LOCAL DIFFERENTIAL ALGEBRA[1]

BY

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1. **Introduction.** The theory of systems of algebraic differential equations developed by Ritt and his school [3] and now known under the name of Differential Algebra ignores the values taken by a function at any particular point. It follows that the solution of initial value problems has no place in it. In the present paper we lay the foundations of a theory in which Ritt's general ideas are supplemented so as to take into account initial conditions. Our starting point is a differential ring the values of whose elements "at a given point" are represented by a homomorphic mapping into an ordinary ring. On this basis we develop theories of ideals and of polynomial ideals and of the corresponding varieties. In particular we obtain a consistency condition for a system of algebraic differential equations with given initial values.

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2. **Localized differential rings.** A *localized differential field* is a system $\Sigma=(F, F_0, H)$ where F is a differential field, F_0 is an ordinary field, and H is a homomorphism into F_0 defined on a subring R of F which is closed with respect to differentiation, such that $1\in R$ and $H(1)=1$. (It should cause no confusion if we use the symbol 1 to denote unity in both F and F_0. The condition $H(1)=1$ is added so as to ensure that the homomorphism is not trivial.)

We shall say that Σ is *regular* if it satisfies the condition

2.1. If for any $a\in R$, $H(a^{(n)})=0$, $n=0, 1, 2, \cdots$ then $a=0$, where $a^{(n)}=D^n a$ is the nth derivative of a, as usual.

We define a *localized differential ring* (*l.d.r.*) as a system $\Omega=(R, R_0, H)$, where R is a differential ring and an integral domain (i.e. with unity), R_0 is an ordinary integral domain and H is a homomorphic mapping of the whole of R into R_0 such that $H(1)=1$. If Ω satisfies 2.1 then it is said to be *regular*. If F and F_0 are the fields of quotients of R and R_0, then $\Sigma=(F, F_0, H)$ is a localized differential field, where H is defined on R as in Ω. Conversely, if for a given localized differential field $\Sigma=(F, F_0, H)$ we take $\Omega=(R, F_0, H)$ where R is the subring of F on which H is defined, then Ω is an l.d.r.

Let $\Omega(R, R_0, H)$ be an l.d.r. such that R_0 is of characteristic 0. It follows that R also is of characteristic 0 (otherwise we should have, for some $p>0$, $0=H(0)=H(1+\cdots+1)=pH(1)=p$ in R_0, contrary to assumption). From Ω we derive a regular l.d.r. as follows. Let J be the set of $a\in R$ such that

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$H(a^{(n)})=0$ for $n=0, 1, 2, \cdots$. Then J is an ideal in R. J is proper (i.e. $J\neq R$) since 1 is not contained in J. Moreover, J is prime. For suppose $a\in R-J$, $b\in R-J$. Then there exist non-negative integers μ, ν such that $H(a^{(n)})=0$ for $n<\mu$, $H(a^{(\mu)})\neq 0$, $H(b^{(n)})=0$ for $n<\nu$, $H(b^{(\nu)})\neq 0$. Consider now $H((ab)^{(\mu+\nu)})$. By Leibnitz' formula, which is valid in Differential algebra,

$$\begin{aligned} H((ab)^{(\mu+\nu)}) &\\ &= H(a^{(\mu+\nu)}b + \cdots + C_{\mu+\nu,\nu}\, a^{(\mu)}b^{(\nu)} + \cdots + ab^{(\mu+\nu)}) \\ &= H(a^{(\mu+\nu)})H(b) + \cdots + C_{\mu+\nu,\nu}\, H(a^{(\mu)})H(b^{(\nu)}) + \cdots + H(a)H(b^{(\mu+\nu)}) \\ &= C_{\mu+\nu,\nu}\, H(a^{(\mu)})H(b^{(\nu)}) \neq 0, \end{aligned}$$

since R_0 is of characteristic 0. Hence $ab\in R-J$, J is prime. Finally, J is a differential ideal, $a\in J$ entails $a'\in J$.

We now define the regular l.d.r. $\Omega^*(R_0^*, R^*, H^*)$ as follows. $R^*=R/J$, $R_0^*=R_0$, while $H^*(a^*)$, where $a^*\in R^*$ so that a^* is a residue class with respect to J, is defined as $H(a)$ where a is an arbitrary element of a^*. It is not difficult to verify that this definition of H^* is unique and that it yields a homomorphism from the ring R^* into R_0^* such that Ω^* is regular.

Now let Ω be a regular l.d.r. such that R_0 (and hence R) is of characteristic 0. Let $\Sigma=(F, F_0, H)$ where F is the field of quotients of R, F_0 is the field of quotients of R_0 and H is defined on R as in Ω. Then we are going to extend the definition of H to a ring $R_1\subseteq F$, $R_1\supseteq R$, such that R_1 is a valuation ring i.e., such that for any a, $b\in R_1$, $a\neq 0$, $b\neq 0$, either ab^{-1} or ba^{-1} is contained in R_1.

Consider the set of all formal power series

$$(2.2)\qquad \sum_{n=k}^{\infty} c_n t^n, \qquad k \text{ any integer,}$$

with coefficients in F_0. Defining addition and multiplication in the obvious way, we obtain a field which will be denoted by F^*. We turn F^* into a differential field by defining

$$D\left(\sum_{n=k}^{\infty} c_n t^n\right) = \sum_{n=k}^{\infty} n c_n t^{n-1} = \sum_{n=k-1}^{\infty} (n+1)c_{n+1}t^n.$$

To every element $a\in R$ we adjoin an element $\sigma(a)$ of F^* by defining

$$\sigma(a) = \sum_{n=0}^{\infty} \frac{1}{n!} H(a^{(n)})t^n.$$

Then $\sigma(a+b)=\sigma(a)+\sigma(b)$ and $\sigma(ab)=\sigma(a)\sigma(b)$ the latter again by Leibnitz formula since

$$\frac{1}{n!}H((ab)^{(n)}) = \sum_{k=0}^{n} \frac{1}{(n-k)!}H(a^{(n-k)})\frac{1}{k!}H(b^{(k)}).$$

Thus, σ is a homomorphism, and since Ω is regular it is even an isomorphism. The ring of images of R by σ will be denoted by R_σ and is isomorphic to R. Moreover, the isomorphism is differential, $\sigma(a') = D\sigma(a)$ since

$$\sigma(a') = \sum_{n=0}^{\infty} \frac{1}{n!} H((a')^{(n)})t^n = \sum_{n=0}^{\infty} \frac{1}{n!} H(a^{(n+1)})t^n$$

while

$$D\sigma(a) = \sum_{n=0}^{\infty} \frac{n}{n!} H(a^{(n)})t^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!} H(a^{(n+1)})t^n.$$

Let R^* be the ring of integral power series in F^*, $R^* \supseteq R_\sigma$. Then R^* is a valuation ring. We define a homomorphism on R^* into F_0 by $H^*(\sum_{n=0}^{\infty} c_n t^n) = c_0$. Let F_σ be the field of quotients of R_σ so that F_σ is isomorphic to F by the extension of σ. Then $R_{1\sigma} = R^* \cap F_\sigma$ is a valuation ring and $\Sigma_\sigma = (F_\sigma, F_0, H^*)$ is a localized differential field if we restrict H^* to $R_{1\sigma}$. Finally, if we define H_1 on $R_1 = \sigma^{-1}R_{1\sigma}$ in F by $H_1(a) = H^*(\sigma(a))$ then we obtain the required extension of H. Thus, $\Sigma_1 = (F, F_0, H_1)$ is a localized differential field such that H_1 is defined on a valuation ring $R_1 \subseteq F$ and is an extension of H.

Let Γ be the valuation group determined by R_1 in F. Since F is isomorphic to F_σ by an isomorphism which maps R_1 on $R_{1\sigma}$ we may regard Γ also as the valuation group determined by the ring of integral power series, $R_{1\sigma}$, in its field of quotients F_σ. But this valuation is given by

$$v\left(\sum_{n=k}^{\infty} c_n t^n\right) = k, \qquad c_k \neq 0, \tag{2.3}$$

and so Γ is simply (isomorphic to) the additive group of integers. As far as the elements of R are concerned we may define this valuation also directly by $v(a) = k$ where k is the smallest integer such that $H(a^{(k)}) \neq 0$.

3. ***S*-perfect ideals.** Let R be a differential ring with unit element, and let S be a multiplicative subset of R which does not include 0. A nonempty subset J of R will be said to be an *S-perfect ideal* if it satisfies the following conditions.

3.1. J is a radical differential (i.e., perfect) ideal.

3.2. If $a, b \in R$, $a \in S$, $ab \in J$ then $b \in J$.

The intersection of a nonempty set of S-perfect ideals is S-perfect. The union of a monotonic set of S-perfect ideals is S-perfect.

Given any set $K \subseteq R$, we shall denote by $K^{1/2}$ the set of all $a \in R$ such that $a^n \in K$ for some $n \geq 1$. Then $K \subseteq K^{1/2}$. In particular, if K is an ideal then $K^{1/2}$ is the radical of K. We shall denote by K_S the set of all $b \in R$ for which there exists an $a\in S$ such that $ab\in K$. In particular, if K is an ideal then K_S is an ideal which includes K and which is called the S-component of K (compare [2, p. 17]). In that case also $K_{SS}=K_S$.

We shall denote the (ordinary) ideal generated by a set $K\subseteq R$ by (K) as usual, while $[K]$, $\{K\}$, $\{K\}_S$ shall denote the differential ideal, the perfect ideal and the S-perfect ideal generated by K, respectively, the latter being the intersection of all S-perfect ideals which include K.

If J is a differential ideal in R, so is J_S. For suppose $b\in J_S$ so that $ab\in J$ for some $a\in S$. Then $(ab)'=ab'+a'b\in J$, and hence $ab'=(ab)'-a'b\in J_S$. Hence $b'\in J_{SS}=J_S$, as required. If J is a radical ideal, so is J_S. Indeed, suppose that $b^n\in J_S$ for some $b\in K$ then $ab^n\in J$ for some $a\in S$. Hence $a^nb^n=(ab)^n\in J$, $ab\in J$, $b\in J_S$, as required. This justifies the notation $\{K\}_S$.

For any ideal $J\subseteq R$,

$$(J_S)^{1/2} = (J^{1/2})_S. \tag{3.3}$$

Indeed, the elements a which belong to the set on the left hand side of this equation are characterized by the fact that for some positive integer n and for some $c\in S$, $ca^n\in J$. On the other hand a belongs to the right hand side if for some $d\in S$, and for some positive integer m, $da\in J^{1/2}$, $d^ma^m\in J$. S is multiplicative and so the two conditions are equivalent.

It is known that if R includes the rational numbers (a "Ritt algebra," [1, p. 12]) and J is a differential ideal then $J^{1/2}$ also is a differential ideal.

Let K be any subset of R. We define a sequence of ideals.

$$K_0 = (K), \qquad K_1 = [K_0], \qquad K_2 = (K_1)^{1/2}, \qquad K_3 = (K_2)_S,$$

$$K_{3j+1} = [K_{3j}], \qquad K_{3j+2} = (K_{3(j+1)})^{1/2}, \qquad K_{3j+3} = (K_{3j+2})_S, \qquad j = 1, 2, \cdots.$$

Then it is clear that $K_n\subseteq\{K\}_S$ for all n, and some reflection shows that $\bigcup_n K_n=\{K\}_S$. More particularly, if R is a Ritt algebra then K_2 and K_3, like K_1, are differential ideals and K_3, like K_2 is a radical ideal. Hence, in that case, K_3 is S-perfect and $K_3=K_4=\cdots=\{K\}_S$, $\{K\}_S=([K]^{1/2})_S$. Thus, by (3.3) $\{K\}_S=([K]_S)^{1/2}$.

3.4. Theorem. *Let K, L be subsets of R. Then*

$$\{K\}_S\{L\}_S \subseteq \{KL\}_S.$$

Proof. Construct the sequence $K_0, K_1, K_2, \cdots$ as above, and construct a similar sequence $L_0, L_1, L_2, \cdots$ for L, Then

$$\{K\}_S = \bigcup_n K_n, \qquad \{L\}_S = \bigcup_n L_n,$$

and so it is sufficient to prove

$$K_nL_n \subseteq \{KL\}_S, \qquad n = 0, 1, 2, \cdots. \tag{3.5}$$

Now $KL\subseteq\{KL\}_S$ and so $K_0L_0=(K)(L)\subseteq\{KL\}_S$. For given $j\geq 0$, suppose that (3.5) has been established for $n=3j$. Then we wish to show that

$$K_{3j+1}L_{3j+1} = [K_n][L_n] \subseteq \{KL\}_S.$$

Now the elements of $[K_n]$, $[L_n]$ are of the form

$$k = q_1k_1^{(i_1)} + \cdots + q_sk_s^{(i_s)} \tag{3.6}$$

and

$$l = r_1l_1^{(j_1)} + \cdots + r_tl_t^{(j_t)}, \tag{3.7}$$

respectively, where q_i, $r_i \in R$, $k_m^{(i_m)} \in K_n$, $l_m^{(j_m)} \in L_n$. But the products $k_m l_p$ all belong to $\{KL\}_S$ by the assumption of our induction. It follows [1, p. 11] that the same applies to all products $k_m^{(i)} l_p^{(j)}$, and hence to all products kl as given by (3.6), (3.7).

