SOME HILBERT SPACES OF ENTIRE FUNCTIONS

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A Hilbert space, whose elements are entire functions, is of especial interest if it satisfies three axioms.

(H1) Whenever $F(z)$ is in the Hilbert space and w is a nonreal zero of $F(z)$, the function $F(z)(z-\bar{w})/(z-w)$ is in the Hilbert space and has the same norm as $F(z)$.

(H2) Whenever w is a nonreal complex number, the linear functional defined on the Hilbert space by $F(z)\to F(w)$, which gives each function in the Hilbert space its value at w, is continuous.

(H3) Whenever $F(z)$ is in the Hilbert space, $F^*(z)=\bar{F}(\bar{z})$ is in the Hilbert space and has the same norm as $F(z)$.

If $E(z)$ is an entire function such that $|E(\bar{z})|<|E(z)|$ for $y>0$, the set $\mathcal{H}(E)$ of entire functions $F(z)$ such that

$$\|F\|_E^2 = \int |F(t)|^2 |E(t)|^{-2} dt < \infty$$

and

$$|F(z)|^2 \leq \|F\|_E^2 \frac{|E(z)|^2 - |E(\bar{z})|^2}{2\pi i(\bar{z}-z)}$$

is a Hilbert space of entire functions which satisfies H1, H2, and H3. By the theorem of [7], every nonzero Hilbert space of entire functions, which satisfies these axioms, is equal isometrically to some such $\mathcal{H}(E)$, where the function $E(z)$ can be chosen in various ways.

THEOREM I. *Let $E_a(z)=C_a(z)-iS_a(z)$ be an entire function such that $|E_a(\bar{z})|<|E_a(z)|$ for $y>0$, where $C_a(z)$ and $S_a(z)$ are entire functions which are real for real z. Let C_0, S_0, C_1, S_1 be real numbers such that $C_0C_1+S_0S_1=1$. Let $E_b(z) = C_b(z) - iS_b(z)$ where $C_b(z) = C_a(z)C_0 - S_a(z)S_1$ and $S_b(z) = C_a(z)S_0 + S_a(z)C_1$. Then, $E_b(z)$ is an entire function such that $|E_b(\bar{z})|<|E_b(z)|$ for $y>0$, and $\mathcal{H}(E_a)=\mathcal{H}(E_b)$ isometrically. If $E(z)$ is an entire function such that $|E(\bar{z})|<|E(z)|$ for $y>0$ and $\mathcal{H}(E_a)=\mathcal{H}(E)$ isometrically, then $E(z)=E_b(z)$ for some such choice of C_0, S_0, C_1, S_1.*

We use the word "Hilbert space," as Halmos [8], to mean "complete inner product space," without any restriction on dimension. We use triangular brackets for the Hilbert space inner product, often with the letter t used like

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a dummy variable of integration within the brackets. For each complex number w,

$$K(w, z) = \frac{\bar{E}(w)E(z) - E(\bar{w})E^*(z)}{2\pi i(\bar{w} - z)} \tag{1}$$

is the unique element of $\mathcal{H}(E)$ such that

$$F(w) = \langle F(t), K(w, t)\rangle_E \tag{2}$$

for each $F(z)$ in $\mathcal{H}(E)$.

By Lemma 7 of [7], there is at most one real number α modulo π such that $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ belongs to $\mathcal{H}(E)$.

THEOREM II. *Let $E(z)$ be an entire function such that $|E(\bar{z})| < |E(z)|$ for $y > 0$, and let $G(z)$ belong to $\mathcal{H}(E)$. A necessary and sufficient condition that the orthogonal complement of $G(z)$ in $\mathcal{H}(E)$ be, in the norm of $\mathcal{H}(E)$, a Hilbert space of entire functions satisfying* (H1), (H2), *and* (H3) *and that for each real number w there is an element $F(z)$ of $\mathcal{H}(E)$ orthogonal to $G(z)$ such that $F(w)/E(w) \neq 0$ is that $G(z)$ be a constant multiple of $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ for some real number α and that $G(z)$ not span $\mathcal{H}(E)$.*

Certain Hilbert spaces, containing only entire functions of exponential type, are of especial interest.

THEOREM III. *Let $E(z)$ be an entire function such that $|E(\bar{z})| < |E(z)|$ for $y > 0$, and let w be a complex number. A necessary and sufficient condition that $[F(z) - F(w)]/(z - w)$ belong to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ is that $E(z)$ have exponential type, and*

$$\int \frac{\log^+ |E(t)|}{1 + t^2}\, dt < \infty \tag{3}$$

and

$$\int (1 + t^2)^{-1} |E(t)|^{-2} dt < \infty. \tag{4}$$

In this special class of Hilbert spaces, certain constructions are possible which are relevant for the later Theorems VI and VII.

THEOREM IV. *Let $E_0(z) = C_0(z) - iS_0(z)$ be an entire function such that $|E_0(\bar{z})| < |E_0(z)|$ for $y > 0$ where $C_0(z)$ and $S_0(z)$ are entire functions which are real for real z. A necessary and sufficient condition that there exist an entire function $E_1(z) = C_1(z) - iS_1(z)$ such that $|E_1(\bar{z})| < |E_1(z)|$ for $y > 0$, where $C_1(z)$ and $S_1(z)$ are entire functions which are real for real z and*

$$C_0(z)C_1(z) + S_0(z)S_1(z) = 1 \tag{5}$$

and

$$\text{(6)} \qquad \operatorname{Re}\left[C_0(z)\bar{C}_1(z) + S_0(z)\bar{S}_1(z)\right] \geqq 1,$$

is that $E_0(z)$ *have exponential type and* (3) *and* (4) *hold for* $E(z)=E_0(z)$. *In this case,* $E_1(z)$ *can be chosen so that* $E_1(iy)=o(yE_0(iy))$ *as* $y\to+\infty$, *and then* $E_1(z)$ *is uniquely determined within an added imaginary multiple of* $E_0(z)$.

The discovery in [7] of Hilbert spaces of entire functions satisfying our axioms came as a result of the formula of [6] for mean squares of entire functions. In case the entire functions are polynomials, these spaces are implicit in the discussions of the Hamburger moment problem by Shohat and Tamarkin [10] and Stone [11]. A comparison of our results with this earlier work led us to conjecture the next theorem.

THEOREM V.A. *Let* $E(z)$ *be an entire function such that* $|E(\bar{z})| < |E(z)|$ *for* $y>0$. *Let* μ *be a non-negative measure on the Borel sets of the real line. A necessary and sufficient condition that*

$$\|F\|_E^2 = \int |F(t)|^2 |E(t)|^{-2} d\mu(t)$$

for every $F(z)$ *in* $\mathcal{H}(E)$ *is that*

$$\text{(7)} \qquad \frac{y}{\pi}\int \frac{d\mu(t)}{(t-x)^2+y^2} = \operatorname{Re}\frac{E(z)+E^*(z)A(z)}{E(z)-E^*(z)A(z)}$$

for $y>0$, *where* $A(z)$ *is defined and analytic for* $y>0$ *and* $|A(z)|\leqq 1$.

THEOREM V.B. *Let* $E(z)$ *be an entire function such that* $|E(\bar{z})| < |E(z)|$ *for* $y>0$. *Let* μ *be a non-negative measure on the Borel sets of the real line and let* $a>0$ *be a number such that*

$$\frac{y}{\pi}\int \frac{d\mu(t)}{(t-x)^2+y^2} + ay = \operatorname{Re}\frac{E(z)+E^*(z)A(z)}{E(z)-E^*(z)A(z)}$$

for $y>0$, *where* $A(z)$ *is defined and analytic for* $y>0$ *and* $|A(z)|\leqq 1$. *Then there is some real number* α (*unique modulo* π) *such that* $e^{i\alpha}E(z)-e^{-i\alpha}E^*(z)$ *belongs to* $\mathcal{H}(E)$ *and there is some nonzero constant multiple* $G(z)$ *of this function such that*

$$\|F\|_E^2 = \int |F(t)|^2 |E(t)|^{-2} d\mu(t) + |\langle F, G\rangle_E|^2$$

for every $F(z)$ *in* $\mathcal{H}(E)$.

THEOREM VI. *Let* $E_0(z)$ *be an entire function of exponential type such that* $|E_0(\bar{z})| < |E_0(z)|$ *for* $y>0$ *and* (3) *and* (4) *hold for* $E(z)=E_0(z)$. *We suppose that there is no real number* α *such that* $e^{i\alpha}E_0(z)-e^{-i\alpha}E_0^*(z)$ *belongs to* $\mathcal{H}(E_0)$. *Let* $E_1(z)$ *correspond to* $E_0(z)$ *as in Theorem* IV, *so that* (5) *and* (6) *hold and*

$E_1(iy) = o(yE_0(iy))$ as $y \to +\infty$. Let μ be a non-negative measure on the Borel sets of the real line. A necessary and sufficient condition $\|F\|_{E_0}^2 = \int |F(t)|^2 d\mu(t)$ for every $F(z)$ in $\mathcal{H}(E_0)$ is that

$$(8) \qquad \frac{y}{\pi}\int \frac{d\mu(t)}{(t-x)^2+y^2} = \operatorname{Re} \frac{E_1(z) + E_1^*(z)A(z)}{E_0(z) - E_0^*(z)A(z)}$$

for $y>0$, where $A(z)$ is defined and analytic for $y>0$ and $|A(z)| \leqq 1$.

These formulas for mean squares of entire functions will be used to find conditions that one Hilbert space of entire functions be contained isometrically in another.

THEOREM VII. *Let $E_a(z) = C_a(z) - iS_a(z)$ be an entire function such that $|E_a(\bar{z})| < |E_a(z)|$ for $y>0$, where $C_a(z)$ and $S_a(z)$ are entire functions which are real for real z. Let $E_0(z) = C_0(z) - iS_0(z)$ and $E_1(z) = C_1(z) - iS_1(z)$ be entire functions such that $|E_0(\bar{z})| \leqq |E_0(z)|$ and $|E_1(\bar{z})| \leqq |E_1(z)|$ for $y>0$, where $C_0(z)$, $S_0(z)$, $C_1(z)$, $S_1(z)$ are entire functions which are real for real z and satisfy (5) and (6). We suppose that if there is some real number α such that $e^{i\alpha}E_a(z) - e^{-i\alpha}E_a^*(z)$ belongs to $\mathcal{H}(E_a)$, then*

$$(9) \qquad E_1(iy)\cos\alpha - iE_0(iy)\sin\alpha = o\{y[E_0(iy)\cos\alpha - iE_1(iy)\sin\alpha]\}$$

as $y \to +\infty$. Let $E_b(z) = C_b(z) - iS_b(z)$ where

$$(10) \qquad \begin{aligned} C_b(z) &= C_a(z)C_0(z) - S_a(z)S_1(z), \\ S_b(z) &= C_a(z)S_0(z) + S_a(z)C_1(z). \end{aligned}$$

Then, $|E_b(\bar{z})| < |E_b(z)|$ for $y>0$ and $\mathcal{H}(E_a)$ is contained isometrically in $\mathcal{H}(E_b)$. Furthermore, $\mathcal{H}(E_a)$ is a separating subspace of $\mathcal{H}(E_b)$ in the sense that for each real w, there is some $F(z)$ in $\mathcal{H}(E_a)$ such that $F(w)/E_b(w) \neq 0$.

THEOREM VIII. *Let $E_a(z)$ and $E(z)$ be entire functions such that $|E_a(\bar{z})| < |E_a(z)|$ and $|E(\bar{z})| < |E(z)|$ for $y>0$. If $\mathcal{H}(E_a)$ is contained isometrically in $\mathcal{H}(E)$ and is a separating subspace of $\mathcal{H}(E)$, then $E(z) = E_b(z)$ for some choice of $E_0(z)$ and $E_1(z)$ as in Theorem* VII.

THEOREM IX. *In the situation of Theorem* VII, *we suppose that there is no real number β such that $e^{i\beta}E_b(z) - e^{-i\beta}E_b^*(z)$ belongs to $\mathcal{H}(E_b)$. A necessary and sufficient condition that*

$$\|F\|_{E_b}^2 = \int |F(t)|^2 |E_a(t)|^{-2} d\mu(t)$$

for every $F(z)$ in $\mathcal{H}(E_b)$ is that

$$(11) \qquad \frac{y}{\pi}\int \frac{d\mu(t)}{(t-x)^2+y^2} = \operatorname{Re} \frac{E_c(z) + E_c^*(z)A(z)}{E_b(z) - E_b^*(z)A(z)}$$

for $y>0$, *where* $E_c(z)=C_c(z)-iS_c(z)$ *and*

$$C_c(z) = C_a(z)C_1(z) - S_a(z)S_0(z),\qquad S_c(z) = C_a(z)S_1(z) + S_a(z)C_0(z) \tag{12}$$

and $A(z)$ *is defined and analytic for* $y>0$ *and* $|A(z)|\leqq 1$.

The discussion of Sturm-Liouville differential equations by Stone [11] and of the Hamburger moment problem by Shohat and Tamarkin [10] and Stone [11] leads to the situation of Theorem VII.

THEOREM X.A. *Let* C_0, S_0, C_1, S_1 *be real numbers such that* $C_0C_1+S_0S_1=1$. *Let* $a(t)$, $b(t)$, $c(t)$ *be locally integrable real-valued functions of* $t\geqq 0$ *such that* $a(t)\geqq 0$, $c(t)\geqq 0$, *and* $b^2(t)\leqq a(t)c(t)$. *For each complex* z, *there exist unique absolutely continuous functions* $C_0(t, z)$, $S_0(t, z)$, $C_1(t, z)$, $S_1(t, z)$ *of* $t\geqq 0$ *such that* $C_0(0, z)=C_0$, $S_0(0, z)=S_0$, $C_1(0, z)=C_1$, $S_1(0, z)=S_1$ *and*

$$\begin{aligned} C_0'(t, z) &= -za(t)S_0(t, z) - zb(t)C_0(t, z),\\ S_0'(t, z) &= zb(t)S_0(t, z) + zc(t)C_0(t, z),\\ C_1'(t, z) &= -zc(t)S_1(t, z) + zb(t)C_1(t, z),\\ S_1'(t, z) &= -zb(t)S_1(t, z) + za(t)C_1(t, z) \end{aligned} \tag{13}$$

a.e. for $t\geqq 0$. *For each* $t\geqq 0$, $C_0(t, z)$, $S_0(t, z)$, $C_1(t, z)$, $S_1(t, z)$ *are entire functions which are real for real* z *and satisfy* (5) *and* (6). *If* $E_0(t, z)=C_0(t, z)-iS_0(t, z)$ *and* $E_1(t, z)=C_1(t, z)-iS_1(t, z)$, *then* $|E_0(t, \bar{z})|\leqq|E_0(t, z)|$ *and* $|E_1(t, \bar{z})|\leqq|E_1(t, z)|$ *for* $y>0$.

THEOREM X.B. *Let* C_0, S_0, C_1, S_1 *be real numbers such that* $C_0C_1+S_0S_1=1$. *Let* a_n, b_n, c_n *be real numbers defined for* $n=0, 1, 2, \cdots$, *such that* $a_n\geqq 0$, $c_n\geqq 0$, *and* $b_n^2=a_nc_n$. *Let* $C_0(n, z)$, $S_0(n, z)$, $C_1(n, z)$, $S_1(n, z)$ *be the polynomials in* z *defined inductively by*

$$\begin{aligned} C_0(n, z) &= C_0 - z\textstyle\sum_{m<n} a_mS_0(m, z) - z\sum_{m<n} b_mC_0(m, z),\\ S_0(n, z) &= S_0 - z\textstyle\sum_{m<n} b_mS_0(m, z) + z\sum_{m<n} c_mC_0(m, z),\\ C_1(n, z) &= C_1 - z\textstyle\sum_{m<n} c_mS_1(m, z) + z\sum_{m<n} b_mC_1(m, z),\\ S_1(n, z) &= S_1 - z\textstyle\sum_{m<n} b_mS_1(m, z) + z\sum_{m<n} aC_1(m, z). \end{aligned} \tag{14}$$

Then for each n, $C_0(n, z)$, $S_0(n, z)$, $C_1(n, z)$, $S_1(n, z)$ *are entire functions which are real for real* z *and satisfy* (5) *and* (6). *If* $E_0(n, z)=C_0(n, z)-iS_0(n, z)$ *and* $E_1(n, z)=C_1(n, z)-iS_1(n, z)$, *then* $|E_0(n, \bar{z})|\leqq|E_0(n, z)|$ *and* $|E_1(n, \bar{z})|\leqq|E_1(n, z)|$ *for* $y>0$.

Our proofs use the Poisson representation of a function positive and harmonic in a half plane, discussed by Loomis and Widder [9]. We use a lemma on linear fractional transformations.

LEMMA 1. *A necessary and sufficient condition that the linear fractional transformation*

$$w = \frac{Az + B}{Cz + D}, \qquad (AD - BC \neq 0),$$

map the upper half plane $i(\bar{z}-z) \geqq 0$ *into the upper half plane* $i(\bar{w}-w) \geqq 0$ *is that* $i(\bar{A}C - A\bar{C}) \geqq 0$, $i(\bar{B}D - B\bar{D}) \geqq 0$, *and* $A\bar{D} - B\bar{C} + \bar{A}D - \bar{B}C \geqq 2|AD - BC|$.

