of [1, Theorem 8]. L^* has continuous first and second derivatives with respect to $u, v_1, \cdots, v_{n-1}$ and, since $(1+u^2)^{1-\epsilon/6} \le (1+|u|)^{2-\epsilon/3}$ for $u \in R_1$, L^* and its derivatives satisfy the order of growth conditions of [1, Theorem 8] on L and its derivatives. By the conclusion of the theorem, θ^* is the minimal positive solution with continuous first derivatives on $I_0 \otimes R_n$ of the system (1.1) and (7.15).

From (7.3), (7.13), (5.7), (5.6) we obtain for $t \in (0, t_0]$, $(u, v_1, \cdots, v_{n-1}) \in R_n$,

$$T_t^0 L^* = A \int_{c_w[0,t]} \exp \left\{ \int_0^t J^n[p, u + y(t - p), v_1 + \Omega^1(p), \cdots, v_{n-1} \right.$$

$$\left. + \Omega^{n-1}(p)]dp \right\}$$

$$\cdot \exp \left\{ B_0[1 + [u + y(t)]^2]^{1-\epsilon/6} \right\} \prod_{j=1}^{n-1} \left\{ 2 + [v_j + \Omega^j(0)]^2 \right\}^{M_j} d_w y.$$

From $J^n \ge -\log H^n$ according to (7.12), and (7.1)

(7.17) $\qquad\qquad T_t^0 L^* \ge A \int_{c_w[0,t]} \prod_{j=1}^{5} G_j d_w y$

where

(7.18) $\qquad\qquad G_1(t) = \exp \left\{ -\int_0^t \log A \, dp \right\},$

(7.19) $\quad G_2(y \mid t, u) = \exp \left\{ -B \int_0^t |u + y(t - p)|^{2-\epsilon} dp \right\},$

$\qquad\qquad G_3(y \mid t, u, v_1, \cdots, v_{n-1})$

(7.20)
$$= \exp \left\{ -M \sum_{j=1}^{n-1} \int_0^t \log [1 + |v_j + \Omega^j(p)|] dp \right\},$$

(7.21) $\quad G_4(y \mid t, u) = \exp \left\{ B_0[1 + [u + y(t)]^2]^{1-\epsilon/6} \right\},$

(7.22) $\quad G_5(y \mid t, u, v_1, \cdots, v_{n-1}, M_1, \cdots, M_{n-1}) = \prod_{j=1}^{n-1} \left\{ 2 + [v_j + \Omega^j(0)]^2 \right\}^{M_j}.$

7.3. Lower bounds for G_1, G_4, G_2.
From (7.18) and from (7.21), by our choice of B_0 according to (7.14)

(7.23) $$G_1(t) \geqq 1/A^{t_0},$$

(7.24) $$G_4(y \mid t, u) \geqq \exp \{ B \mid u + y(t) \mid^{2-\epsilon/2} \}.$$

As for G_2, since $|a+b|^{2-\epsilon} \leqq 4 \{ |a|^{2-\epsilon} + |b|^{2-\epsilon} \}$ for any two real numbers a and b and $0 < \epsilon < 2$,

(7.25) $$-B \int_0^t |u + y(t-p)|^{2-\epsilon} dp \geqq -4Bt_0 |u|^{2-\epsilon} - 4B \int_0^t |y(t-p)|^{2-\epsilon} dp.$$

Let us define

(7.26) $$|||y|||_\beta = \max_{0 \leqq p \leqq \beta} |y(p)|.$$

Then

(7.27) $$-4B \int_0^t |y(t-p)|^{2-\epsilon} dp \geqq -4Bt_0 |||y|||_t^{2-\epsilon},$$

and by (7.19), (7.25), (7.27)

(7.28) $$G_2(y \mid t, u) \geqq \exp \{ -4Bt_0 |u|^{2-\epsilon} \} \exp \{ -4Bt_0 |||y|||_t^{2-\epsilon} \}.$$

