

THE STABLE SUSPENSION OF AN EILENBERG-MACLANE SPACE

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Introduction. In Part I of this paper we set up a spectral sequence for the stable homotopy groups of a countable CW complex X , using the suspension triad sequences for the iterated suspensions $S^r X$. If X is $(n-1)$ -connected, we calculate the differential operator d^1 for the first $n-1$ nontrivial dimensions in terms of the automorphism T_* of $\pi_m(X \times X)$ induced by the map which interchanges factors.

In Part II we continue the study of the behavior of Eilenberg-MacLane spaces under suspension, initiated in [2]. For $K(\pi, n)$, π finitely generated, we calculate the automorphism T_* on the p -primary component ($p \neq 2$) for a range of dimensions. The p -primary component ($p \neq 2$) of the first $n-1$ nontrivial stable homotopy groups can then be read off from the spectral sequence. The corresponding Postnikov invariants are zero, in contrast to those of the single suspension $SK(\pi, n)$; however, this is not true of the 2-primary component.

PART I. THE SPECTRAL SEQUENCE

1. The suspension couple. All spaces considered throughout Part I are to be 1-connected countable CW complexes with fixed base points. Let X be such. The suspension triad sequences for iterated suspensions $S^r X$ (cf. [7] for definitions)

$$\dots \xrightarrow{j_*} \pi_q(S^r X; C_+, C_-) \xrightarrow{\Delta} \pi_{q-2}(S^{r-1} X) \xrightarrow{E} \pi_{q-1}(S^r X) \xrightarrow{j_*} \dots$$

give rise to an exact couple

$$\begin{array}{ccc} A & \xrightarrow{E} & A \\ \Delta \swarrow & & \searrow j_* \\ C & & \end{array}$$

where

$$C_{r,s} = \begin{cases} \pi_{s+2r}(S^r X; C_+, C_-), & r \geq 1 \\ \pi_s(X), & r = 0 \\ 0, & r < 0 \end{cases}$$

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$$A_{r,s} = \begin{cases} \pi_{s+2r}(S^r X), & r \geq 0 \\ 0, & r < 0 \end{cases}$$

and we define $j_* = \text{identity}: A_{0,s} = \pi_s(X) \rightarrow \pi_s(X) = C_{0,s}$. Let $\Pi_m(X)$ denote the stable group $\pi_{m+q}(S^q X)$, q large, and $E_\infty: \pi_m(X) \rightarrow \Pi_m(X)$ the iterated suspension.

(1.1) PROPOSITION. *The spectral sequence associated with the above exact couple [8, §6] converges, and $C_{r,s}^\infty \approx B_{r,s}/B_{r-1,s+1}$, where $B_{r,s}$ is the image of $E_\infty: \pi_{s+2r}(S^r X) \rightarrow \Pi_{s+r}(X)$ (if $r < 0$, $B_{r,s} = 0$).*

The proof is of a standard type.

2. **The operator d^1 .** We first give some definitions.

1. $r: X \times I \rightarrow SX$ identifies $X \times I \cup x_0 \times I$ to a point, where x_0 is the base point. C_+ and C_- denote the images under r of $X \times [1/2, 1]$ and $X \times [0, 1/2]$ respectively.

2. $\phi: (SX, C_+, C_-) \rightarrow (S_1 X \vee S_2 X, S_1 X, S_2 X)$ is defined by

$$\phi(x, s) = \begin{cases} (x, 2s) \in S_2 X & \text{for } 0 \leq s \leq 1/2, \\ (x, 2s - 1) \in S_1 X & \text{for } 1/2 \leq s \leq 1 \end{cases} \quad x \in X,$$

where the subscripts merely distinguish between the two copies of SX .

3. ρ is the reflection of SX in the equator, $\rho(x, s) = (x, 1-s)$.

4. $X \times Y = X \times Y / X \vee Y$ and $\theta: X \times Y \rightarrow X \times Y$ is the identification map.

5. T maps $X \times X$ onto itself by $T(x, x') = (x', x)$.

6. $\eta: S^2(X \times Y) \rightarrow SX \times SY$ is the homeomorphism defined by $\eta((x, y), s, t) = ((x, s), (y, t))$, for $x \in X$, $y \in Y$, and $s, t \in I$.

7. The composition $i'_* \partial: \pi_{m+1}(X \times Y, X \vee Y) \rightarrow \pi_m(X \vee Y) \rightarrow \pi_m(X \vee Y; X, Y)$ is an isomorphism for $m \geq 2$ [3, Corollary 8.3]; Q is the inverse.

Toda [11] defined a generalized Hopf homomorphism for spheres; the analogue for general spaces is

8. $H: \pi_m(S^r X; C_+, C_-) \rightarrow \pi_{m+1}(S^r X \times S^r X)$ is the composition $H = \theta_* Q \phi_*$.

Now the operator $d^1: C_{r+1,s} \rightarrow C_{r,s}$ in the spectral sequence is given by $d^1 = j_* \Delta$ for $r \geq 1$. Consider the diagram

$$\begin{array}{ccc} \pi_{m+1}(S^{r+1} X; C_+, C_-) & \xrightarrow{d^1} & \pi_{m-1}(S^r X; C_+, C_-) \\ H \searrow & & \downarrow H \\ & \pi_m(S^r X \times S^r X) & \\ & & \downarrow \eta_* E^2 \\ & & \pi_{m+2}(S^{r+1} X \times S^{r+1} X). \end{array}$$

(2.1) PROPOSITION. $(\rho \times 1)_* \eta_* E^2 H d^1 = (1 - T_*) H$ for $r \geq 1$.

