ON UNIFORMIZATION OF SETS IN TOPOLOGICAL SPACES

BY

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1. **Introduction.** Let $Q \subset X \times Y$ and P be the projection of Q onto X. A uniformization of Q is a function f on P to Y such that $f \subset Q$, i.e., for every $x \in P$, $(x, f(x)) \in Q$. Clearly with the aid of the axiom of choice, such a function can always be found. The problem, in general, is to find an f that satisfies, in addition, certain conditions. Many authors have considered different aspects of this problem and we mention a few in the bibliography (see also [5, p. 55 and on]).

In case X and Y are Euclidean spaces one important result, due to Luzin [3] and Sierpinski [6] is the following: if Q is Borel, then f can be taken to be the complement of an analytic set in $X \times Y$. Kondô [2] has extended this result to the case where Q is the complement of an analytic set, thereby deriving important properties of projective sets of class 2.

In this paper, we take X and Y to be topological spaces satisfying certain conditions. If Q is compact, then we obtain a uniformization f that is a Borel function (Theorem 4.2). If Q is analytic, then f is a measurable function for a large class of measures on X (Theorem 4.5).

2. **Notation and basic definitions.**

2.1. A non-negative integer n contains all smaller non-negative integers, i.e., $m \in n$ iff $m < n$ and m is a non-negative integer. Thus, 0 denotes both the empty set and the smallest non-negative integer.

2.2. ω denotes the set of all non-negative integers.

2.3. f is a function on A to B iff $f \subset A \times B$ and, for every $x \in A$, there exists a unique y $(=f(x))$ such that $(x, y) \in f$.

2.4. A is Borel H iff A belongs to the smallest family containing H and closed under countable unions and differences of two sets.

2.5. $K(X)$ denotes the family of all compact sets in X.

2.6. A is analytic in X iff, for some Hausdorff space X', A is the continuous image of a $K_{\sigma\delta}(X')$.

2.7. $\mathfrak{A}(X)$ denotes the family of all analytic sets in X.

2.8. If $A \subset X \times Y$, then πA is the projection of A onto X.

2.9. Y satisfies condition (I) iff Y is Hausdorff, regular, has a base whose power is at most that of the first noncountable cardinal, and every family of open sets in Y has a countable subfamily with the same cover.

3. **Preliminary lemmas.**

3.1. *If, for every $n \in \omega$, A_n is compact, $A_{n+1} \subset A_n$, $\bigcap_{n\in\omega} A_n \subset B$, f is a continuous function on B to a Hausdorff space, then*

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$$f\left(\bigcap_{n\in\omega} A_n\right) = \bigcap_{n\in\omega} f(B \cap A_n).$$

Proof. Suppose $y\in\bigcap_{n\in\omega} f(B\cap A_n)$. Then, for every $n\in\omega$, $f^{-1}\{y\}\cap B\cap A_n \neq 0$. Since f is continuous on B, we have $f^{-1}\{y\} = B\cap C$ for some closed set C. Thus, for every $n\in\omega$, $C\cap A_n\neq 0$ and hence

$$f^{-1}\{y\} \cap \bigcap_{n\in\omega} A_n = \bigcap_{n\in\omega} (C \cap A_n) \neq 0$$

i.e., $y\in f(\bigcap_{n\in\omega} A_n)$.

The converse is obvious.

3.2. *Suppose, for every* $i\in I$, $A_i\subset X\times Y$ *and, for every* $x\in X$, *the family* $\bigcup_{i\in I} \{\pi^{-1}\{x\}\cap A_i\}$ *forms a nest of compact sets in the product topology. Then*

$$\bigcap_{i\in I} \pi A_i = \pi \bigcap_{i\in I} A_i.$$

Proof. Let $x\in\bigcap_{i\in I} \pi A_i$. Then, for every $i\in I$, $\pi^{-1}\{x\}\cap A_i\neq 0$ and since the intersection of a nest of nonvoid compact sets is nonvoid, we have

$$\bigcap_{i\in I} (\pi^{-1}\{x\} \cap A_i) \neq 0,$$

i.e.,

$$x \in \pi \bigcap_{i\in I} A_i.$$

The converse is obvious.