7.4. A lower bound for G_3. For any two real numbers a and b, $1 + |a+b| \leqq (1+|a|)(1+|b|)$ so that

(7.29) $$- \int_0^t \log [1 + |v_j + \Omega^i(p)|] dp \geqq \log [1 + |v_j|]^{-t_0} - \int_0^t \log [1 + |\Omega^i(p)|] dp.$$

Now according to our choice of A, B, M, ϵ in the statement of Lemma 10

(7.30) $$|\Omega^j| \leqq H^j(A, B, M, \epsilon) \exp \{ B|||y|||_t^{2-\epsilon} \}, \qquad \text{for } y \in C_w[0, t].$$

Then since $A \geqq 1$, $B > 0$, the right hand of (7.30) is not less than 1, and from the fact that $\log (1+a) \leqq \log 2 + \log a$ for $a \geqq 1$ we have

(7.31) $$- \int_0^t \log [1 + |\Omega^i(p)|] dp \geqq -t_0 \log 2 - \log [H^j(A, B, M, \epsilon)]^{t_0} - t_0 B|||y|||_t^{2-\epsilon}.$$

From (7.20), (7.29), (7.31),

$$G_3(y \mid t, u, v_1, \cdots, v_{n-1})$$

(7.32) $$\geqq \prod_{j=1}^{n-1} [1 + |v_j|]^{-Mt_0} \exp \{ -(n-1)Mt_0 \log 2 \} \prod_{j=1}^{n-1} [H^i(A, B, M, \epsilon)]^{-Mt_0}$$

$$\cdot \exp \{ -(n-1)Mt_0 B|||y|||_t^{2-\epsilon} \}.$$

7.5. A lower bound for G_5. For any two real numbers a and b, $2+a^2$
$\geq 1+|a|$ and $1+|a+b| \geq (1+|a|)(1+|b|)^{-1}$ hold. Thus

$$(7.33) \qquad 2 + [v_j + \Omega^j(0)]^2 \geq (1 + |v_j|)(1 + |\Omega^j(0)|)^{-1}.$$

From our choice of A, B, M, ϵ as stated in Lemma 10,

$$(7.34) \qquad 1 + |\Omega^j(0)| \leq 2H^j(A, B, M, \epsilon) \exp\{B|||y|||_t^{2-\epsilon}\}.$$

From (7.33), (7.34), the fact that $M_j > 0$, (7.22)

$$G_5(y \mid t, u, v_1, \cdots, v_{n-1}, M_1, \cdots, M_{n-1})$$

$$(7.35) \qquad \geq \prod_{j=1}^{n-1} \{[1 + |v_j|]^{M_j} 2^{-M_j} [H^i(A, B, M, \epsilon)]^{-M_j}\}$$

$$\cdot \exp\left\{-B \sum_{j=1}^{n-1} M_j |||y|||_t^{2-\epsilon}\right\}.$$

7.6. A lower bound for $T_t^0 L^*$. Substituting (7.23), (7.28), (7.32), (7.24),
(7.35) in (7.17), we have

$$(7.36) \qquad \overset{0}{T_t} L^* \geq I \qquad \text{on} (0, t_0] \otimes R_n,$$

where

$$(7.37) \qquad I = KG \int_{C_w[0,t]} \exp\{B \mid u + y(t) \mid^{2-\epsilon/2}\} \Gamma d_w y,$$

with

$$(7.38) \quad K(M_1, \cdots, M_{n-1}) = A^{1-t_0} \exp\{-(n-1)Mt_0 \log 2\} \prod_{j=0}^{n-1} 2^{-M_j} > 0,$$

$$G(u, v_1, \cdots, v_{n-1}, M_1, \cdots, M_{n-1})$$

$$(7.39) \qquad = \exp\{-4Bt_0 \mid u \mid^{2-\epsilon}\} \prod_{j=1}^{n-1} \{[1 + |v_j|]^{M_j-Mt_0} [H^i(A, B, M, \epsilon)]^{-Mt_0-M_j}\}$$