Note that if X is $(n-1)$ -connected, then ϕ_* and θ_* are isomorphisms at least for $m < 3n+3r-2$ and epimorphisms for $m = 3n+3r-2$ [9, Corollary 3.3, Corollary 3.5], and hence so is H . E^2 is an isomorphism for $m \leq 4n+4r-2$, while $(\rho \times 1)_*$ and η_* are always such. Hence for $m < 3n+3r-2$, d^1 is given up to isomorphism by the endomorphism $1 - T_*$ of $\pi_{m+2}(S^{r+1}X \times S^{r+1}X)$.

If $X = S^{q-r-1}$, then using the facts that $T_* = (-1)^q \iota_{2q} \circ$ and $(\rho \times 1)_* = -\iota_{2q} \circ$, the proposition reduces to Theorem 3.7 of [11]. Proposition (2.1) can be proved by generalizing Toda's methods. Rather than give the proof, which is geometric, we shall prove an equivalent proposition ((2.8) below) for $r \geq 2$ which is simpler and more enlightening.

The following shows that, since we remain in a range where E^2 is an isomorphism, we need only calculate T_* in the homotopy groups of $X \times X$.

(2.2) PROPOSITION.

$$T_* \eta_* E^2 = -\eta_* E^2 T_* : \pi_m(S^r X \times S^r X) \rightarrow \pi_{m+2}(S^{r+1}X \times S^{r+1}X).$$

The proof follows at once from the definitions.

In order to apply the results of [7], we shall assume for the remainder of this section that X is an $(n-1)$ -connected countable CW complex, $n \geq 2$, with only a single vertex x_0 . Consider the diagram

$$(2.3) \quad \begin{array}{ccccc} \pi_{m+1}(S^{r+1}X; C_+, C_-) & \xrightarrow{\Delta} & \pi_{m-1}(S^r X) & \xrightarrow{j_*} & \pi_{m-1}(S^r X; C_+, C_-) \\ \lambda i_* p_* \uparrow & \nearrow p'_* \partial & \phi'_* Q' & \searrow & \downarrow Q \phi_* \\ \pi_m(S^r X \times S^r X, S^r X \vee S^r X) & & \pi_m(S^r X \times S^r X, S^r X \vee S^r X) & & \end{array}$$

where the homomorphisms are defined as follows: $\phi'_* : \pi_{m-1}(S^r X) \rightarrow \pi_{m-1}(S^r X \vee S^r X)$ is induced by ϕ . Q' is the projection of $\pi_{m-1}(S^r X \vee S^r X)$ onto a direct summand, obtained by injection into the triad group followed by Q . Thus the right-hand triangle in (2.3) is commutative. The left-hand vertical homomorphisms are

$$\begin{aligned} \pi_m(S^r X \times S^r X, S^r X \vee S^r X) &\xrightarrow{p_*} \pi_m((S^r X)_2, S^r X) \xrightarrow{i_*} \pi_m((S^r X)_\infty, S^r X) \\ &\xrightarrow{\lambda} \pi_{m+1}(S^{r+1}X; C_+, C_-). \end{aligned}$$

Here $(S^r X)_q$ denotes the q th space in the reduced product complex [6]; in particular, $(S^r X)_2$ is obtained from $S^r X \times S^r X$ by identifying $(x, x_0) = (x_0, x)$ for all $x \in S^r X$, and p is the identification map. $p' : S^r X \vee S^r X \rightarrow S^r X$ is the restriction of p , i is the inclusion, and λ is the isomorphism denoted by ϕ in [7]. According to (10.1) of [7], $\Delta \lambda = \partial$; hence the left-hand triangle in (2.3) is also commutative. Now λ and Q are isomorphisms for all m , ϕ_* for $m < 3n+3r-2$, and i_* , p_* for $m < 3n+3r-1$, the latter by simple connectivity arguments. Hence in order to calculate $d^1 = j_* \Delta$ for $m < 3n+3r-2$ we need only

find $Q'\phi_* p'_*\partial$. To do this we use the generalized Whitehead products of [5]; we recall briefly their properties.

The join $A * B$ of two countable CW complexes A, B , each with a single vertex, is defined to be the space obtained from $A \times I \times B$ by making the identifications $(a, 0, b) = (a, 0, b')$, $(a, 1, b) = (a', 1, b)$, $(a_0, t, b_0) = (a_0, 1/2, b_0)$ for all $a, a' \in A$, $b, b' \in B$, $t \in I$.

The set of homotopy classes of mappings $X \rightarrow Y$, preserving base points, will be denoted by $\{X, Y\}$. If X is a suspension, this set is in a well-known way a group, Abelian if X is a double suspension.

If $\mu \in \{S^{p+1}A, Y\}$, $\nu \in \{S^{q+1}B, Y\}$ are represented by maps f, g respectively, then the *Whitehead product* $[\mu, \nu] \in \{S^pA * S^qB, Y\}$ is represented by the composition

$$S^pA * S^qB \xrightarrow{w} S^{p+1}A \vee S^{q+1}B \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\phi'} Y$$

where

$$\begin{aligned} w(a, s, b) &= (b, 1 - 2s) && \text{for } s \leq \frac{1}{2}, \quad a \in S^pA, \quad b \in S^qB \\ &= (a, 2s - 1) && \text{for } s \geq \frac{1}{2}. \end{aligned}$$

1. The product is natural: if $f: Y \rightarrow Y'$, then $f_*[\mu, \nu] = [f_*\mu, f_*\nu]$.

Let $A *' B$ denote the usual join, obtained from $A \times I \times B$ by identifying $(a, 0, b) = (a, 0, b')$, $(a, 1, b) = (a', 1, b)$ for all $a, a' \in A$, $b, b' \in B$. Then the (nonreduced) suspension is defined by $S_0A = A *' S^0$, where S^0 is a zero-sphere, and inductively by $S_0^n(A *' B) = (S_0^{n-1}A) *' S^0$. There is a homeomorphism $S_0^{p+q}(A *' B) \rightarrow S_0^pA *' S_0^qB$, obtained by permuting B with the first p copies of S^0 . Taking identification spaces, we obtain a homotopy equivalence $S^{p+q}(A * B) \rightarrow S^pA * S^qB$. Using this, we may suppose that the Whitehead product is represented by a map of a suspension. Then

2. The product is bilinear for $p, q \geq 1$.

Let $\iota_i \in \{S^r X, S^r X \vee S^r X\}$ be the class of the inclusion onto the i th factor, $i = 1, 2$.