Suppose next that (3.5) has been established for $n=3j+1$. Then we have to show that

$$K_{3j+2}L_{3j+2} = (K_n)^{1/2}(L_n)^{1/2} \subseteq \{KL\}_S.$$

Now, given elements k, l, of $(K_n)^{1/2}$, $(L_n)^{1/2}$, respectively, we know that for some positive integers ρ, σ, $k^\sigma \in K_n$, $l^\rho \in L_n (kl)^{\rho+\sigma} \in K_nL_n \subseteq \{KL\}_S$. Hence $kl \in \{KL\}_S$, as required.

Suppose finally that (3.5) has been established for $n=3j+2$. Then we have to show that

$$K_{3(j+1)}L_{3(j+1)} = (K_n)_S(L_n)_S \subseteq \{KL\}_S.$$

Now given elements k, l of $(K_n)_S(L_n)_S$ respectively, there exist a, $b \in S$ such that $ak \in K_n$, $bl \in L_n$. Hence $abkl \in K_nL_n \subseteq \{KL\}_S$, $kl \in \{KL\}_S$. This completes the proof of (3.4).

3.8. Theorem. *Let T be a multiplicative set in R and let J be an S-perfect ideal which excludes T, i.e. such that $J \cap T = 0$. Then there exists an S-perfect ideal J^* which contains J and which is maximal with respect to the exclusion of T.*

Proof. By Zorn's lemma.

If $a, b, c, \cdots$ are elements of R, we denote by $\langle a, b, c, \cdots \rangle$ the set of these elements while the ideal generated by the set will be denoted by $(a, b, c, \cdots)$.

3.9. Theorem. *Let T be a multiplicative system in R and let J be an S-perfect ideal which is maximal with respect to the exclusion of T. Then J is prime.*

Proof. Suppose that the assumptions of the theorem are satisfied but that J is not prime. Then for some a, $b \in R$, $ab \in J$, $b \notin J$, $b \notin J$. It follows that there exist elements t_1, $t_2 \in T$ such that $t_1 \in \{\langle J, a\rangle\}_S$, $t_2 \in \{\langle J, b\rangle\}_S$, where $\langle J, a\rangle = J \cup \langle a\rangle$, $\langle J, b\rangle = J \cup \langle b\rangle$. Hence, by (3.4),

$$t_1t_2 \in \{\langle J, a\rangle\}_S\{\langle J, b\rangle\}_S \subseteq \{\langle J, a\rangle\cdot\langle J, b\rangle\}_S$$
$$\subseteq \{J^2 \cup \langle a\rangle\cdot J \cup J\cdot\langle b\rangle \cup \langle ab\rangle\}_S \subseteq \{J\}_S = J.$$

But $t_1t_2 \in T$ and so we have obtained a contradiction.

An S-perfect ideal is called *reducible* if there exist S-perfect ideals $J_1 \neq J$, $J_2 \neq J$ such that $J = J_1 \cap J_2$. J is called irreducible if it is not reducible.

3.10. THEOREM. *An S-perfect ideal is irreducible if and only if it is prime.*

Proof. Suppose that J is reducible, $J = J_1 \cap J_2$, $J_1 \neq J$, $J_2 \neq J$. Choose $a \in J_1 - J$, $b \in J_2 - J$. Then $ab \in J_1 \cap J_2 = J$, and so J is not prime. Conversely, suppose that J is not prime, $ab \in J$ but $a \notin J$, $b \notin J$. Put $J_1 = \{\langle J, a\rangle\}_S$, $J_2 = \{\langle J, b\rangle\}_S$. Then $J_1 \cap J_2 \supseteq J$. On the other hand, if $c \in J_1 \cap J_2$, then $c \in J_1$, $c \in J_2$ and so, by (3.4)

$$c^2 \in \{\langle J, a\rangle\}_S\{\langle J, b\rangle\}_S \subseteq \{\langle J, a\rangle\cdot\langle J, b\rangle\}_S \subseteq J.$$

But J is a radical ideal and so $c \in J$, and $J_1 \cap J_2 = J$. This proves (3.10).

3.11. THEOREM. *Every S-perfect ideal $J \subseteq R$ is the intersection of S-perfect prime ideals.*

Proof. The theorem is trivial for $J = R$. Suppose then that $J \neq R$ and let $a \in R - J$. Then we have to show that there exists an S-perfect prime ideal $J^* \supseteq J$ such that $a \notin J^*$. Let $T = \langle a, a^2, a^3, \cdots \rangle$. Then T is a multiplicative set such that $T \cap J = 0$ (since J is a radical ideal). Hence, by (3.8), there exists an S-perfect ideal J^* which includes J and which is maximal with respect to the exclusion of T. J^* is prime, by (3.9). This proves (3.11).

Given an S-perfect ideal $J \subseteq R$, we call the S-perfect prime ideal J' *minimal over* J if $J' \supseteq J$ and if for any S-perfect prime ideal J'' such that $J' \supseteq J'' \supseteq J$, we have $J' = J''$.

3.12. THEOREM. *Given the S-perfect ideals J, J^*, such that J^* is prime and includes J, there exists an S-perfect prime ideal J' which is minimal over J and is included in J^*.*

Proof. The intersection of a monotonic set of S-perfect prime ideals is an S-perfect prime ideal. The theorem now follows from Zorn's lemma applied as a minimum principle to the S-perfect prime ideals which include J and are included in J^*.

3.13. THEOREM. *Every S-perfect ideal $J \subseteq R$ is the intersection of S-perfect prime ideals which are minimal over J.*

Proof. Let $a \in R - J$. By (3.11) there exists an S-perfect prime ideal J^* which includes J and excludes a. By (3.12) therefore there exists an S-perfect prime ideal J' which is minimal over J and which is included in J^* and, hence, excludes a. (3.13) now follows immediately.

3.14. THEOREM. *Suppose that R satisfies the finite ascending chain condition for S-perfect ideals. Then every S-perfect ideal in R can be represented as the intersection of a finite number of irreducible* (*i.e. prime—see* (3.10)) *S-perfect ideals.*

Proof. As in standard ideal theory, the representation of an ideal J as an intersection, $J=J_1\cap \cdots \cap J_n$ is irredundant if the omission of any J_i on the right hand side leads to a result different from J. One proves by standard methods:

3.15. THEOREM. *Suppose that R satisfies the finite ascending chain condition for S-perfect ideals. Then every S-perfect ideal J is the irredundant intersection of a finite number of S-perfect prime ideals,* $J=J_1\cap \cdots \cap J_n$. *This intersection is unique except for order.*

The J_i are called the *prime components* of J.

3.16. THEOREM. *Suppose that R satisfies the finite ascending chain condition for S-perfect ideals. Then the set of prime components of an ideal J is identical with the set of S-perfect prime ideals of R which are minimal over J.*

Proof. Let $J=J_1\cap \cdots \cap J_n$ be the representation of J by prime components. Then J_i, $i=1, \cdots, n$, is minimal over J. For suppose $J_i\supseteq J'\supseteq J$ where $J_i\neq J'$, and J' is S-perfect and prime. Then

$$J = J_1 \cap \cdots \cap J_i \cap \cdots \cap J_n = J_1 \cap \cdots \cap J_{i-1} \cap J' \cap \cdots \cap J_n,$$

contrary to the uniqueness of the representation. Again, if J' is prime and minimal over J then $J'\supseteq J\supseteq J_1\cap \cdots \cap J_n$ and so $J'\supseteq J_i$ for at least one J_i. But $J_i\supseteq J$, and J' is minimal over J. Hence, $J'=J$.

4. **Theory of ideals in localized differential rings.** We consider an l.d.r. $\Omega=(R, R_0, H)$ together with a multiplicative subset S of R, $0\notin S$. For any $K\subseteq R$ we denote by $H(K)$ the set of images of K in R_0.

A *bi-ideal* is an ordered pair (J, J_0) of subsets of R and R_0 respectively such that J is an S-perfect ideal in R and J_0 is a radical ideal in R_0 such that $H(J)\subseteq J_0$. Bi-ideals will be denoted by small Greek letters, and the set of all bi-ideals (for given Ω and S) will be denoted by B. B is not enpty since it contains the bi-ideal $\epsilon=(R, R_0)$. Since R and R_0 are integral domains (by definition—see §2 above), the ordered pair $\omega=(\langle 0\rangle, \langle 0\rangle)$ also is a bi-ideal. The bi-ideal $\alpha=(J, J_0)$ is said to be proper if $J_0\neq R_0$.

We introduce a partial ordering in B by defining that $\alpha<\beta$("β includes α") for bi-ideals α and β, $\alpha=(J, J_0)$, $\beta=(K, K_0)$ if $J\subseteq K$ and $J_0\subseteq K_0$. ϵ and ω are upper and lower bounds respectively for the whole of B.

Let $A\subseteq B$, $A\neq 0$, $A=\langle (J_\nu, J_{0\nu})\rangle$, then there exists a greatest lower bound (g.l.b.) for A in B, $A=(\bigcap_\nu J_\nu, \bigcap_\nu J_{0\nu})$. If $A=\langle \alpha, \beta\rangle$ then we write g.l.b. $A=\alpha\cap\beta$. Similarly, we write $\alpha\cup\beta$ for the lowest upper bound of $\langle \alpha, \beta\rangle$, which is the greatest lower bound of all bi-ideals which include both α and β. With these definitions, B becomes a lattice. If $A = \langle J_\nu, J_{0\nu})\rangle$ is a linearly ordered subset of B whose elements are proper bi-ideals then A possesses an l.u.b. which is a proper bi-ideal. This is the bi-ideal $\alpha = (\bigcup_\nu J_\nu, \bigcup_\nu J_{0\nu})$. α is proper since $\bigcup_\nu J_{0\nu}$ does not include $1 \in R_0$.

Let $K \subseteq R$, $K_0 \subseteq R_0$. Then the intersection, α, of all bi-ideals (J, J_0) such that $J \supseteq K$, $J_0 \supseteq K_0$ will be said to be generated by (K, K_0). It will be seen that $\alpha = (\{K\}_S, ((H(\{K\}_S) \cup K_0))^{1/2})$.

4.1. THEOREM. *Every proper bi-ideal α is included in a maximal proper bi-ideal α^*. (α^* is maximal with respect to the property of being a proper bi-ideal).*

Proof by Zorn's lemma.

A bi-ideal (J, J_0) will be said to be *prime* if both J and J_0 are prime.

4.2. THEOREM. *Let $\alpha^* = (J, J_0)$ be a maximal proper bi-ideal. Then α^* is prime.*

(4.2) follows immediately from the more general

4.3. THEOREM. *Let T_0 be a nonempty multiplicative set in R_0 and let the bi-ideal $\alpha^* = (J, J_0)$ be maximal with respect to the exclusion of T_0 from J_0. (That is to say, $J_0 \cap T_0 = 0$, and if for some bi-ideal $\alpha' = (J', J_0')$, $J_0' \cap T_0 = 0$ also, and $\alpha^* \prec \alpha'$ then $\alpha^* = \alpha'$.) Then α^* is prime.*

Indeed, (4.2) follows from (4.3) for $T_0 = \langle 1 \rangle$.

