

# ON UNIQUENESS THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS AND FOR PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE<sup>(1)</sup>

BY

J. B. DIAZ AND W. L. WALTER

**1. Introduction.** The present note contains proofs of uniqueness theorems for the ordinary differential equation  $y' = f(x, y)$ , and for the hyperbolic partial differential equation  $u_{xy} = f(x, y, u, u_x, u_y)$ , under what may be called Nagumo uniqueness conditions. Reference is made to Kamke [5] for description and literature concerning the uniqueness condition on  $f(x, y)$  (which is less restrictive than the Lipschitz condition) which was introduced into the theory of the ordinary differential equation  $y' = f(x, y)$  by Nagumo [1]. In connection with the partial differential equation  $u_{xy} = f(x, y, u, u_x, u_y)$ , uniqueness conditions more general than the Lipschitz condition were introduced by Walter [7].

§2 deals with the ordinary differential equation, first under the assumption of a Lipschitz condition, and then a Nagumo condition. The argument in the case of the Lipschitz condition is included because it appears to be (under the assumptions used) simpler and more direct than that currently employed in textbooks, which usually rely on the theory of the definite integral to some extent (see, however, Carathéodory [2], who treats a more general case). The argument in the case of the Nagumo condition is also independent of the theory of the definite integral, and is to be compared with the proofs in Kamke [5] and Perron [4]. §3 contains an extension of a Nagumo type theorem of Walter [7], for the characteristic boundary value problem for  $u_{xy} = f(x, y, u, u_x, u_y)$ .

The main interest of the present considerations seems to lie, not in the results themselves (although one is led to the improvement of known results, e.g. the theorem in §3), but rather in the elementary method employed, which makes use only of the ordinary Lagrange mean value theorem of the differential calculus and of simple properties of real valued continuous functions. The present approach appears to have been overlooked in the theory of the subject. There is no doubt about the applicability of the method in more general situations, e.g. to systems of ordinary and partial differential equations, and to higher order equations, which will not be considered explicitly here.

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2. **The ordinary differential equation  $y' = f(x, y)$ .** (a) *Lipschitz condition.* Suppose that the real valued function  $f(x, y)$  is defined on the strip  $0 \leq x \leq a$ ,  $-\infty < y < +\infty$ , where  $a$  is positive. A "solution" ("in the classical sense") of the ordinary differential equation  $y' = f(x, y)$  "on the interval  $0 \leq x \leq a$ " will be understood to be a real valued function  $y(x)$ , continuous in the closed interval  $0 \leq x \leq a$  and possessing a finite derivative  $y'(x)$  throughout the open interval  $0 < x < a$ , which satisfies  $y'(x) = f(x, y(x))$  for  $0 < x < a$  (i.e., a "solution" is required to be continuous in the closed interval and to satisfy the ordinary differential equation in the open interval).

The function  $f(x, y)$  will be said to satisfy a Lipschitz condition provided that there is a number  $L \geq 0$  such that  $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$  whenever  $0 \leq x \leq a$  and  $-\infty < y_1, y_2 < +\infty$ .

**THEOREM.** *Let  $f(x, y)$  satisfy a Lipschitz condition on  $0 \leq x \leq a$ ,  $-\infty < y < +\infty$ . Given a real number  $y_0$ , there is at most one solution on  $0 \leq x \leq a$  of the ordinary differential equation  $y' = f(x, y)$  such that  $y(0) = y_0$ .*

**Proof.** Suppose that  $u(x)$  is a solution and that  $v(x)$  is also a solution. Then one has  $u'(x) = f(x, u(x))$  and  $v'(x) = f(x, v(x))$  on  $0 < x < a$ , and  $u(0) = v(0) = y_0$ . Suppose, contrary to what one wants to prove, that the function  $u(x) - v(x)$  (which is known to be continuous on  $0 \leq x \leq a$  and differentiable on  $0 < x < a$  and vanishes at  $x = 0$ ) does not vanish throughout  $0 \leq x \leq a$ . Then there exists a number  $\xi$  with  $0 \leq \xi < a$ , such that the absolute value function  $|u(x) - v(x)| = 0$  for  $0 \leq x \leq \xi$ , and that  $|u(x) - v(x)|$  is not identically zero on any  $x$  interval  $\xi \leq x \leq \xi + l \leq a$ , where  $l > 0$ . Choose  $l > 0$  so small that  $lL < 1$ , where  $L$  is the Lipschitz constant of  $f(x, y)$ . Since the function  $|u(x) - v(x)|$  is continuous and not identically zero on the closed interval  $\xi \leq x \leq \xi + l$ , it has a positive maximum at some number, call it  $\xi_m$ , of this interval (notice that  $\xi < \xi_m$ ). Now, applying first the mean value theorem of the differential calculus to the function  $u - v$  on the interval  $\xi \leq x \leq \xi_m$ , and then using the Lipschitz condition for  $f$ , one obtains, in succession, that