(2.4) LEMMA. *The image of*

$\partial: \pi_m(S^r X \times S^r X, S^r X \vee S^r X) \rightarrow \pi_{m-1}(S^r X \vee S^r X), \quad r \geq 1, m < 3n + 3r - 1,$
consists of the elements $[\iota_1, \iota_2] \circ \alpha$, $\alpha \in \pi_{m-1}(S^{r-1}X * S^{r-1}X)$.

Proof. That $[\iota_1, \iota_2] \circ \alpha \in \text{Image } \partial$ is shown in Theorem 7.13 of [5], where it is also shown that there is an isomorphism

$$\xi: \pi_m(S(S^{r-1}X * S^{r-1}X)) \rightarrow \pi_m(S^r X \times S^r X)$$

such that if α is as above, then there exists $\beta \in \pi_m(S^r X \times S^r X, S^r X \vee S^r X)$ with

$\partial\beta = [\iota_1, \iota_2] \circ \alpha$ and $\theta_*\beta = \xi E\alpha$. Now if θ_* and E are isomorphisms, then given any element β' there exists α such that $\theta_*\beta' = \xi E\alpha$. If β is an element corresponding to α as above, then since θ_* is an isomorphism, $\beta' = \beta$, and therefore $\partial\beta'$ has the desired form. Since θ_* and E are isomorphisms for $m < 3n + 3r - 1$, this proves (2.4).

To calculate $Q'\phi'_*p'_*\partial$: Let $r > 1$, and let $\gamma = [\iota_1, \iota_2] \circ \alpha$ be an element of Image ∂ . If $m < 4n + 4r - 2$, then α is a suspension and composition is distributive. Since $p'_*\gamma = [\iota, \iota] \circ \alpha$,

$$\begin{aligned}\phi'_* p'_* \gamma &= [\iota_1 + \iota_2, \iota_1 + \iota_2] \circ \alpha \\ &= [\iota_1, \iota_1] \circ \alpha + [\iota_1, \iota_2] \circ \alpha + [\iota_2, \iota_1] \circ \alpha + [\iota_2, \iota_2] \circ \alpha \\ &= [\iota_1, \iota_2] \circ \alpha + [\iota_2, \iota_1] \circ \alpha \text{ mod } \ker Q'.\end{aligned}$$

Hence $\phi'_* p'_* : \text{Image } \partial \rightarrow \pi_{m-1}(S^r X \vee S^r X) / \ker Q'$ is given by $1 + T_*$. Since T_* commutes with ∂ and Q' , $Q'\phi'_*p'_*\partial = 1 + T_*$, and

$$(2.5) \quad Q\phi_*d^1\lambda i_*p_* = (1 + T_*) \quad \text{for } r > 1, m < 4n + 4r - 2.$$

In order to express this result in a better form, we introduce the Hopf homomorphism [7, §15]

$$h: \pi_{m+1}(S^{r+1}X; C_+, C_-) \rightarrow \pi_{m+1}(S^r X * {}_0 S^r X)$$

(here $A * {}_0 B$ denotes the join parametrized as $S(A \times B)$). A short calculation using the definition of h shows that

$$(2.6) \quad h = E\theta_*(\lambda i_*p_*)^{-1} \quad \text{for } m < 3n + 3r - 1.$$

If we use T_* to denote the automorphism of $\pi_{m+1}(S^r X * {}_0 S^r X)$ induced by the homeomorphism T of $S^r X \times S^r X$, then (2.5) becomes

$$(2.7) \quad E\theta_*Q\phi_*d^1 = (1 + T_*)h.$$

Recalling that $H = \theta_*Q\phi_*$, we have

$$(2.8) \quad \text{PROPOSITION.}$$

$$\begin{aligned}EHd^1 &= (1 + T_*)h: \pi_{m+1}(S^{r+1}X; C_+, C_-) \rightarrow \pi_{m+1}(S^r X * {}_0 S^r X), \\ &\quad \text{for } r > 1, m < 3n + 3r - 1.\end{aligned}$$

Within this range E and h are isomorphisms, while H is such for $m < 3n + 3r - 2$, so that d^1 is again expressed in terms of T_* , but with the loss of the case $r = 1$.

As a corollary to the above calculations, we have a generalization of G. W. Whitehead's EHP sequence [12]:

$$(2.9) \quad \text{PROPOSITION. Let } X \text{ be } (n-1)\text{-connected, } n \geq 2. \text{ Then there is an exact sequence for } r \geq 1$$

$$\begin{aligned} \pi_{3n+3r-2}(S^r X) &\xrightarrow{E} \cdots \xrightarrow{} \pi_m(S^r X) \xrightarrow{E} \pi_{m+1}(S^{r+1} X) \xrightarrow{E^{-2}H'} \\ &\pi_{m-1}(S^{r-1} X * S^{r-1} X) \xrightarrow{P} \pi_{m-1}(S^r X) \xrightarrow{} \cdots. \end{aligned}$$

Here $H' = h_*$ is James' generalized Hopf invariant [7, §15], $P(\alpha) = [\iota_1, \iota_2] \circ \alpha$, and we denote by E^{-2} the inverse of the composition $E\xi E$: $\pi_{m-1}(S^{r-1} X * S^{r-1} X) \rightarrow \pi_{m+1}(S^r X * {}_0 S^r X)$.

