ONE-PARAMETER SEMIGROUPS IN A SEMIGROUP(1)

BY

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1. **Introduction.** A topological semigroup is a Hausdorff space S with a continuous, associative multiplication. If there is an identity, 1, then $H(1)$ will denote the maximal subgroup of S containing 1; in other words, $H(1)$ is the set of all elements with two-sided inverses. A *one-parameter semigroup* in S is a continuous, one-to-one function σ: $[0, 1]\to S$ such that $\sigma(0)=1$, and $\sigma(a+b)=\sigma(a)\sigma(b)$ for all $a, b\in[0, 1]$ for which $a+b\in[0, 1]$.

In this paper we obtain the following result on the existence of one-parameter semigroups.

THEOREM 1. *Let S be a compact semigroup with identity, and assume that $H(1)$ is not an open set in S. Let there be a neighborhood V of the identity containing no other idempotents. Then S contains a one-parameter semigroup σ such that $\sigma(a)\notin H(1)$ for $0<a\leqq 1$. Moreover, $\sigma(a)=\sigma(b)g$, $g\in H(1)$, implies $a=b$ and $g=1$.*

This is a generalization of a previous result of the authors [4, Theorem A] in which it is assumed that $H(1)$ is a Lie group. Our proof will actually establish the following more general result.

THEOREM 2. *The conclusion of the previous theorem is correct if S is merely assumed to be locally compact, provided $H(1)$ contains a compact subgroup G open in $H(1)$ but not open in S, and provided there is a neighborhood V of G containing no idempotents other than the identity.*

In the last part of the paper it is shown that if S is a compact semigroup with zero and identity and no other idempotents, and if $H(1)$ is not an open set in S, then S is arcwise connected and S contains a subsemigroup that is an arc with 0 and 1 as its endpoints. Examples are given to show that this result need not be true if there are additional idempotents. Finally, a theorem is stated giving conditions when a semigroup can be locally imbedded in a Lie group.

2. **Preliminaries.** The topological closure of a set A will be denoted by A^-. The set-theoretic difference A minus B will be denoted by $A\backslash B$. For the basic properties of topological semigroups see the survey by A. D. Wallace [5].

Let V be the neighborhood of the identity mentioned in the hypotheses of Theorems 1 and 2; without loss of generality we may assume that V has

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compact closure and that V^- contains no idempotents other than the identity. Let N denote the index set of a fundamental system of neighborhoods of the identity, partially ordered by inclusion. N is thus a directed set and we give it the discrete topology. Let $\{x_\alpha\}$ ($\alpha \in N$) denote a net in V^- (for the properties of nets and subnets see [2, Chapter 2]).

Let N^* denote the Stone-Čech compactification of N. Any continuous function f from N to a compact space X admits a unique extension to a continuous function $f^*: N^* \to X$. In particular, define $f: N \to V^-$ by $f(\alpha) = x_\alpha$. Choose a point $b \in N^* \setminus N$ such that b is a limit point of the directed set N, and hold this point b fixed throughout the discussion. A generalized limit is then defined by

$$\text{Lim } x_\alpha = f^*(b).$$

If the ordinary limit exists it will be denoted by $\lim x_\alpha$. If $\{y_\alpha\}$ is another set in V^- indexed by N we define $\text{Lim } y_\alpha$ in a similar manner, using the same point b. The generalized limit has the following properties (Gleason [1]).

(i) $\text{Lim } x_\alpha$ exists for every set in V^- indexed by N.

(ii) $\text{Lim } x_\alpha = \lim x_\alpha$ whenever $\lim x_\alpha$ exists.

(iii) $\text{Lim } x_\alpha y_\alpha = (\text{Lim } x_\alpha)(\text{Lim } y_\alpha)$.

(iv) If $x_\alpha \in C$, then $\text{Lim } x_\alpha \in C^-$.

A *local semigroup* is a Hausdorff space R with a distinguished element **1**, called the identity, a neighborhood V of 1, and a continuous function $m: V \times V \to R$ which is associative whenever the triple products are defined, and which has the property that $m(x, 1) = m(1, x) = x$. We shall write xy instead of $m(x, y)$.