$$\begin{aligned} 0 < |u(\xi_m) - v(\xi_m)| &= |[u(\xi_m) - v(\xi_m)] - [u(\xi) - v(\xi)]| \\ &= |\xi_m - \xi| |u'(\xi^*) - v'(\xi^*)| \\ &= (\xi_m - \xi) |f(\xi^*, u(\xi^*)) - f(\xi^*, v(\xi^*))| \\ &\leq lL |u(\xi^*) - v(\xi^*)|, \end{aligned}$$

where  $\xi^*$  is the "mean value abscissa," satisfying  $\xi < \xi^* < \xi_m \leq \xi + l$ ; and also  $0 \leq lL < 1$ , the last by the choice of  $l$ . Thus the positive number  $|u(\xi_m) - v(\xi_m)|$  is not the maximum value of the continuous function  $|u - v|$  on the interval  $\xi \leq x \leq \xi + l$ , which is a contradiction, and the theorem is proved.

(b) *Nagumo condition.* A real valued function  $f(x, y)$ , of the type considered at the beginning of (a) above, will be said to satisfy a Nagumo condition provided that  $x|f(x, y_1) - f(x, y_2)| \leq |y_1 - y_2|$  whenever  $0 \leq x \leq a$  and  $-\infty < y_1, y_2 < +\infty$ .

**THEOREM.** *Let  $f(x, y)$  satisfy a Nagumo condition on  $0 \leq x \leq a$ ,  $-\infty < y < +\infty$ . Given a real number  $y_0$ , suppose further that  $f$  is continuous in  $(x, y)$  at  $(0, y_0)$  (i.e., that  $\lim_{(x,y) \rightarrow (0,y_0)} f(x, y) = f(0, y_0)$ ). Then there is at most one solution on  $0 \leq x \leq a$  of the ordinary differential equation  $y' = f(x, y)$  such that  $y(0) = y_0$ .*

**Proof.** As before, let  $u(x)$  be a solution and  $v(x)$  be a solution, on  $0 \leq x \leq a$ . Consider the function having the value  $|u(x) - v(x)|/x$  on  $0 < x \leq a$ , and the value zero at  $x = 0$ . It will first be shown that this function is continuous on the closed interval  $0 \leq x \leq a$  (the only real question is the continuity at  $x = 0$ ). (This function will be denoted just by  $|u(x) - v(x)|/x$ .) Using the mean value theorem of the differential calculus (notice that  $u - v$  vanishes at  $x = 0$ ), and the differential equations satisfied by  $u$  and  $v$ , it follows that, for  $0 < x \leq a$ :

$$\begin{aligned} \frac{|u(x) - v(x)|}{x} &= \frac{|[u(x) - v(x)] - [u(0) - v(0)]|}{x} \\ &= |u'(\xi^*) - v'(\xi^*)| \\ &= |f(\xi^*, u(\xi^*)) - f(\xi^*, v(\xi^*))|, \end{aligned}$$

where  $0 < \xi^* < x$ . But  $f(x, y)$  is continuous at  $(0, y_0)$ , therefore

$$\lim_{x \rightarrow 0} \frac{|u(x) - v(x)|}{x} = |f(0, y_0) - f(0, y_0)| = 0,$$

as desired.

Now, suppose, contrary to what one wishes to prove, that the function  $u(x) - v(x)$  is not identically zero on  $0 \leq x \leq a$ . Then there exists a number  $\xi_m > 0$  such that the function  $|u(x) - v(x)|/x$  (which is continuous on  $0 \leq x \leq a$  and is understood to have the value zero at  $x = 0$ ) attains its positive maximum over  $0 \leq x \leq a$  at  $x = \xi_m$ ; and, furthermore

$$\frac{|u(x) - v(x)|}{x} < \frac{|u(\xi_m) - v(\xi_m)|}{\xi_m}$$

whenever  $0 \leq x < \xi_m$ . However, by the mean value theorem of the differential calculus, applied to the function  $u - v$  on the interval  $0 \leq x \leq \xi_m$ , and by the Nagumo condition satisfied by  $f$ , one has that

