CURRENTS AND AREA(1)

BY

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1. Introduction. When a continuous map f of a compact k dimensional manifold X into the Euclidean space R^n is uniformly approximated by smooth maps f_i, the areas of f_i need of course not converge. This is the simple reason for the complexity of the theory of area. Many geometric properties of f have been studied intensively in a search for useful and intuitively appealing concepts which suffice to determine the area of f and which behave properly under uniform approximation. The author believes that this paper makes a decisive contribution toward the natural solution of this problem.

To each of the smooth maps f_i corresponds a measure over X whose values are k dimensional currents in R^n. This measure associates with any continuous real-valued function ψ on X the current $f_{i\#}(X \wedge \psi)$ given by the formula(2)

$$f_{i\#}(X \wedge \psi)(\phi) = \int_X \psi \wedge f_i^{\#}(\phi)$$

whenever ϕ is a differential k form of class ∞ on R^n. The values of this measure are currents of finite mass; its total variation, using mass as norm, equals the area of f_i.

Applying to the limit map f the monotone-light factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "m_f"'] & & R^n \\ & M_f \arrow[ur, "t_f"'] & \end{tikzcd}$$

let μ_i be the m_f image of the measure corresponding to f_i; thus

$$\mu_i(\chi)(\phi) = \int_X (\chi \circ m_f) \wedge f_i^{\#}(\phi)$$

Presented to the Society, June 20, 1959 and October 31, 1959 under the title *On certain measures associated with continuous maps of finite area.* I, II, III; received by the editors August 1, 1960.

(1) This work was supported in part by a Sloan Fellowship.

(2) Of course this formula remains meaningful in case ψ is a $k-j$ form on X and ϕ is a j form on R^n, with $j \leq k$. The supremum of the integral, taking ψ and ϕ with masses not exceeding 1, may be called the j dimensional area of f_i. If $k > n$, when the k dimensional area of f_i equals 0, one can let $j = n$ to obtain the coarea of f_i studied in [F9, §3].

whenever χ is a continuous real-valued function on M_f and ϕ is a differential k form of class ∞ on R^n.

The principal results of this paper may now be summarized as follows:

If f has finite Lebesgue area and either $k=2$ or the range of f has $k+1$ dimensional Hausdorff measure 0, *then*:

(1) *There exists a unique current-valued measure μ over M_f such that for every sequence of smooth maps f_i, which converge uniformly to f and whose areas are bounded, the measures μ_i converge weakly to μ.*

(2) *The total variation of μ is equal to the Lebesgue area of f, and also equal to the integralgeometric M area of f introduced in* [F5].

(3) *μ is the indefinite integral, with respect to k dimensional Hausdorff measure over M_f, of a density function whose values are simple k vectors in R^n with integer norms; these k vectors describe the tangential properties of f, and the multiplicities with which f assumes its values in R^n.*

Much of the present work with currents depends on the recent joint paper [FF] by W. H. Fleming and the author. Hence the terminology of [FF] is readopted here without change. For those facts from the previous theory of Lebesgue area which are used here the reader may consult [F4; F5; F6; F8] and [DF]. It should be noted that this paper eliminates from geometric area theory the need for Morrey's representation theorem, cyclic element theory and the Moore-Roberts-Steenrod characterization of monotone images of 2 dimensional manifolds.

2. **A representation theorem.** The purpose of this section is to establish density properties of certain current-valued measures. Where classical differentiation theory is not applicable, arguments using the relative isoperimetric inequality fill the gap.

2.1. THEOREM. *Suppose*:

(1) *Z is a locally connected compact metric space, $g: Z \to R^n$ is a continuous light mapping, and $\Delta(z, r)$ is the component of z in $g^{-1}\{w: |w-g(z)|<r\}$ whenever $z \in Z$, $r>0$.*

(2) *μ is a countably additive function whose domain is the class of all Borel subsets of Z, and whose range is a class of k dimensional rectifiable currents in R^n; thus*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right)(\phi) = \sum_{i=1}^{\infty} \mu(A_i)(\phi)$$

whenever $\phi \in \boldsymbol{E}^k(R^n)$ and $A_1, A_2, A_3, \cdots$ are disjoint Borel subsets of Z.