Proof of (4.3). Suppose first that J is not prime. Then for some $a_1, a_2 \in R$, $a_1 a_2 \in J$, but $a_1 \notin J$, $a_2 \notin J$. It follows that if $\alpha_1 = (J_1, J_{01})$ is the bi-ideal generated by $((J, a_1), J_0)$ then $J_{01} \cap T_0 \neq 0$ and so $t_1 \in J_{01}$ for some $t_1 \in T_0$. But

$$J_{01} = ((H(\{\langle J, a_1\rangle\}_S) \cup J_0))^{1/2},$$

and so, for some positive integer ρ,

$$t_1^\rho \in (H(\{\langle J, a_1\rangle\}_S) \cup J_0).$$

By a similar argument, there exist an element $t_2 \in T_0$ and a positive integer σ such that

$$t_2^\sigma \in (H(\{\langle J, a_2\rangle\}_S) \cup J_0).$$

Hence, for $t_0 = t_1^\rho t_2^\sigma \in T_0$,

$$\begin{aligned} t_0 &\in (H(\{\langle J, a_1\rangle\}_S) \cup J_0) \cdot (H(\{\langle J, a_2\rangle\}_S) \cup J_0) \\ &\subseteq (H(\{\langle J, a_1\rangle\}_S) \cdot H(\{\langle J, a_2\rangle\}_S) \cup J_0). \end{aligned}$$

But, by (3.4)

$$\begin{aligned} H(\{\langle J, a_1\rangle\}_S) \cdot H(\{\langle J, a_2\rangle\}_S) &= H(\{\langle J, a_1\rangle\}_S \{\langle J, a_2\rangle\}_S) \\ &\subseteq H(\{\langle J, a_1\rangle \cdot \langle J, a_2\rangle\}_S) \subseteq H(\{\langle J, a_1 a_2\rangle\}_S) = H(J). \end{aligned}$$

Hence

$$t_0 \in (H(J) \cup J_0) = J_0,$$

and this is contrary to the assumption that J_0 and T_0 are disjoint. We conclude that J is prime.

Suppose next that J_0 is not prime, $a_1a_2 \in J_0$, $a_1 \notin J_0$, $a_2 \notin J_0$. Consider the bi-ideals $\alpha_1 = (J, (\langle J_0, a_1\rangle)^{1/2})$, $\alpha_2 = (J_1, (\langle J_0, a_2\rangle)^{1/2})$. Since $\alpha^* \prec \alpha_1$, $\alpha^* \neq \alpha_1$, $\alpha^* \prec \alpha_2$, $\alpha \neq \alpha_2$, there exist elements t_1 and t_2 in T_0 such that $t_1 \in (\langle J_0, a_1\rangle)^{1/2}$, $t_2 \in (\langle J_0, a_2\rangle)^{1/2}$, and hence, for certain positive integers ρ and σ, $t_1^\rho \in (\langle J_0, a_1\rangle)$, $t_2^\rho \in (\langle J_0, a_2\rangle)$,

$$t_1^\rho t_2^\rho \in (\langle J_0, a_1\rangle)(\langle J_0, a_2\rangle) \subseteq (\langle J_0, a_1\rangle \cdot \langle J_0, a_2\rangle) \subseteq (\langle J_0, a_1a_2\rangle) = J_0$$

although $t_1^\rho t_2^\rho \in T_0$. It follows that J_0, and hence α^*, is prime.

Combining (4.1) and (4.2) we obtain

4.4. Theorem. *Every proper bi-ideal α is included in a prime proper bi-ideal α^*.*

Now let α be a proper bi-ideal, $\alpha = (J, J_0)$. Then, the g.l.b. of the (non-empty) set of prime proper bi-ideals which include α will be called the *closure* of α and will be denoted by $\bar{\alpha}$. If α, β are proper and $\alpha \prec \beta$ then $\bar{\alpha} \prec \bar{\beta}$. If $\bar{\alpha} = \alpha$ then we say that α is closed. In particular, $\bar{\alpha}$ is closed for any proper α, since $(\bar{\alpha})^- = \bar{\alpha}$. A prime proper bi-ideal is closed.

4.5. Theorem. *Let $\alpha = (J, J_0)$ be a proper bi-ideal, and let $\bar{\alpha} = (\bar{J}, \bar{J}_0)$ be the closure of α. Then $\bar{J}_0 = J_0$.*

Proof. Let $a \in R - J_0$. In order to prove (4.5) it is sufficient to show that there exists a prime proper bi-ideal $\alpha^* = (J^*, J_0^*)$ such that $\alpha \prec \alpha^*$ and $a \notin J_0^*$. Let $T_0 \subseteq R_0$ be the multiplicative set $\langle a, a^2, a^3, \cdots \rangle$. Consider the set A of all bi-ideals (necessarily proper), $\alpha' = (J', J_0')$ such that $\alpha \prec \alpha'$ and $T_0 \cap J_0' = 0$. A is not empty since it includes α. The l.u.b. of any linearly ordered subset of A belongs to A. Hence, by Zorn's Lemma, A includes a maximal element, α^*. α^* is prime, by (4.3).

For the remainder of this section, we shall suppose that R satisfies the finite ascending chain condition for S-perfect ideals. Given a proper bi-ideal $\alpha = (J, J_0)$ and an S-perfect ideal K in R such that $K \supseteq J$, we shall say that K is an *admissible* divisor of J if the bi-ideal generated by (K, J_0) is proper, i.e. if $(H(K) \cup J_0) \neq R_0$.

Now let $\alpha = (J, J_0)$ be any proper bi-ideal and let $\bar{\alpha} = (\bar{J}, \bar{J}_0)$ be its closure.

4.6. Theorem. *The prime components of $\bar{J}$ consist of the admissible prime components of J (i.e. the prime components of J which are admissible divisors of J).*

Proof. Let $J = J_1 \cap \cdots \cap J_n$ be the representation of J as an irredundant intersection of S-perfect prime ideals. Now suppose that $\langle \alpha_\nu \rangle = A$ is the set of proper prime bi-ideals which include α, $\alpha_\nu=(J_\nu, J_{0\nu})$, so that $\bar{\alpha}=(\bigcap_\nu J_\nu, \bigcap_\nu J_{0\nu})$. For each α_ν,

$$(H(J_\nu)\cup J_0)\subseteq(H(J_\nu)\cup J_{0\nu}) = J_{0\nu}\neq R_{01},$$

so that J_ν is an admissible divisor of J. By (3.12), J_ν includes an S-perfect prime ideal J_ν^* which is minimal over J. By (3.16), J_ν^* coincides with a prime component of J, and since $\alpha_\nu^*=(J_\nu^*, J_{0\nu})$ is a proper bi-ideal which includes α, α_ν^* also belongs to A. Then $\bar{J}\subseteq\bigcap_\nu J_\nu^*$ and on the other hand $\bar{J}=\bigcap_\nu J_\nu\supseteq\bigcap_\nu J_\nu^*$ and so $\bar{J}=\bigcap_\nu J_\nu^*$ and $\bar{\alpha}=\bigcap_\nu \alpha_\nu^*$ where the right hand side denotes the g.l.b. of the set $\langle\alpha_\nu^*\rangle$. But $\bar{J}=\bigcap_\nu J_\nu^*$ is a representation of $\bar{J}$ as intersection of S-perfect prime ideals, and since these ideals are minimal over J they are certainly minimal over $\bar{J}\supseteq J$. Thus, only a finite number of the J_ν^*, are different, and they all occur among the prime components of J. Disregarding the repetition of equal components (as is actually implicit in the notation) we conclude that $\bar{J}=\bigcap_\nu J_\nu^*$ is the representation of $\bar{J}$ as irredundant intersection of prime components.

Conversely, if the S-perfect prime ideal J_i is an admissible divisor of J, then we have to show that J_i occurs among the prime components of $\bar{J}$. Let $\bar{J}=J_1\cap\cdots\cap J_k$, $k\leqq n$, where we assume a suitable numbering of the prime components of J. Let J_i be a prime component of J which is an admissible prime divisor, and suppose $k<i\leqq n$. Then $(J_i, (H(J_i)\cup J_0)^{1/2})$ is a proper bi-ideal, and so $(H(J_i)\cup J_0)^{1/2}$ can be extended to a proper radical ideal J_{0i} which is maximal in R_0. Then $\beta=(J_i, J_{0i})$ is a proper prime bi-ideal which includes α and so $\bar{\alpha}<\beta$. Hence $J_i\supseteq J_1\cap\cdots\cap J_k$, and so $J_1\cap\cdots\cap J_k\cap J_i=J_1\cap\cdots\cap J_k$, which is impossible since in that case J_i would be redundant in the representation of J. This completes the proof of (4.6).

4.7. THEOREM. *Let $J_1, \cdots, J_k$ be the admissible prime components of J in the proper bi-ideal $\alpha=(J, J_0)$. Then*

$$J_0 = ((H(J_1)\cup J_0)\cap\cdots\cap(H(J_k)\cup J_0))^{1/2}.$$

Proof. Let $\bar{\alpha}=(\bar{J}, J_0)$, then $\bar{J}=J_1\cap\cdots\cap J_k$, by (4.5) and $\bar{J}_0=J_0$, by (4.6). Clearly

$$J_0\subseteq(H(J_i)\cup J_0),\qquad i=1,\cdots,k. \tag{4.8}$$

Now $\bar{\alpha}=(\bigcap_\nu J_\nu, \bigcap_\nu J_{0\nu})$ where $\langle\alpha_\nu\rangle=\langle(J_\nu, J_{0\nu})\rangle$ is the set of prime proper bi-ideals which include α. As explained previously, every J_ν includes at least one of the J_i, $i=1,\cdots,k$. For this J_i,

$$H(J_i)\subseteq H(J_\nu)\subseteq J_{0\nu},$$

and so

$$(H(J_i)\cup J_0)\subseteq J_{0\nu}.$$

Hence

$$J_0 = \bar{J}_0 = \bigcap_\nu J_{0\nu} \supseteq \bigcap_i (H(J_i) \cup J_0). \tag{4.9}$$

But J_0 is a radical ideal and so (4.9) implies

$$J_0 \supseteq \left(\bigcap_i (H(J_i) \cup J_0) \right)^{1/2}$$

Combining this relation with (4.8), we obtain the conclusion of (4.7).

5. **Extension and polynomial adjunction.** Let $\Omega = (R, R_0, H)$ and $\Omega' = (R', R_0', H')$ be two l.d.r. We say that Ω' is an *extension* of Ω if $R \subseteq R'$, $R_0 \subseteq R_0'$ and $H'(a) = H(a)$ for all $a \in R$.

Given an l.d.r. $\Omega = (R, R_0, H)$ and a positive integer n, we define the l.d.r. $\Omega\{y, z]$ which is obtained by the *adjunction* of n indeterminates to Ω in the following way. Let $R\{y\}$ be the differential ring obtained by the adjunction of the n differential indeterminates $y_1, \cdots, y_n$ to R. Let $R_0[z]$ be the ring which is obtained by adjoining to R_0 the infinite number of indeterminates z_{ik}, $i = 1, \cdots, n$, $k = 0, 1, 2, \cdots$. Let H^* be the (unique) continuation on the whole of $R\{y\}$ of the definitions

$$\begin{aligned} H^*(a) &= H(a), && \text{for } a \in R, \\ H^*(y_i^{(k)}) &= z_{ik}, && \text{for } i = 1, \cdots n,\ k = 0, 1, 2, \cdots, \end{aligned} \tag{5.1}$$

so that H^* is a homomorphism into $R_0[z]$. Then $\Omega\{y, z] = (R\{y\}, R_0[z], H^*)$.

In order to avoid unnecessary complications we shall suppose from now on that R_0 is a field. Accordingly, we shall refer to Ω as an l.d.r. (field) or l.d.f.

Let $\Omega' = (R', R_0', H')$ be any l.d.f. which is an extension of Ω, and let $\eta = (\eta_1, \cdots, \eta_n)$ be an array of n elements of R'. η may be regarded as a point in n-dimensional cartesian space over R', S'^n, say. Let $p\{y\}$ and $q(z)$ be arbitrary elements of $R\{y\}$ and $R_0[z]$ respectively. Substituting η_i for y_i in $p\{y\}$, $i = 1, \cdots, n$, we obtain an expression $p\{\eta\} \in R'$. If $p\{\eta\} = 0$ then we say that η satisfies, or is a zero of, $p\{y\}$, or that $p\{y\}$ vanishes for η. Similarly, substituting $H'(\eta_i^{(k)})$ for z_{ik} in $q(z)$ we obtain an expression in R_0' which we denote by $q(H(\eta))$. If $q(H(\eta)) = 0$ then we say that η satisfies, or is a zero of, $q(z)$, or that $q(z)$ vanishes for η.

Let A be a subset of S'^n. Then the set J of polynomials of $R\{y\}$ which are satisfied by all points of A is a radical differential (i.e. perfect) ideal. Also the polynomials of $R\{y\}$ which reduce to elements of R, other than 0, are clearly not satisfied by the points of A. It follows that if $ap\{\eta\} = 0$ for $a \neq 0$ in R, then $p\{\eta\} = 0$ and so J is S-perfect for $S = R - \langle 0 \rangle$, or for $S =$ any other multiplicative subset of $R - \langle 0 \rangle$.

Again, the set J_0 of $R_0[z]$ which are satisfied by all points of A is a radical ideal. Also, if $p\{y\} \in J$, $p\{\eta\} = 0$, then η satisfies also the polynomial $q(z) = H^*(p\{y\})$ which is obtained from $p\{y\}$ by replacing the coefficients by their homomorphic images in R_0, and by substituting z_{ik} for $y_i^{(k)}$, everywhere. Hence, $H'(J) \subseteq J_0$.