The basic Theorem V results from a study of closed linear isometric transformations in Hilbert space. Given a closed linear isometric transformation in a Hilbert space, we study the unitary extensions in a possibly larger Hilbert space. The next three lemmas set notation.

LEMMA 2. *Let* $\mathcal{H}$ *be a Hilbert space and let* U *be a unitary transformation in* $\mathcal{H}$, *and let* $\mathcal{C}$ *be a closed subspace of* $\mathcal{H}$. *For each complex number* w *in the unit disk* $|w| < 1$, *there exists a unique bounded operator* $B(w)$ *on* $\mathcal{C}$ *such that*

$$\left\langle \frac{1 + wB(w)}{1 - wB(w)}\, a,\, c \right\rangle = \left\langle \frac{1 + wU^*}{1 - wU^*}\, a,\, c \right\rangle \tag{15}$$

for a *and* c *in* $\mathcal{C}$. *The operator valued function* $B(z)$ *is defined and analytic for* $|z| < 1$ *and* $\|B(z)\| \leqq 1$.

In the statement of Lemma 2, we use a star for the adjoint and the norm symbol for the operator norm.

LEMMA 3. *Let* $\mathcal{C}$ *be a Hilbert space. For each* $i = 1, 2$, *let* $\mathcal{H}_i$ *be a Hilbert space containing* $\mathcal{C}$, *let* U_i *be a unitary transformation in* $\mathcal{H}_i$, *and let* $B_i(z)$ *be defined as in Lemma* 2. *Suppose that the smallest closed subspace of* $\mathcal{H}_i$, *which contains* $\mathcal{C}$ *and reduces* U_i, *is all of* $\mathcal{H}_i$. *If* $B_1(z) = B_2(z)$, *there exists a unique linear isometric transformation* T *of* $\mathcal{H}_1$ *onto* $\mathcal{H}_2$, *which is the identity on* $\mathcal{C}$, *and such that for* f *in* $\mathcal{H}_1$, $TU_1f = U_2Tf$.

LEMMA 4. *Let* $\mathcal{C}$ *be a Hilbert space and let* $B(z)$ *be an operator valued function, defined and analytic for* $|z| < 1$, *such that* $\|B(z)\| \leqq 1$. *Then, there exists a Hilbert space* $\mathcal{H}(B)$ *containing* $\mathcal{C}$ *and a unitary transformation* U *in* $\mathcal{H}(B)$ *such that* (15) *holds for* a *and* c *in* $\mathcal{C}$ *and* $|w| < 1$, *and such that the smallest closed subspace of* $\mathcal{H}(B)$, *which contains* $\mathcal{C}$ *and reduces* U, *is all of* $\mathcal{H}(B)$.

In the situation of Lemma 4, we may construct a closed linear isometric transformation S in $\mathcal{H}(B)$ by restricting U to elements f of $\mathcal{H}(B)$ such that Uf is orthogonal to $\mathcal{C}$. Then, $\mathcal{C}$ is the orthogonal complement of the range of S. The transformation U is a unitary extension of S in $\mathcal{H}(B)$. We will look at unitary extensions of S in a Hilbert space containing $\mathcal{H}(B)$.

THEOREM XI. *Let* $\mathcal{C}$ *be a Hilbert space and let* $B_1(z)$ *and* $B_2(z)$ *be operator valued functions, defined and analytic for* $|z| < 1$, *such that* $\|B_i(z)\| \leqq 1$ $(i = 1, 2)$. *A necessary condition that there exist a linear isometric transformation* T *of*

$\mathcal{H}(B_2)$ into $\mathcal{H}(B_1)$, which is the identity on $\mathcal{C}$, and such that U_1Tf is orthogonal to $\mathcal{C}$ and $U_1Tf = TU_2f$, whenever f is in $\mathcal{H}(B_2)$ and U_2f is orthogonal to $\mathcal{C}$, is that $B_1(z) = B_2(z)B_3(z)$, for some operator valued function $B_3(z)$, defined and analytic for $|z| < 1$, such that $\|B_3(z)\| \leq 1$.

We were unable to show that the necessary condition of Theorem XI is, in general, sufficient, but it is in a special case. Let $\mathcal{C}(z)$ be the Hilbert space of all power series $f(z) = \sum a_n z^n$ with coefficients a_n in $\mathcal{C}$ such that $\|f\|^2 = \sum \|a_n\|^2 < \infty$. The index of summation ranges in the non-negative integers.

THEOREM XII. *If, in the situation of Theorem* XI, *$B_2(z)c$, $zB_2(z)c$, $z^2B_2(z)c, \cdots$, is an orthonormal set in $\mathcal{C}(z)$ for every unit vector c in $\mathcal{C}$, the necessary condition of Theorem* XI *is also sufficient.*

LEMMA 5. *Let $\mathcal{C}$ be a Hilbert space and let $f(z) = \sum a_n z^n$ be in $\mathcal{C}(z)$. Then, for $|w| < 1$, the series $f(w) = \sum a_n w^n$ converges in the metric of $\mathcal{C}$ and $(1 - |w|^2)\|f(w)\|^2 \leq \|f\|^2$. The formula*

$$f(w) = \left\langle f(t), \frac{1}{1 - t\bar{w}} \right\rangle \tag{16}$$

is valid in the sense that for each c in $\mathcal{C}$, $(1 - z\bar{w})^{-1}c = \sum \bar{w}^n c z^n$ is in $\mathcal{C}(z)$ and $\langle f(w), c\rangle = \langle f(t), (1 - t\bar{w})^{-1}c\rangle$ with the inner product taken in $\mathcal{C}(z)$.

Lemma 5 allows us to think of our power series as $\mathcal{C}$-valued functions defined and analytic in the unit disk $|z| < 1$. We use the symbol $f(z)$ ambiguously for the formal power series or for the analytic function.

LEMMA 6. *Let $\mathcal{C}$ be a Hilbert space and let $\mathcal{H}$ be a closed subspace of $\mathcal{C}(z)$ containing $\mathcal{C}$ and with the property that $f(z)/z$ belongs to $\mathcal{H}$ whenever $f(z)$ belongs to $\mathcal{H}$ and $f(0) = 0$. Then, the orthogonal complement $\mathcal{M}$ of $\mathcal{H}$ in $\mathcal{C}(z)$ is orthogonal to $\mathcal{C}$ and has the property that $zg(z)$ belongs to $\mathcal{M}$ whenever $g(z)$ belongs to $\mathcal{M}$.*

LEMMA 7. *Let $\mathcal{C}$ be a Hilbert space and let $\mathcal{M}$ be a closed subspace of $\mathcal{C}(z)$ orthogonal to $\mathcal{C}$ and with the property that $zg(z)$ belongs to $\mathcal{M}$ whenever $g(z)$ belongs to $\mathcal{M}$. Then, the orthogonal complement $\mathcal{H}$ of $\mathcal{M}$ in $\mathcal{C}(z)$ is a closed subspace of $\mathcal{C}(z)$ containing $\mathcal{C}$ and with the property that $f(z)/z$ belongs to $\mathcal{H}$ whenever $f(z)$ belongs to $\mathcal{H}$ and $f(0) = 0$.*

LEMMA 8. *Let $\mathcal{C}$ be a Hilbert space and let $\mathcal{H}$ and $\mathcal{M}$ be as in Lemmas* 6 *and* 7. *A necessary and sufficient condition that an element $f(z)$ of $\mathcal{C}(z)$ belong to $\mathcal{H}$ and be orthogonal to every element $g(z)$ of $\mathcal{H}$ such that $zg(z)$ belongs to $\mathcal{H}$ is that $zf(z)$ belong to $\mathcal{M}$ and be orthogonal to every series $zh(z)$ where $h(z)$ belongs to $\mathcal{M}$.*

By an operator, we mean a bounded linear transformation of $\mathcal{C}$ into itself.

LEMMA 9. *Let $\mathcal{C}$ be a Hilbert space and let $B(z) = \sum B_n z^n$ be a formal power series with operator coefficients such that $B(z)c$, $zB(z)c$, $z^2B(z)c, \cdots$, is an* *orthonormal set in* $\mathcal{C}(z)$ *for every unit vector c in* $\mathcal{C}$. *If* $f(z)$ *is in* $\mathcal{C}(z)$, *the formal power series product* $g(z)=B(z)f(z)$ *is in* $\mathcal{C}(z)$ *and* $\|g\|=\|f\|$.

LEMMA 10. *In the situation of Lemma* 9, *let* $\mathfrak{M}(B)$ *be the set of formal power series of the form* $g(z)=zB(z)f(z)$ *where* $f(z)$ *ranges in* $\mathcal{C}(z)$. *Then,* $\mathfrak{M}(B)$ *is a closed subspace of* $\mathcal{C}(z)$ *orthogonal to* $\mathcal{C}$ *and with the property that* $zg(z)$ *belongs to* $\mathfrak{M}(B)$ *whenever* $g(z)$ *belongs to* $\mathfrak{M}(B)$. *An element* $g(z)$ *of* $\mathfrak{M}(B)$ *is orthogonal to every series* $zh(z)$, *where* $h(z)$ *ranges in* $\mathfrak{M}(B)$, *if and only if* $g(z)=zB(z)c$ *for some c in* $\mathcal{C}$, *and then* $\|g\|=\|c\|$. *The series* $B(w)=\sum B_n w^n$ *converges in the operator norm for* $|w|<1$ *and* $\|B(w)\|\leqq 1$. *If* $f(z)$ *is in* $\mathfrak{M}(B)$,

$$f(w) = \left\langle f(t), \frac{t\bar{w}B(t)B^*(w)}{1 - t\bar{w}} \right\rangle \tag{17}$$

for $|w|<1$. *The interpretation is that for each c in* $\mathcal{C}$, $z\bar{w}(1-z\bar{w})^{-1}B(z)B^*(w)c$ *is in* $\mathfrak{M}(B)$ *and* $\langle f(w), c\rangle = \langle f(t), t\bar{w}(1-t\bar{w})^{-1}B(t)B^*(w)c\rangle$.

LEMMA 11. *In the situation of Lemmas* 9 *and* 10, *let* $\mathcal{H}(B)$ *be the orthogonal complement of* $\mathfrak{M}(B)$ *in* $\mathcal{C}(z)$. *Then,* $\mathcal{H}(B)$ *is a closed subspace of* $\mathcal{C}(z)$ *containing* $\mathcal{C}$ *and with the property that* $f(z)/z$ *belongs to* $\mathcal{H}(B)$ *whenever* $f(z)$ *belongs to* $\mathcal{H}(B)$ *and* $f(0)=0$. *For each c in* $\mathcal{C}$, $B(z)c$ *belongs to* $\mathcal{H}(B)$ *and is orthogonal to every element* $g(z)$ *of* $\mathcal{H}(B)$ *such that* $zg(z)$ *belongs to* $\mathcal{H}(B)$. *An element* $f(z)$ *of* $\mathcal{H}(B)$ *which is orthogonal to every element* $g(z)$ *of* $\mathcal{H}(B)$ *such that* $zg(z)$ *belongs to* $\mathcal{H}(B)$ *is of the form* $f(z)=B(z)c$ *for some unique c in* $\mathcal{C}$, *and* $\|f\|=\|c\|$. *The transformation U in* $\mathcal{H}(B)$, *whose adjoint is defined by*

$$U^*f(z) = (f(z) - f(0))/z + B(z)f(0) \tag{18}$$

is unitary. If $f(z)$ *is in* $\mathcal{H}(B)$, *then*

$$f(w) = \left\langle f(t), \frac{1 - t\bar{w}B(t)B^*(w)}{1 - t\bar{w}} \right\rangle \tag{19}$$

for $|w|<1$. *The interpretation is that for each c in* $\mathcal{C}$,

$$(1 - z\bar{w})^{-1}c - z\bar{w}(1 - z\bar{w})^{-1}B(z)B^*(w)c$$

is in $\mathcal{H}(B)$ *and*

$$\langle f(w), c\rangle = \langle f(t), (1 - t\bar{w})^{-1}c - t\bar{w}(1 - t\bar{w})^{-1}B(t)B^*(w)c\rangle.$$

The $\mathcal{H}(B)$ notation of Lemma 11 is reconciled with the notation of Lemma 4 by the next lemma.

LEMMA 12. *In the situation of Lemmas* 9, 10, *and* 11, *we have for* $|w|<1$ *and a, c in* $\mathcal{C}$,

$$\frac{U^*}{1 - wU^*}\, a = \frac{zB(z) - wB(w)}{z - w}\, \frac{1}{1 - wB(w)}\, a \tag{20}$$

and (15) *holds.*

Proof of Theorem I. The hypotheses imply that

$$\frac{\bar{E}_b(w)E_b(z) - E_b(\bar{w})E_b^*(z)}{2\pi i(\bar{w} - z)} = \frac{\bar{E}_a(w)E_a(z) - E_a(\bar{w})E_a^*(z)}{2\pi i(\bar{w} - z)}.$$

On choosing $w=z$, it follows that $|E_b(\bar{z})| < |E_b(z)|$ for $y>0$ since $|E_a(\bar{z})| < |E_a(z)|$ for $y>0$ by hypothesis. So $\mathcal{H}(E_b)$ is defined and $K_b(w, z) = K_a(w, z)$. Because of the orthogonal sets in the proof of the theorem of [7], $\mathcal{H}(E_a)$ and $\mathcal{H}(E_b)$ are isometrically equal.

On the other hand, if $\mathcal{H}(E) = \mathcal{H}(E_a)$ isometrically, $K(w, z) = K_a(w, z)$ or in other words $\bar{S}(w)C(z) - \bar{C}(w)S(z) = \bar{S}_a(w)C_a(z) - \bar{C}_a(w)S_a(z)$. By the discussion of [7], $C(z)S'(z) - S(z)C'(z) > 0$ when z is real and $E(z) \neq 0$. So $C(z)$ and $S(z)$ are linearly independent. So there exist unique complex numbers C_0, S_0, C_1, S_1 such that $C(z) = C_a(z)C_0 - S_a(z)S_1$ and $S(z) = C_a(z)S_0 + S_a(z)C_1$. Since $C(z)$, $S(z)$, $C_a(z)$, $S_a(z)$ are real for real z, C_0, S_0, C_1, S_1 are all real. Substitution in (1) yields $K(w, z) = (C_0C_1 + S_0S_1)K_a(w, z)$ and hence $C_0C_1 + S_0S_1 = 1$.

Proof of Theorem II, **the sufficiency.** With this choice of $G(z)$, we have the identity $G^*(w)G(z) = G(w)G^*(z)$. It follows that the orthogonal complement of $G(z)$ satisfies H3. The axiom H2 follows from the properties of $\mathcal{H}(E)$. To verify H1, note from (1) that

$$K(w, z)G(w) - K(w, w)G(z) = [K(\bar{w}, z)G(\bar{w}) - K(\bar{w}, \bar{w})G(z)](z - \bar{w})/(z - w).$$

Let $f(z)$ be any element of $\mathcal{H}(E)$ orthogonal to $G(z)$ and with a nonreal zero w. To verify H1, we need only show that $F(z)(z-\bar{w})/(z-w)$ is orthogonal to $G(z)$. As a result of the axiom H1 for $\mathcal{H}(E)$ and the last identity,

$$\langle F(t)(t - \bar{w})/(t - w), K(\bar{w}, t)G(\bar{w}) - K(\bar{w}, \bar{w})G(t)\rangle = \langle F(t), K(w, t)G(w) - K(w, w)G(t)\rangle$$

with the inner product taken in $\mathcal{H}(E)$. Since w is a zero of $F(z)$ and $\bar{w}$ is a zero of $F(z)(z-\bar{w})/(z-w)$ and $K(w, w) = K(\bar{w}, \bar{w}) > 0$, the expansion of these inner products yields

$$\langle F(t)(t - \bar{w})/(t - w), G(t)\rangle = \langle F(t), G(t)\rangle = 0,$$

which verifies H1.

Let w be any fixed real number and choose $F(z) = K(w, z)E^{-1}(w) - G(w)E^{-1}(w)G(z)\|G\|^{-2}$, which belongs to $\mathcal{H}(E)$ because of (1), and is orthogonal to $G(z)$ because of (2). By (2), $\|F\|^2 = F(w)/E(w) \neq 0$ unless $F(z)$ vanishes identically. It is seen by (1) that this will not happen unless $G(z)$ spans $\mathcal{H}(E)$, a case excluded by hypothesis.

Proof of Theorem II, **the necessity.** We claim first of all that $G^*(w)G(z) = G(w)G^*(z)$. For $\langle G^*, G\rangle G(z) - \langle G, G\rangle G^*(z)$ is in $\mathcal{H}(E)$ and is orthogonal to $G(z)$. Since the orthogonal complement of $G(z)$ satisfies (H3) by hypothesis, this function is orthogonal to $G^*(z)$, and hence by linearity, to itself. Therefore, it is identically zero and the stated identity follows.