$$\begin{aligned} 0 < \frac{|u(\xi_m) - v(\xi_m)|}{\xi_m} &= \frac{|[u(\xi_m) - v(\xi_m)] - [u(0) - v(0)]|}{\xi_m} \\ &= |u'(\xi^*) - v'(\xi^*)| \\ &= |f(\xi^*, u(\xi^*)) - f(\xi^*, v(\xi^*))| \\ &\leq \frac{|u(\xi^*) - v(\xi^*)|}{\xi^*}, \end{aligned}$$

where the mean value abscissa  $\xi^*$  satisfies  $0 < \xi^* < \xi_m$ . The last inequality is in direct contradiction to the way in which  $\xi_m$  was chosen, and the theorem is proved.

REMARK. The continuity requirement on  $f(x, y)$  made in the last theorem; namely:  $\lim_{(x,y) \rightarrow (0,y_0)} f(x, y) = f(0, y_0)$ , is essential for the validity of the uniqueness theorem. This may be readily seen from the following example. Take  $y_0 = 0$ , for simplicity, and define  $f(x, y)$ , for  $0 \leq x \leq a$ , by

$$f(x, y) = \begin{cases} 1, & \text{for } y > x, \\ y/x, & \text{for } 0 < y \leq x, \\ 0, & \text{for } y \leq 0. \end{cases}$$

The solutions are  $y = Cx$ , where  $0 \leq C \leq 1$ . It is to be noticed that, in order for the example to be valid, the term "solution" must be understood in the sense specified at the beginning of this section. If this term, "solution," is relaxed to mean that, in addition,  $y(x)$  has a finite right hand derivative at  $x = 0$  (let it be denoted by  $y'(0)$ ) which satisfies the ordinary differential equation there (i.e.,  $y'(0) = f(0, y(0))$ ); then, in this modified sense of the word, there is only one solution through  $(0, 0)$  in the example. However, if the word "solution" is understood in this modified sense, then the theorem still holds, *but* the hypothesis concerning the continuity of  $f$  is now superfluous. Because, if  $u(x)$  is a solution, and  $v(x)$  is also a solution (in the modified sense of the word) such that  $u(0) = v(0) = y_0$ , then it follows that  $\lim_{x \rightarrow 0} |u(x) - v(x)|/x = 0$  directly from the fact that  $u'(0) = v'(0)$ , without using the continuity of  $f$  at  $(0, y_0)$ .

3. **The partial differential equation**  $u_{xy} = f(x, y, u, u_x, u_y)$ . Suppose that the real valued function  $f(x, y, u, p, q)$  is defined for  $0 \leq x \leq a, 0 \leq y \leq b, -\infty < u, p, q < +\infty$ , where  $a$  and  $b$  are positive. A "solution" ("in the classical sense") of the hyperbolic partial differential equation  $u_{xy} = f(x, y, u, u_x, u_y)$  "on the rectangle  $0 \leq x \leq a, 0 \leq y \leq b$ " will be understood to be a real valued function  $u(x, y)$  which is continuous, together with its two first order partial derivatives  $u_x$  and  $u_y$ , and its mixed second order partial derivative  $u_{xy}$ , throughout the whole closed rectangle  $0 \leq x \leq a, 0 \leq y \leq b$ . Clearly, by a classical theorem of H. A. Schwarz, any solution  $u(x, y)$  has the property that  $u_{yx}$  exists and equals  $u_{xy}$  on  $0 \leq x \leq a, 0 \leq y \leq b$ .

The function  $f(x, y, u, p, q)$  will be said to satisfy a Nagumo condition provided that, for each  $(x, y)$  in the rectangle  $0 \leq x \leq a, 0 \leq y \leq b$ , there exist numbers  $\alpha(x, y) \geq 0, \beta(x, y) \geq 0, \gamma(x, y) \geq 0$ , with  $\alpha(x, y) + \beta(x, y) + \gamma(x, y) = 1$ , such that

$$xy |f(x, y, u_1, p_1, q_1) - f(x, y, u_2, p_2, q_2)| \leq \alpha(x, y) |u_1 - u_2| + \beta(x, y) x |p_1 - p_2| + \gamma(x, y) y |q_1 - q_2|,$$