4. Uniformization of sets.

4.1. Theorem. *Suppose* X *is any space*; H *is any family of subsets of* X; Y *satisfies condition* (I); $Q\subset X\times Y$; *for every* $x\in X$, $\pi^{-1}\{x\}\cap Q$ *is compact*; *for every closed set* B *in* Y, $\pi((X\times B)\cap Q)\in H$. *Then there exists a function* f *on* πQ *such that* $f\subset Q$ *and, for any open set* V *in* Y, $f^{-1}(V)$ *is Borel* H.

Proof. Let Ω be the first noncountable ordinal and $\bigcup_{i\in\Omega} \{U_i\}$ be a base for Y. Let, for $\alpha\subset X\times Y$ and $\beta\subset X\times Y$,

$$P(\alpha, \beta) = (\alpha \cap \beta) \cup [((X - \pi(\alpha \cap \beta)) \times Y) \cap \beta].$$

Next, by recursion for every $i\in\Omega$, we define A_i as follows:

$$A_0 = Q,$$
$$A_{i+1} = P(X \times \overline{U}_i, A_i),$$
$$A_i = \bigcap_{j<i} A_j \qquad \text{if } i \text{ is a limit ordinal},$$

where $\overline{U}_i$ denotes the closure of U_i. Finally, we set

$$f = \bigcap_{i \in \Omega} A_i$$

and check the following points.

(i) $P(\alpha, \beta) \subset \beta$ and $\pi P(\alpha, \beta) = \pi\beta$.

(ii) If $i \in \Omega$, $j \in \Omega$, $j < i$, then $A_i \subset A_j$.

(iii) If $i \in \Omega$ and $x \in X$, then $\pi^{-1}\{x\} \cap A_i$ is compact.

(iv) If B' is closed in Y, $B = X \times B'$ and $I \subset \Omega$, then

$$\pi\left(B \cap \bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} \pi(B \cap A_i)$$

Proof. Apply 3.2.

(v) If $i \in \Omega$ and $i < j \in \Omega$ then

$$\pi((X \times \overline{U}_i) \cap A_i) = \pi((X \times \overline{U}_i) \cap A_j) = \pi((X \times \overline{U}_i) \cap f)$$

Proof. For the first equality, we use induction on j. Let

$$B = \pi((X \times \overline{U}_i) \cap A_i)$$

then

$$(B \times Y) \cap A_{i+1} \subset B \times \overline{U}_i.$$

Hence, in view of (ii), if $i < j \in \Omega$, we have

$$(B \times Y) \cap A_j \subset (B \times \overline{U}_i) \cap A_j$$

and

$$(B \times \overline{U}_j) \cap A_j \subset (B \times \overline{U}_i \cap \overline{U}_j) \cap A_j.$$

Now, if the first equality holds for a given such j, then

$$\begin{aligned}
&\pi((X \times \overline{U}_i) \cap A_{j+1}) \\
&\quad = \pi((X \times (\overline{U}_i \cap \overline{U}_j)) \cap A_j) \cup (\pi((X \times \overline{U}_i) \cap A_j) - \pi((X \times \overline{U}_j) \cap A_j)) \\
&\quad = \pi((B \times (\overline{U}_i \cap \overline{U}_j)) \cap A_j) \cup (\pi((B \times \overline{U}_i) \cap A_j) \\
&\qquad - \pi((B \times (\overline{U}_i \cap \overline{U}_j)) \cap A_j)) \\
&\quad = \pi((B \times \overline{U}_i) \cap A_j) \\
&\quad = B
\end{aligned}$$

so that the equality holds for $j+1$. If j is a limit ordinal and the equality holds for all j' such that $i < j' < j$, then by (iv) it also holds for j. The second equality follows from the first and (iv).

(vi) $\pi f = \pi Q$.

Proof. Apply (i), (iv) and induction or take $U_0 = Y$ and apply (v).

(vii) f is a function.