Proof. Let $m < 3n + 3r - 1$ throughout. It follows from the suspension triad sequence

$$\cdots \rightarrow \pi_m(S^r X) \rightarrow \pi_{m+1}(S^{r+1} X) \xrightarrow{j_*} \pi_{m+1}(S^{r+1} X; C_+, C_-) \xrightarrow{\Delta} \pi_{m-1}(S^r X) \rightarrow \cdots$$

that we need only prove that $PE^{-2}h = \Delta$; i.e. that $\Delta h^{-1}E\xi E = P$. According to Theorem 7.13 of [5], for $\alpha \in \pi_{m-1}(S^{r-1} X * S^{r-1} X)$, $\partial\theta_*^{-1}\xi E\alpha = [\iota_1, \iota_2] \circ \alpha$. Hence

$$\begin{aligned} \Delta h^{-1}E\xi E\alpha &= \Delta \lambda i_* p_* \theta_*^{-1} E^{-1} E\xi E\alpha && \text{by (2.6)} \\ &= p'_* \partial\theta_*^{-1} \xi E\alpha && \text{by (2.3)} \\ &= p'_* [\iota_1, \iota_2] \circ \alpha = [\iota_1, \iota_2] \circ \alpha = P(\alpha). \end{aligned}$$

This proves (2.9).

3. An example. As a simple example of the use of the spectral sequence we take $X = S^n$, $n > 2$. It is well-known that $T_* = (-1)^m$ in $\pi_q(S^m \times S^m)$, and that $S^m \times S^m$ is homeomorphic to S^{2m} . Hence it follows from (2.1) that for $s < 3n + r - 3$,

$$C_{r,s} = \pi_{s+2r}(S^{n+r}; C_+, C_-) \approx \pi_{s+2r+1}(S^{2n+2r}) \quad \text{for } r > 0,$$

while for $r > 1$ and $s < 3n + r - 4$, $d_{r,s}^1: C_{r,s} \rightarrow C_{r-1,s}$ is given up to isomorphism by the endomorphism $1 - (-1)^{n+r}$ of $\pi_{s+2r+1}(S^{2n+2r})$. Recall also that $C_{0,s} = \pi_s(S^n)$.

Let \mathcal{C}_2 be the class of finite Abelian groups with order a power of 2. We distinguish two cases:

1. If n is odd, then $C_{r,s}^2 \in \mathcal{C}_2$ for $r > 0$. $d_{2,s}^1: C_{2,s} \rightarrow C_{1,s}$ is a \mathcal{C}_2 -epimorphism, and the fact that $d^1 d^1 = 0$ implies that $d_{1,s}^1 = 0 \bmod \mathcal{C}_2$. Therefore $C_{0,s}^2 \approx C_{0,s} = \pi_s(S^n) \bmod \mathcal{C}_2$.

2. If n is even, then $C_{r,s}^2 \in \mathcal{C}_2$ for $r > 1$. If $s < 3n - 3$, then we may calculate $d_{1,s}^1$ by noting that $j_* d_{1,s}^1 = j_* \Delta$ is the operator $d_{2,s-2}^1$ in the spectral sequence for S^{n-1} . The latter is a \mathcal{C}_2 -isomorphism, and hence $d_{1,s}^1$ is a \mathcal{C}_2 -monomorphism. Therefore $C_{1,s}^2 \in \mathcal{C}_2$, and $C_{0,s}^2 \approx C_{0,s}/d_{1,s}^1 C_{1,s} \approx \pi_s(S^n)/G \bmod \mathcal{C}_2$, where G is \mathcal{C}_2 -isomorphic to $\pi_{s+3}(S^{2n+2}) \approx \pi_s(S^{2n-1})$. Hence we have

(3.1) **PROPOSITION.** *The stable suspension $E_\infty: \pi_s(S^n) \rightarrow \Pi_s(S^n)$ is a \mathcal{C}_2 -isomorphism if n is odd and $s < 3n - 2$, and a \mathcal{C}_2 -epimorphism with kernel \mathcal{C}_2 -isomorphic to $\pi_s(S^{2n-1})$ if n is even and $s < 3n - 3$.*

This is a known result. The restrictions on s can be improved slightly by using less superficial information about H .

PART II. THE STABLE SUSPENSION OF AN EILENBERG-MACLANE SPACE

4. On Postnikov systems. All spaces in this section are to have the homotopy type of CW complexes.

Let E be 1-connected. Recall that a *Postnikov system* for E consists of fiberings $p_n: E_n \rightarrow E_{n-1}$, $n = 3, 4, \dots$, together with maps $h_n: E \rightarrow E_n$ such that h_n induces homotopy isomorphisms in dimensions $\leq n$, $\pi_i(E_n) = 0$ for $i > n$, and $p_n h_n = h_{n-1}$. Recall also that each fibering $E_n \rightarrow E_{n-1}$ may be taken as the fibre space induced from a contractible fibre space over $K(\pi_n, n+1)$, $\pi_n = \pi_n(E)$, by a map $k_n: E_{n-1} \rightarrow K(\pi_n, n+1)$ whose homotopy class is determined by the Postnikov invariant⁽¹⁾ $k_n \in H^{n+1}(E_{n-1}, \pi_n)$. If the homotopy groups of E are finite direct sums, $\pi_n \approx \pi_n^1 + \dots + \pi_n^r$, then we may take $K(\pi_n, n+1) = X_i K(\pi_n^i, n+1)$, and as a contractible fibre space we may take the product of contractible fibre spaces $\mathcal{O}_i \rightarrow K(\pi_n^i, n+1)$. One verifies immediately that, with these choices, the resulting fibering $p_n: E_n \rightarrow E_{n-1}$ is a composition

$$E_n = E_n^r \rightarrow E_n^{r-1} \rightarrow \dots \rightarrow E_n^0 = E_{n-1}$$

in which the fibering $p_n^s: E_n^s \rightarrow E_n^{s-1}$ is induced from $\mathcal{O}_s \rightarrow K(\pi_n^s, n+1)$ by the map $k_n^s = \pi_n^s k_n p_n^1 \dots p_n^{s-1}$, where $\pi_n^s: X_i K(\pi_n^i, n+1) \rightarrow K(\pi_n^s, n+1)$ is the projection.