We conclude that (J, J_0) is a bi-ideal in $\Omega\{y, z]$ if we choose for S any multiplicative subset of $R-\langle 0\rangle$. Now the set of bi-ideals in $\Omega\{y, z]$ is the same for all S which include the non-units of R (i.e. the $a\in R$, $a\neq 0$, such that $a^{-1}\notin R$) for if ab belongs to an ideal for a unit a, then $b=a^{-1}(ab)$ belongs to that ideal anyhow. Accordingly we shall from now on take S to be the set $R-\langle 0\rangle$.

The above conclusion still holds if instead of considering a subset of S'^n for given Ω' we consider sets of points η over different l.d.f. which are extensions of Ω. We may then regard η as the composite array $(\Omega'; \eta_1, \cdots, \eta_n)$. The absence of a clear delimitation for S^n, the totality of these points (compare [3, p. 21] for the corresponding problem in standard Differential Algebra) can be overcome without difficulty by supposing that the individual elements which occur in the Ω' under consideration belong to a predetermined but sufficiently large pool (see [4, pp. 146–153] for a general setting for this argument). For the theory which follows the pool is sufficiently large if it is of infinite cardinal number exceeding the cardinal numbers of both R and R_0. Accepting this, we may talk of the set of "all points" η which satisfy a given bi-ideal $\alpha=(J, J_0)$ in $\Omega\{y, z]$, i.e. such that $p\{\eta\}=0$ for all $p\{y\}\in J$ and $q(H(\eta))=0$ for all $q(z)\in J_0$. This set of points will be called the variety of α, V, and we write $\alpha\rightarrow V$. *A variety* is by definition the variety of some bi-ideal in $\Omega\{y, z]$. Conversely, given a set of points A (in particular, a variety) we define the bi-ideal $\alpha=(J, J_0)$ of A, $A\rightarrow\alpha$, by taking J as the set of polynomials in $R\{y\}$ which are satisfied by all points of A, and J_0 as the set of polynomials in $R_0[z]$ which are satisfied by the same points.

Let $\alpha=(J, J_0)$ be a bi-ideal in $\Omega\{y, z]$. A point η will be said to be a generic point of the bi-ideal if $p\{\eta\}=0$ for all $p\in J$, $p\{\eta\}\neq 0$ for all $p\in R\{y\}-J$, $q(H(\eta))=0$ for all $q\in J_0$, and $q(H(\eta))\neq 0$ for all $q\in R_0[z]-J_0$. It will be seen that if α possesses a generic point then α is prime. Conversely, we are going to prove—

5.2. THEOREM. *Let $\alpha=(J, J_0)$ be a prime proper bi-ideal in $\Omega\{y, z]$. Then α possesses a generic point η.*

Proof. In order to construct η, we consider the l.d.f. $\Omega'=(R', R_0', H')$ which is obtained as follows. R' is the differential quotient ring $R\{y\}/J$. R' is an integral domain which is an extension of R, since J does not contain any elements of R except 0. (If there existed an $a\in R\cap J$, $a\neq 0$, then $a\in S$, and so $1\in J$. Hence $H(1)=1\in J_0$, $J_0=R_0$, contrary to the assumption that α is proper.) Next, define R_1 as the quotient ring R_0/J_0. Then R_1 is an extension of R_0, since R_0 is a field and J_0 is a proper ideal in $R_0[z]$. Let R_0' be the field of quotients of R_1 so that R_0' also is an extension of R_0.

Finally, the homomorphic mapping, H', of R' into R_0' is defined in the following way. Let $a\in R'$ so that a is a residue class in $R\{y\}$ with respect to J. Let $p\in a$, then we define $H'(a)$ as the residue class of R_0 with respect to J_0 which contains the element $H^*(p)=q$. Thus, $H'(a)$ is an element of R_1 and hence of $R_0' \supseteq R_1$.

In order to see that this definition is unique, let $p_1 \equiv p_2$ mod J in $R\{y\}$. Then we have to show that $H^*(p_1) \equiv H^*(p_2)$ mod J_0 in $R_0[z]$. But the assumption is that $p_1 - p_2 \in J$, and the required conclusion that $H^*(p_1) - H^*(p_2) = H^*(p_1 - p_2) \in J_0$, and this follows immediately from the definition of a bi-ideal. Moreover, H' coincides with H on R since for $a \in R$, $H'(a) = H^*(a) = H(a)$ by the definition of H^* in $\Omega\{y, z]$. (Strictly speaking, H' is not defined for a but for the residue class which contains a.) We call Ω' the *quotient l.d.f.* $\Omega\{y, z\}/\alpha$.

Now let $\eta_1, \cdots, \eta_n$, be the elements of R which are the residue classes with respect to J of $y_1, \cdots, y_n$ respectively. Then $\eta = (\Omega'; \eta_1, \cdots, \eta_n)$ is a generic point of α.

It is indeed clear that $p\{\eta\} = 0$ for $p\{y\} \in R\{y\}$ if and only if $p\{y\} \in J$. Now let $q(z) \in R[z]$. Then $q(H'(\eta))$ contains the polynomial $q(z)$ and so $q(H'(\eta)) = 0$ if and only if $q(z) \in J_0$. This completes the proof of (5.2).

5.3. THEOREM. *The intersection of a set of varieties is a variety.*

Proof. Let $\langle V_\mu \rangle$ be a set of varieties where for each μ, V_μ is the variety of a bi-ideal α_μ. Let α be the l.u.b. of $\langle \alpha_\mu \rangle$ i.e. the g.l.b. of all bi-ideals β and that $\alpha_\mu < \beta$ for all α_μ. The set of such β is not empty since it includes $\epsilon = (R\{y\}, R_0[z])$. Let

$$\alpha_\mu = (J_\mu, J_{0\mu}), \qquad \alpha = (J, J_0), \qquad V = \bigcap_\mu V_\mu.$$

Then $J_\mu \subseteq J$, $J_{0\mu} \subseteq J_0$ and so every point which belongs to the variety of α satisfies the polynomial of all J_μ, $J_{0\mu}$ and hence belongs to V. In order to establish that V is the variety of α, we therefore have to show only that every point of V satisfies the polynomials of J and J_0. Now

$$J = \left\{ \bigcup_\mu J_\mu \right\} S, \qquad J_0 = \left(H^*(J) \cup \bigcup_\mu J_{0\mu} \right)^{1/2}.$$

Hence, putting $\bigcup_\mu J_\mu = K$, $(K) = K_0$, $[K_0] = K_1$, $(K_1)^{1/2} = K_2$, $(K_2)_S = K_3$, $[K_3] = K_4$, etc., as in §3, we have $J = \bigcup_n K_n$. Let $\eta \in V$, then η satisfies the polynomials of K since $\eta \in V_\mu$ for all μ. By examining the passage from K to K_0, from K_0 to K_1, etc., we see in turn that η satisfies the polynomials of $K_0, K_1, \cdots$, and finally, of J. It follows that η satisfies also the polynomials $q(z)$ of $H^*(J)$. Since η satisfies the elements of all $J_{0\mu}$, we may therefore conclude that it satisfies the elements of the ideal $(H^*(J) \cup \bigcup_\mu J_{0\mu})$ and, hence, of its radical, which is J_0. This completes the proof of (5.3).

A suitable example shows that the union of two varieties is not necessarily a variety.

5.4. THEOREM. *Let α be a proper bi-ideal in $\Omega\{y, z]$, let $\bar{\alpha} = (\bar{J}, \bar{J}_0)$ be the closure of α, and let $p\{y\} \in R\{y\}$. In order that $p\{y\}$ be satisfied by all points*

η which belong to the variety of α, it is necessary and sufficient that $p\{y\}\in\bar{J}$.

Proof. If $p\{y\}\notin\bar{J}$ then, by the definition of $\bar{\alpha}$, there exists a proper and prime bi-ideal $\alpha^*=(J^*, J_0^*)$ which includes α, such that $p\{y\}\notin J^*$. Now by (5.2), α^* possesses a generic point η, and this point satisfies all polynomials of $J\subseteq J^*$ while at the same time $p\{\eta\}\neq 0$. This shows that the condition of the theorem is necessary.

Conversely, suppose that $p\{\eta\}\neq 0$ for some η which belongs to the α_η variety of α. Let $\alpha_\eta=(J_\eta, J_{0\eta})$ be the bi-ideal of $\langle\eta\rangle$. Thus, J_η consists of all elements of $R\{y\}$ which are satisfied by η and $J_{0\eta}$ consists of all elements of $R_0[z]$ which are satisfied by that point. It follows that α_η is proper and prime and $\alpha<\alpha_\eta$. But $p\{y\}$ does not belong to J_η, and so it cannot belong to $\bar{J}$ (which is the intersection of a set including J_η). This completes the proof of (5.4).

5.5. Theorem. *Given α, $\bar{\alpha}$ as in* (5.4), *let $q(z)\in R_0[z]$. In order that $q(z)$ be satisfied by all points which belong to the variety of α, it is necessary and sufficient that $q(z)\in J_0$.*

Proof similar to that of (5.4), taking into account that $\bar{J}_0=J_0$.

We may sum up (5.4) and (5.5) by the relation

$$\alpha\rightarrow V\rightarrow\bar{\alpha} \tag{5.6}$$

which holds for all proper bi-ideals α. Moreover, if for any improper bi-ideal α we define $\bar{\alpha}=\epsilon=(R, R_0)$ then (5.6) still holds with V the empty variety. With this definition ϵ is closed and $(\bar{\alpha})^-=\bar{\alpha}$ for *all* bi-ideals α.

Moreover, $\alpha<\beta$ now entails $\bar{\alpha}<\bar{\beta}$ for all α, β. On the other hand, if α is proper, then V is not empty and $\bar{\alpha}$ also is proper (by (4.4)).

(5.6) shows that if α is closed then it is the bi-ideal of its own variety and this is the case in particular if α is prime. Conversely, if α is the bi-ideal of its own variety, i.e. if $\alpha\rightarrow V\rightarrow\alpha$, then by (5.6), $\alpha=\bar{\alpha}$, α is closed. Thus, the relation $\alpha\rightarrow V\rightarrow\alpha$ establishes a one-one correspondence between the set of all varieties, and the set of closed bi-ideals. Also, if $\alpha_1\rightarrow V_1\rightarrow\alpha_1$, $\alpha_2\rightarrow V_2\rightarrow\alpha_2$ and $\alpha_1<\alpha_2$ then $V_2\subseteq V_1$.

Let K and K_0 be sets of polynomials in $R\{y\}$ and $R[z]$ respectively. What is the condition that the given differential equations (i.e. the elements of K) possess a solution with the given "initial conditions" (i.e. the elements of K_0) in some l.d.f. which is an extension of Ω? If such a solution exists then we say that the system (K, K_0) is consistent.

A well-known metamathematical argument (compare [4]) permits us to read off the result that if (K', K_0') is consistent for all finite subsets K', K_0' of K and K_0 respectively, then (K, K_0) also is consistent. However, this can be deduced without difficulty also from the mathematics of the present paper.

The system (K, K_0) is consistent if and only if the variety of the bi-ideal α generated by (K, K_0), $\alpha=(J, J_0)$ say, is not empty, and this is the case precisely when α is proper, $J_0 \neq R_0$. But $J_0 = (H^*(\{K\}_S) \cup K_0)^{1/2}$ and this is different from R_0 precisely when $(H^*(\{K\}_S) \cup K_0)$ is different from R_0, i.e. precisely when the latter ideal does not contain 1. Hence

5.7. THEOREM. *In order that the system* (K, K_0) *be consistent,* $K \subseteq R\{y\}$, $K_0 \subseteq R_0[z]$, *it is necessary and sufficient that*

$$1 \notin (H^*(\{K\}_S) \cup K_0).$$

If R is a Ritt algebra, then by a remark in §3, $\{K\}_S = ([K]_S)^{1/2}$. Also, for any set $L \in R\{y\}$, we have quite generally, $H^*(L^{1/2}) \subseteq (H^*(L))^{1/2}$. Hence, in the present case,

$$J_0 = (H^*([K]_S) \cup K_0)^{1/2} \subseteq ((H^*([K]_S))^{1/2} \cup K_0)^{1/2} = (H^*([K]_S) \cup K_0)^{1/2} \subseteq J_0$$

or

$$J_0 = (H^*([K]_S) \cup K_0)^{1/2}. \tag{5.8}$$

Accordingly we may replace (5.7) by

5.9. THEOREM. *Let R be a Ritt algebra (i.e., R includes the field of rational numbers). Then in order that the system* (K, K_0) *be consistent,* $K \subseteq R\{y\}$, $K_0 \subseteq R_0[z]$, *it is necessary and sufficient that*

$$1 \notin (H^*([K]_S) \cup K_0). \tag{5.10}$$

We shall now apply this test to the standard initial value problem of the theory of systems of differential equations.