(3) *The total $\boldsymbol{M}$ variation of μ is finite; hence a finite Borel measure $\|\mu\|$ over Z is defined by the formula*

$$\|\mu\|(A) = \sup\left\{\sum_{i=1}^{\infty} \boldsymbol{M}[\mu(B_i)]: B_1, B_2, B_3, \cdots \text{ are disjoint Borel subsets of } A\right\}.$$

(4) *For each Borel subset A of Z,*

$$\operatorname{spt} \mu(A) \subset g(\operatorname{Clos} A), \qquad \operatorname{spt} \partial\mu(A) \subset g(\operatorname{Bdry} A).$$

Then there exists a Baire function $v\colon Z \to \mathbf{\Lambda}_k(R^n)$ with the following properties:

(5) *For $\|\mu\|$ almost all z in Z, $v(z)$ is a simple k-vector, $|v(z)|$ is an integer, and*

$$\phi[g(z)][v(z)] = \lim_{r\to 0+} \alpha(k)^{-1} r^{-k} \mu[\Delta(z, r)](\phi)$$

whenever $\phi \in \boldsymbol{E}^k(R^n)$.

(6) *If A is a Borel subset of Z and $\phi \in \boldsymbol{E}^k(R^n)$, then*

$$\mu(A)(\phi) = \int_{R^n} \phi(y) \left[\sum_{z \in A \cap g^{-1}\{y\}} v(z) \right] d\boldsymbol{H}^k y.$$

Proof. Where possible define $v(z) \in \mathbf{\Lambda}_k(R^n)$ so that

$$f[v(z)] = \lim_{r\to 0+} \alpha(k)^{-1} r^{-k} \mu[\Delta(z, r)](\phi)$$

whenever $f \in \mathbf{\Lambda}^k(R^n)$, $\phi \in \boldsymbol{E}^k(R^n)$, $\phi(x) = f$ for $x \in R^n$. In case the limit fails to exist for some f, let $v(z) = 0$. Applying classical arguments to the real valued measures $\mu(\cdot)(\phi)$ one sees that the components of v are Baire functions.

We let $\gamma = g_\#(\|\mu\|)$, so that $\gamma(Y) = \|\mu\| [g^{-1}(Y)]$ for every Borel subset Y of R^n, and divide the remainder of the argument into eleven parts.

PART 1. *If A and Y are Borel subsets of Z and R^n, then*

$$\mu(A) \cap Y = \mu[A \cap g^{-1}(Y)].$$

Proof. For a fixed set A, both members of the above equation are countably additive with respect to Y. Hence it suffices to verify the equation in the special case when Y is closed.

Let $K_1, K_2, K_3 \cdots$ be closed sets whose union is $R^n - Y$ and observe that

$$\operatorname{spt} \mu[A \cap g^{-1}(Y)] \subset Y, \qquad \operatorname{spt} \mu[A \cap g^{-1}(K_i)] \subset K_i$$

$$\mu[A \cap g^{-1}(Y)] = (\mu[A \cap g^{-1}(Y)] + \mu[A \cap g^{-1}(K_i)]) \cap Y,$$

$$\mu(A) \cap Y - \mu[A \cap g^{-1}(Y)] = (\mu[A - g^{-1}(Y \cup K_i)]) \cap Y,$$

$$\boldsymbol{M}(\mu(A) \cap Y - \mu[A \cap g^{-1}(Y)]) \leqq \|\mu\| [A - g^{-1}(Y \cup K_i)] \to 0 \text{ as } i \to \infty.$$

PART 2. *If $A \subset B$ are Borel subsets of Z, then*

$$\|\mu\|(A) - \boldsymbol{M}[\mu(A)] \leqq \|\mu\|(B) - \boldsymbol{M}[\mu(B)].$$

Proof. $\mu(B) = \mu(A) + \mu(B - A)$, hence

$$\boldsymbol{M}[\mu(B)] - \boldsymbol{M}[\mu(A)] \leqq \boldsymbol{M}[\mu(B - A)] \leqq \|\mu\|(B - A) = \|\mu\|(B) - \|\mu\|(A).$$

PART 3. *There exists a countable family F of k dimensional proper regular submanifolds of class* 1 *of R^n such that*

$$\gamma(R^n - \bigcup F) = 0.$$

Proof. Suppose $\epsilon > 0$.