Let G denote the set of all elements of V having inverses in V, in other words, $x \in G$ if and only if $x \in V$ and there is an element $y \in V$ such that $xy = yx = 1$. R is said to be an *adequate local semigroup* if G is a group (not merely a local group). We shall need the following result (see [4, Theorem A]).

Let R be a compact adequate local semigroup whose maximal subgroup G is a Lie group. If G is not an open set in R, and if there exists a neighborhood of the identity containing no other idempotents, then there exists a one-parameter semigroup σ such that $\sigma(a) \notin G$ for $a > 0$. Moreover, if $\sigma(a) = \sigma(b)g$ for some $g \in G$, then $a = b$ and $g = 1$.

(The last sentence was not included in the statement of Theorem A but may be found in §4.4.4 of [4].)

3. **Proof of Theorem** 1. Throughout this section we assume the hypotheses of the theorem. As remarked in §2, we may assume that V is open and has compact closure, and that V^- contains no idempotents other than 1. Let $x_\alpha \to 1$ ($x_\alpha \in V \setminus H(1)$, $\alpha \in N$).

LEMMA 1. *For each $\alpha \in N$ there is a positive integer $n(\alpha)$ such that*

$$x_\alpha^k \in V, \qquad (1 \leqq k \leqq n(\alpha)),$$
(1)
$$x_\alpha^{n(\alpha)+1} \notin V.$$

Proof. Let $x \in V\backslash H(1)$ and assume that $x^n \in V$ $(n=1, 2, \cdots)$. Then $\Gamma(x) = \{x, x^2, x^3, \cdots\}^- \subset V^-$. But $\Gamma(x)$ is a compact semigroup and hence contains an idempotent. This can only be 1, and therefore $\Gamma(x)$ is a group and $x \in H(1)$, which is a contradiction. (See [3] and [5] for the properties of $\Gamma(x)$.)

Let τ: $[0, 1] \rightarrow V^-$ be defined by

$$\tau(a) = \text{Lim } x_\alpha^{[an(\alpha)]}$$

where $[s]$ denotes the greatest integer less than or equal to s. (x^0 is defined to be 1 for all $x \in V$.)

LEMMA 2. *The function τ has the following properties.*
(i) $\tau(0)=1$,
(ii) $\tau(1) \in V^-\backslash V$,
(iii) $\tau(a+b)=\tau(a)\tau(b)$ *for* $a, b, a+b \in [0, 1]$.

Proof. (i) is obvious.

(ii) Since $x_\alpha \rightarrow 1$ we have $\tau(1) = \text{Lim } x^{[n(\alpha)]} = \text{Lim } x^{[n(\alpha)]+1}$; the result now follows from (1).

(iii) Fix a and b and let $c(\alpha) = [an(\alpha)+bn(\alpha)]$. Then $c(\alpha) = [an(\alpha)] + [bn(\alpha)] + \epsilon(\alpha)$, where $\epsilon(\alpha) = 0$ or 1. Thus $\tau(a+b) = \text{Lim } x_\alpha^{c(\alpha)} = \text{Lim } x_\alpha^{[an(\alpha)]} \text{ Lim } x_\alpha^{[bn(\alpha)]} \text{ Lim } x_\alpha^{\epsilon(\alpha)}$ from which the result follows.

The function τ is not necessarily continuous. However, in the local semigroup $R^* = \{\tau([0, 1])\}^-$ we shall be able to construct a one-parameter semigroup. Notice that R^* is abelian and that $R^* \subset V^-$ (and therefore R^* contains no idempotents except the identity).

LEMMA 3. *Let $Q = \bigcap_{a>0} \{\tau([0, a])\}^-$. Then Q is a compact abelian subgroup of $H(1)$.*

Proof. By (iii) of Lemma 2, $Q^2 \subset Q$, so Q is a compact abelian semigroup. Furthermore, $1 \in Q$ and Q contains no other idempotents (since $Q \subset V^-$). Thus Q is a group, and therefore $Q \subset H(1)$.