5.11. THEOREM. *Given an l.d.f.* $\Omega = (R, R_0, H)$ *such that R is a Ritt algebra, let*

$$y_i' = P_i(y_1, \cdots, y_n), \qquad i = 1, \cdots, n, \tag{5.12}$$

be a set of differential equations with the "initial conditions"

$$z_{i0} = a_i, \qquad i = 1, \cdots, n, \tag{5.13}$$

where $P_i(y_1, \cdots, y_n) \in R[y_1, \cdots, y_n]$, $a_i \in R_0$. *Then* (5.12), (5.13) *possesses a solution in an l.d.f. which is an extension of* Ω.

Notice that $z_{i0} = H^*(y_i)$; also, that since R is a Ritt algebra R_0 is a field of characteristic 0.

Proof of (5.11). We apply (5.8) for $K = \langle y_i' - P_i \rangle$, $K_0 = \langle z_{i0} - a_i \rangle$, $i = 1, \cdots, n$. Since any particular element of $(H^*([K]_S) \cup K_0)$ must be obtainable from K, K_0 by the application of a finite number of operations, it follows that if (5.12), (5.13) is not consistent then there exist positive integers $k_1, \cdots, k_n$ such that

$$1 \in (H(J_S') \cup (z_{10} - a_1, \cdots, z_{n0} - a_n)) \tag{5.14}$$

where

$$J' = (y_1' - P_1, \cdots, y_1^{(k1)} - P_1^{(k1-1)}, \cdots, y_n' - P_n, \cdots, y_n^{(k_n)} - P_n^{(k_n-1)}).$$

That is to say, there exist polynomials $t_1, \cdots, t_k \in H(J_S')$, $q_1, \cdots, q_k$, $r_1, \cdots, r_n \in R_0[z]$ such that

$$(5.15) \quad 1 = q_1 t_1 + \cdots + q_k t_k + r_1(z_{10} - a_1) + \cdots + r_n(z_{n0} - a_n).$$

Also, for $i = 1, \cdots, k$, $t_i = H(T_i)$ where for certain $s_i \in S$,

$$(5.16) \quad s_i T_i = \sum_{j,l} p_{ijl}(y_j^{(l)} - P_j^{(l-1)}), \quad p_{ijl} \in R\{y\}, \quad 1 \leqq j \leqq n, 1 \leqq l \leqq k_j.$$

Now replace every $y_j^{(l)}$ by z_{jl} in (5.16), including the p_{ijl}. This does not affect the s_i since $s_i \in S \subseteq R$. (S is the set of nonunits in R). We star the resulting polynomials, so

$$(5.17) \quad s_i T_i^* = \sum_{j,l} p_{ijl}^*(z_{jl} - P_j^{*(l-1)}).$$

Thus, t_i is obtained from T_i^* by replacing the *coefficients* by the corresponding elements of R_0. Let $p(z)$ be an ordinary polynomial of the z_{ik} with *coefficients in R*. Then if we substitute $p(z)$ for *a particular* z_{jl} on the right hand side of (5.17) the result will still be divisible by s_i, and so we may still define T_i^{**} as the polynomial which is obtained from T_i^* by the substitution in question. If then we make the corresponding substitution in (5.15), i.e., if we replace z_j in (5.15) by $q(z)$ which is obtained from $p(z)$ by passing to the homomorphic images of the coefficients then we still obtain an identity. In particular, the polynomials t_i are turned into polynomials t_i^{**} which are obtained from T_i^{**} by replacing the coefficients as in $p(z)$.

We now carry out the following substitutions. First we set all z_{jl} for $l > k_j$ equal to zero, if any such z_{jl} occur in q_i, r_i, p_{ijl}. Next we select a greatest k_j (for $j = 1$, say) and we replace z_{1,k_1} by $P_1^{*(k-1)}$ in (5.17), making the corresponding substitution in (5.15). This eliminates the terms involving $z_{1,k_1} - P_1^{*(k_1-1)}$ in (5.17). Continuing in this way, we dispose of the remaining $z_{jl} - P_j^{*(l-1)}$ in (5.17), ($\geqq 1$) and hence of the t_i in (5.15).

Accordingly, we are left with an identity

$$1 = r_1^*(z_{10} - a_1) + \cdots + r_n^*(z_{n0} - a_n),$$

and such an identity is impossible as we see by substituting $a_1, \cdots, a_n$ for $z_{10}, \cdots, z_{n0}$ respectively. This completes the proof of (5.11).

5.18. THEOREM. *If R is a Ritt algebra, then $R\{y\}$ satisfies the finite ascending chain condition for S-perfect ideals.*

Proof. Let F be the field of quotients of R. Let J be an S-perfect ideal in $R\{y\}$ and let J^* be the set of elements of $F\{y\}$ which are of the form $a^{-1}p\{y\}$, $p\{y\} \in J$, $a \in R$, i.e. $a \in S$ since $S = R - \langle 0 \rangle$. We maintain that J^* is a perfect ideal in $F\{y\}$.

It is in fact not difficult to see that J^* is an ideal in $F\{y\}$. Now if $a^{-1}p\in J^*$, $a\in S, p\in J$, then $p'a-a'p\in J$ since J is a differential ideal, and so $a^{-2}(p'a-a'p) = (a^{-1}p)'\in J^*$, J^* also is a differential ideal. Again, suppose $q^n\in J^*$, where $q=a^{-1}p$, $a\in S$, $p\in R\{y\}$. Then $p^n=a^nq^n\in J^*$ and so $p^n=b^{-1}r$ with $b\in S$, $r\in J$. Hence $b^np^n\in J$, $bp\in J$ since J is a radical ideal, and so $p\in J$ and $q=a^{-1}p\in J^*$. This shows that J^* is a perfect ideal. Clearly, J^* is the perfect ideal generated by J in $F\{y\}$.

Now suppose that there exists a strictly ascending infinite chain of ideals in $R\{y\}$,

$$J_1\subseteq J_2\subseteq J_3\subseteq\cdots,\qquad J_i\neq J_{i+1},\qquad i=1,2,\cdots.$$

Let J_i^* be the perfect ideal generated by J_i in $F\{y\}$, as above. We are going to show that $\langle J_i^*\rangle$ also constitutes a strictly ascending chain.

Since $J_i\subseteq J_{i+1}$, clearly $J_i^*\subseteq J_{i+1}^*$. Now suppose that for some positive integer i, $J_i^*=J_{i+1}^*$. Let $p\{y\}\in J_{i+1}-J_i$ then $p\in J_{i+1}^*$ and so, by assumption, $p\in J_i^*$. Hence $p=a^{-1}q$ where $q\in J_i$, $a\in S$. But $p\in J_{i+1}\subseteq R\{y\}$ and $q=ap\in J_i$. Since J_i is S-perfect it follows that $p\in J_i$, and this is contrary to the choice of p. We conclude that $\langle J_i^*\rangle$ constitutes a strictly increasing chain. But this is impossible since by Ritt's basis theorem, the perfect ideals of $F\{y\}$ satisfy the finite ascending chain condition. This proves (5.18).

It follows that if $\Omega=(R, R_0, H)$ is an l.d.f. such that R is a Ritt algebra then Theorem 4.6 applies to $\Omega\{y, z]$. We make use of this fact in order to derive some further information concerning the following problem.

Let $K\subseteq R\{y\}$, $K_0\subseteq R_0[z]$ be sets of polynomials in $R\{y\}$ and $R_0[z]$ respectively, R a Ritt algebra, and let $p\{y\}$ be an additional polynomial in $R\{y\}$. Under what conditions is it true that $p\{y\}$ vanishes for all joint zeros of K and K_0?

Let $\alpha=(J, J_0)$ be the bi-ideal generated by (K, K_0), $J=\{K\}_S=([K]_S)^{1/2}$, $J_0=(H^*(J)\cup K_0)^{1/2}$, and let $\bar\alpha=(\bar J, \bar J_0)$ be the closure of α. $p\{y\}$ is satisfied by all joint zeros of (K, K_0) if and only if $p\in\bar J$ (Theorem 5.4 for a proper bi-ideal α, trivial for improper α). Suppose that we have determined the prime components J_i of J, $i=1,\cdots,k$, $J=J_1\cap\cdots\cap J_k$. Then $p\in\bar J$ if and only if $p\in J_i$ for all admissible prime components of J_1 i.e., if for all $i=1,\cdots,k$ either $p\in J_i$ or $(H^*(J_i)\cup J_0)=R_0[z]$ (or both). Now $(H^*(J_i)\cup J_0) = (H^*(J_i)\cup(H^*(J)\cup K_0))^{1/2}=R_0[z]$ only if there exists an identity

$$1=ra+\sum r_ia_i \tag{5.19}$$

where r, $r_i\in R_0[z]$, $a\in H^*(J_i)$ and $a_i^{\rho_i}\in(H^*(J)\cup K_0)$ for certain positive integers ρ_i. By raising (5.19) to a sufficiently high power, we then find

$$1\in(H^*(J_i)\cup(H^*(J)\cup K_0))=(H^*(J_i)\cup K_0).$$

Conversely if this relation is satisfied then $(H(J_i)\cup J_0)=R_0[z]$. We have proved, for the case that R is a Ritt algebra

5.20. THEOREM. *Let* $\{K\}_S=([K]_S)^{1/2}=J_1\cap\cdots\cap J_k$ *be the representation of* $\{K\}_S$ *as intersection of its prime components* J_i. *In order that the polynomial* $p\{y\}\in R\{y\}$ *vanish for all joint zeros of the sets* $K\subseteq R\{y\}$ *and* $K_1\subseteq R_0[z]$ *it is necessary and sufficient that, for* $i=1,\cdots,k$, *either* $p\in J_i$ *or* $1\in(H^*(J_i)\cup K_0)$.

The corresponding problem for a polynomial $q(z)\in R_0[z]$ (and for given K, K_0) has been settled already by Theorem 5.5. If in particular R is a Ritt algebra, then we may make use of (5.8) to obtain the following *Nullstellensatz.*

5.21. THEOREM. *Given sets of polynomials* $K\subseteq R\{y\}$, $K_0\subseteq R_0[z]$, *and a polynomial* $q(z)\in R[z]$. *In order that* $q(z)$ *vanish for all joint zeros of* K *and* K_0 *it is necessary and sufficient that there exist a positive integer* ρ *such that*

$$q^\rho\in(H^*([K]_S)\cup K_0).$$

(5.21) is a generalization of (5.9).

To conclude this section we give an example which shows that we may indeed have $\bar{\alpha}\neq\alpha$, or, which is the same, that the prime components of J are not always admissible.

Let R and R_0 be given by the field of all algebraic numbers, the differentiation in R being defined by $a'=0$ throughout, and the homomorphism H by the identity, $H(a)=a$. Then $\Omega=(R, R_0, H)$ is an l.d.f. (with R a Ritt algebra). Consider $\Omega\{y, z\}$ for $n=1$, $y_1=y$, $z_{ik}=z_k$. Consider the bi-ideal $\alpha=(J, J_0)$ generated by $K=\langle y(y-1), y', y'', \cdots\rangle$, $K_0=\langle z_0, z_1, z_2, \cdots\rangle$. Since R is a field, we have $\{K\}_S=\{K\}$. We maintain that $J=J_1\cap J_2$ where

$$J_1=(y, y', y'', \cdots),$$
$$J_2=(y-1, y', y'', \cdots),$$

is the representation of J as intersection of its prime components. It is in fact not difficult to see that both J_1 and J_2 are differential ideals and, at the same time, prime, and hence, perfect. Clearly $J_1\supseteq J$, $J_2\supseteq J$, and so $J_1\cap J_2\supseteq J$. On the other hand suppose that $p\in J_1\cap J_2$, $p=p\{y\}=p(y, y', y'', \cdots)$. Then

$$p=\sum_{i=0} r_i y^{(i)}=\sum_{i=1} s_i y^{(i)}+s(y-1) \tag{5.22}$$

where r_i, s_i, s are elements of $R\{y\}$, regarded as ordinary polynomials of the variables $y, y', y'', \cdots$. By (5.22)

$$r_0 y\equiv s(y-1)\mod J_3$$

where $J_3=(y', y'', \cdots)$. It follows that $s\equiv s_0 y \mod J_3$ for some $s_0\in R[y]$ and so $p\equiv s_0 y(y-1)\mod J_3$, $p\in J$.