$$(7.40) \qquad \Gamma(y \mid M_1, \cdots, M_{n-1}) = \exp\{-B_1|||y|||_t^{2-\epsilon}\},$$

$$(7.41) \qquad B_1 = 4Bt_0 + (n-1)Mt_0 B + B \sum_{j=1}^{n-1} M_j > 0.$$

So far the only condition on $M_1, \cdots, M_{n-1}$ has been that they be posi-
tive and as long as this condition is satisfied (7.36) holds. We now proceed
to choose such positive values for $M_1, \cdots, M_{n-1}$ that the powers of

$$1+|v_1|, \cdots, 1+|v_{n-1}| \text{ in } G \text{ are all equal to } M.$$

7.7. **Choice of values for the parameters** $M_1, M_2, \cdots, M_{n-1}$. From (7.1),
(7.39)

$$G(u, v_1, \cdots, v_{n-1}, M_1, \cdots, M_{n-1})$$

(7.42)
$$= A^{-(n-1)Mt_0}\left\{\prod_{j=1}^{n-1} A^{-M_j}\right\} \exp\left\{-B_1\,|\,u\,|^{2-\epsilon}\right\}$$

$$\cdot \left\{\prod_{j=1}^{n-1}[1 + |v_j|]^{M_j - Mt_0}\right\}\left\{\prod_{j=1}^{n-1}\prod_{l=1}^{j-1}[1 + |v_l|]^{-M^2t_0 - MM_j}\right\}.$$

From (7.42), it follows that if we make the choice

$$M_{n-1} = M + Mt_0,$$
$$M_{n-2} = M + Mt_0 + M^2t_0 + MM_{n-1},$$
$$M_{n-3} = M + Mt_0 + 2M^2t_0 + MM_{n-1} + MM_{n-2},$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$M_1 \;\; = M + Mt_0 + (n-2)M^2t_0 + MM_{n-1} + MM_{n-2} + \cdots + MM_2,$$

then $M_1, \cdots, M_{n-1}$ are all positive and the powers of $1 + |v_1|, \cdots, 1 + |v_{n-1}|$
in (7.42) all become M. From now on $M_1, \cdots, M_{n-1}$ will stand for these
values. Now from (7.42)

$$G(u, v_1, \cdots, v_{n-1}, M_1, \cdots, M_{n-1})$$

(7.43)
$$= A^{-(n-1)Mt_0}\left\{\prod_{j=1}^{n-1} A^{-M_j}\right\} \exp\left\{-B_1\,|\,u\,|^{2-\epsilon}\right\} \prod_{j=1}^{n-1}[1 + |v_j|]^M,$$

and from (7.37), (7.43)

$$I = K_1 \exp\left\{-B_1\,|\,u\,|^{2-\epsilon}\right\} \prod_{j=1}^{n-1}[1 + |v_j|]^M$$

(7.44)
$$\cdot \int_{C[0,t]} \exp\left\{B\,|\,u + y(t)\,|^{2-\epsilon/2}\right\}\Gamma(y|\;)d_w y$$

where

(7.45)
$$K_1 = KA^{-(n-1)Mt_0}\prod_{j=1}^{n-1} A^{-M_j} > 0.$$

It remains to estimate the Wiener integral (7.44).

7.8. **Lower bound of the Wiener integral.** Let us decompose the Wiener
integral I given by (7.44) into two Wiener integrals I^+, I^- over the two
intervals $C_w^+[0, t]$, $C_w^-[0, t]$ of the Wiener space $C[0, t]$ where

$$C_w^+[0, t] = \{y \in C_w[0, t] \mid y(t) \geq 0\}, \qquad C_w^-[0, t] = \{y \in C_w[0, t] \mid y(t) < 0\},$$