whenever  $-\infty < u_1, u_2, p_1, p_2, q_1, q_2 < +\infty$ ; and also

$$x | f(x, 0, u, p, q_1) - f(x, 0, u, p, q_2) | \leq | q_1 - q_2 |,$$

when  $0 \leq x \leq a$ ,  $-\infty < u, p, q_1, q_2 < +\infty$ , and

$$y | f(0, y, u, p_1, q) - f(0, y, u, p_2, q) | \leq | p_1 - p_2 |,$$

when  $0 \leq y \leq b$ ,  $-\infty < u, p_1, p_2, q < +\infty$ . It is remarked that, if  $f(x, y, u, p, q)$  happens to be, in addition, continuous in the variable  $y$  at  $y=0$ , for each fixed  $(x, u, p, q)$ , then the first inequality in the definition of the Nagumo condition actually implies the next inequality, in which  $y=0$ . A similar statement applies to the inequality for  $x=0$  in the definition above. In particular, if  $f$  is continuous for all  $(x, y, u, p, q)$  involved, and  $\alpha, \beta, \gamma$  are constants, one has the Nagumo condition of [7].

**THEOREM.** *Let  $f(x, y, u, p, q)$  satisfy a Nagumo condition on  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ ,  $-\infty < u, p, q < +\infty$ , with  $\alpha(x, y), \beta(x, y), \gamma(x, y)$  continuous on the closed rectangle  $0 \leq x \leq a, 0 \leq y \leq b$ . Given a real valued function  $\sigma(x)$ , defined on  $0 \leq x \leq a$ , and a real valued function  $\tau(y)$ , defined on  $0 \leq y \leq b$ , with  $\sigma(0) = \tau(0)$ , there is at most one solution on  $0 \leq x \leq a, 0 \leq y \leq b$ , of the partial differential equation  $u_{xy} = f(x, y, u, u_x, u_y)$  such that  $u(x, 0) = \sigma(x)$ , for  $0 \leq x \leq a$  and  $u(0, y) = \tau(y)$ , for  $0 \leq y \leq b$ .*

**REMARK.** A similar uniqueness theorem, but under much milder assumptions on the solutions, may be proved for the partial differential equation  $u_{xy} = f(x, y, u)$ , which is, in this sense, more an "analogue" of  $y' = f(x, y)$  than the equation  $u_{xy} = f(x, y, u, u_x, u_y)$  considered here.

**Proof.** (The argument follows, with some essential modifications, the general outline of that of the theorem in (b) of §2.) Suppose that  $u(x, y)$  is a solution and that  $v(x, y)$  is also a solution on  $0 \leq x \leq a, 0 \leq y \leq b$ . Then  $u(x, 0) = v(x, 0) = \sigma(x)$ , and  $u_x(x, 0) = v_x(x, 0) = \sigma'(x)$ , for  $0 \leq x \leq a$ . It will be shown next, using the result of (b) of §2, that also  $u_y(x, 0) = v_y(x, 0)$  for  $0 \leq x \leq a$ . This follows at once from the remark that  $u_y(x, 0)$  and  $v_y(x, 0)$  are solutions of the same ordinary differential equation (in  $x$ ) obtained when one puts  $y=0$  in the partial differential equations satisfied by  $u$  and  $v$ ; and also satisfy the same "initial" condition at  $x=0$ , namely  $u_y(0, 0) = v_y(0, 0) = \tau'(0)$ . Further, the "right hand side" of this ordinary differential equation:

$$w'(x) = f(x, 0, \sigma(x), \sigma'(x), w(x)),$$

satisfies the Nagumo condition of §2, because

$$x | f(x, 0, \sigma(x), \sigma'(x), w) - f(x, 0, \sigma(x), \sigma'(x), w_1) | \leq | w - w_1 |$$

for  $0 \leq x \leq a$ , and  $-\infty < w, w_1 < +\infty$ . Hence  $u_y(x, 0) = v_y(x, 0)$ , for  $0 \leq x \leq a$ , as desired.

Similarly, one has that

$$u(0, y) = v(0, y) = \tau(y),$$

$$u_y(0, y) = v_y(0, y) = \tau'(y),$$

and that

$$u_x(0, y) = v_x(0, y),$$

for  $0 \leq y \leq b$ . Consequently, from the partial differential equation satisfied by  $u(x, y)$ :

$$u_{xy}(x, y) = f(x, y, u(x, y)u_x(x, y), u_y(x, y)),$$

(and from the corresponding equation for  $v(x, y)$ ) it follows also that

$$u_{xy}(x, 0) = v_{xy}(x, 0),$$

for  $0 \leq x \leq a$ , and that

$$u_{xy}(0, y) = v_{xy}(0, y),$$

for  $0 \leq y \leq b$ .