LEMMA 4. *If $a_n \rightarrow a$ $(a_n, a \in [0, 1])$, then all cluster points of $\{\tau(a_n)\}$ are contained in the set $\tau(a)Q$.*

Proof. We may assume $a_n > a$ for all n, or $a_n < a$ for all n. In the first case, $\tau(a_n) = \tau(a_n - a)\tau(a)$ and the result follows since $a_n - a \rightarrow 0$ and therefore $\tau(a_n - a)$ clusters in Q. In the second case $\tau(a) = \tau(a_n)\tau(a - a_n)$. Let x be a cluster point of $\{\tau(a_n)\}$. By taking subnets we may assume that $\tau(a_n) \rightarrow x$, and that $\tau(a - a_n)$ is convergent, say to an element y (necessarily in Q). Then $\tau(a) = xy$ or $x = \tau(a)y^{-1}$.

LEMMA 5. *If* $a>0(a\in[0, 1])$, *then* $\tau(a)\notin H(1)$.

Proof. Assume the statement were false, so $\tau(a)=g\in H(1)$ for some $a>0$. Let $x=\tau(a/2)$, so that $x^2=g$. Let $y=xg^{-1}$, $z=g^{-1}x$. Then $xy=1$, $zx=1$ and so x has both a left and a right inverse and therefore $x\in H(1)$. Similarly, $\tau(a/2^n)\in H(1)$ $(n=0, 1, 2, \cdots)$.

Choose dyadic rationals r_n such that $r_na\rightarrow 1^-$. Then by Lemma 4, $\tau(r_na)$ clusters in $\tau(1)Q$. But since $\tau(r_na)\in H(1)$ for all n this implies that $\tau(1)\in H(1)$, contrary to Lemma 2 (ii).

LEMMA 6. *There exists* $a_0>0$ *such that* τ *is one-to-one on* $[0, a_0]$. *Moreover,* $\tau(a)=\tau(b)g$ *with* $a\leqq a_0$ *and* $g\in Q$ *implies* $a=b$.

Proof. Let $c_0=\inf\{c\in[0, 1]: \tau(c)=\tau(b)g \text{ for some } b>c, g\in Q\}$. It will be sufficient to prove that $c_0>0$, for then a_0 can be taken to be any positive number smaller than c_0.

Assume $c_0=0$. Then there exist $c_n<b_n$, $c_n\rightarrow 0$, $g_n\in Q$ such that $\tau(c_n)=\tau(b_n)g_n$. Let $d_n=b_n-c_n$. Then

$$(2)\qquad \tau(c_n) = \tau(c_n)\tau(d_n)g_n = \tau(c_n)\tau(d_n)^m g_n^m = \tau(c_n)\tau(md_n)g_n^m$$

for every positive integer m such that $md_n\leqq 1$. There exist integers $m(n)$ such that $m(n)d_n$ converges to a number d, $1/2\leqq d\leqq 1$. By taking subnets we may assume

$$\tau(c_n) \rightarrow h \in Q,$$
$$\tau(m(n)d_n) \rightarrow z \in \tau(d)Q \text{ (Lemma 4)},$$
$$g_n^{m(n)} \rightarrow g \in Q.$$

From (2) we have $h=hzg$ or $z\in Q$, which is a contradiction to Lemma 5.

Let $R=\tau([0, a_0])Q$. R is a closed set by Lemma 4, and therefore R is a compact abelian adequate local semigroup. Also by Lemma 4, Q is not an open set in R. If H is a closed subgroup of Q then the factor semigroup R/H is an adequate local semigroup whose maximal subgroup is Q/H.