each with a measure $1/2$. Then

(7.46) $$I = I^+ + I^-, \qquad\qquad\qquad \text{on } I_0 \otimes R_n$$

and from the fact that the integrand of I is positive

(7.47) $$I > I^+, \qquad I > I^-, \qquad\qquad \text{on } I_0 \otimes R_n.$$

We now proceed to give a lower estimate for I^+ when $u \geq 0$ and a lower estimate for I^- when $u < 0$. Consider I^+. Since $y(t)$ in its integrand is non-negative, $|u + y(t)| \geq u$ for $u \geq 0$ and

$$\exp\{B\,|\,u + y(t)\,|^{2-\epsilon/2}\} \geq \exp\{B\,|\,u\,|^{2-\epsilon/2}\}, \qquad \text{for } u \geq 0,$$

and consequently from (7.44)

(7.48)
$$I^+ \geq K_1 \exp\{-B_1\,|\,u\,|^{2-\epsilon}\}$$
$$\cdot \exp\{B\,|\,u\,|^{2-\epsilon/2}\} \prod_{j=1}^{n-1} [1 + |\,v_j\,|\,]^M \int_{C_w^+[0,t]} \Gamma(y\,|\quad)d_w y, \quad \text{for } u \geq 0.$$

Since for $B, B_1 > 0$ there exists a positive constant K_2 such that

(7.49) $\quad \exp\{B\,|\,u\,|^{2-\epsilon/2}\}\exp\{-B_1\,|\,u\,|^{2-\epsilon}\} \geq K_2 \exp\{B\,|\,u\,|^{2-\epsilon}\}, \quad \text{for } u \in R_1,$

we have

(7.50) $\quad I^+ \geq K_3 \exp\{B\,|\,u\,|^{2-\epsilon}\} \prod_{j=1}^{n-1} [1 + |\,v_j\,|\,]^M \int_{C_w^+[0,t]} \Gamma(y\,|\quad)d_w y, \text{ for } u \geq 0$

where $K_3 = K_1 K_2 > 0$. In estimating the Wiener integral in the right side of (7.50) we first write

(7.51) $$\int_{C_w^+[0,t]} \Gamma(y\,|\quad)d_w y = \int_{C_w[0,t]} \frac{1}{2}\{1 + \mathrm{sgn}\,[y(t)]\}\Gamma(y\,|\quad)d_w y$$

and then reduce the integral to one over $C_w[0, t]$ by using the transformation formula[8]

$$\int_{C_w[a,b]} F[y]d_w y = \int_{C_w[0,1]} F\left[(b-a)^{1/2}x\left[\frac{(\cdot) - a}{b - a}\right]\right]d_w x.$$

Then

$$\int_{C_w[0,t]} \frac{1}{2}\{1 + \mathrm{sgn}\,[y(t)]\}\Gamma(y\,|\quad)d_w y$$
$$= \int_{C_w[0,1]} \frac{1}{2}\{1 + \mathrm{sgn}\,[x(1)]\}\Gamma\left[t^{1/2}x\left(\frac{(\cdot)}{t}\right)\right]d_w x.$$

From (7.40)

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(8) See [3].

$$\Gamma\left[t^{1/2}x\left(\frac{(\cdot)}{t}\right)\right] = \exp\left\{-B_1\left|\left|\left|t^{1/2}x\left(\frac{(\cdot)}{t}\right)\right|\right|\right|_t^{2-\epsilon}\right\}$$

$$\geqq \exp\left\{-B_1 t_0^{1-\epsilon/2}|||x|||^{2-\epsilon}\right\}$$

where by definition $|||x||| = |||x|||_1$. Thus

(7.52)
$$\int_{C_w^+[0,t]} \Gamma(y|\ )d_w y$$

$$\geqq \int_{C_w[0,1]} \frac{1}{2}\{1 + \text{sgn}\,[x(1)]\}\exp\left\{-B_1 t_0^{1-\epsilon/2}|||x|||^{2-\epsilon}\right\}d_w x.$$

Since $\int_{C_w[0,1]} \exp\{B|||x|||^{2-\epsilon}\}d_w x$ exists as a finite number for any real number B and $2 > \epsilon > 0$ [9] the Wiener integral in the right side of (7.52) is finite. Let us call it K_4. Then from (7.50), (7.52), with $K_5 = K_3 K_4 > 0$