Next, consider the auxiliary function (call it  $F(x, y)$  to abbreviate the writing) defined on the rectangle  $0 \leq x \leq a, 0 \leq y \leq b$ , which has the value

$$\alpha(x, y) \frac{|u(x, y) - v(x, y)|}{xy}$$

$$+ \beta(x, y) \frac{|u_x(x, y) - v_x(x, y)|}{y} + \gamma(x, y) \frac{|u_y(x, y) - v_y(x, y)|}{x}$$

when  $0 < x \leq a$ , and  $0 < y \leq b$ ; and has the value zero when either  $x = 0$  and  $0 \leq y \leq b$ , or when  $0 \leq x \leq a$  and  $y = 0$ . It will now be shown that the auxiliary function  $F(x, y)$  is continuous in the whole closed rectangle. Clearly,  $F(x, y)$  is continuous on  $0 < x \leq a, 0 < y \leq b$ , it only remains to verify that

$$\lim_{(x,y) \rightarrow (\bar{x}, 0)} F(x, y) = 0, \quad 0 \leq \bar{x} \leq a,$$

and that

$$\lim_{(x,y) \rightarrow (0, \bar{y})} F(x, y) = 0, \quad 0 \leq \bar{y} \leq b.$$

Since the functions  $\alpha, \beta$ , and  $\gamma$  are continuous on the closed rectangle, it suffices to prove that the three functions

$$\frac{u(x, y) - v(x, y)}{xy}, \quad \frac{u_x(x, y) - v_x(x, y)}{y} \quad \text{and} \quad \frac{u_y(x, y) - v_y(x, y)}{x},$$

where  $0 < x \leq a, 0 < y \leq b$ , approach zero as  $(x, y)$  approaches  $(\bar{x}, 0)$ , with  $0 \leq \bar{x} \leq a$ , or as  $(x, y)$  approaches  $(0, \bar{y})$ , with  $0 \leq \bar{y} \leq b$ .

One has, by the ordinary mean value theorem of the differential calculus, applied to the function  $u(x, y) - v(x, y)$ , regarded as a function of  $y$ , for fixed  $x > 0$  (recall that  $u(x, 0) - v(x, 0) = 0$ ), that

$$\frac{u(x, y) - v(x, y)}{xy} = \frac{u_y(x, \eta) - v_y(x, \eta)}{x},$$

where  $0 < \eta < y$ . Now, applying the mean value theorem of the differential calculus to the function  $u_y(x, \eta) - v_y(x, \eta)$ , regarded as a function of  $x$ , for fixed  $\eta$  (recall that  $u_y(0, \eta) - v_y(0, \eta) = 0$ ), one obtains

$$\frac{u(x, y) - v(x, y)}{xy} = u_{yx}(\xi, \eta) - v_{yx}(\xi, \eta),$$

where  $0 < \xi < x$ ,  $0 < \eta < y$ . (This last equation is a mean value theorem for the differential operator  $\partial^2/\partial y \partial x$ , see Goursat [3].) Proceeding in a similar manner, one is led to the equality

$$\frac{u_x(x, y) - v_x(x, y)}{y} = u_{xy}(x, \eta^*) - v_{xy}(x, \eta^*),$$

where  $0 < \eta^* < y$ , and to the equation

$$\frac{u_y(x, y) - v_y(x, y)}{x} = u_{yx}(\xi^*, y) - v_{yx}(\xi^*, y),$$

where  $0 < \xi^* < x$ . But the function  $u_{xy} - v_{xy}$  is continuous on the closed rectangle and vanishes when  $x=0$  and when  $y=0$ . Thus the three functions  $(u-v)/xy$ ,  $(u_x-v_x)/y$  and  $(u_y-v_y)/x$  tend to zero as asserted; and this, in turn, implies the desired continuity of the auxiliary function  $F(x, y)$ .