J_1 and J_2 are both proper divisors of J. Also, J_1 is admissible since $J_0=(H^*(J)\cup K_0)^{1/2}=(z_0, z_1, z_2, \cdots)$, and so $H(J_1)\subseteq(H^*(y), H^*(y'), \cdots)$

$=(z_0, z_1, \cdots)=J_0$. On the other hand, J_2 is not admissible since $(H^*(J_2)\cup J_0)$ contains both z_0 and $H^*(y-1)=z_0-1$, and hence contains $z_0-(z_0-1)=1$.

6. **Regular localized differential rings.** Let D be a given domain (open region) in the complex plane and let z_0 be a point in D. Let R be the ring of functions which are analytic in D, R_0 the field of complex numbers, and H the homomorphism on R into R_0 which is defined by

$$H(f(z)) = f(z_0)$$

for all $f(z)\in R$. With these definitions, $\Omega=(R, R_0, H)$ becomes an l.d.f. with the additional property that $H(f^{(n)})=0$, $n=0, 1, 2, \cdots$ implies $f(z)=0$ in R. Thus, Ω is regular according to the definition given in §2. The above example shows that the case of a regular ring is of particular interest, although the alternative possibility cannot be ruled out either. For example, let R be the ring of functions $f(z)$ which are analytic in the finite complex plane except possibly at the origin, and such that $\lim f(z)$ exists as $z\downarrow 0$ (z_0 tends to zero along the positive real axis). Let R_0 be the field of complex numbers and define

$$H(f(z)) = \lim_{z\downarrow 0} f(z).$$

Then $\Omega=(R, R_0, H)$ is again an l.d.f. However, Ω is not regular since R includes the function

$$f(z) = \exp\left(-\frac{1}{z^2}\right).$$

$f(z)$ is different from 0 in R although

$$H(f^{(n)}(z)) = \lim_{z\downarrow 0} f(z) = 0 \qquad \text{for } n = 0, 1, 2, \cdots.$$

However, if we start with a regular Ω then it is natural to investigate under what conditions there exists a zero in a *regular* extension of Ω, for a given system of equations and initial conditions in Ω.

Consider then a regular l.d.r. $\Omega=(R, R_0, H)$ together with a multiplicative set $S\subseteq R-\langle 0\rangle$ as in §3, with the additional conditions

6.1. R is a Ritt algebra.

6.2. For all $a\in S$, either $a'=0$ or $a'\in S$, and if

$$a_0 = H(a) \neq 0 \quad \text{then} \quad a_0^{-1} \in R_0.$$

Since $H(1)=1$, $H(n)=n$ for all integers, and further $H(a)=a$ for all rational a. Thus R_0 also contains the rational numbers.

Let J_0 be a radical ideal in R_0. By the *expansion* of J_0 in R, $E(J_0)\subseteq R$, we mean the set of $a\in R$ such that $H(a^{(n)})\in J_0$ for all $n\geqq 0$.

6.3. THEOREM. *$E=E(J_0)$ is an S-perfect ideal in R.*

Proof. E is not empty since $0 \in E$. If $a, b \in E$, i.e., $H(a^{(n)}), H(b^{(n)}) \in J_0$, $n=0, 1, 2, \cdots$, then $H((a+b)^{(n)}) = H(a^{(n)}) + H(b^{(n)}) \in J_0$; if $a \in E, r \in R$, then

$$H((ra)^{(n)}) = H(r^{(n)})H(a) + C_{n,1}H(r^{(n-1)})H(a') + \cdots + H(r)H(a^{(n)})$$

belongs to J_0 for $n \geqq 0$, $ra \in E$. It follows that E is an ideal.

If $a \in E$ then $b = a' \in E$ since $H(b^{(n)}) = H(a^{(n+1)}) \in J_0$, $n \geqq 0$, so that E is a differential ideal. In order to establish that E is its own radical it is sufficient to show that $a^2 \in E$ entails $a \in E$. Now suppose that $a^2 \in E$, $H(a^2)^{(n)}) \in J_0$ for all n, but $a \notin E$. Iu that case, let m be the smallest integer such that $H(a^{(m)}) \notin J_0$. Consider

$$H((a^2)^{(2m)}) = H(a^{(2m)})H(a) + (C_{2m,1})H(a^{(2m-1)})H(a') + \cdots$$
$$+ C_{2m,m}(H(a^{(m)}))^2 + \cdots + H(a)H(a^{(2m)}).$$

This identity shows that

$$C_{2m,m}(H(a^{(m)}))^2 \in J_0$$

(since all the other terms occurring in the identity belong to J_0). Since R_0 includes the rationals, and J_0 is a radical ideal, by assumption, we conclude that $H(a^{(m)}) \in J_0$. This contradicts the definition of m and proves that E is a radical ideal.

Finally we have to show that if $ab \in E$, $a \in S$ then $b \in E$. Now if $a \in S$, then $a \neq 0$ and so, by the regularity of Ω there exists a smallest integer m such that $H(a^{(m)}) \neq 0$. Also, if $b \notin E$ there exists a smallest n such that $H(b^{(n)}) \notin J_0$.

Consider

$$H((ab)^{(m+n)}) = H(a^{(m+n)})H(b) + \cdots + (C_{m+n,n})H(a^{(m)})H(b^{(n)}) + \cdots$$
$$+ H(a)H(b^{(m+n)}).$$

As before, this identity shows that

$$(C_{m+n,n})H(a^{(m)})H(b^{(n)}) \in J_0.$$

But $C_{m+n,n} \in R_0$ and $(H(a^{(m)}))^{-1} \in R_0$ and so $H(b^{(n)}) \in J_0$, contrary to assumption. This completes the proof of (6.3).

A bi-ideal α will be called *regular* if $H(a^{(n)}) \in J_0$ for all n entails $a \in J$. Given any radical ideal J_0 in R_0, $(E(J_0), J_0)$ is a regular bi-ideal, by (6.3), and since $H(E(J_0)) \subseteq J_0$. Given any regular bi-ideal (J, J_0), we must have $E(J_0) \subseteq J$, by definition. Conversely, if $a \in J$, then $a^{(n)} \in J$ for all n, hence $H(a^{(n)}) \in J_0$, $a \in E(J_0)$. Hence

6.4. Theorem. *A bi-ideal $\alpha_0 = (J, J_0)$ is regular if and only if $J = E(J_0)$. Thus α_0 is determined entirely by J_0, and we write $\alpha_0 = \rho(J_0)$.*

6.5. Theorem. *For any bi-ideal $\alpha = (J, J_0)$ we have*

$$\alpha \prec \rho(J_0).$$

Proof. We have to show that $J \subseteq E(J_0)$ and this follows immediately from the fact that $a \in J$ entails $a^{(n)} \in J$, $H(a^{(n)}) \in J_0$, $n = 0, 1, 2, \cdots$

6.6. THEOREM. *If J_0 is prime then $\rho(J_0)$ also is prime.*

Proof. We have to show that $E = E(J_0)$ is prime. Suppose on the contrary that $ab \in E$, $a \notin E$, $b \notin E$. Then $H((ab)^{(n)}) \in J_0$, for all n while for some smallest integers k and l, $H(a^{(k)}) \notin J_0$, $H(b^{(l)}) \notin J_0$. Considering $H((ab)^{(k+l)})$, we obtain as before,

$$C_{k+l,n} H(a^{(k)}) H(b^{(l)}) \in J_0.$$

But J_0 is prime and so one of the factors $H(a^{(k)})$, $H(b^{(l)})$ belongs to J_0. This contradicts the definition of k and l and proves (6.6).

6.7. THEOREM. *Let $\langle J_{0\nu} \rangle$ be a set of radical ideals in R_0 and let $\langle E(J_{0\nu}) \rangle$ be the corresponding expansions in R. Then*

$$\bigcap_\nu E(J_{0\nu}) = E\left(\bigcap_\nu J_{0\nu} \right).$$

Proof. If $a \in R$ belongs to the left hand side then $a \in E(J_{0\nu})$ for all ν, $H(a^{(n)}) \in J_{0\nu}$ for all n and ν, hence $H(a^{(n)}) \in \bigcap_\nu J_{0\nu}$, $a \in E(\bigcap_\nu J_{0\nu})$. Conversely, if $a \in E(\bigcap_\nu J_{0\nu})$ $H(a^{(n)}) \in \bigcap_\nu J_{0\nu} \subseteq J_{0\nu}$ for all ν, $a \in E(J_{0\nu})$ for all ν, hence $a \in \bigcap_\nu E(J_{0\nu})$.

6.8. THEOREM. *The intersection (g.l.b.) of a set of regular bi-ideals is a regular bi-ideal.*

Proof. Let the set be $\langle \alpha_\nu \rangle$, $\alpha_\nu = (E(J_{0\nu}), J_{0\nu})$. Then $\bigcap_\nu \alpha_\nu = (\bigcap_\nu E(J_{0\nu}), \bigcap_\nu J_{0\nu}) = (E(\bigcap_\nu J_{0\nu}), \bigcap_\nu J_{0\nu})$.

Let $J_0 \in R_0$ be a radical ideal in R_0. Then J_0 can be represented by a (finite or infinite) intersection of proper prime ideals, $J_0 = \bigcap_\nu J_{0\nu}$. It follows that $\rho(J_0)$ is equal to $\bigcap_\nu \rho(J_{0\nu}) = (\bigcap_\nu E(J_{0\nu}), \bigcap_\nu J_{0\nu})$. Since all the $\rho(J_{0\nu})$ are proper prime bi-ideals we conclude that $\rho(J_0)$ is closed. Hence

6.9. THEOREM. *A proper and regular bi-ideal is closed. It is the intersection of a set of proper bi-ideals which are prime and regular.*

The only regular bi-ideal which is not proper is $\epsilon = (R, R_0)$. If we adopt the convention $\bar{\epsilon} = \epsilon$ (see the preceding section), then we may omit the word "proper" in the formulation of (6.9).

Now let K, K_0 be sets of elements of R, R_0 respectively, and let α be the regular bi-ideal generated by (K, K_0). Thus, α is the intersection of all regular bi-ideals $(E(J_{0\nu}), J_{0\nu})$ such that $K \subseteq E(J_{0\nu})$, $K_0 \subseteq J_{0\nu}$. Then it will be seen without difficulty that

$$\alpha = (E(J_0), J_0) \quad \text{where} \quad J_0 = (H(\{K\}_S) \cup K_0)^{1/2}.$$

But R is a Ritt algebra and so (compare (5.8) where this relation is formulated for a special case), $J_0=(H([K]_S)\cup K_0)^{1/2}$. Hence

$$(6.10)\qquad \alpha = (E((H([K]_S) \cup K_0)^{1/2}), \qquad (H([K]_S) \cup K_0)^{1/2}).$$

Thus, the regular bi-ideal generated by (K,K_0) is proper if and only if the bi-ideal generated by (K, K_0) has the same property, i.e., if and only if $(H([K]_S)\cup K_0)\neq R_0$.

Let J be an S-perfect ideal in R. It is not difficult to see that quite generally, $J\subseteq E((H(J))^{1/2})$. We say that J is *an expansion* if there exists a radical ideal $J_0\subseteq R_0$ such that J is the expansion of J_0, $J=E(J_0)$. If so then

$$J = E(J_0) \supseteq E((H(J))^{1/2}) \quad \text{since} \quad (H(J))^{1/2} \subseteq J_0.$$

Hence $J=E((H(J))^{1/2})$. But if this relation is satisfied then J is the expansion of the radical ideal $(H(J))^{1/2}$. Hence

6.11. THEOREM. *An S-perfect ideal J in R is an expansion if and only if*

$$J = E((H(J))^{1/2}).$$

The intersection of a set of expansions in R is an expansion by (6.8).

Let J be an expansion in R. We shall say that J is *irreducible by expansions* if $J=J_1\cap J_2$, where J_1 and J_2 are expansions, entails $J_1=J$ or $J_2=J$.