Choose disjoint Borel sets $B_1, B_2, B_3, \cdots$ for which

$$\bigcup_{i=1}^{\infty} B_i = Z, \qquad \sum_{i=1}^{\infty} \boldsymbol{M}[\mu(B_i)] > \|\mu\|(Z) - \epsilon,$$

then apply [FF, 8.16] to secure countable families $G_1, G_2, G_3, \cdots$ of k dimensional proper regular submanifolds of class 1 of R^n such that

$$\|\mu(B_i)\|(R^n - \bigcup G_i) = 0 \quad \text{for } i = 1, 2, 3, \cdots,$$

and consider the family

$$G = \bigcup_{i=1}^{\infty} G_i.$$

Letting $A_i = B_i \cap g^{-1}(R^n - \bigcup G_i)$ one sees from Part 1 that

$$\mu(A_i) = \mu(B_i) \cap (R^n - \bigcup G_i) = 0 \quad \text{for } i = 1, 2, 3, \cdots,$$

and uses Part 2 to obtain

$$\epsilon > \sum_{i=1}^{\infty} (\|\mu\|(B_i) - \boldsymbol{M}[\mu(B_i)]) \geqq \sum_{i=1}^{\infty} \|\mu\|(A_i)$$

$$\geqq \sum_{i=1}^{\infty} \|\mu\|[B_i \cap g^{-1}(R^n - \bigcup G)] = \gamma(R^n - \bigcup G).$$

PART 4. *If Y is a Borel subset of R^n for which $\boldsymbol{H}^k(Y) = 0$, then $\gamma(Y) = 0$.*

Proof. For each Borel set $B \subset g^{-1}(Y)$ one sees from Part 1 and [FF, 8.16] that

$$\mu(B) = \mu[B \cap g^{-1}(Y)] = \mu(B) \cap Y = 0.$$

PART 5. *For γ almost every y in R^n there exists an $M \in F$ such that $y \in M$ and $\Theta^k(\gamma, R^n - M, y) = 0$.*

Proof. For each $M \in F$ it follows from [F3, 3.2] and Part 4 that $\Theta^k(\gamma, R^n - M, y) = 0$ for $\boldsymbol{H}^k$ and γ almost all y in M.

PART 6. $\Theta^k(\gamma, y) < \infty$ *for $\boldsymbol{H}^k$ and γ almost all y in R^n.*

Proof. If $M \in F$, then

$$\Theta^k(\boldsymbol{H}^k, M, y) = 1, \qquad \Theta^k(\gamma, y) = \frac{d\gamma}{d(\boldsymbol{H}^k \cap M)}(y) < \infty$$

for $\boldsymbol{H}^k$ and γ almost all y in M. Moreover [F3, 3.2] implies that

$$\Theta^k(\gamma, y) = \Theta^k(\gamma, \bigcup F, y) = 0$$

for H^k and γ almost all y in $R^n - \bigcup F$.

PART 7. *Suppose* $y \in M \in F$,

$$\Theta^k(\gamma, y) < \infty, \qquad \Theta^k(\gamma, R^n - M, y) = 0,$$

P is an oriented k dimensional plane through y tangent to M, and

$$W_r = R^n \cap \{w: |w - y| < r\} \quad \textit{for } r > 0.$$

Then to each open subset V of Z, such that $y \notin g(\text{Bdry } V)$, corresponds a unique integer $m(V)$ for which

$$\lim_{r \to 0+} r^{-k} F[\mu(V) \cap W_r - m(V) \cdot (P \cap W_r)] = 0.$$

In fact if $\zeta > 0$ and $\nu \geqq 1$ are as in [FF, 8.18], *and if*

$$0 < \epsilon < \inf\{\zeta, \alpha(k)/3\}, \quad 0 < t < 1,$$

$$\nu\alpha(k)\Theta^k(\gamma, y)(t^{-k} - 1) < \epsilon < \alpha(k)t^k/3,$$

then there exists a $\rho > 0$ such that

$$\begin{aligned} F[\mu(V) \cap W_r - m(V) \cdot (P \cap W_r)] &\leqq \nu t^{-k} \|\mu\| (V \cap g^{-1}[W_r - (M \cap W_{tr})]) \\ &\quad + 2rt^{-k} \|\mu\| [V \cap g^{-1}(W_r)] \leqq \epsilon r^k \end{aligned}$$

whenever $0 < r < \rho$, V is an open subset of Z and $g(\text{Bdry } V) \subset R^n - W_r$.

Proof. Observing that

$$\lim_{r \to 0+} r^{-k} \gamma(W_r) = \alpha(k) \Theta^k(\gamma, y),$$

$$\lim_{r \to 0+} r^{-k} \gamma(W_r - W_{tr}) = \alpha(k) \Theta^k(\gamma, y)(1 - t^k),$$

$$\lim_{r \to 0+} r^{-k} \gamma(W_r - M) = 0,$$

choose $\rho > 0$ so that

$$\nu t^{-k} r^{-k} \gamma[W_r - (M \cap W_{tr})] + 2rt^{-k} r^{-k} \gamma(W_r) < \epsilon$$

and the conclusion of [FF, 8.19] holds whenever $0 < r \leqq \rho$.