Now for the remainder of the proof. Suppose, contrary to what one wants to prove, that the function  $u(x, y) - v(x, y)$  is not identically zero on the closed rectangle  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ . Then the auxiliary continuous function  $F(x, y)$  described above must have a positive maximum at a point  $(\xi_m, \eta_m)$  such that  $0 < \xi_m \leq a$ ,  $0 < \eta_m \leq b$ ; and that, *further*,  $F(x, y) < F(\xi_m, \eta_m)$  whenever  $0 \leq x + y < \xi_m + \eta_m$  with  $0 \leq x \leq a$  and  $0 \leq y \leq b$ . (To see this, consider the set of points  $(x, y)$  with  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ , at which the value of  $F$  equals the positive maximum of  $F$ , and choose  $(\xi_m, \eta_m)$  as a point of this set for which the continuous function  $x + y$  has a minimum, which must necessarily be positive.)

At this stage one gets a contradiction by applying the mean value theorem twice to the  $\alpha$  "term" of  $F(\xi_m, \eta_m)$ , once to each of the other two terms ( $\beta$  and  $\gamma$ ) of  $F(\xi_m, \eta_m)$ , and then using the Nagumo condition which is satisfied by  $f$ . (It is to be recalled that the three functions  $u-v$ ,  $u_x-v_x$ , and  $u_y-v_y$  vanish for  $0 \leq x \leq a$ ,  $y=0$  and for  $x=0$ ,  $0 \leq y \leq b$ .) One obtains, for example, that

$$\begin{aligned}
\frac{|u(\xi_m, \eta_m) - v(\xi_m, \eta_m)|}{\xi_m \eta_m} &= |u_{xy}(\xi, \eta) - v_{xy}(\xi, \eta)| \\
&= |f(\xi, \eta, u(\xi, \eta), u_x(\xi, \eta), u_y(\xi, \eta)) \\
&\quad - f(\xi, \eta, v(\xi, \eta), v_x(\xi, \eta), v_y(\xi, \eta))| \\
&\leq \alpha(\xi, \eta) \frac{|u(\xi, \eta) - v(\xi, \eta)|}{\xi \eta} \\
&\quad + \beta(\xi, \eta) \frac{|u_x(\xi, \eta) - v_x(\xi, \eta)|}{\eta} \\
&\quad + \gamma(\xi, \eta) \frac{|u_y(\xi, \eta) - v_y(\xi, \eta)|}{\xi} = F(\xi, \eta),
\end{aligned}$$

where  $0 < \xi < \xi_m$  and  $0 < \eta < \eta_m$ . Similarly

$$\begin{aligned}
|u_x(\xi_m, \eta_m) - v_x(\xi_m, \eta_m)| &= |u_{xy}(\xi_m, \eta^*) - v_{xy}(\xi_m, \eta^*)| \\
&= |f(\xi_m, \eta^*, u(\xi_m, \eta^*), u_x(\xi_m, \eta^*), u_y(\xi_m, \eta^*)) \\
&\quad - f(\xi_m, \eta^*, v(\xi_m, \eta^*), v_x(\xi_m, \eta^*), v_y(\xi_m, \eta^*))| \\
&\leq \alpha(\xi_m, \eta^*) \frac{|u(\xi_m, \eta^*) - v(\xi_m, \eta^*)|}{\xi_m \eta^*} \\
&\quad + \beta(\xi_m, \eta^*) \frac{|u_x(\xi_m, \eta^*) - v_x(\xi_m, \eta^*)|}{\eta^*} \\
&\quad + \gamma(\xi_m, \eta^*) \frac{|u_y(\xi_m, \eta^*) - v_y(\xi_m, \eta^*)|}{\xi^*} = F(\xi_m, \eta^*),
\end{aligned}$$

where  $0 < \eta^* < \eta_m$ ; and also

$$|u_y(\xi_m, \eta_m) - v_y(\xi_m, \eta_m)| \leq F(\xi^*, \eta_m),$$

where  $0 < \xi^* < \xi_m$ .

Putting together the last three inequalities, and using the definition of the auxiliary function  $F$ , one obtains

$$F(\xi_m, \eta_m) \leq \alpha(\xi_m, \eta_m)F(\xi, \eta) + \beta(\xi_m, \eta_m)F(\xi_m, \eta^*) + \gamma(\xi_m, \eta_m)F(\xi^*, \eta_m),$$

where  $\xi + \eta$ ,  $\xi_m + \eta^*$  and  $\xi^* + \eta_m$  are less than  $\xi_m + \eta_m$ . This contradicts the choice of  $(\xi_m, \eta_m)$ , since by the definition of  $(\xi_m, \eta_m)$  one must have

$$\left. \begin{aligned}
F(\xi, \eta) \\
F(\xi_m, \eta^*) \\
F(\xi^*, \eta_m)
\end{aligned} \right\} < F(\xi_m, \eta_m);$$

while, at the same time

$$\left. \begin{aligned} \alpha(\xi_m, \eta_m) \\ \beta(\xi_m, \eta_m) \\ \gamma(\xi_m, \eta_m) \end{aligned} \right\} \geq 0,$$

and

$$\alpha(\xi_m, \eta_m) + \beta(\xi_m, \eta_m) + \gamma(\xi_m, \eta_m) = 1,$$

and the proof is complete.