We shall again call the element $k_n^s \in H^{n+1}(E_n^{s-1}, \pi_n^s)$ determined by the map k_n^s a *Postnikov invariant*; it is equal to the transgression of the fundamental class in the fibering $E_n^s \rightarrow E_n^{s-1}$, and is related to k_n by

$$(4.1) \quad k_n^s = \pi_{n*}^s(p_n^1 \dots p_n^{s-1})^* k_n$$

where $\pi_{n*}^s: H^{n+1}(E_n^{s-1}, \pi_n) \rightarrow H^{n+1}(E_n^{s-1}, \pi_n^s)$ is the coefficient homomorphism induced by projection. By reindexing the summands π_n^s for each fixed n , we may assume that for some $t = t(n)$

$$(4.2) \quad k_n^1, \dots, k_n^t \text{ are } \neq 0, \quad \text{while} \quad k_n^{t+1} = \dots = k_n^r = 0.$$

We shall always assume that Postnikov systems are constructed in the above fashion.

The fibre F_n^s of $E_n^s \rightarrow E_n^{s-1}$ is a $K(\pi_n^s, n)$ -space, and the group $\pi_n^s = \pi_n(F_n^s)$ will be identified with a summand of $\pi_n(E_n^s)$ under injection. Lemma (4.2) of [2] generalizes to

(1) In the classifying bundle for $K(\pi, n)$ we shall identify the groups of the fibre $\pi_n(K(\pi, n)) = H_n(K(\pi, n))$ with those of the base $\pi_{n+1}(K(\pi, n+1)) = H_{n+1}(K(\pi, n+1))$ under the Hurewicz homomorphism and the homology suspension. Then the above definition of k_n is the negative of the usual definition (as the obstruction to a cross-section).

(4.3) LEMMA. $k_n^s = 0$ if and only if the Hurewicz homomorphism $\mathfrak{H}: \pi_n(E_n^s) \rightarrow H_n(E_n^s)$ maps π_n^s monomorphically onto a direct summand.

Let $h_n^s = p_n^{s+1} \cdots p_n^r h_n: E \rightarrow E_n^s$.

(4.4) LEMMA. In the above Postnikov system for E , if $k_n^{t+1} = \cdots = k_n^r = 0$, then for any m the m -fold suspension maps the summand $\ker h_{n+m}^t \subset \pi_n(E)$, which is isomorphic to $\pi_n^{t+1} + \cdots + \pi_n^r$, monomorphically onto a direct summand G of $\pi_{n+m}(S^m E)$; and there is a Postnikov system for $S^m E$

$$\cdots \rightarrow (S^m E)_{n+m} = (S^m E)_{n+m}^{r'} \rightarrow (S^m E)_{n+m}^{r'-1} \rightarrow \cdots$$

in which $G = \ker h_{n+m}^{r'-1}$ and $k_{n+m}^{r'}(S^m E) = 0$.

The proof is straightforward, using the fact that

$$E_n = E_n^t \times K(\pi_n^{t+1} + \cdots + \pi_n^r, n).$$

5. The stable homotopy groups. Let π be a finitely generated Abelian group, written as $\pi = \sum_{1 \leq i \leq m} G_i$, where each G_i is cyclic of prime power or infinite order. Then $K(\pi, n)$ has the homotopy type of the product $X_{1 \leq i \leq m} K_i$, where $K_i = K(G_i, n)$ is a countable CW complex. According to Theorem 19 of [10],

$$(5.1) \quad S(XK_i) \equiv VS(K_{i_1} \times \cdots \times K_{i_r}),$$

where there is one term in the wedge for each index set (i_1, \dots, i_r) such that $1 \leq i_1 < \cdots < i_r \leq m$, and r ranges from 1 to m . Hence, using iterated suspension, we conclude that the Postnikov space

$$(5.2) \quad [S^r(XK_i)]_{3n+r-1} \equiv [VS^r(K_{i_1} \times \cdots \times K_{i_r})]_{3n+r-1},$$

where $1 \leq i \leq m$, $1 \leq j < k \leq m$. Since the stable homotopy group of a wedge of spaces is the direct sum of the stable homotopy groups of the spaces,

$$(5.3) \quad \Pi_q(K(\pi, n)) \approx \sum \Pi_q(K(G_i, n)) + \sum \Pi_q(K(G_j, n) \times K(G_k, n))$$

for $q \leq 3n-1$ in which i, j, k have the same range as before. Now

$$\Pi_q(K(G_i, n) \times K(G_j, n)) \approx \pi_q(K(G_i, n) \times K(G_j, n)) \approx \tilde{H}_{q-n}(G_i, n; G_j),$$

the first isomorphism because we are in the stable range, the second by Theorem 6.1 of [13]. Therefore in order to find $\Pi_q(K(\pi, n))$, $q \leq 3n-1$, we need only find $\Pi_q(K(G_i, n))$.

For the remainder of the paper, G is to be a cyclic group of prime-power or infinite order. We shall use the fibering of [2] together with the previous lemmas on Postnikov systems to calculate T_* in the p -primary component, $p \neq 2$, of $\pi_q(K(G, n) \times K(G, n))$, $q \leq 3n-1$, and shall then use the spectral sequence to find the p -primary component of $\Pi_q(K(G, n))$.