Now suppose V is an open subset of Z, $y \notin g(\text{Bdry } V)$, let

$$G = \{r: 0 < r \leqq \inf\{\rho, \text{distance } [y, g(\text{Bdry } V)]\}\},$$

and let H be the subset of G consisting of those points where $\gamma(W_r)$ is differentiable with respect to r; clearly $L_1(G - H) = 0$.

If $r \in H$, then spt $\partial\mu(V) \subset R^n - W_r$ and the proof of [FF, 3.9], with $T = \mu(V)$ and omitting all references to ∂T, shows (see also [FF, 8.14]) that

$$\mu(V) \cap W_r \in I_k(\text{Clos } W_r),$$

$$\text{spt } \partial[\mu(V) \cap W_r] \subset \text{Bdry } W_r.$$

Choosing f according to [FF, 8.19] one obtains

$$X = f_\#[\mu(V) \cap W_r] \in I_k(\text{Clos } W_r)$$

with $\partial X = \partial[\mu(V) \cap W_r]$ and

$$\begin{aligned} r^{-k}\|X\|(R^n - P) &\leqq r^{-k}t^{-k}\|\mu(V)\|[W_r - (M \cap W_{tr})] \\ &\leqq r^{-k}t^{-k}\gamma[W_r - (M \cap W_{tr})] < \epsilon < \zeta. \end{aligned}$$

Accordingly [FF, 8.18] yields an integer $m_r(V)$ for which

$$\begin{aligned} \boldsymbol{M}[X - m_r(V)\cdot(P \cap W_r)] &\leqq \nu\|X\|(R^n - P) \\ &\leqq \nu t^{-k}\|\mu\|(V \cap g^{-1}[W_r - (M \cap W_{tr})]). \end{aligned}$$

Letting h be the linear homotopy from f to the identity map of R^n, one also finds that

$$\begin{gathered} \mu(V) \cap W_r - X = \partial h_\#(I \times [\mu(V) \cap W_r]), \\ \boldsymbol{M}[h_\#(I \times [\mu(V) \cap W_r])] \leqq 2rt^{-k}\|\mu(V)\|(W_r), \end{gathered}$$

hence

$$\begin{aligned} \boldsymbol{F}[\mu(V) \cap W_r - m_r(V)\cdot(P \cap W_r)] &\leqq \nu t^{-k}\|\mu\|(V \cap g^{-1}[W_r - (M \cap W_{tr})]) \\ &\quad + 2rt^{-k}\|\mu\|[V \cap g^{-1}(W_r)] < \epsilon r^k. \end{aligned}$$

Moreover, since $\epsilon r^k < \alpha(k)r^k/3 = \boldsymbol{M}(P \cap W_r)/3$, the integer $m_r(V)$ is uniquely characterized by the preceding inequality.

Next it will be shown that

$$m_r(V) = m_s(V) \quad \text{whenever} \quad r \in H, s \in H, tr < s < r.$$

In fact

$$\begin{gathered} \boldsymbol{M}[\mu(V) \cap W_r - \mu(V) \cap W_s] \leqq \gamma(W_r - W_{tr}) < \epsilon r^k, \\ \boldsymbol{F}[m_r(V)\cdot(P \cap W_r) - m_s(V)\cdot(P \cap W_s)] < 3\epsilon r^k, \end{gathered}$$

and the assumption $m_r(V) \neq m_s(V)$ would imply that

$$\alpha(k)s^k = \boldsymbol{M}(P \cap W_s) < 3\epsilon r^k, \qquad \alpha(k)t^k/3 < \epsilon.$$

It is now obvious that $m_r(V)$ has the same value, say $m(V)$, for all $r \in H$. Hence the desired inequality has been proved in case $r \in H$. By left continuity, it remains valid for all $r \in G$.