Consider the family F of all pairs (H, σ) where H is a closed subgroup of Q and σ is a one-parameter semigroup in R/H, σ: $[0, a_0]\rightarrow R/H$, such that

$$(3)\qquad \sigma(t) \in \pi(\tau(t)Q), \qquad (0 \leqq t \leqq a_0),$$

and

$$(4)\qquad \sigma(a) = \sigma(b)g \quad (g \in Q/H) \quad \text{implies} \quad a = b \quad \text{and} \quad g = 1.$$

Here π denotes the natural projection homomorphism from R to R/H. We shall consider all one-parameter semigroups to be defined on the interval $[0, a_0]$. This is no restriction since we may always change parameters and pass to the interval $[0, 1]$. Notice that condition (3) implies that

$$\sigma(t) \notin Q/H, \qquad (t > 0). \tag{5}$$

The family F is not empty since we may take $H=Q$ in which case R/Q is just the arc $[0, a_0]$, in other words, R/Q is itself a one-parameter semigroup.

We introduce a partial order into F by saying that $(H, \sigma) \leqq (H_1, \sigma_1)$ if

(a) $H_1 \subset H$,

(b) $p\sigma_1(t) = \sigma(t)$, $(0 \leqq t \leqq a_0)$,

where p is the natural projection from R/H_1 to R/H. Notice that $p^{-1}(1)$ is isomorphic to H/H_1. (The symbol 1 will be used indiscriminately to denote the identity of any semigroup with which we are dealing.)

Let $(H_\alpha, \sigma_\alpha)$ be a linearly ordered subfamily, and let $K = \bigcap_\alpha H_\alpha$. One sees easily that there is a one-parameter semigroup σ in R/K such that $(H_\alpha, \sigma_\alpha) \leqq (K, \sigma)$ for all $(H_\alpha, \sigma_\alpha)$ in the subfamily. Indeed, let $R_\alpha = R/H_\alpha$. Then R/K is the inverse limit of the R_α, and σ may be taken to be the inverse limit of the $\sigma_\alpha(t)$. It is easily verified that (3) and (4) hold for σ. Now let $(H_\alpha, \sigma_\alpha)$ be a maximal linearly ordered subfamily. The proof of Theorem 1 will be complete if we can show that in this case $K = \{1\}$. This will follow from the following lemma.

Lemma 7. *Let $(K, \sigma) \in F$. Let $K_1 \subset K$ be a closed subgroup such that K/K_1 is a Lie group. Then there is a one-parameter semigroup σ_1 in R/K_1 such that $(K, \sigma) \leqq (K_1, \sigma_1)$.*

Proof. Let p denote the natural projection from R/K_1 to R/K. Let $R' = p^{-1}(\sigma([0, a_0]))$. Then R' is an adequate local semigroup contained in R/K_1, and the maximal group of R' is isomorphic to K/K_1. We may apply the result stated at the end of §2 to conclude that there is a one-parameter semigroup γ: $[0, 1] \rightarrow R'$ such that if $\gamma(a) = \gamma(b)g (g \in p^{-1}(1))$, then $a = b$ and $g = 1$. Then $\gamma(1) \in \pi_1(\tau(a_1)Q)$ for some a_1 $(0 < a_1 \leqq a_0)$, where π_1 denotes the natural projection from R to R/K_1.

Define $\sigma_1(t)$: $[0, a_0] \rightarrow R'$ by

$$\sigma_1(t) = \gamma(t/a_1), \qquad (0 \leqq t \leqq a_1),$$

$$\sigma_1(t) = (\sigma_1(a_1))^k \sigma_1(r), \qquad (t \leqq a_0,\ t = ka_1 + r).$$

Then $p\sigma_1(a_1) = \sigma(a_1)$. From this it follows easily that $p\sigma_1(t) = \sigma(t)$ $(0 \leqq t \leqq a_0)$.

It only remains to show that σ_1 satisfies (4). Assume that $\sigma_1(a) = \sigma_1(b)g$ for some $g \in Q/K_1$. Applying p we have $\sigma(a) = \sigma(b)p(g)$. But (4) holds for σ and therefore $a = b$ and $p(g) = 1$. This means that $g \in p^{-1}(1)$, but γ was so chosen that this implies $g = 1$. This completes the proof.

4. **Applications.** An (I)-semigroup is a semigroup on an arc such that one endpoint is a zero and the other is an identity. The general structure is given in [4, Theorem B]. In particular, if an (I)-semigroup has no idempotents except the two endpoints, then it is isomorphic to one of the following:

(1) $[0, 1]$ with the usual multiplication of real numbers,

(2) $[1/2, 1]$ with the product $x\cdot y$ defined by $x\cdot y=\max(1/2, xy)$, where xy denotes the ordinary multiplication of real numbers.