6.12. THEOREM. *An expansion J is irreducible by expansions if and only if it is prime.*

Proof. Let $J=E(J_0)$ and suppose that $J=J_1\cap J_2$ where $J_1\neq J$, $J_2\neq J$. Then, for $a\in J_1-J$, $b\in J_2-J$ we have $ab\in J_1\cap J_2$ and so J cannot be prime. Conversely, suppose that J is not prime $ab\in J$, $a\notin J$, $b\notin J$ for some $a, b\in R$. Then there exist smallest integers i, j, such that $H(a^{(i)})\neq J_0$, $H(b^{(j)})\neq J_0$. Consider the following two radical ideals in R_0,

$$J_{01} = (J_0 \cup \langle H(a), H(a'), H(a''), \cdots \rangle)^{1/2},$$
$$J_{02} = (J_0 \cup \langle H(b), H(b'), H(b''), \cdots \rangle)^{1/2}.$$

Then $J_{01}\cap J_{02}$ is a radical ideal in R_0 and $J_{01}\cap J_{02}\supseteq J_0$. On the other hand, if $c\in J_{01}$, $c\in J_{02}$ then for certain positive integers ρ, σ,

$$c^\rho \in (J_0 \cup \langle H(a), H(a'), H(a''), \cdots \rangle) = J'_{01},$$
$$c^\sigma \in (J_0 \cup \langle H(b), H(b'), H(b''), \cdots \rangle) = J'_{02},$$

and so

$$c^{\rho+\sigma} \in J'_{01}\cdot J'_{02} \subseteq (J_0 \cup \langle H(a)H(b), \cdots, H(a^{(i)})H(b^{(j)}), \cdots \rangle)$$
$$= (J_0 \cup \langle H(ab), \cdots, H(a^{(i)}b^{(j)}), \cdots \rangle).$$

But $ab\in E(J_0)$ and $E(J_0)$ is a radical differential ideal. Hence [1, p.11]

$a^{(i)}b^{(j)}\in E(J_0)$ for all i, j, $H(a^{(i)}b^{(j)})\in J_0$. Hence $c^{\rho+\sigma}\in J_0$, $c\in J_0$, $J_{01}\cap J_{02}=J_0$ and so

$$J = E(J_0) = E(J_{01}\cap J_{02}) = E(J_{01})\cap E(J_{02}).$$

Moreover, $E(J_{01})\neq J$ since $a\in E(J_{01})$ but $a\notin J$, and similarly $E(J_{02})\neq J$. Hence, J is reducible by expansions.

Combining (3.10) with (6.12) we obtain

6.13. THEOREM. *An expansion is reducible if and only if it is reducible by expansions.*

Now suppose that R satisfies the finite ascending chain condition for S-perfect ideals. This implies that R satisfies the finite ascending chain condition for expansions. It follows, in the usual way, that every expansion J in R is the irredundant intersection of a set of irreducible expansions (irreducible by expansions and hence, irreducible). These are prime, by (6.12), and therefore coincide with the prime components of J. Hence

6.14. THEOREM. *Suppose that R satisfies the finite ascending chain condition for S-perfect ideals. Then the prime components of an expansion J in R are expansions.*

6.15. THEOREM. *Let $\alpha=(E(J_0), J_0)$ be a proper and regular bi-ideal in $\Omega=(R, R_0, H)$, where R satisfies the finite ascending chain condition for S-perfect ideals. Then the prime components of $E(J_0)$ are admissible.*

This follows directly from (6.9).

Now let $\Omega=(R, R_0, H)$ be a regular l.d.r. Construct the l.d.r. $\Omega\{y, z]$ $=(R\{y\}, R_0[z], H^*)$ as in §4.

6.16. THEOREM. *$\Omega\{y, z]$ is regular.*

Proof. It is sufficient to prove (6.16) for $n=1$, $y_1=y$, $z_{1i}=z_i$. Let $p\{y\}\in R\{y\}$, $p\{y\}\neq 0$ and let $q_k(z)=H^*((p\{y\})^{(k)})$, $k=0, 1, 2, \cdots$. Then we have to show that for some integer k, $q_k(z)\neq 0$. Let $\langle a_\nu\rangle$ be the set of coefficients of $p\{y\}$, $a_\nu\in R$, and let k be the smallest integer such that for some ν, $H(a_\nu^{(k)})\neq 0$. If we write Y_ν for the products of powers of $y, y', y'', \cdots$ in $p\{y\}$, such that $H(a_\nu^{(k)})\neq 0$, and Z_ν for the corresponding products of powers of z_i, $Z_\nu=H^*(Y)$, we then have

$$H^*((p\{y\})^{(k)}) = H^*\left(\left(\sum_\nu a_\nu Y_\nu\right)^{(k)}\right) = \sum_\nu H(a_\nu^{(k)})Z_\nu$$

since the remaining terms involve factors $H(a_\mu^{(l)})$, $l<k$, which vanish by assumption. Now the Z_ν are different for different ν and since at least one of these monomials appears with a coefficient different from zero, we have $H^*((p\{y\})^{(k)})\neq 0$, as required.

Now suppose that Ω is a regular l.d.f. and R is a Ritt algebra. Let S be the set $R-\langle 0\rangle$, then S satisfies (6.2) since R is a differential ring and R_0 is a field. It follows that the theory of expansions and regular bi-ideals developed above is applicable. R satisfies the finite ascending chain condition by (5.18).

As mentioned at the beginning of this section, we may, or may not wish to restrict the zeros of a given set of differential equations with initial conditions to extensions $\Omega'=(R', R_0', H)$ which, like Ω, are regular. In the latter case, the theory of varieties given in §5 is applicable. We shall now consider the corresponding theory for regular extensions Ω'. Again, we introduce the inessential restriction that R_0' is a field.

Given a regular l.d.f. Ω, we shall say that the point $\eta=(\Omega'; \eta_1, \cdots, \eta_n)$ is regular if Ω' is an extension of Ω and a regular l.d.f. For given $n\geq 1$, the space of all regular points η will be denoted by S^{*n}, so that $S^{*n}\subseteq S^n$. (Compare the beginning of §5 for a discussion of the legitimacy of these sets.)

Restricting the variety of a bi-ideal α in $\Omega\{y, z]$ to S^{*n} we obtain what will be called a *regular* variety V^*. We say that V^* is the regular variety of α, $\alpha\rightarrow^* V^*$. Given a subset A of S^{*n} we have as before that the sets of polynomials in $R\{y\}$ and $R_0[z]$ which are satisfied by all points of A constitute a bi-ideal $\alpha=(J, J_0)$. Moreover, since A consists of regular points, we now have that if $H^*((p\{\eta\})^{(n)})=0$ for all $\eta\in A$, then $p\{\eta\}=0$ for all $\eta\in A$. Thus, α is regular, $J=E(J_0)$.

If $\alpha=(J, J_0)$ is a proper bi-ideal which is prime and regular, then the construction of (5.2) yields a regular point. Hence

6.17. THEOREM. *A regular proper bi-ideal is prime if and only if it possesses a generic point which is regular.*

6.18. THEOREM. *The intersection of a set of regular varieties is a regular variety.*

Proof. Let $\langle V_\mu^*\rangle$ be a set of regular varieties. There exist bi-ideals α_μ and varieties V_μ such that $\alpha_\mu\rightarrow V_\mu$ and $V_\mu^*=V_\mu\cap S^{*n}$. Then $V=\bigcap_\mu V_\mu$ is a variety, by (5.3), and

$$V^* = V\cap S^{*n} = \bigcap_\mu V_\mu \cap S^{*n} = \bigcap_\mu (V_\mu\cap S^{*n}) = \bigcap_\mu V_\mu^*$$

is a regular variety.

6.19. THEOREM. *Every regular variety V^* is the regular variety of a regular bi-ideal α, $\alpha\rightarrow^* V^*$.*

Proof. There exists a bi-ideal $\alpha=(J, J_0)$ and a variety V and that $\alpha\rightarrow V$, $V^*=V\cap S^{*n}$. Let $\eta\in V^*$, then η satisfies all polynomials of $E(J_0)$. For if $H^*(p^{(n)}(y))=q_n(z)$, $n=0, 1, 2, \cdots$, and $q_n(H(\eta))=0$, then $p(\eta)=0$ (since $\eta_1, \cdots, \eta_n$ belong to a regular l.d.f.). It follows that V^* is the regular variety of the regular bi-ideal $\alpha^*=(E(J_0), J_0)$, which includes α.

6.20. THEOREM. *The union of two regular varieties is a regular variety.*

Proof. Let V_1^* and V_2^* be two regular varieties. By (6.19), there exist regular bi-ideals $\alpha_1=(E(J_{01}),\ J_{01})$ and $\alpha_2=(E(J_{02}),\ J_{02})$ such that $\alpha_1\rightarrow^* V_1^*$, $\alpha_2\rightarrow^* V_2^*$. We propose to show that $V_1^*\cup V_2^*$ is the regular variety of $\alpha_1\cap\alpha_2 =(E(J_{01})\cap E(J_{02}),\ J_{01}\cap J_{02})=(E(J_{01}\cap J_{02}),\ J_{01}\cap J_{02})$.

Clearly $V_1^*\cup V_2^*$ consist of regular points only and all these points are satisfied by the polynomials of $R\{y\}$ which belong to both $E(J_{01})$ and $E(J_{02})$ and by the polynomials of $R_0[z]$ which belong to both J_{01} and J_{02}. Now let $\eta=(\Omega';\ \eta_1,\ \cdots,\ \eta_n)$ be a regular point which belongs neither to V_1^* nor to V_2^*. Since η does not belong to V_1^* there either exists a polynomial $p_1\{y\}\in E(J_{01})$ such that $p\{\eta\}\neq 0$ or there exists a polynomial $q_1(z)\in J_{01}$ such that $q_1(H(\eta))\neq 0$. But in the former case, taking into account that Ω' is regular, we have for some integer n that $q_{1n}(H(\eta))\neq 0$ where $q_{1n}(z)=H((p\{y\})^{(n)})$. Thus, in any case there exists a polynomial $q_1(z)\in J_{01}$ such that $q_1(H(\eta))\neq 0$ and, similarly, here exists a polynomial $q_2(z)\in J_{02}$ such that $q_2(H(\eta))\neq 0$. Now consider the polynomial $q(z)=q_1(z)q_2(z)$. Clearly $q(z)\in J_{01}\cap J_{02}$. But $q(H(\eta)) =q_1(H(\eta))q_2(H(\eta))\neq 0$ and so η does not belong to the regular variety of $\alpha_1\cap\alpha_2$. This proves that $V_1^*\cup V_2^*$ is indeed the regular variety of $\alpha_1\cap\alpha_2$, as asserted.

6.21. THEOREM. *Let $\alpha=(J,\ J_0)$ be a regular bi-ideal and let $p\{y\}\in R\{y\}$ be a differential polynomial which is satisfied by all points that belong to the regular variety of α. Then $p\{y\}\in J$.*

Proof. If α is not proper then $J_0=R_0[z]$ and so $J=E(R_0[z])=R\{y\}$. Hence $p\{y\}\in J$. If α is proper then by (6.9), $\alpha=\bigcap_\nu \alpha_\nu$ where the $\alpha_\nu =(E(J_{0\nu}),\ J_{0\nu})$ are regular bi-ideals which are proper and prime. It follows that if $p\{y\}$ does not belong to J, then $p\{y\}$ does not belong to $E(J_{0\nu})$, for some ν. Now α_ν possesses a generic point η which is regular. η satisfies all polynomials of $J_{0\nu}$, $E(J_{0\nu})$ and hence of J, J_0. Thus η belongs to the regular variety of α. But η does not satisfy $p\{y\}$, contrary to assumption. This proves (6.21).

We prove by the same method

6.22. THEOREM. *Let $\alpha=(J,\ J_0)$ be a regular bi-ideal, and let $q(z)\in R_0[z]$ be satisfied by all points that belong to the regular variety of α. Then $q(z)\in J_0$.*

Combining (6.21) and (6.22) we obtain

6.23. THEOREM. *Let $\alpha=(J,\ J_0)$ be a regular bi-ideal. Then*

$$\alpha\overset{*}{\rightarrow} V^*\rightarrow\alpha.$$

The following theorem corresponds to, and is more definite than (5.9).

6.24. THEOREM. *In order that a system of differential equations and initial* *conditions*, (K, K_0), *possess a regular zero it is necessary and sufficient that*

$$1 \notin (H^*([K]) \cup K_0).$$

Proof. $p \in [K]_S$ implies $ap \in [K]$ with $a \neq 0 \in S$. Then $H^*((ap)^{(k)}) = H(a^{(k)})H^*(p) \in H^*([K])$ where n is the smallest integer for which $H(a^{(k)}) \neq 0$. Hence $H^*(p) \in H^*([K])$, and so

$$H^*([K]_S) = H^*([K]). \tag{6.25}$$

Moreover, we are dealing with a Ritt algebra and so (6.24) is equivalent to $1 \in J_0$ where $J_0 = (H^*(\{K_S\}) \cup K_0)^{1/2}$.

Now the regular bi-ideal generated by (K, K_0) is $\alpha = (E(J_0), J_0)$ so that condition (6.24) states that J_0, and hence α, is proper. If so, there exists a maximal proper ideal J_0' in $R_0[z]$, such that $J_0 \subseteq J_0'$. Then J_0' is prime, and the same therefore applies to $\alpha' = \rho(J_0')$, by (6.6). By (6.17) there exists a regular point η which satisfies all polynomials of α, and hence of K and K_0. It follows that the condition of (6.24) is sufficient. Necessity follows from (5.9).