We claim next that for each nonreal complex number w, $[K(\bar{w}, z)G(\bar{w})-K(\bar{w}, \bar{w})G(z)](z-w)/(z-\bar{w})$ is a constant multiple of $K(w, z)G(w)-K(w, w)G(z)$. For let $F(z)$ be any element of $\mathcal{H}(E)$ orthogonal to $G(z)$ and such that $F(w)=0$. Then $F(z)(z-\bar{w})/(z-w)$ is orthogonal to $G(z)$ by our hypotheses, and it vanishes at $\bar{w}$. So

$$\langle [K(\bar{w}, z)G(\bar{w}) - K(\bar{w}, \bar{w})G(z)](z - w)/(z - \bar{w}), F(z)\rangle$$
$$= \langle K(\bar{w}, z)G(w) - K(\bar{w}, \bar{w})G(z), F(z)(z - w)/(z - \bar{w})\rangle = 0.$$

It follows that $[K(\bar{w}, z)G(\bar{w})-K(\bar{w}, \bar{w})G(z)](z-w)/(z-\bar{w})$ depends linearly on $K(w, z)$ and $G(z)$. Since $K(w, z)G(w)-K(w, w)G(z)$ is, within a constant factor, the only linear combination of these two functions which vanishes at w, there is some number $A=A(w)$ such that

$$[K(\bar{w}, z)G(\bar{w}) - K(\bar{w}, \bar{w})G(z)](z - w)/(z - \bar{w}) = A[K(w, z)G(w) - K(w, w)G(z)]$$

as claimed.

From the last formula and (1) we find that there is a linear function $L(z)$ and complex constants, a and b, not both zero, such that $L(z)G(z)=aE(z)+bE^*(z)$. If $L(z)$ is not a constant, it has a zero $\bar{w}$ and $G(z)$ is a constant multiple of

$$[\bar{E}(w)E(z) - E(\bar{w})E^*(z)]/(\bar{w} - z) = 2\pi i K(w, z),$$

if w is not a real zero of $E(z)$, or $G(z)$ is a constant multiple of

$$[\bar{E}(w)E(z) - E(\bar{w})E^*(z)]/[E(w)(\bar{w} - z)] = 2\pi i K(w, z)/E(w),$$

if w is a real zero of $E(z)$. Because of (2), our hypotheses rule out both of these possibilities. Therefore, $L(z)$ is a constant. The necessity now follows from the fact that a and b must be chosen so that $G^*(w)G(z)=G(w)G^*(z)$.

Proof of Theorem III, the necessity. By hypothesis, $[F(z)-F(w)]/(z-w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. There must be some choice of $F(z)$ such that $F(w)\neq 0$, for otherwise we can show inductively that $F(z)/(z-w)^n$ belongs to $\mathcal{H}(E)$ for every $n=0, 1, 2, \cdots$, and hence $F(z)$ is zero by analyticity. If $F(z)$ is chosen so that $F(w)\neq 0$, then by the definition of $\mathcal{H}(E)$

$$A = \int |F(t) - F(w)|^2 |t - w|^{-2} |E(t)|^{-2} dt < \infty,$$

$$B = \int |F(t)|^2 |E(t)|^{-2} dt < \infty,$$

$$|F(z) - F(w)|^2 |z - w|^{-2} \leqq A[|E(z)|^2 - |E(\bar{z})|^2]/[2\pi i(\bar{z} - z)],$$
$$|F(z)|^2 \leqq B[|E(z)|^2 - |E(\bar{z})|^2]/[2\pi i(\bar{z} - z)].$$

Since A and B are finite, (4) follows, and $y|(z+i)E(z)|^{-2}$ is bounded above for $y>0$. By approximating from large semi-circles, it is easy to establish Cauchy's formula

$$(z + i)^{-1}E^{-1}(z) = (2\pi i)^{-1}\int (t + 1)^{-1}E^{-1}(t)(t - z)^{-1}dt$$

in the upper half plane $y>0$. Since for $y>0$,

$$(z + i)^{-1} = (2\pi i)^{-1}\int (t + i)^{-1}(t - z)^{-1}dt$$

for the same reason,

$$\int \frac{E(z) - E(t)}{z - t}(t + i)^{-1}E^{-1}(t)dt = 0$$

for $y>0$, and hence for all complex z since the left hand side is an entire function of z. Elementary estimates from this formula, similar to those for the lemma of [2], show that $E(z)$ has exponential type and that (3) holds.

Proof of Theorem III, **the sufficiency.** Let $F(z)$ be in $\mathcal{H}(E)$ and let $G(z)=[F(z)-F(w)]/(z-w)$. Since $F(z)$ belongs to $\mathcal{H}(E)$, $\int|F(t)|^2|E(t)|^{-2}dt<\infty$, and since (4) holds, $\int|G(t)|^2|E(t)|^{-2}dt<\infty$. Since $E(z)$ is of exponential type by hypothesis, elementary estimates from the definition of $\mathcal{H}(E)$ show that $F(z)$ and hence $G(z)$ is of exponential type. Since (3) holds, we see similarly that $\int(1+t^2)^{-1}\log^+|G(t)|dt<\infty$. It follows from (3) and from Boas, [1, pp. 85 and 93] that $\int(1+t^2)^{-1}|\log|G(t)E^{-1}(t)||dt<\infty$ and for some number $a\geqq 0$,

$$\log|G(x + iy)E^{-1}(x + iy)| \leqq ay + \frac{y}{\pi}\int\frac{\log|G(t)E^{-1}(t)|\,dt}{(t - x)^2 + y^2}$$

for $y>0$. Since $F(z)$ belongs to $\mathcal{H}(E)$,

$$\limsup_{y\to+\infty} y^{-1}\log|F(iy)E^{-1}(iy)| \leqq 0.$$

By Boas [1, p. 76], we have as a general property of entire functions of exponential type,

$$\limsup_{y\to-\infty}|y|^{-1}\log|E(iy)| + \limsup_{y\to+\infty} y^{-1}\log|E(iy)| \geqq 0.$$

Since $|E(\bar z)|<|E(z)|$ for $y>0$ by hypothesis, the second limit cannot be negative. It follows that we can choose $a=0$. Since for $r>0$, r^2 is a convex function of $\log r$, it follows by Jensen's inequality, as in Boas [1, p. 100], that

$$|G(x+iy)|^2|E(x+iy)|^{-2} \leqq \frac{y}{\pi}\int \frac{|G(t)|^2|E(t)|^{-2}dt}{(t-x)^2+y^2}$$

for $y>0$, and hence

$$\pi y|G(z)|^2|E(z)|^{-2} \leqq \int |G(t)|^2|E(t)|^{-2}dt$$

is bounded above for $y>0$. It is now easy to establish the Cauchy formula

$$G(z)E^{-1}(z) = (2\pi i)^{-1}\int G(t)E^{-1}(t)(t-z)^{-1}dt$$

for $y>0$, and hence

$$0 = (2\pi i)^{-1}\int G(t)E^{-1}(t)(t-z)^{-1}dt$$

for $y<0$. A similar argument shows that these last two formulas hold with $G(z)$ replaced by $G^*(z)$. It follows that $G(z)=\int G(t)\overline{K}(z,t)|E(t)|^{-2}dt$ for $y>0$ and for $y<0$, and hence by continuity for all complex z. One now sees by the Schwarz inequality that $G(z)$ belongs to $\mathcal{H}(E)$.

Proof of Lemma 1. We can write $i(\bar{w}-w)=[P(x)+Q(y)]|Cz+D|^{-2}$ where $P(x)=i(\overline{A}C-A\overline{C})x^2+i(\overline{A}D-B\overline{C}-A\overline{D}+\overline{B}C)x+i(\overline{B}D-B\overline{D})$ and $Q(y)=i(\overline{A}C-A\overline{C})y^2+(\overline{A}D-B\overline{C}+A\overline{D}-\overline{B}C)y$. Since

$$\begin{aligned}4i(\overline{A}C-A\overline{C})i(\overline{B}D-B\overline{D}) - |\overline{A}D-B\overline{C}-A\overline{D}+\overline{B}C|^2 \\ = |\overline{A}D-B\overline{C}+A\overline{D}-\overline{B}C|^2-4|AD-BC|^2,\end{aligned}$$

the sufficiency follows, for the hypotheses imply that $P(x)\geqq 0$ for all real x and $Q(y)\geqq 0$ for all $y\geqq 0$. For the necessity, we must have $P(x)\geqq 0$ for all real x, since we can choose $y=0$, and hence $i(\overline{A}C-A\overline{C})\geqq 0$, $i(\overline{B}D-B\overline{D})\geqq 0$, and $|\overline{A}D-B\overline{C}+A\overline{D}-\overline{B}C|\geqq 2|AD-BC|$. These conditions in themselves make the real axis map into the upper half plane. To make the upper half plane map above rather than below, we must have $Q(y)$ increasing when $y=0$, and hence $\overline{A}D+A\overline{D}-B\overline{C}-\overline{B}C\geqq 0$.

Proof of Theorem IV, **the necessity.** The hypotheses imply that $i[\overline{S}_0(z)C_0(z)-\overline{C}_0(z)S_0(z)]\geqq 0$ and $i[\overline{S}_1(z)C_1(z)-\overline{C}_1(z)S_1(z)]\geqq 0$ for $y>0$. These inequalities, together with (5) and (6), imply that

$$i[\overline{S}_0(z)C_1(z)-\overline{C}_0(z)S_1(z)]\geqq 0$$

and

$$i[\overline{S}_1(z)C_0(z)-\overline{C}_1(z)S_0(z)]\geqq 0$$

for $y>0$. Therefore,

$$E_0(z)\bar{E}_1(z) + E_1(z)\bar{E}_0(z)$$
$$= 2\ \mathrm{Re}\ [C_0(z)\bar{C}_1(z) + S_0(z)\bar{S}_1(z)] + i[\bar{S}_1(z)C_0(z) - \bar{C}_1(z)S_0(z)]$$
$$+ i[\bar{S}_0(z)C_1(z) - \bar{C}_0(z)S_1(z)] \geqq 2$$

for $y \geqq 0$. Therefore, $|E_0(z)E_1(z)| \geqq 1$ for $y \geqq 0$, and both $E_0(z)$ and $E_1(z)$ are without zeros there.

The hypotheses (5) and (6) imply that $\mathrm{Re}[E_1(z)/E_0(z)] > 0$ for $y > 0$ and $\mathrm{Re}[E_1(z)/E_0(z)] = |E_0(z)|^{-2}$ when z is real. By the Poisson representation of a function positive and harmonic in a half plane, (4) follows for $E(z) = E_0(z)$, and there is some number $a \geqq 0$ such that

$$\frac{y}{\pi}\int \frac{|E_0(t)|^{-2}}{(t-x)^2 + y^2}\,dt + ay = \mathrm{Re}\,\frac{E_1(z)}{E_0(z)}$$

for $y > 0$. If in addition, $E_1(iy) = o(yE_0(iy))$ as $y \to +\infty$, we can conclude that $a = 0$ and this formula implies the uniqueness part of the theorem. Since $\mathrm{Re}[E_0(z)\bar{E}_1(z)] \geqq 1$ for $y > 0$, $\mathrm{Re}[E_1(z)/E_0(z)] \geqq |E_0(z)|^{-2}$ for $y > 0$. Therefore,

$$|E_0(x+iy)|^{-2} \leqq \frac{y}{\pi}\int \frac{|E_0(t)|^{-2}}{(t-x)^2 + y^2}\,dt + ay$$

for $y > 0$. Arguments similar to the proof of sufficiency for Theorem III show that

$$\int \frac{E_0(z) - E_0(t)}{z - t}\,(t+i)^{-1}\bar{E}_0^{-1}(t)\,dt = 0$$

for all complex z, and $E_0(z)$ is of exponential type, and (3) holds.

Proof of Theorem IV, **the sufficiency.** For each complex number w, $[E_0(z) - E_0(w)]/(z - w]$ belongs to $\mathcal{H}(E_0)$, for the proof of sufficiency for Theorem III applies with $G(z)$ taken equal to this function. We will show that there is an entire function $E_1(z)$ such that

$$(21)\quad \frac{1}{\pi}\left\langle \frac{E_0(t) - E_0(z)}{t - z}, \frac{E_0(t) - E_0(w)}{t - w}\right\rangle = \frac{\bar{E}_0(w)E_1(z) + \bar{E}_1(w)E_0(z) - 2}{i(\bar{w} - z)}$$

for all complex z and w, with the inner product taken in $\mathcal{H}(E_0)$. Later, we will show that this function $E_1(z)$ has the required properties.

By Lemma 7 of [7], there is at most one real number α modulo π such that $e^{i\alpha}E_0(z) - e^{-i\alpha}E_0^*(z)$ belongs to $\mathcal{H}(E_0)$. For simplicity in the calculations, we suppose that $E_0(z) - E_0^*(z) = -iS_0(z)$ is not in $\mathcal{H}(E_0)$. We use the formula of [6] with $\alpha = 0$, so that

$$\|F\|^2_{E_0} = \pi \sum S_0(t) = 0\,\frac{|F(t)|^2}{C_0(t)S_0'(t)}$$

for every $F(z)$ in $\mathcal{H}(E_0)$. If $E_0(z)-E_0^*(z)$ does belong to $\mathcal{H}(E_0)$, a similar argument can be made with a different choice of α. Therefore,

$$\frac{1}{\pi}\left\langle \frac{E_0(t)-E_0(z)}{t-z}, \frac{E_0(t)-E_0(w)}{t-w}\right\rangle$$

$$= \sum \frac{C_0(t)}{S_0'(t)(t-z)(t-\bar{w})} - E_0(z)\sum \frac{1}{S_0'(t)(t-z)(t-\bar{w})}$$

$$- \bar{E}_0(w)\sum \frac{1}{S_0'(t)(t-z)(t-\bar{w})}$$

$$+ E_0(z)\bar{E}_0(w)\sum \frac{1}{C_0(t)S_0'(t)(t-z)(t-\bar{w})}$$

whenever $S_0(z)\neq 0$ and $S_0(w)\neq 0$, where t ranges in the zeros of $S_0(z)$ and each sum is absolutely convergent. By the first formula in the proof of Lemma 3 of [6] and the hypothesis that $S_0(z)$ does not belong to $\mathcal{H}(E_0)$,

$$\frac{C_0(z)}{S_0(z)} - \frac{C_0(\bar{w})}{S_0(\bar{w})} = (\bar{w}-z)\sum \frac{C_0(t)}{S_0'(t)} \frac{1}{(t-z)(t-\bar{w})}$$

when $S_0(z)\neq 0$ and $S_0(w)\neq 0$. By the methods used in the proof of Lemma 2 of [5], it is easily shown that

$$\frac{1}{S_0(z)} - \frac{1}{S_0(\bar{w})} = (\bar{w}-z)\sum \frac{1}{S_0'(t)(t-z)(t-\bar{w})}$$

when $S_0(z)\neq 0$ and $S_0(w)\neq 0$, and this formula is submitted without explicit proof. Clearly,

$$G(z) = S_0(z)\sum \frac{1+tz}{(1+t^2)(t-z)} \frac{1}{C_0(t)S_0'(t)}$$

is an entire function which is real for real z and

$$\frac{G(z)}{S_0(z)} - \frac{G(\bar{w})}{S_0(\bar{w})} = (\bar{w}-z)\sum \frac{1}{C_0(t)S_0'(t)(t-z)(t-\bar{w})}$$

when $S_0(z)\neq 0$ and $S_0(w)\neq 0$. Formula (21) now follows with $E_1(z) = [iE_0(z)G(z)-1]/S_0(z)$, which is an entire function because the left hand side of (21) is an entire function of z.

Since when $\bar{w}=z$, the numerator of the fraction on the right hand side of (21) must vanish by continuity, we have the identity $E_0(z)E_1^*(z)+E_1^*(z)E_0(z) =2$, which is equivalent to (5).