Proof. Suppose $(x, y)\in f$ and $(x, y')\in f$ and $y\neq y'$. Let $y\in U_i$, $y'\in U_j$, $\overline{U}_i\cap\overline{U}_j=0$, $i<j$. Then by (v) we have

$$x \in \pi((X \times \overline{U}_i) \cap A_i) \quad \text{and} \quad x \in \pi((X \times \overline{U}_j) \cap A_j),$$

but

$$A_j \subset A_{i+1} = P(X \times \overline{U}_i, A_i) \subset ((X \times \overline{U}_i) \cap A_i) \cup ((X - \{x\}) \times Y).$$

Hence

$$(X \times \overline{U}_j) \cap A_j \subset (X - \{x\}) \times Y,$$

i.e.

$$x \notin \pi((X \times \overline{U}_j) \cap A_j)$$

in contradiction to the above.

(viii) If $i\in\Omega$ and B' is closed in Y and $B=X\times B'$ then $\pi(B\cap A_i)$ is Borel H.

Proof. We use induction on i. It is clearly true for $i=0$. Suppose it is true for j and $i=j+1$. Then:

$$\begin{aligned}
&\pi(B \cap A_i)\\
&\quad = \pi(B \cap P(X \times \overline{U}_j, A_j))\\
&\quad = \pi(B \cap (X \times \overline{U}_j) \cap A_j)\\
&\qquad \cup\, \pi(B \cap ((X - \pi((X \times \overline{U}_j) \cap A_j)) \times Y) \cap A_j)\\
&\quad = \pi(B \cap (X \times \overline{U}_j) \cap A_j) \cup (\pi(B \cap A_j) - \pi((X \times \overline{U}_j) \cap A_j))
\end{aligned}$$

so that $\pi(B\cap A_i)$ is Borel H.

If i is a limit ordinal and the result is true for all $j<i$, by (iv), we have

$$\pi(B \cap A_i) = \pi\left(\bigcap_{j<i} (B \cap A_j)\right) = \bigcap_{j<i} \pi(B \cap A_j)$$

and the last set is Borel H since the set of all $j<i$ is countable.

(ix) If V is open in Y, then $f^{-1}(V)$ is Borel H.

Proof. Let

$$I = \{i \in \Omega: \overline{U}_i \subset V\}.$$

Then there exists a countable $I'\subset I$ such that

$$\bigcup_{i\in I'} U_i = \bigcup_{i\in I} U_i = V.$$

Hence

$$\bigcup_{i\in I'} \overline{U}_i = V.$$

Then

$$f^{-1}(V) = \pi((X \times V) \cap f) = \bigcup_{i \in I'} \pi((X \times \overline{U}_i) \cap f)$$
$$= (by(\text{v})) = \bigcup_{i \in I'} \pi((X \times \overline{U}_i) \cap A_i)$$

so that $f^{-1}(V)$ is Borel H, in view of (viii).

4.2. COROLLARY. *Suppose X is any topological space; Y satisfies condition* (I); *Q is compact in $X \times Y$. Then there exists a function f on πQ such that $f \subset Q$ and, for every open set V in Y, $f^{-1}(V)$ is Borel $K(X)$. (If in addition, Q is metrizable then f can be chosen to be of Baire first class.)*

4.3. THEOREM. *Suppose X and Y are Hausdorff and Q is analytic in $X \times Y$. Then there exists Q' such that: $Q' \subset Q$; $\pi Q' = \pi Q$; for every $x \in X$, $\pi^{-1}\{x\} \cap Q'$ is compact; for every closed set B in $X \times Y$, $\pi(B \cap Q')$ is Borel $\mathfrak{A}(X)$.*

Proof. Since Q is analytic in $X \times Y$, there exists a Hausdorff space Z such that Q is the continuous image of a $K_{\sigma\delta}(Z)$. For every $n \in \omega$ and $i \in \omega$, let $d(n, i)$ be compact in Z,

$$D = \bigcap_{n \in \omega} \bigcup_{i \in \omega} d(n, i),$$

f be a continuous function on D, and $Q = f(D)$. Next, let S_n be the set of all $(n+1)$-tuples $k = (k_0, \cdots, k_n)$ with $k_i \in \omega$. For every $n \in \omega$ and $k \in S_n$, let

$$\Delta(k) = f\left(D \cap \bigcap_{i \in (n+1)} d(i, k_i)\right),$$

and then define $A(k)$ by recursion on n as follows: if $k \in S_0$, then

$$A(k) = \Delta(k) \cap \left(\left(X - \bigcup_{j \in k} \pi\Delta(j)\right) \times Y\right).$$