Let X be a countable CW complex which is a $K(G, n+1)$ -space, and let

$K = \Omega X$, the space of loops on X . Then the fibering $F \rightarrow E \rightarrow X$ described in [2] is such that there are homotopy equivalences $q: E \rightarrow SK$, $h: K * K \rightarrow F$, and the following diagram commutes [2, (4.4)]:

$$(5.4) \quad \begin{array}{ccc} \pi_i(F) & \xrightarrow{i_*} & \pi_i(E) \\ \mathfrak{H} \downarrow & & \downarrow \mathfrak{H}' \\ H_i(F) & & H_i(E) \\ \mu \downarrow \approx & & \approx \downarrow \nu \\ H_{i-1}(K \times K, K \vee K) & \xrightarrow{\rho} & H_{i-1}(K) \\ \lambda_p \downarrow & & \downarrow \lambda'_p \\ H_{i-1}(K \times K, K \vee K; Z_p) & \xrightarrow{\rho} & H_{i-1}(K, Z_p) \\ \omega \downarrow & & \nearrow m_* \\ H_{i-n-1}(K, Z_p) \otimes H_n(K, Z_p) & + & \\ & & H_n(K, Z_p) \otimes H_{i-n-1}(K, Z_p) \end{array}$$

Here \mathfrak{H} , \mathfrak{H}' are Hurewicz homomorphisms; λ_p , λ'_p denote reduction mod p ; ω is a projection onto a direct summand; and m_* is induced by multiplication in K . $\nu = E^{-1}q_*$, μ is the composition

$$\begin{aligned} H_*(F) &\xrightarrow{h_*^{-1}} H_*(K * K) \xrightarrow{\phi_*} H_*(S(K \times K)) \xrightarrow{j_*} H_*(S(K \times K), S(K \vee K)) \\ &\xrightarrow{E^{-1}} H_{i-1}(K \times K, K \vee K) \end{aligned}$$

and ϕ_* and ρ are defined in [2]. Of these homomorphisms we need only the following information: for $2n+2 < i \leq 3n$, the p -primary component $\pi_i(F)_p$ is a Z_p -module which is mapped monomorphically by $\chi' = \omega \lambda_p \mu \mathfrak{H}$, the image of χ' is invariant under T_* and i_* is an isomorphism for $i > n+1$. For the remainder of this section let $2n+2 < i \leq 3n$ and let p be an odd prime. Let $\mathfrak{H}'_p = \lambda_p \mathfrak{H}'_i : \pi_i(E) \rightarrow H_i(E, Z_p)$. We shall use (5.4) to prove:

(5.5) LEMMA. A basis g_1, \dots, g_s for $\pi_i(E)_p$, can be chosen such that g_1, \dots, g_r is a basis for $\ker \mathfrak{H}'_p$ and $\chi(g_j)$ has the form $b \otimes \gamma_j \pm \gamma_j \otimes b$ for $j=1, \dots, s$, where $\chi = \chi' i_*^{-1}$ and $b \in H_n(K, Z_p)$ is a generator.

Proof. Suppose that we have chosen g_1, \dots, g_q satisfying the lemma. Let g be an element which is not in the subgroup $[g_1, \dots, g_q]$ spanned by g_1, \dots, g_q , with the further restriction that $g \in \ker \mathfrak{H}'_p$ provided $\ker \mathfrak{H}'_p \subset [g_1, \dots, g_q]$. Then $\chi(g) = b \otimes \gamma_2 + \gamma_1 \otimes b$ for some γ_1, γ_2 . Since $T_* \chi(g) = (-1)^{ni} (b \otimes \gamma_2 + \gamma_1 \otimes b)$, by invariance of $\chi(\pi_i(E))$ under T_* , there exists $g' \in \pi_i(E)_p$ such that $\chi(g') = b \otimes \gamma_2 + \gamma_1 \otimes b$. Then

$$\begin{aligned}\chi(g + g') &= b \otimes (\gamma_1 + \gamma_2) + (\gamma_1 + \gamma_2) \otimes b, \\ \chi(g - g') &= b \otimes (\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2) \otimes b.\end{aligned}$$

Also, it cannot be the case that both $g+g'$ and $g-g'$ lie in $[g_1, \dots, g_q]$; for then their sum $2g \in [g_1, \dots, g_q]$ and since $p \neq 2$, g also lies in this subgroup, which violates the hypothesis. Hence we have found at least one new basis element with the desired image under χ , and it only remains to show that if $g \in \ker \mathfrak{H}'_p$, then so are $g+g'$ and $g-g'$. But if $g \in \ker \mathfrak{H}'_p$, then

$$(5.6) \quad \lambda'_p \nu \mathfrak{H}'(g') = m_* \chi(g') = \pm m_* \chi(g) = \pm \lambda'_p \nu \mathfrak{H}'(g) = 0.$$

Hence $\mathfrak{H}'_p(g') = 0$, and both $g+g'$ and $g-g'$ lie in $\ker \mathfrak{H}'_p$.

We have thus shown that the partial basis g_1, \dots, g_q can be extended to satisfy the conditions of the lemma. Since $\pi_i(E)_p$ is finitely generated, this proves (5.5).

Let $g \in \pi_i(E)_p$ be one of the generators in Lemma (5.5), with $\chi(g) = b \otimes \gamma + (-1)^q \gamma \otimes b$. If $g \in \ker \mathfrak{H}'_p$, then

$$0 = \lambda'_p \nu \mathfrak{H}'(g) = m_* \chi(g) = b \cdot \gamma + (-1)^q \gamma \cdot b = (1 + (-1)^{q+n}) b \cdot \gamma.$$

Now $b \cdot \gamma \neq 0$ [4, Theorem 3], so $(-1)^q = -(-1)^{n_i}$. Thus

$$T_* \chi(g) = (-1)^{n_i} (\gamma \otimes b - (-1)^{n_i} (b \otimes \gamma)) = -\chi(g).$$

On the other hand, if $g \notin \ker \mathfrak{H}'_p$, then

$$0 \neq \lambda'_p \nu \mathfrak{H}'(g) = m_* \chi(g) = (1 + (-1)^{q+n}) b \cdot \gamma,$$

so $(-1)^q = (-1)^{n_i}$. Then $T_* \chi(g) = \chi(g)$.