By formula (2),

$$(22)\quad \frac{1}{\pi}\left\langle \frac{E_0(t)-E_0(z)}{t-z}, \frac{E_0^*(w)E_0(t)-E_0(w)E_0^*(t)}{t-w}\right\rangle = 2i\frac{E_0(\bar{w})-E_0(z)}{\bar{w}-z}.$$

From this formula and (21) it follows that

$$\frac{1}{\pi}\left\langle \frac{E_0(t)-E_0(z)}{t-z}, \frac{E_0^*(t)-E_0^*(w)}{t-w}\right\rangle = \frac{E_0(\bar{w})E_1(z)-E_1(\bar{w})E_0(z)}{i(\bar{w}-z)}.$$

It follows from (21) and (22) by the properties of conjugation that

$$(23)\quad \frac{1}{\pi}\left\langle \frac{E_0^*(t)-E_0^*(z)}{t-z}, \frac{E_0(t)-E_0(w)}{t-w}\right\rangle = \frac{\overline{E}_1(w)E_0^*(z)-\overline{E}_0(w)E_1^*(z)}{i(\bar{w}-z)}$$

and

$$(24)\quad \frac{1}{\pi}\left\langle \frac{E_0^*(t)-E_0^*(z)}{t-z}, \frac{E_0^*(t)-E_0^*(w)}{t-w}\right\rangle = \frac{E_0(\bar{w})E_1^*(z)+E_1(\bar{w})E_0^*(z)-2}{-i(\bar{w}-z)}.$$

By (21), (22), (23), and (24), for every complex number w,

$$\begin{aligned}&\frac{\overline{E}_0(z)E_1(z)+\overline{E}_1(z)E_0(z)-2}{i(\bar{z}-z)} + w\frac{\overline{E}_1(z)E_0^*(z)-\overline{E}_0(z)E_1^*(z)}{i(\bar{z}-z)}\\ &\qquad + \bar{w}\frac{E_0(\bar{z})E_1(z)-E_1(\bar{z})E_0(z)}{i(\bar{z}-z)} + w\bar{w}\frac{E_0(\bar{z})E_1^*(z)+E_1(\bar{z})E_0^*(z)-2}{-i(\bar{z}-z)}\\ &\qquad = \left\| \frac{E_0(t)-E_0(z)}{t-z} + w\frac{E_0^*(t)-E_0^*(z)}{t-z}\right\|^2 \geqq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&-[\overline{E}_0(z)E_1(z)+\overline{E}_1(z)E_0(z)-2][E_0(\bar{z})E_1^*(z)+E_1(\bar{z})E_0^*(z)-2]\\ &-[\overline{E}_1(z)E_0^*(z)-\overline{E}_0(z)E_1^*(z)][E_1(z)E_0(\bar{z})-E_0(z)E_1(\bar{z})] \geqq 0,\end{aligned}$$

an inequality which reduces to (6) on simplifying. Since

$$\begin{aligned}&i[\overline{S}_0(z)C_0(z)-\overline{C}_0(z)S_0(z)]i[\overline{S}_1(z)C_1(z)-\overline{C}_1(z)S_1(z)]\\ &\qquad = |C_0(z)\overline{C}_1(z)-S_0(z)\overline{S}_1(z)|^2-1 \geqq 0\end{aligned}$$

by (5) and (6) and since $i[\overline{S}_0(z)C_0(z)-\overline{C}_0(z)S_0(z)]\geqq 0$ for $y>0$ because $|E_0(\bar{z})|<|E_0(z)|$ for $y>0$ by hypothesis, we have $i[\overline{S}_1(z)C_1(z)-\overline{C}_1(z)S_1(z)]\geqq 0$ for $y>0$, which implies that $|E_1(\bar{z})|\leqq|E_1(z)|$ for $y>0$. By the Lebesgue dominated convergence theorem,

$$\lim_{y\to+\infty} E_0^{-1}(iy)\left\langle \frac{E_0(t)-E_0(iy)}{t-iy}, \frac{E_0(t)-E_0(w)}{t-w}\right\rangle = 0.$$

Therefore, (21) implies that $E_1(iy)=o(yE_0(iy))$ as $y\to+\infty$.

Proof of Lemma 2. Since U is a unitary transformation in $\mathcal{H}$, the self adjoint part of the transformation

$$(1 + wU^*)/(1 - wU^*) = 1 + 2wU^{-1} + 2w^2U^{-2} + \cdots$$

is non-negative. We have as an estimate of the operator norm

$$\|(1 + wU^*)/(1 - wU^*)\| \leqq 1 + 2|w| + 2|w|^2 + \cdots = (1 + |w|)/(1 - |w|).$$

For a and c in $\mathcal{C}$,

$$|\langle(1 + wU^*)(1 - wU^*)^{-1}a, c\rangle| \leqq (1 + |w|)(1 - |w|)^{-1}\|a\| \|c\|.$$

It follows that there is a bounded operator $\phi(w)$ on $\mathcal{C}$ such that for a and c in $\mathcal{C}$,

$$\langle\phi(w)a, c\rangle = \langle(1 + wU^*)(1 - wU^*)^{-1}a, c\rangle,$$

and $\|\phi(w)\| \leqq (1+|w|)/(1-|w|)$. The power series expansion $\langle\phi(w)a, c\rangle = \langle a, c\rangle + 2w\langle U^{-1}a, c\rangle + \cdots$ shows that $\langle\phi(w)a, c\rangle$ is an analytic complex valued function of w with a Taylor series bounded by $2\|a\| \|c\|$. It follows that $\phi(w)$ is an analytic operator valued function of w with a Taylor series bounded by 2. Since the self adjoint part of $(1+wU^*)/(1-wU^*)$ is non-negative, for each c in $\mathcal{C}$, the real part of $\langle\phi(w)c, c\rangle$ is non-negative. Therefore, $\phi(w)+\phi^*(w) \geqq 0$ for $|w| < 1$. On substituting $w=0$, we find that $\phi(0)=1$ is the identity operator on $\mathcal{C}$.

Let $B(z)=[\phi(z)-1][\phi(z)+1]^{-1}z^{-1}$. Since $\phi(0)=1$, $[\phi(z)-1]z^{-1}$ is defined and analytic for $|z| <1$. Since $\phi(z)+\phi^*(z) \geqq 0$ for $|z| <1$, $\|\phi(z)c+c\| \geqq \|c\|$ for every c in $\mathcal{C}$ when $|z| <1$ and hence $[\phi(z)+1]^{-1}$ is defined and analytic for $|z| <1$. So, $B(z)$ is defined and analytic for $|z| <1$. Solving for $\phi(z)$, we find that $\phi(z)=[1+zB(z)]/[1-zB(z)]$. Let c be any element of $\mathcal{C}$ and let $a=c-wB(w)c$, an element of $\mathcal{C}$ which depends on both c and w. Then, $\phi(w)a=c+wB(w)c$ and $\|c\|^2-\|wB(w)c\|^2=\text{Re}\,\langle\phi(w)a, a\rangle \geqq 0$. Therefore, $\|wB(w)c\|^2 \leqq \|c\|^2$ for every c in $\mathcal{C}$. By Schwarz's inequality,

$$|\langle w_1B(w_1)c, w_2B(w_2)c\rangle| \leqq \|c\|^2$$

for $|w_1| <1$ and $|w_2| <1$. Since this inner product is an analytic function of w_1, we have by Schwarz's lemma, $|\langle B(w_1)c, w_2B(w_2)c\rangle| \leqq \|c\|^2$. By the same argument applied to the complex conjugate as a function of w_2,

$$|\langle B(w_1)c, B(w_2)c\rangle| \leqq \|c\|^2.$$

On choosing $w_1=w_2=w$, we have $\|B(w)c\| \leqq \|c\|$, which by the arbitrariness of c means that $\|B(w)\| \leqq 1$.

Proof of Lemma 3. By hypothesis,

$$\langle(1 + wU_1^*)(1 - wU_1^*)^{-1}a, c\rangle_1 = \langle(1 + wU_2^*)(1 - wU_2^*)^{-1}a, c\rangle_2.$$

By expanding each side as a power series in w and comparing coefficients term by term, we find that

$$\langle U_1^{-n}a, c\rangle_1 = \langle U_2^{-n}a, c\rangle, \qquad \text{for } n = 0, 1, 2, \cdots.$$

The isometric character of U_1 and U_2 can be used to show that the formula remains valid when the sign on the exponents is reversed. It can also be used to show that if m and n are integers,

$$\langle U_1^m a, U_1^n c\rangle_1 = \langle U_2^m a, U_2^n c\rangle_2.$$

Let $P(z) = \sum a_n z^n$ be any formal sum of integral powers of z with coefficients in $\mathcal{C}$, all but a finite number of them being equal to zero. By what we have shown $\| \sum U_1^n a_n \|_1^2 = \sum \langle U_1^m a_m, U_1^n a_n\rangle_1 = \sum \langle U_2^m a_m, U_2^n a_n\rangle_2 = \| \sum U_2^n a_n\|_2^2$. By the hypotheses on T, we must have $T(\sum U_1^n a_n) = \sum U_2^n a_n$. On the other hand, the previous formula shows that there is such a transformation T, defined at least on finite combinations $\sum U_1^n a_n$ so as to be linear and isometric, to act identically on $\mathcal{C}$, and to satisfy $TU_1 = U_2 T$. But the finite combinations of the form $\sum U_1^n a_n$ above are dense in $\mathcal{H}_1$ by our hypotheses, since the closure is a closed subspace of $\mathcal{H}_1$ which contains $\mathcal{C}$ and reduces U_1. The continuous extension to $\mathcal{H}_1$ is easily seen to have the required properties.

Proof of Lemma 4. We will use an integration theory with operator valued measures. Let S be a compact Hausdorff space, which in our application will be the reals modulo 2π. By an operator valued probability measure μ on the Borel sets of S, we mean a countably additive function $\mu(E)$ of Borel subsets E of S, whose values are non-negative operators on $\mathcal{C}$, and whose value for S is the identity operator. By "countably additive," we mean that the measure of the empty set is the zero operator and that if (E_n) is a disjoint sequence of Borel subsets of S, $\mu(\bigcup E_n) = \sum \mu(E_n)$. The interpretation is that for every c in $\mathcal{C}$,

$$\langle \mu(\bigcup E_n)c, c\rangle = \sum \langle \mu(E_n)c, c\rangle.$$

In particular, for each unit vector c in $\mathcal{C}$, $\langle \mu(E)c, c\rangle$ is a probability measure of Borel subsets E of S, in the ordinary numerical sense. Let $f(\theta)$ be a Borel measurable, complex valued function of θ in S, such that $\int |f(\theta)| d\langle \mu(\theta)c, c\rangle$ remains bounded as c ranges in the unit vectors of $\mathcal{C}$. We define $\int f(\theta)d\mu(\theta)$ to be the unique operator such that for each unit vector c in $\mathcal{C}$,

$$\left\langle \int f(\theta)d\mu(\theta)c, c\right\rangle = \int f(\theta)d\langle \mu(\theta)c, c\rangle.$$

The integration theory is used to generalize a theorem of Herglotz. Let $\phi(z)$ be an operator valued function, defined and analytic for $|z| < 1$, such that $\phi(z) + \phi^*(z) \geqq 0$ and $\phi(0) = 1$. Then there exists a unique operator valued probability measure μ on the Borel sets of the reals modulo 2π such that

$$\phi(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$

For let c be any unit vector in $\mathcal{C}$. Then the complex valued function $\langle \phi(z)c, c\rangle$ is defined and analytic for $|z| < 1$, and Re $\langle \phi(z)c, c\rangle \geqq 0$ and $\langle \phi(0)c, c\rangle = 1$. By the usual form of the Herglotz theorem for complex valued functions, there is a unique probability measure μ_c on the Borel sets of the reals modulo 2π such that

$$\langle \phi(z)c, c\rangle = \int (e^{i\theta} + z)(e^{i\theta} - z)^{-1} d\mu_c(\theta).$$

For every Borel subset E of the reals modulo 2π, consider the quadratic form Φ_E on $\mathcal{C}$ such that for each unit vector c, $\Phi_E(c) = \mu_c(E)$. Since Φ_E is bounded, there is an operator $\mu(E)$ such that $\langle \mu(E)c, c\rangle = \Phi_E(c)$ for every c in $\mathcal{C}$. Obviously, μ is the required operator valued, countably additive function of sets.

The Herglotz theorem will be applied to $\phi(z) = [1 + zB(z)]/[1 - zB(z)]$. Since $\|B(z)\| \leqq 1$, this function is defined and analytic for $|z| < 1$. The formulas in the proof of Lemma 1 show that $\phi(z) + \phi^*(z) \geqq 0$. On substituting 0 for z, we find that $\phi(0) = 1$. Therefore, the Herglotz theorem is applicable to produce an operator valued probability measure μ. We have

$$\int ((e^{i\theta} + z)/(e^{i\theta} - z)) d\mu(\theta) = [1 - zB(z)]/[1 - zB(z)]$$

for $|z| < 1$. We now construct a Hilbert space $L^2(\mu)$ associated with μ.

A function defined on the reals modulo 2π will be called a simple function if it assumes only a finite number of values, and if the set on which it assumes each value is a Borel set. If f and g are simple $\mathcal{C}$-valued functions, we will use μ to define a complex number $\langle f, g\rangle_\mu$, which serves to define a Hilbert space inner product. Let $E_1, \cdots, E_r$ be disjoint Borel sets with corresponding characteristic functions $\chi_{E_1}, \cdots, \chi_{E_r}$ and let $a_1, \cdots, a_r$ and $b_1, \cdots, b_r$ be elements of $\mathcal{C}$, all chosen so that $f = a_1\chi_{E_1} + \cdots + a_r\chi_{E_r}$ and $g = b_1\chi_{E_1} + \cdots + b_r\chi_{E_r}$. Define

$$\langle f, g\rangle_\mu = \langle \mu(E_1)a_1, b_1\rangle + \cdots + \langle \mu(E_r)a_r, b_r\rangle,$$

using the inner product in $\mathcal{C}$. The definition does not depend on the choice of representation by characteristic functions. If f, g, and h are simple $\mathcal{C}$-valued functions and if α and β are complex numbers, $\langle \alpha f + \beta g, h\rangle_\mu = \alpha\langle f, h\rangle_\mu + \beta\langle g, h\rangle_\mu$, $\langle g, f\rangle_\mu = \langle f, g\rangle_\mu^-$, and $\|f\|_\mu^2 = \langle f, f\rangle_\mu \geqq 0$. To define $L^2(\mu)$, identify any two simple $\mathcal{C}$-valued functions f, g such that $\|g - f\|_\mu = 0$, and let $L^2(\mu)$ be the completion of the resulting inner product space.

Every element c of $\mathcal{C}$ may be thought of as a constant function with that value. The resulting simple function determines an element of $L^2(\mu)$. This transformation of $\mathcal{C}$ into $L^2(\mu)$ is isometric, and we think of $\mathcal{C}$ as contained in $L^2(\mu)$. Since there is no danger of confusion, we drop the subscripts from the inner products.

If E is a Borel subset of the reals modulo 2π and if f and g are simple $\mathcal{C}$-valued functions, the products $\chi_E f$ and $\chi_E g$ are simple $\mathcal{C}$-valued functions, and $\langle \chi_E f, g\rangle = \langle f, \chi_E g\rangle$, and $\|\chi_E f\| \leqq \|f\|$. It follows that the linear transformation, multiplication by χ_E, of simple $\mathcal{C}$-valued functions into simple $\mathcal{C}$-valued functions determines a linear transformation $P(E)$ of $L^2(\mu)$ into itself. This transformation in $L^2(\mu)$ is an orthogonal projection. The transformation $E \to P(E)$ is a spectral measure. Every Borel measurable, complex valued function of θ in the reals modulo 2π now defines a linear transformation in $L^2(\mu)$ according to the Stone operational calculus [6]. Let U be the transformation corresponding to $e^{i\theta}$, which is a unitary transformation in $L^2(\mu)$.

Let a and c be in $\mathcal{C}$ and let $|w| < 1$. By the Stone operational calculus,

$$\langle (1 + wU^*)(1 - wU^*)^{-1}a, c\rangle = \int (1 + w\bar{e}^{i\theta})(1 - w\bar{e}^{i\theta})^{-1} d\langle P(\theta)a, c\rangle.$$

By construction, for each Borel set E, $\langle P(E)a, c\rangle = \langle \mu(E)a, c\rangle$. Therefore,

$$\begin{aligned}\langle (1 + wU^*)(1 - wU^*)^{-1}a, c\rangle &= \int (e^{i\theta} + w)(e^{i\theta} - w)^{-1} d\langle \mu(\theta)a, c\rangle \\ &= \langle [1 + wB(w)][1 - wB(w)]^{-1}a, c\rangle.\end{aligned}$$

We now show that the smallest closed subspace of $L^2(\mu)$ which contains $\mathcal{C}$ and reduces U is all of $L^2(\mu)$, by showing that if f is in $L^2(\mu)$ and $\langle f, U^n c\rangle = 0$ for every c in $\mathcal{C}$ and integral n, then $f = 0$. For then

$$\int (e^{i\theta} + w)(e^{i\theta} - w)^{-1} d\langle P(\theta)c, f\rangle = \langle (U + w)(U - w)^{-1}c, f\rangle$$

vanishes identically for every c in $\mathcal{C}$ and $|w| < 1$. By the uniqueness part of the Herglotz theorem, $\langle P(E)c, f\rangle = 0$ for every Borel set E. Therefore, f is orthogonal to (the image of) every simple $\mathcal{C}$-valued function. Since these are dense in $L^2(\mu)$ by construction, $f = 0$. The lemma follows on choosing $\mathcal{H}(B) = L^2(\mu)$.

Proof of Theorem XI. By hypothesis there exists a linear isometric transformation T of $\mathcal{H}(B_2)$ into $\mathcal{H}(B_1)$ with certain properties. Without loss of generality we can suppose that $\mathcal{H}(B_2)$ is actually contained in $\mathcal{H}(B_1)$ and that T is the inclusion.

Let U be the unitary extension of U_2 in $\mathcal{H}(B_1)$ which is the identity on the orthogonal complement of $\mathcal{H}(B_2)$. Let V be the unitary transformation in $\mathcal{H}(B_1)$ such that $U_1 = UV$. It has the property that $Vf = f$ whenever f is in $\mathcal{H}(B_2)$ and Uf is orthogonal to $\mathcal{C}$.