If $k \in S_{n+1}$, then

$$A(k) = A(k_0, \cdots, k_n) \cap \Delta(k)$$
$$\cap \left(\left(X - \bigcup_{j \in k_{n+}} \pi(A(k_0, \cdots, k_n) \cap \Delta(k_0, \cdots, k_n, j)\right) \times Y\right).$$

Finally, we set

$$A_n' = \bigcup_{k \in S_n} A(k),$$
$$Q' = \bigcap_{n \in \omega} A_n',$$

and check the following points.

(i) If $n \in \omega$, $k \in S_n$, $k' \in S_n$, $k \neq k'$, then

$$\pi A(k) \cap \pi A(k') = 0.$$

(ii) For every $n \in \omega$ and $k \in S_n$,

$$A(k) \subset \Delta(k) = \bigcup_{j\in\omega} \Delta(k_0, \cdots, k_n, j),$$

and

$$Q = \bigcup_{j\in\omega} \Delta(j).$$

(iii) For every $n \in \omega$, $\pi A_n' = \pi Q$.

Proof. We use induction on n. Let $x \in \pi Q$. Then by (ii)

$$x \in \bigcup_{j\in\omega} \pi\Delta(j).$$

Let j_0 be the first integer such that $x \in \pi\Delta(j_0)$. Then,

$$x \in \pi A(j_0) \subset \pi A_0'.$$

Next, suppose for some $k \in S_n$, $x \in \pi A(k)$. Then by (ii)

$$x \in \bigcup_{j\in\omega} \pi(A(k) \cap \Delta(k_0, \cdots, k_n, j)).$$

Let j_0 be the first integer such that

$$x \in \pi(A(k) \cap \Delta(k_0, \cdots, k_n, j_0)).$$

Then

$$x \in \pi A(k_0, \cdots, k_n, j_0) \subset \pi A_{n+1}'.$$

Thus, for every $n \in \omega$, $\pi Q \subset \pi A_n'$. The converse is immediate.

(iv) For every $x \in X$ there exists a unique sequence k such that for every $n \in \omega$:

$$\pi^{-1}\{x\} \cap A_n' = \pi^{-1}\{x\} \cap A(k_0, \cdots, k_n) = \pi^{-1}\{x\} \cap \Delta(k_0, \cdots, k_n).$$

Proof. Use induction on n and (i).

(v) If B is closed in $X \times Y$ then

$$\pi(B \cap Q') = \bigcap_{n\in\omega} \pi(B \cap A_n').$$

Proof. Suppose

$$x \in \bigcap_{n\in\omega} \pi(B \cap A_n').$$

Let k be the sequence given by (iv). Then for every $n \in \omega$:

$$0 \neq \pi^{-1}\{x\} \cap B \cap A_n' = \pi^{-1}\{x\} \cap B \cap \Delta(k_0, \cdots, k_n).$$

Since f is continuous on D and $\pi^{-1}\{x\}\cap B$ is closed, there exists a closed set C in Z such that

$$f^{-1}(\pi^{-1}\{x\} \cap B) = D \cap C.$$

Recalling the definition of $\Delta(k_0, \cdots, k_n)$ we conclude

$$D \cap C \cap \bigcap_{i\in(n+1)} d(i, k_i) \neq 0.$$

Since the $d(i, k_i)$ are compact, we have

$$\bigcap_{n\in\omega} \left(C \cap \bigcap_{i\in(n+1)} d(i, k_i)\right) \neq 0.$$

Therefore

$$\pi^{-1}\{x\} \cap B \cap f\left(\bigcap_{n\in\omega} \bigcap_{i\in(n+1)} d(i, k_i)\right) \neq 0$$

and, by 3.1,

$$\pi^{-1}\{x\} \cap B \cap \bigcap_{n\in\omega} f\left(D \cap \bigcap_{i\in(n+1)} d(i, k_i)\right) \neq 0$$

i.e.,

$$\bigcap_{n\in\omega} (\pi^{-1}\{x\} \cap B \cap \Delta(k_0, \cdots, k_n)) \neq 0$$

Hence, by (iv)

$$\bigcap_{n\in\omega} (\pi^{-1}\{x\} \cap B \cap A_n') \neq 0$$

i.e.,

$$x \in \pi \bigcap_{n\in\omega} (B \cap A_n') = \pi(B \cap Q').$$

Thus,

$$\bigcap_{n\in\omega} \pi(B \cap A_n') \subset \pi(B \cap Q').$$

The converse is immediate.