We may express the condition $g \in \ker \mathfrak{H}'_p$ in another way by considering a Postnikov system in the sense of §4 for SK , up to the term E_{3n} , in which the groups π_i^t are cyclic of prime order (recall that $2n+2 < i \leq 3n$), and $k_i^s \neq 0$ for $s \leq t$, $k_i^s = 0$ for $t < s \leq r$. Suppose that $\pi_i^q = Z_p$ for some $q \leq t$. Then by (4.3), since $k_i^q \neq 0$, $\mathfrak{H}' : \pi_i^q \rightarrow H_i(E_i^q)$ is not a monomorphism onto a direct summand, and hence \mathfrak{H}'_p maps π_i^q to zero.

Now let $p : E_i^t \rightarrow E_i^t$ be the projection, with fibre F , and let $g : E_i^t \rightarrow E_i^t$ be a cross-section. Then

$$\pi_i(E_i^t) = g_* \pi_i(E_i^t) + i_* \pi_i(F).$$

According to the above, \mathfrak{H}'_p is zero on π_i^q for $q \leq t$, and hence on the p -primary component $g_* \pi_i(E_i^t)_p$. On the other hand, the Postnikov invariant of the fibering $E_i^t \rightarrow E_i^t$ is zero, and hence by (4.3) \mathfrak{H}' is a monomorphism of $\ker p_* = i_* \pi_i(F)_p$ onto a direct summand, and therefore so is \mathfrak{H}'_p .

Thus d_i , which is the dimension of $\ker \mathfrak{H}'_p$ in $\pi_i(E)_p$, or equivalently the dimension of $\ker \mathfrak{H}'_p$ in $\pi_i(E_i^t)_p$, is equal to the number of cyclic invariants $k_i^q \neq 0 \pmod{p}$.

From the value of T_* on the image of χ we can obtain that of T_* on $\pi_{i-1}(K \times K)_p$ by noting that the image of the monomorphism

$$\pi_{i-1}(K \times K)_p \xrightarrow{\lambda_p \mathcal{H}} H_{i-1}(K \times K, Z_p) \xrightarrow{\approx} H_{i-1}(K \times K, K \vee K; Z_p)$$

followed by ω is equal to the image of χ , and that T_* commutes with this composition. The above calculations yield:

(5.7) PROPOSITION. *Let $K = K(G, n)$, G cyclic of prime power or infinite order. Then $\pi_{i-1}(K \times K)_p$, $2n+2 < i \leq 3n$, admits a cyclic decomposition such that $T_* = -1$ on a submodule of dimension d_i and $T_* = 1$ on a supplementary submodule. Here $d_i = \dim \ker \mathcal{H}'_p: \pi_i(SK)_p \rightarrow H_i(SK, Z_p)$, which is also equal to the number of (cyclic) Postnikov invariants $k_i^*(SK) \not\equiv 0 \pmod{p}$.*

For the lowest two dimensions, $\mathcal{H}: \pi_{2n+q}(K \times K) \rightarrow H_{2n+q}(K \times K)$ is an isomorphism, $q = 0, 1$; elementary calculation in the homology groups gives:

(5.8) ADDENDUM. $T_* = (-1)^{n+q}$ in $\pi_{2n+q}(K \times K)$, $q = 0, 1$.

(5.9) PROPOSITION. *The stable suspension in the p -primary component $E_\infty: \pi_i(SK)_p \rightarrow \Pi_{i-1}(K)_p$ is an epimorphism for $i \leq 3n$, p an odd prime. If in addition $i > 2n+2$, then $\ker E_\infty$ has dimension d_i . If $i = 2n+2$, then $\ker E_\infty = 0$ if n is odd, and $\Pi_{i-1}(K)_p = 0$ if n is even. If $i = 2n+1$, then $\ker E_\infty = 0$ if n is even, and $\Pi_{i-1}(K)_p = 0$ if n is odd.*

Using (2.1) and the above values for T_* , the result for $i < 3n$ can be read off from the spectral sequence. For $i = 3n$, note that

$$\pi_{3n}(SK; C_+, C_-) \approx \pi_{3n}(SK) \approx \pi_{3n}(K * K) \approx \pi_{3n+1}(SK \times SK),$$

the first two isomorphisms coming from the suspension triad sequence and the fibering of [2] respectively. Hence $H: \pi_{3n}(SK; C_+, C_-) \rightarrow \pi_{3n+1}(SK \times SK)$ is an epimorphism of a finite group onto an isomorphic copy, and is therefore an isomorphism. This allows (2.1) to be applied in one higher dimension than would otherwise be possible.

We shall use a Postnikov system for $SK(G, n)$ as in §4, with $E_{n+1} = K(G, n+1)$, in order to determine which cyclic invariants are nonzero mod p . Recall from [2] that the invariants are known in the form $k_i^* = \tilde{\beta}_i^{s-1} * \tilde{k}_i^*$, where $\tilde{k}_i^* \in H^{i+1}(K(G, n+1), \pi_i^*)$ and $\tilde{\beta}_i^{s-1}: E_i^{s-1} \rightarrow K(G, n+1)$ is the composite fibering. There are two cases: 1. If $G = Z_p$, then $\pi_{2n+1} = Z_{p'} = \pi_{2n+2}$, where $\pi_i = \pi_i(SK(G, n))$, while the higher homotopy groups within the range we consider are Z_p -modules. The classes \tilde{k}_{2n+1} and \tilde{k}_{2n+2} mod p are $b \cdot b$ and $b \cdot \beta(p')b$ respectively, where $b \in H^{n+1}(K(G, n+1), Z_p)$ is the basic class mod p , and $\beta(p')$ is the f th order Bockstein. The \tilde{k}_i^* for $2n+2 < i \leq 3n$ are of the form $b \cdot \theta_i^*$, where θ_i^* runs through a basis for $H^{i-n}(K(G, n+1), Z_p)$. 2. If $G = Z$, then $\pi_{2n+1} = Z$ and $\pi_{2n+2} = 0$, while the higher homotopy groups within the range we consider are finite, with p -primary components which are Z_p -modules. \tilde{k}_{2n+1} mod p is equal to $b \cdot b$, while the \tilde{k}_i^* mod p for $2n+2 < i \leq 3n$ are of the

form $b \cdot \text{St}_p^I b$, where $\text{St}_p^I b$ runs through those elements of the Cartan basis [4] for $H^{i-n}(K(G, n+1), Z_p)$ such that the Steenrod operation St_p^I does not end with a Bockstein. Let \overline{H}^{i-n} denote the subgroup with this basis.