Let $\mathfrak{D}$ be the set of elements f of $\mathcal{H}(B_2)$ such that Uf is orthogonal to $\mathfrak{C}$, and let $\mathfrak{B}$ be the orthogonal complement of $\mathfrak{D}$ in $\mathcal{H}(B_2)$. The transformation U takes $\mathfrak{B}$ isometrically onto $\mathfrak{C}$. Let $A(z)$ be the operator valued function, defined and analytic for $|z| < 1$, such that $\|A(z)\| \leqq 1$ and

$$\langle [1 + wA(w)][1 - wA(w)]^{-1}a, c\rangle = \langle U^*a, (1 + \bar{w}V)(1 - \bar{w}V)^{-1}U^*c\rangle$$

for a and c in $\mathfrak{C}$ and $|w| < 1$. The existence of $A(z)$ follows by the proof of Lemma 2 with obvious changes. We will prove the necessity by showing that we can choose $B_3(z) = A(z)$.

First, we make a calculation with the adjoint $(UV)^* = V^*U^*$. An arbitrary element of $\mathcal{H}(B_1)$ can be written $f+g$ where f is in $\mathcal{H}(B_2)$ and g is orthogonal to $\mathcal{H}(B_2)$. By the definition of U, $U^*(f+g) = U^*f+g$. Let k be the projection of f into $\mathfrak{C}$. Since U is unitary, U^*k is the projection of U^*f into $\mathfrak{B}$. So $U^*(f+g) = U^*(f-k) + U^*k + g$ where $U^*(f-k)$ is in $\mathfrak{D}$. Since V is the identity on $\mathfrak{D}$, $(UV)^*(f+g) = U^*(f-k) + V^*U^*k + V^*g$.

Let a be in $\mathfrak{C}$ and $|w| < 1$. We claim that there is an element g of $\mathcal{H}(B_1)$ orthogonal to $\mathcal{H}(B_2)$ such that

$$(UV)^*[1 - w(UV)^*]^{-1}a = (U - w)^{-1}[e - wB_2(w)e] + g$$

where e is in $\mathfrak{C}$ and $e = [1 - wA(w)B_2(w)]^{-1}A(w)a$. Since UV is a unitary transformation and $|w| < 1$, $[1 - w(UV)^*]^{-1}$ is a bounded transformation of $\mathcal{H}(B_1)$ into itself. The left hand side of the proposed formula makes sense. We will verify the formula by showing that

$$(UV)^*a = [1 - w(UV)]^*(U - w)^{-1}[e - wB_2(w)e] + [1 - w(UV)^*]g.$$

It can be seen from the definition of $B_2(w)$ that the projection of $(U-w)^{-1}[e - wB_2(w)e]$ into $\mathfrak{C}$ is $B_2(w)e$. On the other hand,

$$(U - w)^{-1}[e - wB_2(w)e] - B_2(w)e = (U - w)^{-1}[e - UB_2(w)e].$$

By the formula of the last paragraph, we must show that

$$V^*U^*a = U^*e - wV^*U^*B_2(w)e + (1 - wV^*)g.$$

On letting V act on each side, we find that the formula is equivalent to

$$U^*a + U^*wB_2(w)e - wU^*e = (V - w)(g + U^*e),$$

or

$$(V - w)^{-1}U^*[a + wB_2(w) - we] = g + U^*e.$$

Each side has a projection of zero into $\mathfrak{D}$. By our freedom in the choice of g, we need only verify that each side has the same projection into $\mathfrak{B}$. This will be the case if we can show that for each d in $\mathfrak{C}$, left and right hand sides have the same inner product with U^*d. Since $(V+w)/(V-w) = 1 + 2w/(V-w)$, it is sufficient to show that

$$\langle (V+w)(V-w)^{-1}U^*[a+wB_2(w)e-we],\ U^*d\rangle = \langle a+wB_2(w)e+we, d\rangle.$$

Since $[(V+w)/(V-w)]^* = (1+\bar{w}V)/(1-\bar{w}V)$, the definition of $A(z)$ makes this formula equivalent to

$$\langle [1+wA(w)][1-wA(w)]^{-1}[a+wB_2(w)e-we],\ d\rangle = \langle a+wB_2(w)e+we,\ d\rangle.$$

But it is easily verified that with our choice of e,

$$[1+wA(w)][a+wB_2(w)e-we] = [1-wA(w)][a+wB_2(w)+we]$$

is indeed satisfied.

By the results of the last paragraph and the definition of $B_2(z)$,

$$\langle (UV)^*[1-w(UV)^*]^{-1}a,\ c\rangle = \langle B_2(w)[1-wA(w)B_2(w)]^{-1}A(w)a,\ c\rangle.$$

Since

$$B_2(w)[1-wA(w)B_2(w)]^{-1}A(w) = B_2(w)A(w)[1-wB_2(w)A(w)]^{-1},$$

$$\langle [1+w(UV)^*][1-w(UV)^*]^{-1}a,\ c\rangle = \langle [1+wB_2(w)A(w)][1-wB_2(w)A(w)]^{-1}a,\ c\rangle.$$

It follows that $B_1(w) = B_2(w)A(w)$ and that we can choose $B_3(z) = A(z)$.

Proof of Lemma 5. The series defining $f(w)$ is dominated by $\|f(w)\| \leqq \sum \|a_n\| \, |w|^n$. By the Schwarz inequality, $\|f(w)\|^2 \leqq (\sum \|a_n\|^2)(\sum |w|^{2n}) = \|f\|^2(1-|w|^2)^{-1}$. So, $(1-|w|^2)\|f(w)\|^2 \leqq \|f\|^2$ as claimed. For each c in $\mathcal{C}$, $(1-\bar{w}z)^{-1}c = \sum c\bar{w}^n z^n$ belongs to $\mathcal{C}(z)$ because $\sum \|c\|^2 |w|^{2n} = \|c\|^2(1-|w|^2)^{-1}$ converges. By the definition of the inner product in $\mathcal{C}(z)$, $\langle f(w), c\rangle = \langle f(t), (1-t\bar{w})^{-1}c\rangle$.

Proof of Lemma 6. Let $g(z)$ be in $\mathfrak{M}$. Let $f(z)$ be in $\mathfrak{H}$. By hypothesis $(f(z)-f(0))/z$ belongs to $\mathfrak{H}$. Therefore, $(f(z)-f(0))/z$ is orthogonal to $g(z)$ and hence $f(z)-f(0)$ is orthogonal to $zg(z)$. Since $zg(z)$ is orthogonal to $f(0)$, $zg(z)$ is orthogonal to $f(z)$. By the arbitrariness of $f(z)$, $zg(z)$ is in $\mathfrak{M}$.

Proof of Lemma 7. Let $f(z)$ be an element of $\mathfrak{H}$ such that $f(0)=0$. Then $f(z)/z$ belongs to $\mathcal{C}(z)$. Let $g(z)$ be in $\mathfrak{M}$. The $zg(z)$ belongs to $\mathfrak{M}$ by hypothesis and so $f(z)$ is orthogonal to $zg(z)$, and so $f(z)/z$ is orthogonal to $g(z)$. By the arbitrariness of $g(z)$, $f(z)/z$ belongs to $\mathfrak{H}$.

Proof of Lemma 8, **the necessity.** Let $g(z)$ be any element of $\mathfrak{H}$. Then, $(g(z)-g(0))/z$ is in $\mathfrak{H}$ by hypothesis, and so is its product by z. Therefore, $f(z)$ is orthogonal to $(g(z)-g(0))/z$ and so $zf(z)$ is orthogonal to $g(z)-g(0)$ and so $zf(z)$ is orthogonal to $g(z)$. By the arbitrariness of $g(z)$, $zf(z)$ is in $\mathfrak{M}$. Let $h(z)$ be in $\mathfrak{M}$. Since $f(z)$ is in $\mathfrak{H}$, $f(z)$ is orthogonal to $h(z)$ and so $zf(z)$ is orthogonal to $zh(z)$.

Proof of Lemma 8, **the sufficiency.** If $h(z)$ is in $\mathfrak{M}$, then $zf(z)$ is orthogonal to $zh(z)$ by hypothesis, and so $f(z)$ is orthogonal to $h(z)$. By the arbitrariness of $h(z)$, $f(z)$ is in $\mathfrak{H}$. If $g(z)$ is in $\mathfrak{H}$ and $zg(z)$ is also in $\mathfrak{H}$, then $zf(z)$ is orthogonal to $zg(z)$ by our hypotheses and so $f(z)$ is orthogonal to $g(z)$.

Proof of Lemma 9. Let $f(z)=\sum a_nz^n$. Then $g(z)=\sum B(z)a_nz^n$ where the hypotheses imply that $(B(z)a_nz^n)$, $n=0, 1, 2, \cdots$, is an orthogonal set in $\mathcal{C}(z)$, and $\|B(t)a_nt^n\|=\|a_n\|$. By the definition of the norm in $\mathcal{C}(z)$, $\|f\|^2=\sum\|a_n\|^2$. By Parseval's formula, $g(z)$ is in $\mathcal{C}(z)$ and $\|g\|=\|f\|$.

Proof of Lemma 10. By Lemma 9, $\mathfrak{M}(B)$ is a well defined closed subspace of $\mathcal{C}(z)$. Obviously, $zg(z)$ belongs to $\mathfrak{M}(B)$ whenever $g(z)$ belongs to $\mathfrak{M}(B)$, and $\mathfrak{M}(B)$ is orthogonal to $\mathcal{C}$. If $g(z)=zB(z)f(z)$ is in $\mathfrak{M}(B)$ and is orthogonal to every element $zh(z)$ where $h(z)$ is in $\mathfrak{M}(B)$, we can write $h(z)=zB(z)k(z)$ where $k(z)$ is in $\mathcal{C}(z)$ and $\langle g(t), th(t)\rangle=\langle f(t), tk(t)\rangle=0$ by Lemma 9, since a linear isometric transformation preserves inner products. By the arbitrariness of $k(z)$, $f(z)=c$ is a constant. Conversely, if $f(z)$ is a constant, the same formulas show that $g(z)$ is orthogonal to $zh(z)$ for every $h(z)$ in $\mathfrak{M}(B)$.

The hypothesis that $\|B(t)c\|^2=\|c\|^2$ or that $\sum\|B_nc\|^2=\|c\|^2$ implies that $\|B_nc\|\leqq\|c\|$ for every c in $\mathcal{C}$, and so $\|B_n\|\leqq 1$. It follows that $B(w)=\sum B_nw^n$ converges in the operator norm for $|w|<1$ and $\|B(w)\|\leqq\sum|w|^n=(1-|w|)^{-1}$.

Actually, $\|B(w)\|\leqq 1$ for $|w|<1$. To see this, we claim first that for $|w_1|<1$, $|w_2|<1$ and for a, c in $\mathcal{C}$,

$$\left\langle\frac{1-B(w_1)B^*(w_2)}{1-w_1\bar{w}_2}a, c\right\rangle=\left\langle\frac{1-B(t)B^*(w_2)}{1-t\bar{w}_2}a, \frac{1-B(t)B^*(w_1)}{1-t\bar{w}_1}c\right\rangle.$$

This formula makes sense since for $|w|<1$ and d in $\mathcal{C}$, $(1-z\bar{w})^{-1}d$ is in $\mathcal{C}(z)$ by Lemma 5, and hence $B(z)(1-z\bar{w})^{-1}B^*(w)d$ is in $\mathcal{C}(z)$ by Lemma 9. The desired formula follows on expanding the right hand side using these same two lemmas. On taking $w_1=w_2=w$ and $a=c$ in this formula, we find that

$$\frac{\|c\|^2-\|B^*(w)c\|^2}{1-|w|^2}=\left\|\frac{1-B(t)B^*(w)}{1-t\bar{w}}c\right\|^2\geqq 0.$$

So $\|B^*(w)c\|\leqq\|c\|$ for every c in $\mathcal{C}$, and hence $\|B(w)\|\leqq 1$.

Proof of Lemma 11. The lemma follows from Lemmas 7, 8, and 10.

Proof of Lemma 12. Since U is unitary and $|w|<1$, $[1-wU^*]^{-1}$ is defined. Since $\|B(w)\|\leqq 1$ and $|w|<1$, $c=[1-wB(w)]^{-1}a$ is a well-defined element of $\mathcal{C}$. By Lemma 11, $f(z)=B(z)c$ is in $\mathcal{H}(B)$. Let $f(z)=\sum a_nz^n$ and let $f_m(z)=\sum a_{m+n}z^n$ for every $m=0, 1, 2, \cdots$. Since $f_0(z)=f(z)$ and $zf_{m+1}(z)=f_m(z)-f_m(0)$, we see inductively by Lemma 11 that $f_m(z)$ belongs to $\mathcal{H}(B)$ for every m. It is clear that $\|f_m\|\leqq\|f\|$. Therefore, for $|w|<1$, $[zf(z)-wf(w)]\cdot(z-w)^{-1}=\sum w^mf_m(z)$ converges in the metric of $\mathcal{C}(z)$ and defines an element of $\mathcal{H}(B)$. In other words, $[zB(z)-wB(w)](z-w)^{-1}[1-wB(w)]^{-1}a$ belongs to $\mathcal{H}(B)$. We will prove the lemma by showing that

$$U^*a=[1-wU^*][zB(z)-wB(w)](z-w)^{-1}[1-wB(w)]^{-1}a$$

or equivalently that

$$\begin{aligned} B(z)a = {} & [zB(z) - wB(w)](z - w)^{-1}[1 - wB(w)]^{-1}a \\ & - w[B(z) - B(w)](z - w)^{-1}[1 - wB(w)]^{-1}a \\ & - B(z)B(w)[1 - wB(w)]^{-1}a. \end{aligned}$$

A few simple calculations show that this last formula is an identity. Formula (15) follows on using the identity $(1+wU^*)/(1-wU^*) = 1+2wU^*/(1-wU^*)$.

Proof of Theorem XII. By hypothesis, $B_1(z) = B_2(z)B_3(z)$ where the operator valued function $B_3(z)$ is defined and analytic for $|z| < 1$, and $\|B_3(z)\| \leqq 1$. Let $\mathcal{H}(B_3)$ be a Hilbert space constructed for $B_3(z)$ according to Lemma 4, and let U_3 be the corresponding unitary transformation in $\mathcal{H}(B_3)$.

Now construct a Hilbert space $\mathcal{H}$ containing $\mathcal{H}(B_2)$ isometrically, with these properties. There is a linear isometric transformation L of $\mathcal{H}(B_3)$ into $\mathcal{H}$ whose restriction to $\mathcal{C}$ agrees with U_2^* and whose restriction to the orthogonal complemenent of $\mathcal{C}$ in $\mathcal{H}(B_3)$ maps onto the orthogonal complement of $\mathcal{H}(B_2)$ in $\mathcal{H}$. Let U be the unitary extension of U_2 in $\mathcal{H}$ which is the identity on the orthogonal complement of $\mathcal{H}(B_2)$. Let V be the unitary transformation in $\mathcal{H}$ such that $Vf = LU_3L^{-1}f$ whenever f is in the range of L, and which is the identity on the orthogonal complement of the range of L. By the proof of Theorem XI, the composed unitary transformation UV in $\mathcal{H}$ has the property that

$$\langle [1 + w(UV)^*][1 - w(UV)^*]^{-1}a, c\rangle = \langle [1 + wB_1(w)][1 - wB_1(w)]^{-1}a, c\rangle$$

for a and c in $\mathcal{C}$ and $|w| < 1$. The main problem is to show that the smallest closed subspace of $\mathcal{H}$, which contains $\mathcal{C}$ and reduces UV, contains $\mathcal{H}(B_2)$.

To see this, let $\mathcal{H}_4$ be a Hilbert space containing $\mathcal{H}$ isometrically, with these properties. There is a linear isometric transformation M of $\mathcal{H}(0)$ into $\mathcal{H}_4$ whose restriction to $\mathcal{C}$ agrees with $(UV)^*$ and whose restriction to the orthogonal complement of $\mathcal{C}$ in $\mathcal{H}(0)$ maps onto the orthogonal complement of $\mathcal{H}$ in $\mathcal{H}_4$. Let U_4 be the unitary extension of UV in $\mathcal{H}_4$ which is the identity on the orthogonal complement of $\mathcal{H}$. Let V_4 be the unitary transformation in $\mathcal{H}_4$ such that $V_4f = MU_0M^{-}f^{1}$ whenever f is in the range of M, and which is the identity on the orthogonal complement of the range of M. By the proof of Theorem XI, the composed unitary transformation U_4V_4 has the property that

$$\langle [1 + w(U_4V_4)^*][1 - w(U_4V_4)^*]^{-1}a, c\rangle = \langle a, c\rangle$$

for a and c in $\mathcal{C}$ and $|w| < 1$, since $B_1(w)\cdot 0 = 0$. We will show that $\mathcal{H}(B_2)$ is contained in the smallest closed subspace of $\mathcal{H}$, which contains $\mathcal{C}$ and reduces UV, by showing that it is contained in the smallest closed subspace of $\mathcal{H}_4$, which contains $\mathcal{C}$ and reduces U_4V_4.