**4. Concluding remarks.** It was mentioned in the introduction that the method of proof employed in this note is also applicable to more complicated problems for ordinary differential equations and for partial differential equations of hyperbolic type. In lieu of the formulation of the general case, which involves a nonlinear integral equation of Volterra type in several variables, and is both cumbersome and not transparent, only several particular examples will be discussed.

(a) Consider the initial value problem for the ordinary differential equation of  $n$ th order:  $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$ ;  $0 \leq x \leq a$ , where the values  $y^{(\nu)}(0) = y_\nu$  ( $\nu = 0, \dots, n-1$ ) are assigned. One can prove uniqueness with a method which is essentially that of §2, provided that  $f$  satisfies the following condition of Nagumo type:

$$x^n |f(x, z, z_1, \dots, z_{n-1}) - f(x, \bar{z}, \bar{z}_1, \dots, \bar{z}_{n-1})| \leq \sum_{\nu=0}^{n-1} (n - \nu) |\alpha_\nu(x) x^\nu| |z_\nu - \bar{z}_\nu|$$

for  $0 \leq x \leq a$  and  $-\infty < z, \bar{z}, z_1, \bar{z}_1, \dots, z_{n-1}, \bar{z}_{n-1} < +\infty$ , where the functions  $\alpha_\nu(x)$  are supposed to be nonnegative and continuous on  $0 \leq x \leq a$  and  $\sum_{\nu=0}^{n-1} \alpha_\nu(x) = 1$ . (For the proof, use the auxiliary function which is zero for  $x=0$  and equals  $\sum_{\nu=0}^{n-1} (n - \nu) |\alpha_\nu(x) x^{\nu-n}| |u^{(\nu)}(x) - v^{(\nu)}(x)|$  for  $0 < x \leq a$ , where  $u(x)$  is a solution and  $v(x)$  is also a solution.)

(b) In the characteristic initial value problem for the equation  $u_{xy} = f(x, y, z, u, u_x, u_y, u_z, u_{xy}, u_{xz}, u_{yz})$ , one looks for a solution  $u = u(x, y, z)$  in a parallelepiped  $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$ , which takes on prescribed initial data  $u(x, y, 0) = \sigma(x, y), u(0, y, z) = \tau(y, z)$ , and  $u(x, 0, z) = \phi(x, z)$ . In this case the uniqueness condition of Nagumo type reads as follows:

$$\begin{aligned} xyz |f(x, y, z, u, u_x, u_y, u_z, u_{xy}, u_{xz}, u_{yz}) - f(x, y, z, \bar{u}, \bar{u}_x, \bar{u}_y, \bar{u}_z, \bar{u}_{xy}, \bar{u}_{xz}, \bar{u}_{yz})| \\ \leq \alpha_1 |u - \bar{u}| + \alpha_2 x |u_x - \bar{u}_x| + \alpha_3 y |u_y - \bar{u}_y| + \alpha_4 z |u_z - \bar{u}_z| \\ + \alpha_5 xy |u_{xy} - \bar{u}_{xy}| + \alpha_6 xz |u_{xz} - \bar{u}_{xz}| + \alpha_7 yz |u_{yz} - \bar{u}_{yz}|, \end{aligned}$$

where  $\alpha_i$  ( $i = 1, \dots, 7$ ) are supposed to be continuous nonnegative functions of  $(x, y, z)$  on  $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$ , whose sum is equal to unity. If  $f$  is required to be continuous in  $(x, y, z)$  for each (fixed) set of its remaining independent variables, then uniqueness can be proved along the lines of §3. If this assumption about  $f$  is not made, then in the formulation of the Nagumo

condition one needs additional inequalities, similar to those of §3, which ensure the uniqueness of the partial derivatives  $u_x, u_y, u_z, u_{xy}, u_{xz}, u_{yz}$  on the three edges through  $(0, 0, 0)$  and on the faces  $x=0, y=0$  and  $z=0$  of the parallelepiped; the exact formulation is left to the reader.