Thus in order to determine which k_i^s are nonzero mod p , we must find the intersection of $H^{n+1} \cdot H^{i-n}$, or $H^{n+1} \cdot \overline{H}^{i-n}$, with the kernel of

$$\tilde{p}_{i-1}^*: H^{i+1}(K(G, n+1), Z_p) \rightarrow H^{i+1}(E_{i-1}, Z_p).$$

Since $\tilde{p}_{i-1} = p_{2n+1}^1 \cdots p_{i-1}^r$, we need only compute the kernel of each p_j^{s*} in turn. Now the sequence

$$H^q(K(\pi_j^s, j), Z_p) \xrightarrow{\tau} H^{q+1}(E_j^{s-1}, Z_p) \xrightarrow{\tilde{p}_j^{s*}} H^{q+1}(E_j^s, Z_p)$$

is exact within the range we are considering, and if $j > 2n+2$, then either π_j^s has order p , or has order prime to p , in which case we may disregard it. If $\pi_j^s = Z_p$, then $H^*(K(\pi_j^s, j), Z_p)$ is, in the stable range, just $A(b_j^s)$, where A is the mod p Steenrod algebra and b_j^s is the basic class. Thus

$$\ker p_j^{s*} = \tau A(b_j^s) = A\tau(b_j^s) = A(k_j^s) = A(\tilde{p}_j^{s-1*} \bar{k}_j^s) = \tilde{p}_j^{s-1*} A(\bar{k}_j^s).$$

For the lower dimensions there are two cases:

1. $G = Z_{p^f}$. $H^*(K(Z_{p^f}, j), Z_p)$ is, in the stable range, just $A(b_j) + A(\beta(p^f)b_j)$, where b_j is the basic class mod p (actually we need only use the subalgebra of A generated by those elements which do not have a Bockstein on the right). The kernel of p_{2n+1}^* is therefore $A(b \cdot b) + A(b \cdot \beta(p^f)b)$ if n is odd, and $A(b \cdot b)$ if n is even. Similarly the kernel of $(p_{2n+1}p_{2n+2})^*$ is

$$A(b \cdot b) + A(b \cdot \beta(p^f)b) + A\beta(p^f)(b \cdot \beta(p^f)b).$$

The last term has zero intersection with $H^{n+1} \cdot H^*$, and may therefore be omitted.

2. $G = Z$. Then $\ker p_{2n+1}^* = A(b \cdot b)$.

Putting together the above calculations, we have:

(5.10) PROPOSITION. For $2n+2 < i \leq 3n$, p an odd prime, d_i is the dimension of the following Z_p -module, which is a factor module of a submodule of $H^{i+1}(K(G, n+1), Z_p)$:

$$H^{n+1} \cdot H^{i-n} / A \left(H^{n+1} \cdot \sum_{n+1}^{i-n-1} H^i \right) \cap H^{n+1} \cdot H^{i-n}$$

where A is the mod p Steenrod algebra, and

1. If $G = Z_{p^f}$, then $H^i = H^i(K(G, n+1), Z_p)$;

2. If $G = Z$, then H^i is the subgroup of $H^i(K(G, n+1), Z_p)$ with Cartan basis elements [4] $\text{St}_p^I b$ such that the Steenrod operation St_p^I does not end with a Bockstein on the left.

The explicit calculation of d_i is purely mechanical.

(5.11) EXAMPLE. For m large, the first few stable groups $\Pi_r(K(Z_p, m))$ are:

r	m odd	m even
m	Z_p	Z_p
$2m$	0	Z_p
$2m+1$	Z_p	0
$2m+2(p-1)$	Z_p	0
$2m+2(p-1)+1$	Z_p	Z_p

(5.12) PROPOSITION. *The stable Postnikov invariants $k_{i-1+r}(S^r(K(\pi, n)))$, $i \leq 3n$, r large, are all zero mod C_2 , the class of finite Abelian groups with order a power of 2.*

Proof. Let $2n+2 < i \leq 3n$, and let p be an odd prime. By (4.4), those Z_p -summands π_i^s of $\pi_i(SK(\pi, n))$ corresponding to zero invariants k_i^s suspend monomorphically, and with a choice of Postnikov system for $S^r K(\pi, n)$, the corresponding invariants $k_{i-1+r}^s(S^r K(\pi, n))$ are zero. But by (5.9), the stable suspension $E_\infty: \pi_i(SK(\pi, n))_p \rightarrow \Pi_{i-1}(K(\pi, n))_p$ maps the remaining summands to zero, and is an epimorphism. This proves (5.12) for $i > 2n+2$; in the bottom dimensions the proof is essentially the same.

The situation is no longer so simple for the 2-primary component. Firstly, E_∞ is not necessarily an epimorphism on $\pi_r(SK(\pi, n))_2$; for example, it can be seen from the spectral sequence that the cokernel of E_∞ acting on $\pi_{2n+2}(SK(Z_2, n))$ is Z_2 . Secondly, the Postnikov invariants are not all zero. A short calculation shows that

$$k_{2n+2}(S^2 K(Z_2, n)) = Sq^{n+1}b \in H^{2n+3}(K(Z_2, n+2), Z_2).$$

This is already in the stable range.

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