In fact, let f be any element of $\mathcal{H}(B_2)$. Let f_m, $m = 0, 1, 2, \cdots$, be the elements of $\mathcal{H}(B_2)$ defined inductively by $f_0 = f$ and $U_2f_{m+1} = f_m - a_m$, where a_m is the projection of f_m into $\mathcal{C}$. In view of Lemmas 11 and 12 and our hypotheses, $\mathcal{H}(B_2)$ can be mapped isometrically into $\mathcal{C}(z)$, where it is seen that $\|f\|^2=\sum\|a_m\|^2$. On the other hand, the imbedding of $\mathcal{H}(B_2)$ in $\mathcal{H}$ is such that $UVf_{m+1}=f_m-a_m$ for every m, and the imbedding of $\mathcal{H}$ in $\mathcal{H}_4$ is such that $U_4V_4f_{m+1}=f_m-a_m$ for every m. Since the smallest closed subspace of $\mathcal{H}_4$ which contains $\mathcal{C}$ and reduces U_4V_4 is an $\mathcal{H}(0)$, the formula $\|f\|^2=\sum\|a_m\|^2$ shows that f belongs to this subspace of $\mathcal{H}_4$. By the arbitrariness of f, $\mathcal{H}(B_2)$ is contained in the smallest closed subspace of $\mathcal{H}_4$ which contains $\mathcal{C}$ and reduces U_4V_4.

By Lemma 3, there exists a linear isometric transformation T_0 of $\mathcal{H}$ into $\mathcal{H}(B_1)$ which is the identity on $\mathcal{C}$ and such that $T_0UVf=U_1T_0f$ for f in $\mathcal{H}$. If f is in $\mathcal{H}(B_2)$ and U_2f is orthogonal to $\mathcal{C}$, $UVf=U_2f$. Therefore, we may choose T to be the restriction of T_0 to $\mathcal{H}(B_2)$.

Proof of Theorem V.A. We consider first the case that $E(-i)=0$. The function $\phi(z)=i(1+z)/(1-z)$ maps the unit disk $|z|<1$ conformally onto the upper half plane $y>0$ in such a way that the origin goes into the point i. Let $\mathcal{H}$ be the Hilbert space of functions $f(z)$, defined and analytic in the unit disk $|z|<1$, of the form

$$f(z) = \pi^{1/2}[1 - i\phi(z)]F(\phi(z))/E(\phi(z))$$

for some $F(z)$ in $\mathcal{H}(E)$, with $\|f\|=\|F\|_E$. Define $B(z)$ for $|z|<1$ by $zB(z)=E^*(\phi(z))/E(\phi(z))$. Since we have supposed that $E(-i)=0$, $B(z)$ is defined and analytic for $|z|<1$. On conformal mapping, formula (2) becomes the statement that for $|w|<1$ and $f(z)$ in $\mathcal{H}$

$$f(w) = \left\langle f(t), \frac{1 - t\bar{w}B(t)\bar{B}(w)}{1 - t\bar{w}} \right\rangle. \tag{25}$$

Let ν, corresponding to μ, be the non-negative measure on the Borel sets of the reals modulo 2π, with mass zero at the origin, such that for every μ-integrable function $h(x)$ of real x,

$$\pi^{-1}\int (1 + t^2)^{-1}h(t)d\mu(t) = \int h(\phi(\theta))d\nu(\theta).$$

A necessary and sufficient condition that $\|F\|_E^2=\int|F(t)|^2|E(t)|^{-2}d\mu(t)$ for every $F(z)$ in $\mathcal{H}(E)$ is that $\|f\|^2=\int|f(e^{i\theta})|^2d\nu(\theta)$ for every $f(z)$ in $\mathcal{H}$.

Since $|E(\bar{z})|<|E(z)|$ for $y>0$, $|B(z)|\leq 1$ for $|z|<1$, and since $E(z)$ is analytic across the real axis, $B(z)$ is analytic across the unit circle except for a singularity at $z=1$, and $|B(z)|=1$ for $|z|=1$ except at $z=1$ where it is undefined. Let $\mathcal{C}$ be the complex numbers thought of as a Hilbert space with $\|c\|=|c|$ for every complex number c. We may think of $B(z)$ as an operator valued analytic function as in Lemma 9, since the operators are just complex numbers. The hypotheses of Lemma 9 are satisfied because $|B(e^{i\theta})|=1$ a.e. for $0\leq\theta\leq 2\pi$ and we can use Parseval's formula for Fourier series to compute the inner products. Let $\mathcal{H}(B)$ be defined as in Lemma 11. We claim that $\mathcal{H}=\mathcal{H}(B)$.

Since $\|F\|_E^2 = \int |F(t)|^2 |E(t)|^{-2}dt$ for every $F(z)$ in $\mathcal{H}(E)$, $\|f\|^2 = (2\pi)^{-1}\int |f(e^{i\theta})|^2 d\theta$ for every $f(z)$ in $\mathcal{H}$. By Parseval's formula for Fourier series, $\mathcal{H}$ is contained isometrically in $\mathcal{C}(z)$. For $f(z)$ in $\mathcal{H}$,

$$\langle f(t), t\bar{w}(1 - t\bar{w})^{-1}B(t)\bar{B}(w)\rangle = 0$$

for $|w| < 1$, with the inner product taken in $\mathcal{C}(z)$, since (25) holds and by Cauchy's formula $f(w) = \langle f(t), (1-t\bar{w})^{-1}\rangle$. By the arbitrariness of w, $f(z)$ is orthogonal to $zB(z)$, $z^2B(z)$, $\cdots$, in $\mathcal{C}(z)$. Therefore, $f(z)$ is orthogonal to $\mathfrak{M}(B)$ and so is contained in $\mathcal{H}(B)$. So $\mathcal{H}$ is contained in $\mathcal{H}(B)$. Let $g(z)$ be any element of $\mathcal{H}(B)$ orthogonal to $\mathcal{H}$. Since $[1-z\bar{w}B(z)\bar{B}(w)]/(1-z\bar{w})$ belongs to $\mathcal{H}$ for $|w| < 1$, we see from (25) that $g(w)=0$ for $|w| < 1$, and hence that the power series $g(z)$ vanishes identically. Since $\mathcal{H}$ is complete and so a closed subspace of $\mathcal{H}(B)$, it is all of $\mathcal{H}(B)$.

Let U be the unitary transformation in $\mathcal{H}(B)$ of Lemma 11. Then $U^*f(z) = f(z)/z$ whenever $f(z)$ is in $\mathcal{H}(B)$ and $f(0)=0$. We will match the situation of Theorem XI with $\mathcal{H}(B_2)=\mathcal{H}(B)$ and $U_2 = U$.

Let $L^2(\nu)$ be the Hilbert space of all (a.e. equivalence classes of) complex valued measurable functions $f(e^{i\theta})$ defined for $0 \leqq \theta \leqq 2\pi$ such that $\|f\|_2^2 = \int |f(e^{i\theta})|^2 d\nu(\theta) < \infty$ and let U_1 be the unitary transformation defined in $L^2(\nu)$ by $U_1 f(e^{i\theta}) = e^{i\theta}f(e^{i\theta})$ a.e. Since 1 is an element of norm 1 in $\mathcal{H}$, a necessary condition that $\|f\|^2 = \int |f(e^{i\theta})|^2 d\nu(\theta)$ for every $f(z)$ in $\mathcal{H}$ is that $\int d\nu(\theta) = 1$. If this is the case, there is a complex valued function $B_1(z)$, defined and analytic for $|z| < 1$, such that $|B_1(z)| \leqq 1$ and

$$\int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu(\theta) = \frac{1 + zB_1(z)}{1 - zB_1(z)}$$

for $|z| < 1$. We may look upon $B_1(z)$ as an operator valued function since the operators on $\mathcal{C}$ are complex numbers, and we see that $L^2(\nu) = \mathcal{H}(B_1)$ in the notation of Lemma 4.

If $\|f\|^2 = \int |f(e^{i\theta})|^2 d\nu(\theta)$ for every $f(z)$ in $\mathcal{H}$, the transformation T of $\mathcal{H}(B_2)$ into $\mathcal{H}(B_1)$ defined by $T: f(z) \to f(e^{i\theta})$ is linear and isometric and $T1 = 1$. Whenever $f(z)$ is in $\mathcal{H}(B_2)$ and $U_2 f$ is orthogonal to $\mathcal{C}$, $U_2 f(z) = zf(z)$ and hence $TU_2 f = U_1 Tf$. By Theorem XI, $B_1(z) = B_2(z)B_3(z)$ for some complex valued function $B_3(z)$, defined and analytic for $|z| < 1$, such that $|B_3(z)| \leqq 1$.

On the other hand, suppose that $B_1(z) = B_2(z)B_3(z)$ for some complex valued function $B_3(z)$, defined and analytic for $|z| < 1$, such that $|B_3(z)| \leqq 1$. We have already verified the hypotheses of Theorem XII. By that theorem, there exists a linear isometric transformation T of $\mathcal{H}$ into $L^2(\nu)$ such that $T(1) = 1$ and $T(zf(z)) = e^{i\theta}T(f(z))$ whenever $f(z)$ and $zf(z)$ are in $\mathcal{H}$.

Let $f(z) = \sum a_n z^n$ be in $\mathcal{H}$. We claim that $Tf = \sum a_n e^{in\theta}$ with convergence in the metric of $L^2(\nu)$. To see this, let $f_m(z) = \sum a_{m+n} z^n$ for $m = 0, 1, 2, \cdots$.

We see inductively from Lemma 11 that $f_m(z)$ is in $\mathcal{H}$ for every m and $zf_{m+1}(z)=f_m(z)-a_m$. Therefore, $e^{i\theta}Tf_{m+1}(e^{i\theta})=Tf_m(e^{i\theta})-a_m$ for every m, and so $Tf(e^{i\theta})=\sum_{n<m}e^{in\theta}a_n+e^{im\theta}Tf_m(e^{i\theta})$. Since $f_m(z)\to 0$ in the metric of $\mathcal{H}$ as $m\to\infty$, and since T is isometric, $Tf_m\to 0$ in the metric of $L^2(\nu)$, and $Tf(e^{i\theta}) = \sum e^{in\theta}a_n$ as claimed.

Since $f(z)$ is analytic across the unit circle $|z|=1$ except for a singularity at $z=1$, $f(e^{i\theta})=\sum a_n e^{in\theta}$ converges uniformly in any interval $-\pi\leqq\theta\leqq\epsilon$ or $\epsilon\leqq\theta\leqq\pi$ where $0<\epsilon<\pi$. Since ν has mass 0 at $\theta=0$ by construction, the last paragraph shows that $T:f(z)\to f(e^{i\theta})$ a.e. with respect to ν. Since T is isometric, $\|f\|^2=\int|f(e^{i\theta})|^2d\nu(\theta)$.

We have now seen that a necessary and sufficient condition that $\|f\|^2 =\int|f(e^{i\theta})|^2d\nu(\theta)$ for every $f(z)$ in $\mathcal{H}$ is that $\int d\nu(\theta)<\infty$ and

$$\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\nu(\theta) = \frac{1+zB(z)B_3(z)}{1-zB(z)B_3(z)}$$

for $|z|<1$, where $B_3(z)$ is defined and analytic for $|z|<1$, and $|B_3(z)|\leqq 1$. Corresponding to such a $B_3(z)$, let $A(z)$ be the function, defined and analytic for $y>0$, such that $B_3(z)=A(\phi(z))$. On conformal mapping, we find that a necessary and sufficient condition that $\|F\|_E^2=\int|F(t)|^2|E(t)|^{-2}d\mu(t)$ for every $F(z)$ in $\mathcal{H}(E)$ is that $\int(1+t^2)^{-1}|E(t)|^{-2}d\mu(t)<\infty$ and

$$\int \frac{1+tz}{t-z}\,\frac{d\mu(t)}{1+t^2} = i\pi\,\frac{E(z)+E^*(z)A(z)}{E(z)-E^*(z)A(z)}$$

for $y>0$, where $A(z)$ is defined and analytic for $y>0$ and $|A(z)|\leqq 1$. Because we have supposed $E(-i)=0$, this statement is equivalent to Theorem V.A in that case.

In general, if $E(z)$ has a nonreal zero s, Theorem V.A can be proved by the method just used in the case $s=-i$. The major change is to alter the linear fractional transformation to $\phi(z)=[-sz+\bar{s}]/(1-z)$. The case in which $E(z)$ has no nonreal zeros cannot be handled that way, and we will be concerned with it for the rest of the proof. Since $E^*(z)/E(z)$ is analytic and without zeros for $y\geqq 0$ and is bounded by 1 there and has absolute value 1 when z is real, we see by Boas [1, p. 92] that $E^*(z)/E(z)=be^{2iaz}$ for some $a>0$ and $|b|=1$. Therefore, $E(z)=e^{-iaz}G(z)$ where $G(z)$ is an entire function which is real for real z and has only real zeros. Since $G(z)$ must be a divisor of every $F(z)$ in $\mathcal{H}(E)$, there is no loss of generality in supposing that $G(z)=1$, for otherwise we can consider a new Hilbert space of entire functions of the form $F(z)/G(z)$ where $F(z)$ ranges in $\mathcal{H}(E)$.

To see the necessity in this case, define for each $h>0$, $E_h(z)=C_h(z)-iS_h(z)$ where $C_h(z)=e^{-h}\cos az$ and $S_h(z)=e^h\sin az$. By Theorem I, $|E_h(\bar{z})|<|E_h(z)|$ for $y>0$ and $\mathcal{H}(E_h)=\mathcal{H}(E)$ isometrically. The function $E_h(z)$ has a zero for $z=iy$ where $2ay=-\log(e^h+e^{-h})+\log(e^h-e^{-h})<0$. If $\|F\|_E^2 =\int|F(t)|^2|E(t)|^{-2}d\mu(t)$ for every $F(z)$ in $\mathcal{H}(E)$, then

$$\|F\|_E^2 = \int |F(t)|^2 |E_h(t)|^{-2} |E_h(t)|^2 d\mu(t)$$

for every $F(z)$ in $\mathcal{H}(E)$ since $|E(z)| = 1$ when z is real. By the cases in which the necessity has already been established,

$$\frac{y}{\pi}\int \frac{|E_h(t)|^2 d\mu(t)}{(t-x)^2+y^2} = \operatorname{Re} \frac{E_h(z) + E_h^*(z)A_h(z)}{E_h(z) - E_h^*(z)A_h(z)}$$

for $y>0$, where $A_h(z)$ is defined and analytic for $y>0$, and $|A_h(z)| \leqq 1$. It follows by partial fraction decompositions that

$$\frac{i(\bar{w}-z)}{\pi}\int \frac{|E_h(t)|^2 d\mu(t)}{(t-z)(t-\bar{w})} = \frac{E_h(z) + E_h^*(z)A_h(z)}{E_h(z) - E_h^*(z)A_h(z)} + \frac{\bar{E}_h(w) + E_h(\bar{w})\bar{A}_h(w)}{\bar{E}_h(w) - E_h(\bar{w})\bar{A}_h(w)}$$

for $i(\bar{z}-z)>0$ and $i(\bar{w}-w)>0$. Let w be held fixed. Since $|A_h(w)| \leqq 1$, there is a sequence h_n such that $h_n \searrow 0$ and $\lim A_{h_n}(w)$ exists. It follows from the last formula that $A(z) = \lim A_{h_n}(z)$ converges uniformly on every compact subset of the half plane $y>0$ and that

$$\frac{i(\bar{w}-z)}{\pi}\int \frac{d\mu(t)}{(t-z)(t-\bar{w})} = \frac{E(z) + E^*(z)A(z)}{E(z) - E^*(z)A(z)} + \frac{\bar{E}(w) + E(\bar{w})\bar{A}(w)}{\bar{E}(w) - E(\bar{w})\bar{A}(w)}$$

for $i(\bar{z}-z)>0$, where $A(z)$ is defined and analytic for $y>0$ and $|A(z)| \leqq 1$. The convergence problems are easily handled since $\int(1+t^2)^{-1}d\mu(t) < \infty$ and $E_h(z) \to E(z)$ boundedly on the real axis and uniformly on compact subsets of the complex plane, and $|E(z)| = 1$ when z is real. Clearly, the limit function $A(z)$ does not depend on the choice of w and the last formula holds for all z and w with $i(\bar{z}-z)>0$ and $i(\bar{w}-w)>0$. On choosing $w=z$, we get (7), which completes the proof of necessity in the case that $E(z) = e^{-iaz}$.