THEOREM 3. *Let S be a compact semigroup with zero and identity and no other idempotents, and assume that $H(1)$ is not an open set. Then S contains an (I)-subsemigroup J, and S is arcwise connected.*

For the proof of this theorem we shall need the following lemma.

LEMMA 8. *Let S be a compact semigroup with zero and identity and no other idempotents. Let $y\notin H(1)$. Then $y^n\to 0$, and if $yz=y$ for any $z\notin H(1)$ then $y=0$.*

Proof. Let $\Gamma(y)=\{y, y^2, y^3, \cdots\}^-$. This is a compact semigroup and therefore contains an idempotent. As in the proof of Lemma 1, this idempotent cannot be the identity. But it is known [3, Theorem 1] that if $0\in\Gamma(y)$ then $y^n\to 0$.

If $yz=y$, then $yz^2=yz=y$, and so $yz^n=y$. But $z^n\to 0$ and so $y=0$.

We do not know whether $yz=y$ is possible for some $z\neq 1$ in $H(1)$.

Proof of Theorem 3. Let σ be the one-parameter semigroup of Theorem 1. Define $\sigma(t)$ for $t\in[1, 2]$ by

$$\sigma(t) = \sigma(1)\sigma(t-1), \qquad (1 \leqq t \leqq 2). \tag{6}$$

It is easy to see that if $a, b, a+b\in[0, 2]$ then $\sigma(a+b)=\sigma(a)\sigma(b)$, and that $\sigma(t)\notin H(1)$ for $t>0$.

Assume that $\sigma(a)=\sigma(b)$ $(a<b)$. Then $\sigma(a)=\sigma(a)\sigma(b-a)$, and so by Lemma 8 $\sigma(a)=0$. Thus either σ is one-to-one on $[0, 2]$, or $\sigma(a)=0$ for some a.

We may successively extend σ into the intervals $[2, 3]$, $[3, 4]$, etc., so that σ is defined for $0\leqq t<\infty$. By Lemma 8, $\lim \sigma(t)=0$ $(t\to\infty)$. Thus, either σ is one-to-one on $[0, \infty]$, or there is a finite number a such that $\sigma(a)=0$ and then $\sigma(t)=0$ for all $t\geqq a$. If we let a' be the smallest such a, then σ is one-to-one on $[0, a']$. In any case, $J=\sigma([0, \infty])$ is the required (I)-semigroup.

To show that S is arcwise connected we observe that if $x\neq 0$ then xJ is an arc from x to 0.

We conjecture that J is a homomorphic retract of S. This would imply that S has a "character" (a homomorphism onto an (I)-semigroup).

If S contains an additional idempotent the result need not be true, even when S is connected. For example, let S be the points of the form $(1+1/(1+x))\exp(2\pi i x)$ $(0\leqq x<\infty)$ together with the unit disc in the complex plane. We multiply as follows:

$$\left(1+\frac{1}{1+x}\right)e^{2\pi i x}\cdot\left(1+\frac{1}{1+y}\right)e^{2\pi i y} = \left(1+\frac{1}{1+x+y}\right)e^{2\pi i(x+y)},$$

$$\rho_1 e^{2\pi i x}\cdot\rho_2 e^{2\pi i y} = \rho_1\rho_2 e^{2\pi i(x+y)}, \qquad (\rho_1, \rho_2 \leqq 1),$$

$$\left(1+\frac{1}{1+x}\right)e^{2\pi i x}\cdot\rho e^{2\pi i y} = \rho e^{2\pi i(x+y)},$$

and take multiplication to be commutative. This is a ray winding asymptotically about the unit disc. The identity ($2e^{2\pi i0}=2$) of the ray is an identity for S, while the zero of the unit disc is a zero for S. In this case there is no (I)-semigroup, in fact no arc at all, from zero to the identity, although a one-parameter semigroup does exist.