(vi) If B is either closed or analytic in $X \times Y$ then, for every $n\in\omega$ and $k\in S_n$, $\pi(B\cap A(k))$ is Borel $\mathfrak{A}(X)$.

Proof. We use induction on n. We first observe that the intersection of two analytic sets, the intersection of an analytic set with a closed set, and the continuous image of an analytic set are all analytic. Thus, for every $n\in\omega$ and $k\in S_n$, $\pi(B\cap\Delta(k))$ is analytic in X. Now, if $k\in S_0$, since

$$\pi(B \cap A(k)) = \pi(B \cap \Delta(k)) - \bigcup_{j\in k} \pi\Delta(j)$$

we see that $\pi(B\cap A(k))$ is the difference of two analytic sets in X. Next, suppose the result holds for a given $n\in\omega$ and let $k\in S_{n+1}$. Then

$$\pi(B \cap \Delta(k_0, \cdots, k_n, j) \cap A(k_0, \cdots, k_n))$$

is Borel $\mathfrak{A}(X)$ for every $j\in\omega$, and since

$$\pi(B \cap A(k)) = \pi(B \cap \Delta(k) \cap A(k_0, \cdots, k_n)) - \bigcap_{j\in k_{n+1}} \pi(A(k_0, \cdots, k_n) \cap \Delta(k_0, \cdots, k_n, j)),$$

we see that $\pi(B\cap A(k))$ is the difference of two Borel $\mathfrak{A}(X)$ sets.

(vii) For every closed set B in $X\times Y$, $\pi(B\cap Q')$ is Borel $\mathfrak{A}(X)$.

Proof. By (v) we have:

$$\pi(B \cap Q') = \bigcap_{n\in\omega} \pi(B \cap A_n') = \bigcap_{n\in\omega} \bigcup_{k\in S_n} \pi(B \cap A(k)).$$

Since S_n is countable, the result follows from (vi).

(viii) For every $x\in X$, $\pi^{-1}\{x\}\cap Q'$ is compact.

Proof. Let k be the sequence given by (iv). Then

$$\pi^{-1}\{x\} \cap Q' = \bigcap_{n\in\omega} (\pi^{-1}\{x\} \cap A_n') = \bigcap_{n\in\omega} (\pi^{-1}\{x\} \cap \Delta(k_0, \cdots, k_n))$$

$$= \pi^{-1}\{x\} \cap \bigcap_{n\in\omega} f\left(D \cap \bigcap_{i\in(n+1)} d(i, k_i)\right) = \text{(by 3.1)}$$

$$= \pi^{-1}\{x\} \cap f\left(\bigcap_{i\in\omega} d(i, k_i)\right)$$

so that $\pi^{-1}\{x\}\cap Q'$ is the intersection of a closed set with a compact set.

4.4. COROLLARY. *Suppose X is Hausdorff, Y satisfies condition* (I), *and Q is analytic in $X\times Y$. Then there exists a function f on πQ such that $f\subset Q$ and, for every open set V in Y, $f^{-1}(V)$ is Borel $\mathfrak{A}(X)$.*

4.5. COROLLARY. *Suppose X is Hausdorff, Y satisfies condition* (I), *and Q is analytic in $X\times Y$. Then there exists a function f on πQ such that $f\subset Q$ and, for every open set V in Y, $f^{-1}(V)$ is μ-measurable for all Carathéodory outer measures μ on X such that closed sets are μ-measurable.*

Proof. In this case (see [7]), the analytic sets in X can be obtained by applying Souslin's operation $\mathfrak{a}$ to the family of closed sets in X. Hence they are μ-measurable whenever closed sets are μ-measurable. (See e.g., S. Saks, *Theory of the integral*, p. 50.)

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