For the sufficiency when $E(z) = e^{-iaz}$, suppose (7) holds where $A(z)$ is defined and analytic for $y>0$ and $|A(z)| \leqq 1$. Define $E_h(z)$ for $h>0$ as in the proof of necessity. Let μ_h be the non-negative measure on the Borel sets of the real line such that

$$\frac{y}{\pi}\int \frac{d\mu_h(t)}{(t-x)^2+y^2} = \operatorname{Re} \frac{E_h(z) + E_h^*(z)A(z)}{E_h(z) - E_h^*(z)A(z)}$$

for $y>0$. This measure is given by the Poisson representation of a function positive and harmonic in a half plane since the right hand side is $o(y)$ as $z = iy$ where $y \to +\infty$, because

$$\lim_{y\to+\infty} E_h^*(iy)/E_h(iy) = (e^{-h} - e^h)/(e^{-h} + e^h)$$

has absolute value less than 1. If $f(t)$ is a continuous function of real t which is $o(t^2)$ as $|t| \to \infty$,

$$\int f(t)d\mu(t) = \lim_{h\searrow 0} \int f(t)d\mu_h(t).$$

This follows because $(1+t^2)f(t)$ can be approximated uniformly by finite linear combinations of the functions $(1+t^2)/[(t-x)^2+y^2]$ where $y\neq 0$ as a result of properties of the Poisson kernel. Since $e^{-h}\leq |E_h(z)| \leq e^h$ for real z, $e^{-2h}\|F\|_E^2 \leq \int |F(t)|^2 d\mu_h(t) \leq e^{2h}\|F\|_E^2$ for every $F(z)$ in $\mathcal{H}(E)$. Therefore, if $F(z)$ is in $\mathcal{H}(E)$ and if $F(t)=o(t)$ as $|t|\to\infty$, $\int |F(t)|^2 d\mu(t) = \|F\|_E^2$. By Fourier analysis, such $F(z)$ are dense in $\mathcal{H}(E)$ for the norm metric, and so the same formula holds for every $F(z)$ in $\mathcal{H}(E)$. Since $|E(z)|=1$ for real z, this completes the proof of sufficiency in this case.

Proof of Theorem V.B. Consider first the case $E(-i)=0$. We proceed as for Theorem V.A except that ν, corresponding to μ, is to be the non-negative measure on the Borel sets of the reals modulo 2π, with mass a at the origin, such that for every μ-integrable function $f(x)$ of real x, $\pi^{-1}\int (1+t^2)^{-1}h(t)d\mu(t) = \int h(\phi(\theta))d\nu(\theta)$. Then, ν is a probability measure such that

$$\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\nu(\theta) = \frac{1+zB(z)B_3(z)}{1-zB(z)B_3(z)}$$

for $|z|<1$, where $B_3(z)=A(\phi(z))$. As before, there is a linear isometric transformation T of $\mathcal{H}$ into $L^2(\nu)$, and if $f(z)=\sum a_n z^n$ is in $\mathcal{H}$, $Tf(e^{i\theta})=\sum a_n e^{in\theta}$ with convergence in the metric of $L^2(\nu)$, and $Tf(e^{i\theta})=f(e^{i\theta})$ a.e. with respect to ν for $\theta\neq 0$. Therefore, $\|f\|^2=\int_{\theta\neq 0}|f(e^{i\theta})|^2 d\nu(\theta)+a|Tf(1)|^2$ for every $f(z)$ in $\mathcal{H}$. It follows on conformal mapping that there is a continuous linear functional L on $\mathcal{H}(E)$ such that

$$\|F\|_E^2 = \int |F(t)|^2 |E(t)|^{-2} d\mu(t) + |LF|^2$$

for every $F(z)$ in $\mathcal{H}(E)$. By the Riesz representation of a continuous linear functional on a Hilbert space, there is an element $G(z)$ of $\mathcal{H}(E)$ such that $LF=\langle F, G\rangle$ for every $F(z)$ in $\mathcal{H}(E)$. Since $a>0$ by hypothesis, we see from Theorem V.A that $G(z)$ is not identically zero. Since the orthogonal complement of $G(z)$ in $\mathcal{H}(E)$ is just the set of elements $F(z)$ of $\mathcal{H}(E)$ such that $\|F\|_E^2=\int |F(t)|^2|E(t)|^{-2}d\mu(t)$, it is, in the norm of $\mathcal{H}(E)$, a Hilbert space of entire functions which satisfies (H1), (H2), and (H3).

First let us investigate the case in which there is a real number w such that the orthogonal complement of $G(z)$ is the set of elements $F(z)$ of $\mathcal{H}(E)$ such that $F(w)/E(w)=0$. We will see that $G(z)$ spans $\mathcal{H}(E)$ in this case. For there clearly is a non-negative measure μ_1 on the Borel sets of the real line such that $\mu_1(S)=\mu(S)$ for every Borel set S which does not contain w, and $\|F\|^2=\int |F(t)|^2|E(t)|^{-2}d\mu_1(t)$ for every $F(z)$ in $\mathcal{H}(E)$. By Theorem V.A and the hypothesis that $E(-i)=0$,

$$\int \frac{1 + tz}{t - z} d\mu_1(t) = i\pi \frac{E(z) + E^*(z)A_1(z)}{E(z) - E^*(z)A_1(z)}$$

for $y>0$, where $A_1(z)$ is defined and analytic for $y>0$ and $|A_1(z)| \leqq 1$. But for some real number e,

$$\int \frac{1 + tz}{t - z} \frac{d\mu_1(t)}{1 + t^2} = \int \frac{1 + tz}{t - z} \frac{d\mu(t)}{1 + t^2} + e \frac{1 + wz}{w - z}$$

for $y>0$, and so

$$i\pi \frac{E(z) + E^*(z)A_1(z)}{E(z) - E^*(z)A_1(z)} + \pi az = i\pi \frac{E(z) + E^*(z)A(z)}{E(z) - E^*(z)A(z)} + e \frac{1 + wz}{w - z}$$

for $y>0$. It follows that $E(z)$ cannot have a nonreal zero z_0 other than $z_0 = -i$, for substitute $z = \bar{z}_0$ in this formula. Similarly, the zero of $E(z)$ at $-i$ must be simple, for compute the derivative of each side at $z = i$. It follows from the factorization given by Boas [1, p. 92] that

$$E^*(z)/E(z) = b(z - i)(z + i)^{-1}e^{2icz}$$

where $|b| = 1$ and $c \geqq 0$. But it is equally clear from the same formula that $E^*(iy)/E(iy)$ is not $o(1)$ as $y \to +\infty$ since $a>0$ by hypothesis. Therefore, $c=0$. So there is an entire function $J(z)$, which is real for real z and has only real zeros, such that $E(z) = (z+i)J(z)$. It is clear that $\mathcal{H}(E)$ is one dimensional and hence is spanned by $G(z)$. The theorem is obvious in this case.

Otherwise, there is no real number w such that $F(w)/E(w) = 0$ for every $F(z)$ in $\mathcal{H}(E)$ orthogonal to $G(z)$. By Theorem II, there is some real number α such that $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ belongs to $\mathcal{H}(E)$, and $G(z)$ is a constant multiple of this function. This completes the proof of the theorem in the case $E(-i) = 0$. A similar argument applies if $E(z)$ has any nonreal zero.

The case in which $E(z)$ has no nonreal zeros cannot occur here, for as we saw in the proof of Theorem V.A, $E^*(z)/E(z) = be^{2icz}$ where $|b| = 1$ and $c>0$. Since $E^*(iy)/E(iy) = o(1)$ as $y \to +\infty$,

$$[E(iy) + E^*(iy)A(iy)]/[E(iy) - E^*(iy)A(iy)] = o(y)$$

as $y \to +\infty$. Since the left hand side of (7) is $o(y)$ when $x=0$ and $y \to +\infty$, by the Lebesgue dominated convergence theorem, this contradicts the hypothesis that $a>0$.

Proof of Theorem VI, **the necessity.** By hypothesis there is a non-negative measure μ on the Borel sets of the real line such that $\|F\|_{E_0}^2 = \int |F(t)|^2 d\mu(t)$ for every $F(z)$ in $\mathcal{H}(E_0)$. Because of (5), $E_0(z)$ is without real zeros. By Theorem V.A,

$$\frac{y}{\pi} \int \frac{|E_0(t)|^2 d\mu(t)}{(t - x)^2 + y^2} = \text{Re} \frac{E_0(z) + E_0^*(z)A(z)}{E_0(z) - E_0^*(z)A(z)}$$

for $y>0$, where $A(z)$ is defined and analytic for $y>0$ and $|A(z)|\leqq 1$. By partial fraction decompositions, we find that

$$(26)\quad \frac{i(\bar{w}-z)}{\pi}\int\frac{E_0(t)\bar{E}_0(t)d\mu(t)}{(t-z)(t-\bar{w})}=\frac{E_0(z)+E_0^*(z)A(z)}{E_0(z)-E_0^*(z)A(z)}+\frac{\bar{E}_0(w)+E_0(\bar{w})\bar{A}(w)}{\bar{E}_0(w)-E_0(\bar{w})\bar{A}(w)}$$

for $i(\bar{z}-z)>0$ and $i(\bar{w}-w)>0$. By symmetry, the same formula holds whenever z and w are not real, if we define $A(z)$ for $y<0$ so that $A^*(z)A(z)=1$.

We claim that whenever z and w are not real

$$(27)\quad \frac{i(\bar{w}-z)}{\pi}\int\frac{\bar{E}_0(t)d\mu(t)}{(t-z)(t-\bar{w})}=\frac{2}{E_0(z)-E_0^*(z)A(z)}-\frac{2}{E_0(\bar{w})-\bar{E}_0(w)A(\bar{w})}.$$

If $E_0(z)$ has a nonreal zero s, this formula follows from formula (26) with w replaced by s, and from formula (26) with z replaced by $\bar{w}$ and w replaced by s, and from formula (21) with w replaced by s, and from the partial fraction decomposition

$$(\bar{w}-z)(t-z)^{-1}(t-\bar{w})^{-1}$$
$$=(\bar{s}-z)(t-z)^{-1}(t-\bar{s})^{-1}-(\bar{s}-w)(t-\bar{w})^{-1}(t-\bar{s})^{-1}.$$

If $E_0(z)$ has no nonreal zeros, formula (27) can be obtained by an approximation technique as in the proof of Theorem VA.

If in formula (27), we take the complex conjugate of each side and interchange z and w, we find

$$(28)\quad \frac{i(\bar{w}-z)}{\pi}\int\frac{E_0(t)d\mu(t)}{(t-z)(t-\bar{w})}=\frac{2}{\bar{E}_0(w)-E_0(\bar{w})\bar{A}(w)}-\frac{2}{E_0^*(z)-E_0(z)A^*(z)}$$

when z and w are not real. If follows from (21), (26), (27), and (28), using (5), that

$$(29)\quad \frac{i(\bar{w}-z)}{\pi}\int\frac{d\mu(t)}{(t-z)(t-\bar{w})}=\frac{E_1(z)+E_1^*(z)A(z)}{E_0(z)-E_0^*(z)A(z)}+\frac{\bar{E}_1(w)+E_1(\bar{w})\bar{A}(w)}{\bar{E}_0(w)-E_0(\bar{w})\bar{A}(w)}$$

when z and w ard not real. Formula (8) follows on choosing $w=z$.

Proof of Theorem VI, the sufficiency. By hypothesis, μ is a non-negative measure on the Borel sets of the real line such that formula (8) holds, where $A(z)$ is defined and analytic for $y>0$ and $|A(z)|\leqq 1$. Consider the right hand side of (7) with this choice of $A(z)$ and $E(z)=E_0(z)$. Since this function is positive and harmonic for $y>0$ and since $E_0(z)$ is without real zeros, there is a non-negative measure ν on the Borel sets of the real line and a number $a\geqq 0$ such that

$$\frac{y}{\pi}\int\frac{|E_0(t)|^2d\nu(t)}{(t-x)^2+y^2}+ay=\operatorname{Re}\frac{E_0(z)+E_0^*(z)A(z)}{E_0(z)-E_0^*(z)A(z)}$$

for $y>0$. By Theorem V.B and the hypothesis that there is no real number α such that $e^{i\alpha}E_0(z)-e^{-i\alpha}E_0^*(z)$ belongs to $\mathcal{H}(E_0)$, $a=0$. By the proof of necessity, formula (8) holds with μ replaced by ν, with the same choice of $A(z)$. Therefore, $\mu=\nu$ and the sufficiency follows from Theorem V.A.

Proof of Theorem VII. Since

$$\begin{aligned}\frac{\bar{E}_b(w)E_b(z) - E_b(\bar{w})E_b^*(z)}{2\pi i(\bar{w}-z)} &- \frac{\bar{E}_a(w)E_a(z) - E_a(\bar{w})E_a^*(z)}{2\pi i(\bar{w}-z)} \\ &= \bar{C}_a(w)C_a(z)\,\frac{\bar{S}_0(w)C_0(z) - \bar{C}_0(w)S_0(z)}{\pi(\bar{w}-z)} \\ &\quad - \bar{C}_a(w)S_a(z)\,\frac{\bar{C}_0(w)C_1(z) + \bar{S}_0(w)S_1(z) - 1}{\pi(\bar{w}-z)} \\ &\quad + \bar{S}_a(w)C_a(z)\,\frac{\bar{C}_1(w)C_0(z) + \bar{S}_1(w)S_0(z) - 1}{\pi(\bar{w}-z)} \\ &\quad + \bar{S}_a(w)S_a(z)\,\frac{\bar{S}_1(w)C_1(z) - \bar{C}_1(w)S_1(z)}{\pi(\bar{w}-z)}\end{aligned}$$

where

$$\begin{aligned}&[\bar{S}_0(w)C_0(z) - \bar{C}_0(w)S_0(z)][\bar{S}_1(w)C_1(z) - \bar{C}_1(w)S_1(z)] \\ &\qquad + [\bar{C}_0(w)C_1(z) + \bar{S}_0(w)S_1(z) - 1][\bar{C}_1(w)C_0(z) + \bar{S}_1(w)S_0(z) - 1] \\ &\quad = - [\bar{C}_0(w)C_1(z) + \bar{S}_0(w)S_1(z) + \bar{C}_1(w)C_0(z) + \bar{S}_1(w)S_0(z) - 2],\end{aligned}$$

our hypotheses imply that

$$\frac{\bar{E}_b(w)E_b(z) - E_b(\bar{w})E_b^*(z)}{2\pi i(\bar{w}-z)} \geqq \frac{\bar{E}_a(w)E_a(z) - E_a(\bar{w})E_a^*(z)}{2\pi i(\bar{w}-z)} > 0$$

when $w=z$ is not real. Therefore, $|E_b(\bar{z})| < |E_b(z)|$ for $y>0$. From the same inequality and the definition of $\mathcal{H}(E_a)$, we see that

$$|F(z)|^2 \leqq \|F\|_{E_a}^2 \frac{|E_b(z)|^2 - |E_b(\bar{z})|^2}{2\pi i(\bar{z}-z)} \tag{30}$$

for every $F(z)$ in $\mathcal{H}(E_a)$.

Our hypotheses imply that Re $[E_0(z)\bar{E}_1(z)]>0$ for $y>0$, and so $A(z) = [E_1(z)-E_0(z)]/[E_1(z)+E_0(z)]$ is defined and analytic for $y>0$ and $|A(z)| \leqq 1$. Therefore,

$$\operatorname{Re} \frac{E_a(z) + E_a^*(z)A(z)}{E_a(z) - E_a^*(z)A(z)} > 0$$

for $y>0$, and because of (5), this function has value $|E_a(z)|^2|E_b(z)|^{-2}$ when

z is real. By the Poisson representation of a function positive and harmonic in a half plane, there is a number $a \geqq 0$ such that

$$\frac{y}{\pi}\int \frac{|E_a(t)|^2}{|E_b(t)|^2}\frac{dt}{(t-x)^2+y^2} + ay = \operatorname{Re}\frac{E_a(z)+E_a^*(z)A(z)}{E_a(z)-E_a^*(z)A(z)} \tag{31}$$

for $y>0$. If there is no real number α such that $e^{i\alpha}E_a(z)-e^{-i\alpha}E_a^*(z)$ belongs to $\mathcal{H}(E_a)$, then $a=0$ by Theorem V.B. If there is a real number α such that $e^{i\alpha}E_a(z)-e^{-i\alpha}E_a^*(z)$ belongs to $\mathcal{H}(E_a)$, then

$$\lim_{y\to\infty} E_a^*(iy)/E_a(iy) = e^{2i\alpha},$$

then

$$a = \lim_{y\to+\infty} y^{-1}\operatorname{Re}\frac{E_a(iy)+E_a^*(iy)A(iy)}{E_a(iy)-E_a^*(iy)A(iy)}$$

$$= \lim_{y\to+\infty} y^{-1}\operatorname{Re}\frac{E_1(iy)\cos\alpha - iE_0(iy)\sin\alpha}{E_0(iy)\cos\alpha - iE_1(iy)\sin\alpha} = 0$$

by (9). By (31) and Theorem V.A, $\|F\|_{E_a}^2 = \int |F(t)|^2 |E_b(t)|^{-2}dt$ for every $F(z)$ in $\mathcal{H}(E_a)$. It now follows from (30) that $\mathcal{H}(E_a)$ is contained isometrically in $\mathcal{H}(E_b)$. By (10) and the analyticity of $E_0(z)$ and $E_1(z)$ on the real axis, every real zero of $E_a(z)$ is a real zero of $E_b(z)$ of the same multiplicity. It follows that for every real w, $K_a(w,z)/E_b(z)\neq 0$ when $z=w$.