We now give another example to illustrate the possibilities when there are additional idempotents. In this example S is the Cantor set endowed with a multiplication which will be described presently. The endpoint 0 will be a zero, 1 will be an identity, and every neighborhood of the identity will contain additional idempotents.

Let $x\in S$. Then x admits a triadic expansion in which only the digits 0 and 2 are used:

$$x = \sum_{1}^{\infty} \epsilon_n/3^n, \qquad (\epsilon_n = 0 \text{ or } 2). \tag{7}$$

Let the indices in which two occurs be $n_1 < n_2 < \cdots$, and let $J^* = \{1, 2, 3, \cdots, \infty\}$. Define a function f_x: $J^*\to J^*$ by:

$$\begin{aligned} f_x(j) &= n_j, \qquad (j \neq \infty), \\ f_x(\infty) &= \infty. \end{aligned} \tag{8}$$

This definition is not complete if the expansion (7) has only a finite number of two's. In this case the indices for which a two occurs have a greatest member, n_N. We complete the definition of f_x by

$$f_x(j) = \infty, \qquad (j > N). \tag{8$'$}$$

In any case the function f_x has the following property:

(P) *f_x is strictly increasing as long as it is finite-valued; once it takes the value ∞ it is constantly equal to ∞ thereafter.*

Conversely, let g: $J^*\to J^*$ satisfy (P), and let $x=\sum 2/3^{g(i)}$. Then $x\in S$ and $g=f_x$. So there is a one-to-one correspondence between the Cantor set S and the set of functions satisfying (P). But the functions form a semigroup under composition, and we take this as the multiplication in S (continuity is easily checked).

The idempotents are precisely the elements 1 and $e_n=2/3+2/3^2+\cdots+2/3^n$; they converge to 1. The multiplication is order-preserving on one side only: $x<y$ implies $cx\leqq cy$ for all $c\in S$.

Another corollary to Theorems 1 and 2 is the following result (whose proof we omit).

THEOREM 4. *Let S be a locally compact semigroup with identity and assume S is an n-dimensional manifold with regular boundary in some neighborhood of* 1. *If the hypotheses of Theorem* 2 *are satisfied and if* $\dim H(1)=n-1$, *then S can be locally imbedded in a Lie group.*

This result need not be true if dim $H(1) < n-1$, as the following example shows.

Let I_1 be $[0, 1]$ with ordinary multiplication, and let $I_2 = [0, 1]$ with the multiplication $x \cdot y = \min(x, y)$. Let $S = \{(x, y): x \in I_1, y \in I_2, y \geqq x\}$ and give S the coordinatewise multiplication. Then S satisfies the hypotheses of Theorem 4 except that dim $H(1) = 0$. However, every neighborhood of 1 contains elements for which the cancellation law fails, and so no neighborhood of the identity can be imbedded in a group.

5. **Concluding remarks.** The hypotheses of Theorem 1 are not necessary and sufficient for the existence of a one-parameter semigroup. Let us agree to say that an element x is *Archimedean* with respect to a neighborhood V of the identity ($x \in V$) if some power of x is not in V. We originally conjectured that if S is a compact semigroup with identity then a necessary and sufficient condition for the existence of a one-parameter semigroup (with $\sigma(t) \notin H(1)$ for $t > 0$) is the existence of a neighborhood V of 1 and a net x_a in V, $x_a \rightarrow 1$, such that each element x_a is Archimedean with respect to V. The main difficulty in proving this seems to be in showing that Q is a group (Lemma 3). A. M. Gleason has given us an example of a semigroup with an Archimedean collection for which Q is not a group. Although this example does not necessarily kill the conjecture, it certainly throws doubt on it.

We further conjecture that if S is a locally compact connected semigroup with identity, and if there is a neighborhood of 1 containing no other idempotents, then S contains a one-parameter semigroup.

Finally, we note that R. J. Koch [6] has an unpublished result showing that every compact connected semigroup with identity contains a generalized arc, but an example of R. P. Hunter shows that there need not be any arc having the identity as one endpoint. These results will appear soon.

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