Again, if a polynomial $q(z) \in R_0[z]$ is satisfied by all regular zeros of sets K and K_0, and hence by all regular zeros of the bi-ideal $(E(J_0), J_0)$, where J_0 is defined as above, then $q(z) \in J_0$ by (6.22). Hence, taking into account (5.8), (5.21), and (6.25), we have

6.26. THEOREM. *In order that $q(z) \in R_0[z]$ be satisfied by all regular zeros of (K, K_0) it is necessary and sufficient that there exist a positive integer ρ such that*

$$q^\rho \in (H^*([K]) \cup K_0).$$

Similarly, from (6.21)

6.27. THEOREM. *In order that $p\{y\} \in R\{y\}$ be satisfied by all regular zeros of (K, K_0) it is necessary and sufficient that there exist positive integers $\sigma(n)$, $n = 0, 1, 2, \cdots$, such that*

$$(H^*((p\{y\})^{(n)}))^{\sigma(n)} \in (H^*([K]) \cup K_0).$$

Since in a regular l.d.r. an element a of R is determined completely by the values of the images of $a^{(n)}$ in R_0, $n = 0, 1, 2, \cdots$ it is in fact not surprising (and in keeping with the determination of a solution of a differential equation by means of the values of the derivatives at a given point) that the above condition is formulated entirely in terms of elements of $R_0[z]$, including the H^*-images of the given elements of $R\{y\}$ and of their derivatives.

(6.24) in conjunction with (5.9) shows that if the polynomials of sets K, K_0 possess a joint zero at all, then they possess a joint zero which is regular. The same conclusion can be obtained more directly by means of the construction by which we derived a regular Ω^* from a given Ω in §2. Similarly, from (5.21) and (6.26), if $q(z) \in R_0[z]$ vanishes for all joint *regular* zeros of K, K_0, then $q(z)$ vanishes for *all* joint zeros of K, K_0.

We shall say that a system (K, K_0) possesses a *unique* (regular) solution, if it possesses a joint (regular) zero, and if any (regular) l.d.f. which is an extension of Ω does not contain more than one zero of (K, K_0).

6.28. THEOREM. *Given a regular l.d.f. Ω, let*

$$(6.29) \qquad y_i' = P_i(y_1, \cdots, y_n), \qquad z_{i0} = a_i, \qquad i = 1, \cdots, n,$$

be a set of differential equations with initial conditions, as in (5.11). *Then* (6.29) *possesses a unique regular solution.*

Proof. The existence of a solution follows from (6.24) in conjunction with (5.11) (or the arguments used in the proof of (5.11)). To see that the solution is unique, let Ω' be a regular l.d.f. which is an extension of Ω, $\Omega' = (R', R_0', H')$. Let $y_i = \eta_i \in R'$ and $y_i = \bar{\eta}_i \in R'$, $i = 1, \cdots, n$, be two solutions of (6.29). Then we have to show that $\eta_i = \bar{\eta}_i$. Since Ω' is regular, this will be proved if we establish that

$$(6.30) \qquad H'(\eta_i^{(k)}) = H'(\bar{\eta}_i^{(k)}) \qquad \text{for } k = 0, 1, 2, \cdots.$$

Now (6.30) holds for $k = 0$ since $H'(\eta_i) = H'(\eta_i) = a_i$. Also, let $Q_i(z_{01}, \cdots, z_{0n}) = H'(P_i(y_1, \cdots, y_n))$, then, by (6.29),

$$H'(\eta_i') = H'(\bar{\eta}_i') = Q_i(a_1, \cdots, a_n).$$

Next, we differentiate $y_i' = P_i$ and obtain

$$y_i'' = \frac{\partial P_i}{\partial y_1} y_1' + \cdots + \frac{\partial P_i}{\partial y_n} y_n', \qquad i = 1, \cdots, n.$$

Passing to the images in R_0' we see that the right hand side is determined uniquely and conclude that $H'(\eta_i'') = H'(\bar{\eta}_i'')$. Continuing in this way, we prove (6.30) and hence (6.28).

Next, we consider the case of a simple equation (of higher order) in one variable. We write y for y_1, z_i for z_{1i}. Let then

$$(6.31) \qquad p\{y\} \equiv p(y, y', \cdots, y^{(n)}) = 0$$

be a differential polynomial equation with coefficients in R, and let

$$(6.32) \qquad z_i \equiv H^*(y^{(i)}) = a_i, \qquad a_i \in R_0, \qquad i = 0, \cdots, n,$$

be a set of initial conditions. Let $s\{y\} = \partial P/\partial y^{(n)}$ be the separant of $p\{y\}$ and let $q(z_0, \cdots, z_n) \equiv q(z) = H^*(p\{y\})$ and $t(z_0, \cdots, z_n) \equiv t(z) = H^*(s\{y\})$.

6.33. THEOREM. *The differential equation* (6.31) *with initial conditions* (6.32) *has a unique regular solution provided*

$$(6.34) \qquad q(a_0, \cdots a_n) = 0, \qquad t(a_0, \cdots, a_n) \neq 0.$$

Proof. To prove existence, we make use of 6.24. If no solution exists then for some non-negative integer k,

$$1 \in (H(J_S') \cup (z_0 - a_0, \cdots, z_n - a_n))$$

where J' is the ideal

$$(p(y, y', \cdots, y^{(n)})), (p(y, y', \cdots, y^{(n)}))', \cdots, (p(y, y', \cdots, y^{(n)}))^{(k)}).$$

Thus, there exist polynomials $q_1, \cdots, q_m, r_0, \cdots, r_n \in R_0[z]$, $T_i \in J$ such that

$$(6.35) \quad 1 = q_1 t_1 + \cdots + q_m t_m + r_0(z_0 - a_0) + \cdots + r_n(z_n - a_n)$$

where $t_i = H^*(T_i)$ and

$$(6.36) \quad T_i = \sum_{j=0}^{k} p_{ij}(p\{y\})^j, \qquad p_{ij} \in R\{y\}, \qquad i = 1, \cdots, n.$$

We set all $y^{(l)}$, $l > n+k$ which occur in the p_{ij} (if any) equal to zero, as well as the corresponding z_l in the q_i, r_i. Thus we may suppose that the highest derivative of y which appears in (6.36) is $y^{(n+k)}$, which occurs in

$$(6.37) \quad (p\{y\})^{(j)} = s\{y\}y^{(n+j)} + P_j\{y\}, \qquad j = 1, 2, \cdots$$

for $j = k$. $P_j\{y\}$ is a differential polynomial of order less then $n+j$.

We multiply (6.36) by a suitable power of $s\{y\}$ so that we may write the result as a polynomial of $y, y', \cdots, y^{(n+k-1)}$ and $s\{y\}y^{(n+k)} = Y$ (in place of $y^{(n+k)}$) and we multiply (6.35) similarly by a power of $t(z) = H^*(s\{y\})$ so that we can replace $t(z)z_{n+k}$ everywhere by a variable Z. We then substitute $-P_k\{y\}$ for Y everywhere in the modified (6.36), and we substitute $-H(p_k\{y\})$ for Z. This eliminates $y^{(n+k)}$ and z_{n+k}. Repeating this process we finally replace (6.35) by

$$(6.38) \quad (t(z))^\mu = q_1^* t_1^* + \cdots + q_m^* t_m^* + r_0^*(z_0 - a_0) + \cdots + r_n^*(z_n - a_n)$$

where q_i^*, $r_i^* \in R_0[z]$, $t_i = H^*(T_i)$ and

$$(6.39) \quad T_i^* = p_i\{y\}p\{y\}, \qquad i = 1, \cdots, m.$$

Hence

$$(6.40) \quad (t(z))^\mu = w(z)q(z) + r_0^*(z_0 - a_0) + \cdots + r_n^*(z_n - a_n).$$

Substituting $a_0, \cdots, a_n$ for $z_0, \cdots, z_n$ in (6.40), we obtain zero on the right hand side, and so $t(a_0, \cdots, a_n) = 0$, contrary to (6.34). This proves the existence part of the theorem. Uniqueness follows from the fact that the images of (6.37) determine z_{n+j} uniquely in terms of $a_0, \cdots, a_n, j = 1, 2, \cdots$.

Concluding this section we note a regular l.d.r. satisfies the condition:

$$(6.42) \quad \text{If } a' = 0,\ H(a) = 0 \quad \text{for} \quad a \in R \quad \text{then} \quad a = 0.$$

Indeed, since $a' = 0$, $H(a^{(n)}) = 0$ for $n \geqq 1$, and this together with $H(a) = 0$ implies $a = 0$. On the other hand (compare the second example at the beginning of this section) (6.42) may well be satisfied in an l.d.f. which is not regular. At the same time (6.42) does not necessarily hold in an arbitrary l.d.f., as is shown by the following example.

Let R_0 be the field of rational functions of an indeterminate x with complex coefficients and let R be the ring of polynomials of two indeterminates x, y with complex coefficients. For any $f(x, y) \in R$, we define differentiation by partial differentiation with respect to x, $f' = \partial f/\partial x$. On the other hand, we define the homomorphism H from R to R_0 by $H(f(x, y)) = f(x, 0)$. Then the polynomial $f(x, y) = y$ satisfies the conditions $y' = 0$, $H(y) = 0$, yet $y \neq 0$ in R. Thus, (6.42) is an independent condition which is weaker than regularity. It may well deserve separate attention.

7. **Reflexive localized differential rings.** So far we have assumed that the homomorphism takes values in a ring, or field, which is not otherwise related to the given differential ring. We shall say that $\Omega = (R, R_0, H)$ is *reflexive* if R_0 coincides with the ring of constants of R and such that $H(a) = a$ for all $a \in R_0$, an assumption which is implicit in the classical theory of differential equations.

Given a reflexive Ω, we might now wish to develop a theory of reflexive extensions, and a corresponding theory of ideals and varieties. However, for regular Ω at least, it seems to be more convenient to reduce the consideration of reflexive extensions to the theory developed previously by means of the following theorem.

7.1. THEOREM. *Let $\Omega = (R, R_0, H)$ be a regular and reflexive l.d.f. and let $\Omega' = (R', R_0', H')$ be a regular l.d.f. which is an extension of Ω. Then there exists a regular and reflexive l.d.f., $\Omega'' = (R'', R_0'', H'')$, which is an extension of Ω such that $R_0' = R_0''$ and such that R' is (differentially) isomorphic to a subring R_τ of R'' under an isomorphism τ satisfying the condition:*

$$(7.2) \qquad \text{If } \tau(a) = b \quad \text{and} \quad H'(a) = a_0, \quad \text{then} \quad H''(b) = a_0.$$

Proof. Let R^* be the differential ring of integral power series $\sum_{n=0}^{\infty} c_n t^n$ with coefficients in R_0' (compare §2) and define $H^*(\sum_{n=0}^{\infty} c_n t^n) = c_0$. Then $\Omega^* = (R^*, R_0', H^*)$ is an l.d.f. Moreover R_0' coincides (by an obvious identification) with the ring of constants of R^* and every constant corresponds to itself under the homomorphism H^*. Thus, Ω^* is regular and reflexive. Also R^* includes a subring R_σ which is isomorphic to R'. R_σ is given by the set of elements

$$(7.3) \qquad \sigma(a) = \sum_{n=0} \frac{1}{n!} H'(a^{(n)}) t^n \qquad \text{for } a \in R'.$$

Under σ, there correspond to the elements $b \in R \leqq R'$ certain elements $b \in R_\sigma$, $\sigma(a) = b$, which constitute a differential ring R_1. We replace the elements of R_1 in R^* by the corresponding elements of R to obtain a ring R'', and we modify H^* accordingly, yielding a homomorphism H''. Then $\Omega'' = (R'', R_0'', H'')$, with $R_0'' = R_0'$ is an extension of Ω'. Also, the constants of R correspond to themselves, by (7.3), ($\sigma(a) = H(a) = a$) and so Ω'' is reflexive. Finally, (7.3) shows that if we define $\tau(a) = a$ for all $a \in R$, and $\tau(a) = \sigma(a)$ for all $a \in R' - R$ then the images of $\tau(a)$ in R'' constitute a subring R_τ of R'' such that (7.2) is satisfied. This proves (7.1).

It follows that every system of polynomials and initial conditions with coefficients in Ω which has a zero in a regular extension Ω' also has a zero in a regular and reflexive extension Ω''. Similarly, the generic point η constructed in connection with (5.2) and (6.17) may be supposed to belong to a regular and reflexive extension of Ω. Thus, the entire theory of regular varieties developed above still holds if we consider only points which belong to regular and reflexive extensions of a given regular and reflexive Ω.

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