Proof of Theorem VIII. It is easy to see from the definitions that for each complex number w, $[E_a(z)E(w)-E(z)E_a(w)]/(z-w)$ belongs to $\mathcal{H}(E)$ as a function of z. There is an entire function $E_c(z)$ such that

$$\frac{1}{\pi}\left\langle \frac{E_a(t)E(z)-E(t)E_a(z)}{t-z}, \frac{E_a(t)E(w)-E(t)E_a(w)}{t-w}\right\rangle = \frac{\bar{E}_c(w)E(z)+\bar{E}(w)E_c(z)-2\bar{E}_a(w)E_a(z)}{i(\bar{w}-z)} \tag{32}$$

with the inner product taken in $\mathcal{H}(E)$. Since the proof is exactly analogous to that of formula (21), it is omitted. It follows from (32) that

$$\frac{1}{\pi}\left\langle \frac{E_a(t)E(z)-E(t)E_a(z)}{t-z}, \frac{E_a(t)E^*(w)-E^*(t)E_a(w)}{t-w}\right\rangle = \frac{E(\bar{w})E_c(z)-E_c(\bar{w})E(z)}{i(\bar{w}-z)}, \tag{33}$$

$$\frac{1}{\pi}\left\langle \frac{E_a(t)E^*(z)-E^*(t)E_a(z)}{t-z}, \frac{E_a(t)E(w)-E(t)E_a(w)}{t-w}\right\rangle = \frac{\bar{E}_c(w)E^*(z)-\bar{E}(w)E_c^*(z)}{i(\bar{w}-z)}, \tag{34}$$

$$\frac{1}{\pi}\left\langle \frac{E_a(t)E^*(z) - E^*(t)E_a(z)}{t-z}, \frac{E_a(t)E^*(w) - E^*(t)E_a(w)}{t-w}\right\rangle \tag{35}$$
$$= -\frac{E_c(\bar{w})E^*(z) + E(\bar{w})E_c^*(z) - 2\bar{E}_a(w)E_a(z)}{i(\bar{w}-z)},$$

in the same way that (22), (23), and (24) followed from (21). Since the numerator on the right hand side of (32) must vanish for continuity when $\bar{w}=z$, we find that

$$E_c^*(z)E_b(z) + E_b^*(z)E_c(z) = 2E_a^*(z)E_a(z). \tag{36}$$

For each real number β, let $-2iS(\beta, z)=e^{i\beta}E(z)-e^{-i\beta}E^*(z)$. It follows from (32), (33), (34), and (35) that

$$\frac{2(\bar{w}-z)}{\pi}\left\langle \frac{E_a(t)S(\beta, z) - S(\beta, t)E_a(z)}{t-z}, \frac{E_a(t)S(\beta, w) - S(\beta, t)E_a(w)}{t-w}\right\rangle \tag{37}$$
$$= [e^{i\beta}E_c(z) + e^{-i\beta}E_c^*(z)]\bar{S}(\beta, w) - S(\beta, z)[\bar{e}^{i\beta}\bar{E}_c(w) + e^{i\beta}E_c(\bar{w})].$$

Let $\mu(\beta)$ be the measure on the Borel sets of the real line which is supported at the points t such that $e^{i\beta}E(t)$, or its first nonvanishing derivative, is real, and which has mass $2\pi i|E(t)|^2[E(t)\bar{E}'(t)-\bar{E}(t)E'(t)]^{-1}$ at each such point t. By the formula of [6], $\|F\|_E^2=\int|F(t)|^2|E(t)|^{-2}d\mu(\beta, t)$ for every $F(z)$ in $\mathcal{H}(E)$, if $S(\beta, z)$ does not belong to $\mathcal{H}(E)$. In this formula, β is just an index on the measure, and t is the dummy variable of integration. Formula (37) now becomes

$$\frac{i(\bar{w}-z)}{\pi}\int \frac{|E_a(t)|^2}{|E(t)|^2}\frac{d\mu(\beta, t)}{(t-z)(t-\bar{w})} = i[e^{i\beta}E_c(z) + e^{-i\beta}E_c^*(z)]/[2S(\beta, z)] \tag{38}$$
$$- i[e^{-i\beta}\bar{E}_c(w) + e^{i\beta}E_c(\bar{w})]/[2\bar{S}(\beta,w)]$$

whenever $S(\beta, z)\neq 0$ and $S(\beta, w)\neq 0$, if $S(\beta, z)$ does not belong to $\mathcal{H}(E)$. Since $\mathcal{H}(E_a)$ is contained isometrically in $\mathcal{H}(E)$,

$$\|F\|_{E_a}^2 = \int |F(t)|^2|E(t)|^{-2}d\mu(\beta, t)$$

for every $F(z)$ in $\mathcal{H}(E_a)$, if $S(\beta, z)$ does not belong to $\mathcal{H}(E)$. Since $\mathcal{H}(E_a)$ is a separating subspace of $\mathcal{H}(E)$ by hypothesis, $E_a(z)/E(z)\neq 0$ when z is real. By Theorem V.A,

$$\frac{y}{\pi}\int \frac{|E_a(t)|^2}{|E(t)|^2}\frac{d\mu(\beta, t)}{(t-x)^2+y^2} = \operatorname{Re}\frac{E_a(z) + E_a^*(z)A(\beta, z)}{E_a(z) - E_a^*(z)A(\beta, z)}$$

for $y>0$, where $A(\beta, z)$ is defined and analytic for $y>0$ and $|A(\beta, z)|\leqq 1$, if $S(\beta, z)$ does not belong to $\mathcal{H}(E)$. Therefore,

$$\frac{i(\bar{w}-z)}{\pi}\int \frac{|E_a(t)|^2}{|E(t)|^2}\frac{d\mu(\beta,t)}{(t-z)(t-\bar{w})} \tag{39}$$
$$= \frac{E_a(z)+E_a^*(z)A(\beta,z)}{E_a(z)-E_a^*(z)A(\beta,z)} + \frac{\bar{E}_a(w)+E_a(\bar{w})\bar{A}(\beta,w)}{\bar{E}_a(w)-E_a(\bar{w})\bar{A}(\beta,w)}$$

for $i(\bar{z}-z)>0$ and $i(\bar{w}-w)>0$, if $S(\beta,z)$ does not belong to $\mathcal{H}(E)$. Note from the definition of $E_c(z)$ in (32) that it is unique only within an added imaginary multiple of $E(z)$. Compare the right hand sides of (38) and (39) and choose this added multiple of $E(z)$ in such a way that

$$i\frac{e^{i\beta}E_c(z)+e^{-i\beta}E_c^*(z)}{2S(\beta,z)} = \frac{E_a(z)+E_a^*(z)A(\beta,z)}{E_a(z)-E_a^*(z)A(\beta,z)} \tag{40}$$

for $y>0$, when $S(\beta,z)$ does not belong to $\mathcal{H}(E)$. In other words, if w is a nonreal zero of $E_a^*(z)$, we require that $e^{i\beta}E_c(w)+e^{-i\beta}E_c^*(w)=e^{i\beta}E(w)-e^{-i\beta}E^*(w)$, or if $E_a^*(iy)=o(E_a(iy))$ as $y\to+\infty$, we require that $[e^{i\beta}E_c(iy)+e^{-i\beta}E_c^*(iy)]$ $=o[e^{i\beta}E(iy)-e^{-i\beta}E^*(iy)]$ as $y\to+\infty$. It follows from (37) that this normalization does not depend on the choice of β. From (40) we find that

$$[e^{i\beta}E_c(z)+e^{-i\beta}E_c^*(z)-e^{-i\beta}E(z)+e^{i\beta}E^*(z)]E_a(z)$$
$$= [e^{i\beta}E_c(z)+e^{-i\beta}E_c^*(z)+e^{i\beta}E(z)-e^{-i\beta}E^*(z)]E_a^*(z)A(\beta,z).$$

Since $E_a(z)$ and $S(\beta,z)$ are without zeros for $y>0$, it follows from this formula that

$$W(\beta,z) = [e^{i\beta}E_c(z)+e^{-i\beta}E_c^*(z)+e^{i\beta}E(z)-e^{-i\beta}E^*(z)]/E_a(z)$$

is analytic for $y<0$. It is analytic for $y>0$ because $E_a(z)$ is without zeros there. It is analytic for real z as a result of the hypothesis that $\mathcal{H}(E_a)$ is a separating subspace of $\mathcal{H}(E)$, and this can be seen from formula (32). In other words, $W(\beta,z)$ is an entire function of z and $A(\beta,z)=W^*(\beta,z)/W(\beta,z)$ for $y>0$.

Let $C_0(z)$, $S_0(z)$, $C_1(z)$, $S_1(z)$ be the entire functions which are real for real z such that

$$W(\beta,z) = C_1(z)\cos\beta + S_1(z)\sin\beta + iC_0(z)\sin\beta - iS_0(z)\cos\beta.$$

By the arbitrariness of β, we have (10) with $E_b(z)=E(z)$ and (12). Formula (5) follows from substituting in (36).

Since $|W^*(\beta,z)/W(\beta,z)|\leq 1$ for $y>0$,

$$\operatorname{Re} -i\frac{S_0(z)\cos\beta - C_0(z)\sin\beta}{C_1(z)\cos\beta + S_1(z)\sin\beta} \geqq 0$$

for $y>0$. By the arbitrariness of β, it follows from the proof of Lemma 1 that $|\operatorname{Re}[C_0(z)\bar{C}_1(z)+S_0(z)\bar{S}_1(z)]|\geqq 1$ for $y>0$, and hence by symmetry and continuity, for all z. Since $C_0(z)\bar{C}_1(z)+S_0(z)\bar{S}_1(z)=1\geqq 0$ for real z, we see by continuity that (6) holds. The same considerations show that

$$i[\bar{S}_0(z)C_0(z) - \bar{C}_0(z)S_0(z)] \geqq 0,$$

and

$$i[\bar{S}_1(z)C_1(z) - \bar{C}_1(z)S_1(z)] \geqq 0, \qquad \text{for } y > 0,$$

and hence $|E_0(\bar{z})| \leqq |E_0(z)|$ and $|E_1(\bar{z})| \leqq |E_1(z)|$ for $y>0$. If there is some real number α such that $e^{i\alpha}E_a(z)-e^{-i\alpha}E_a^*(z)$ belongs to $\mathcal{H}(E_a)$, we see from Theorem V.A that formula (31) must hold with $a=0$, and hence from the proof of Theorem VII,

$$\lim_{y\to+\infty} y^{-1}\,\mathrm{Re}\,\frac{E_1(iy)\cos\alpha - iE_0(iy)\sin\alpha}{E_0(iy)\cos\alpha - iE_1(iy)\sin\alpha} = 0$$

which implies (9) by use of the Poisson representation.

Proof of Theorem IX, the necessity. By hypothesis there is a non-negative measure μ on the Borel sets of the real line such that

$$\|F\|^2_{E_b} = \int |F(t)|^2 |E_a(t)|^{-2} d\mu(t)$$

for every $F(z)$ in $\mathcal{H}(E_b)$. We have seen before that because of (5), $E_a(z)/E_b(z) \neq 0$ when z is real. By Theorem V.A,

$$\frac{y}{\pi}\int \frac{|E_b(t)|^2}{|E_a(t)|^2}\frac{d\mu(t)}{(t-x)^2+y^2} = \mathrm{Re}\,\frac{E_b(z)+E_b^*(z)A(z)}{E_b(z)-E_b^*(z)A(z)}$$

for $y>0$, where $A(z)$ is defined and analytic for $y>0$ and $|A(z)|\leqq 1$. It follows by partial fractions that

$$\frac{i(\bar{w}-z)}{\pi}\int \frac{|E_b(t)|^2}{|E_a(t)|^2}\frac{d\mu(t)}{(t-z)(t-\bar{w})} = \frac{E_b(z)+E_b^*(z)A(z)}{E_b(z)-E_b^*(z)A(z)} + \frac{\bar{E}_b(w)+E_b(\bar{w})\bar{A}(w)}{\bar{E}_b(w)-E_b(\bar{w})\bar{A}(w)} \tag{41}$$

when z and w are not real, provided $A(z)$ is extended to the half plane $y<0$ so that $A^*(z)A(z)=1$.

We claim that when z and w are not real

$$\frac{i(\bar{w}-z)}{\pi}\int \frac{\bar{E}_b(t)}{\bar{E}_a(t)}\frac{d\mu(t)}{(t-z)(t-\bar{w})} = \frac{2E_a(z)}{E_b(z)-E_b^*(z)A(z)} - \frac{2E_a(\bar{w})}{E_b(\bar{w})-\bar{E}_b(w)A(\bar{w})} \tag{42}$$

as an absolutely convergent integral. The formula is derived most easily in the case that $E_b(z)$ has a nonreal zero s. We use formula (32) with $E(z)$

$=E_b(z)$ and w replaced by s, and formula (32) again with z replaced by $\bar{w}$ and w replaced by s. Formula (42) follows on using the partial fraction expansion $(\bar{w}-z)(t-z)^{-1}(t-\bar{w})^{-1}=(\bar{s}-z)(t-z)^{-1}(t-\bar{s})^{-1}-(\bar{s}-\bar{w})(t-\bar{w})^{-1}(t-\bar{s})^{-1}$ and (41). The case in which $E_b(z)$ has no nonreal zeros can be obtained as a limiting case of the case with zeros, as in the proof of Theorem V.A; we omit the details.

If in (42), we take the complex conjugate of each side and interchange z and w, we find that when z and w are not real

$$\frac{i(\bar{w}-z)}{\pi}\int \frac{E_b(t)}{E_a(t)}\frac{d\mu(t)}{(t-z)(t-\bar{w})} = \frac{2\overline{E}_a(w)}{\overline{E}_b(w)-E_b(\bar{w})\overline{A}(w)}-\frac{2E_a^*(z)}{E_b^*(z)-E_b(z)A^*(z)}. \tag{43}$$

Formula (11) now follows from (32) with $E(z)=E_b(z)$, and from (41), (42), and (43), using (36).

Proof of Theorem IX, **the sufficiency.** The proof is so closely analogous to the proof of sufficiency for Theorem VI, that it is not included explicitly.

Proof of Theorem X.A. The existence and uniqueness of solutions of the differential equations (13) with the given initial conditions will not be shown, since similar results are presented in Stone [11]. Since C_0, S_0, C_1, S_1 are real, the uniqueness implies that $C_0(t,z)$, $S_0(t,z)$, $C_1(t,z)$, $S_1(t,z)$ are real for real z. By the equations (13), $C_0(t,z)C_1(t,z)+S_0(t,z)S_1(t,z)$ is absolutely continuous with respect to t and has a derivative of zero a.e. This function must then be independent of t and have the value 1 which it has at $t=0$ by hypothesis. This verifies (5). Similarly, for each real number α, we find on computing the derivative of the function

$$\{[\overline{S}_0(t,z)\cos\alpha-\overline{C}_1(t,z)\sin\alpha]\times[C_0(t,z)\cos\alpha+S_1(t,z)\sin\alpha] \\ =[\overline{C}_0(t,z)\cos\alpha+\overline{S}_1(t,z)\sin\alpha]\times[S_0(t,z)\cos\alpha-C_1(t,z)\sin\alpha]\}/(\bar{z}-z)$$

that it is nondecreasing as a function of t. Since it vanishes for $t=0$, it is nonnegative for all values of t. Formula (6) now follows from the arbitrariness of α, as well as the inequalities $|E_0(t,\bar{z})|\leqq|E_0(t,z)|$ and $|E_1(t,\bar{z})|\leqq|E_1(t,z)|$ for $y>0$.

Proof of Theorem X.B. The recurrence relations (14) imply that

$$\begin{aligned} C_0(n+1,z)&=(1-zb_n)C_0(n,z)-za_nS_0(n,z),\\ S_0(n+1,z)&=(1+zb_n)S_0(n,z)+zc_nC_0(n,z),\\ C_1(n+1,z)&=(1+zb_n)C_1(n,z)-zc_nS_1(n,z),\\ S_1(n+1,z)&=(1-zb_n)S_1(n,z)+za_nC_1(n,z). \end{aligned} \tag{44}$$

Since a_n, b_n, c_n are assumed real, we see inductively that $C_0(n,z)$, $S_0(n,z)$, $C_1(n,z)$, $S_1(n,z)$ are real for real z. It follows from (44) that

$$C_0(n+1, z)C_1(n+1, z) + S_0(n+1, z)S_1(n+1, z) = C_0(n, z)C_1(n, z) + S_0(n, z)S_1(n, z).$$

Since $C_0C_1+S_0S_1=1$ by hypothesis, we see inductively that (5) holds for every n. It also follows from (44) that

$$\{[\bar{S}_0(n, z)\cos\alpha - \bar{C}_1(n, z)\sin\alpha] \times [C_0(n, z)\cos\alpha + S_1(n, z)\sin\alpha] - [\bar{C}_0(n, z)\cos\alpha + \bar{S}_1(n, z)\sin\alpha] \times [S_0(n, z)\cos\alpha - C_1(n, z)\sin\alpha]\}/(\bar{z} - z)$$

is a nondecreasing function of n. Since it is zero when $n=0$, it is non-negative for all n. Formula (6) now follows from the arbitrariness of α, as well as the inequalities $|E_0(n, \bar{z})| \leqq |E_0(n, z)|$ and $|E_1(n, \bar{z})| \leqq |E_1(n, z)|$ for $y>0$.

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