(c) An example of a higher order equation in several variables is  $u_{zzzy} = f(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{zz}, u_{zxy})$ . Here one looks (for example) for a solution  $u(x, y)$  in the rectangle  $0 \leq x \leq a, 0 \leq y \leq b$ , which assumes the following prescribed initial values:  $u(x, 0) = \sigma(x)$  and  $u(0, y) = \tau(y), u_x(0, y) = \tau_1(y), u_{xx}(0, y) = \tau_2(y)$ , where  $\sigma(0) = \tau(0), \sigma'(0) = \tau_1(0), \sigma''(0) = \tau_2(0)$ . Uniqueness can be proved if  $f$  satisfies the following condition of Nagumo type:

$$x^3y \left| f(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{zzz}, u_{zxy}) - f(x, y, \bar{u}, \bar{u}_x, \bar{u}_y, \bar{u}_{zzz}, \bar{u}_{zxy}, \bar{u}_{zxy}) \right| \leq 6\alpha_1 |u - \bar{u}| + 2\alpha_2 x |u_x - \bar{u}_x| + 6\alpha_3 y |u_y - \bar{u}_y| + \alpha_4 x^2 |u_{xx} - \bar{u}_{xx}| + 2\alpha_5 xy |u_{xy} - \bar{u}_{xy}| + \alpha_6 x^3 |u_{zzz} - \bar{u}_{zzz}| + \alpha_7 x^2 y |u_{zxy} - \bar{u}_{zxy}|,$$

where again the  $\alpha_i$  are continuous nonnegative functions and  $\alpha_1 + \dots + \alpha_7 = 1$ . As in §2, this condition has to be accompanied by additional conditions on the characteristics  $x=0$  and  $y=0$  if  $f$  is not required to be continuous in  $x$  and  $y$  when its remaining independent variables are held fixed.

(d) It should be noticed that systems of differential equations can be treated in the same manner. If, for example, a system of ordinary differential equations  $y' = f(x, y)$  is given, where  $y$  is a vector in  $R_n: y = (y_1, \dots, y_n)$  and  $f$  is a vector-valued function; and if a norm

$$\|y\| = c_1 |y_1| + \dots + c_n |y_n|, \quad (c_i > 0, i = 1, \dots, n),$$

is defined in  $R_n$ , then the uniqueness condition of Nagumo reads exactly as in §2:  $x\|f(x, y) - f(x, \bar{y})\| \leq \|y - \bar{y}\|$ . The reasoning is precisely the same as in §2, except that the *single* "mean value abscissa"  $\xi^*$  may now be obtained in the following slightly modified manner:

$$\begin{aligned} \frac{\|u(\xi_m) - v(\xi_m)\|}{x} &= \frac{1}{x} \sum_{i=1}^n c_i |u_i(\xi_m) - v_i(\xi_m)| = \frac{1}{x} \sum_{i=1}^n \epsilon_i c_i [u_i(\xi_m) - v_i(\xi_m)] \\ &= \sum_{i=1}^n c_i \epsilon_i [u_i(\xi^*) - v_i(\xi^*)] \leq \|u'(\xi^*) - v'(\xi^*)\|, \end{aligned}$$

where each number  $\epsilon_i$  is either  $+1$  or  $-1$ , as the particular circumstances may demand.

(e) It was shown by Perron [4] that the uniqueness theorem of Nagumo for the ordinary differential equation  $y' = f(x, y)$  is a "best" theorem in the following sense. For every  $C > 1$  there exist bounded continuous functions  $f(x, y)$  for which the inequality  $x\|f(x, y) - f(x, \bar{y})\| \leq C\|y - \bar{y}\|$  holds and for which the initial value problem  $y' = f(x, y); y(0) = y_0$ , has more than one solution. In [7] one of the authors showed that the theorem of §3 is also a

best theorem in a similar sense. An analogous statement can be made about the more general uniqueness theorems indicated in this section. The construction of counter examples resembles that of [4] and [7].

*Added in proof*, May 10, 1960: At the conclusion of a lecture given by one of us at RIAS, in Baltimore, in October 1959, Professor Philip Hartmen kindly drew our attention to results, related to those of §3 above, obtained by his student, Mr. J. Shanahan, which had been submitted to *Mathematische Zeitschrift*.

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UNIVERSITY OF MARYLAND,  
COLLEGE PARK, MARYLAND