(7.53)
$$I^+ \geqq K_5 \exp\{B|u|^{2-\epsilon}\}\prod_{j=1}^{n-1}[1 + |v_j|]^M, \qquad\text{for } u \geqq 0.$$

As for the integrand of I^-, since $y(t)$ is negative, $|u + y(t)| \geqq |u|$ for $u < 0$. Following the same line of argument as for I^+ we obtain

(7.54)
$$I^- \geqq K_5' \exp\{B|u|^{2-\epsilon}\}\prod_{j=1}^{n-1}[1 + |v_j|]^M, \qquad\text{for } u < 0,$$

where $K_5' = K_3 K_4'$ and

$$K_4' = \int_{C[0,1]} \frac{1}{2}\{1 - \text{sgn}\,[x(1)]\}\exp\left\{-B_1 t_0^{1-\epsilon/2}|||x|||^{2-\epsilon}\right\}d_w x,$$

a finite positive number. If we now let $K_6 = \min\{K_5, K_5'\}$, then from (7.47), (7.53), (7.54)

(7.55)
$$I > K_6 \exp\{B|u|^{2-\epsilon}\}\prod_{j=1}^{n-1}[1 + |v_j|]^M \qquad\text{on } (0, t_0] \otimes R_n.$$

From (7.36), (7.55)

(7.56)
$$T_t^0 L^* \geqq K_6 \exp\{B|u|^{2-\epsilon}\}\prod_{j=1}^{n-1}[1 + |v_j|]^M \qquad\text{on } (0, t_0] \otimes R_n.$$

If we let $A_0 = \min\{A, K_6\}$ then from (7.4), (7.13), (7.14), (7.56)

$$T_t^0 L^* \geqq A_0 \exp\{B|u|^{2-\epsilon}\}\prod_{j=1}^{n-1}[1 + |v_j|]^M \qquad\text{on } I_0 \otimes R_n.$$

This completes the proof of Lemma 10.

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[9] See [4].

7.9. **Proof of Theorem** VI. Let us assume without loss of generality that the functions H^k ($k = 1, 2, \cdots, n$) in the hypothesis of the theorem all have the same constants A, B, M, $2 - \epsilon$ which are so large as to satisfy the condition imposed on them in the statement of Lemma 10. Then for $\theta^* = T_t^0 L^*$ with L^* defined by (7.13), the lemma applies.

Now assume that the differential system (1.1), (1.2) has two solutions θ_1, θ_2 with continuous first derivatives and satisfying (7.10) on $I_0 \otimes R_n$. Let $\theta_3 = \theta_1 - \theta_2$ on $I_0 \otimes R_n$. We show that $\theta_3 \equiv 0$ on $I_0 \otimes R_n$. Consider

$$(7.57) \qquad\qquad \theta_4 = \theta^* \pm \frac{A_0}{2A} \theta_3.$$

θ_4 has continuous first derivatives and satisfies (1.1), (7.15) on $I_0 \otimes R_n$. Since θ_1, θ_2 satisfy (7.10), we have $|\theta_3| \leq H^n(2A, B, M, \epsilon)$ on $I_0 \otimes R_n$, $A_0(2A)^{-1}|\theta_3| \leq H^n(A_0, B, M, \epsilon)$ and by (7.57), (7.16) we have $\theta_4 > 0$ on $I_0 \otimes R_n$ for both of the two signs in (7.57). Then since θ^* is the minimal positive solution with continuous first derivatives of (1.1), (7.15), we have $\theta_4 \geq \theta^*$ on $I_0 \otimes R_n$ for both of the two signs in (7.57). This implies that $A_0(2A)^{-1}\theta_3 \equiv 0$ on $I_0 \otimes R_n$ and hence $\theta_3 \equiv 0$ on $I_0 \otimes R_n$. This completes the proof of the theorem.

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