PART 8. *If the conditions of Part 7 hold, V is an open subset of Z and $C(V, r)$ is the family of components of $V \cap g^{-1}(W_r)$, then*

$$m(V) = \sum_{U\in C(V,r)} m(U)$$

whenever $0<r<\rho$ *and* $g(\text{Bdry } V)\subset R^n - W_r$. *Furthermore*

$$\limsup_{r\to 0+} \sum_{U\in C(Z,r)} |m(U)| \leqq \Theta^k(\gamma, y).$$

Proof. Since

$$\mu(V)\cap W_r = \mu[V\cap g^{-1}(W_r)] = \sum_{U\in C(V,r)} \mu(U),$$

one finds that

$$\begin{aligned} &F\left[\mu(V)\cap W_r - \sum_{U\in C(V,r)} m(U)\cdot(P\cap W_r)\right] \\ &\quad\leqq \sum_{U\in C(V,r)} (\nu t^{-k}\|\mu\|[U - g^{-1}(M\cap W_{tr})] + 2rt^{-k}\|\mu\|(U)) \\ &\quad\leqq \nu t^{-k}\|\mu\|(V\cap g^{-1}[W_r - (M\cap W_{tr})]) + 2rt^{-k}\|\mu\|[V\cap g^{-1}(W_r)] \leqq \epsilon r^k. \end{aligned}$$

Similarly one obtains

$$\begin{aligned} \alpha(k)r^k \sum_{U\in C(Z,r)} |m(U)| &\leqq \sum_{U\in C(Z,r)} (F[\mu(U)] + F[\mu(U) - m(U)\cdot(P\cap W_r)]) \\ &\leqq \gamma(W_r) + \epsilon r^k. \end{aligned}$$

PART 9. *If the conditions of Part 7 hold and P is oriented by the simple k-vector ξ with $|\xi| = 1$, then for each $z\in g^{-1}\{y\}$ the conclusion of* (5) *holds with*

$$v(z) = \lim_{r\to 0+} m[\Delta(z, r)]\cdot\xi.$$

Furthermore

$$\Theta^k[\mu(V)\wedge\phi, y] = \phi(y)\left[\sum_{z\in V\cap g^{-1}\{y\}} v(z)\right]$$

whenever V is an open subset of Z, $y\notin \text{Bdry}(V)$, $\phi\in E^k(R^n)$.

Proof. One sees from Part 8 that for $0<r<\rho$ the number of elements of the set

$$D(r) = C(Z, r)\cap \{U: m(U)\neq 0\}$$

does not decrease as r decreases, and is bounded, hence constant for small r, say for $0<r<\delta$. Moreover for $0<s<r<\delta$ the relation

$$\{(U, V): U\in D(r),\ V\in D(s),\ U\supset V\}$$

is an m preserving univalent map of $D(r)$ onto $D(s)$.

Since Z is compact and g is light, the points of $g^{-1}\{y\}$ constitute the components of

$$\bigcap_{r>0} \bigcup C(Z, r).$$

If $0<r<\delta$, each $U\in D(r)$ contains a unique $z\in g^{-1}\{y\}$ such that $m[\Delta(z, s)] = m(U)$ whenever $0<s\leqq r$. All other points $z\in g^{-1}\{y\}$ have the property that $m[\Delta(z, s)]=0$ for all sufficiently small $s>0$.

For $z\in g^{-1}\{y\}$ and $\phi\in E^k(R^n)$ one infers, with the help of Part 7, that

$$\begin{aligned}\phi(y)\left(\lim_{r\to 0+} m[\Delta(z, r)]\cdot\xi\right) &= \lim_{r\to 0+} \alpha(k)^{-1}r^{-k}m[\Delta(z, r)]\int_{P\cap W_r}\phi\\ &= \lim_{r\to 0+} \alpha(k)^{-1}r^{-k}\mu[\Delta(z, r)](\phi).\end{aligned}$$

Furthermore, if V is an open subset of Z and $y\notin g(\text{Bdry } V)$, then

$$\begin{aligned}\Theta^k[\mu(V)\wedge\phi, y] &= \lim_{r\to 0+} \alpha(k)^{-1}r^{-k}m(V)\int_{P\cap W_r}\phi\\ &= \lim_{r\to 0+} \sum_{U\in C(V,r)} m(U)\alpha(k)^{-1}r^{-k}\int_{P\cap W_r}\phi\\ &= \sum_{z\in V\cap g^{-1}\{y\}} \lim_{r\to 0+} m[\Delta(z, r)]\phi(y)(\xi).\end{aligned}$$

PART 10. *If Y is a Borel subset of R^n, then*

$$\int_Y \Theta^k(\gamma, y)dH^k y \leqq \gamma(Y).$$

Proof. Apply [F3, 3.1, 3.2] and Parts 4, 6.

PART 11. *Proof of* (6).

Fix $\phi\in E^k(R^n)$, let C be the class of those Borel subsets A of Z for which the conclusion of (6) holds, and let D be the class of those open subsets V of Z for which $\gamma[g(\text{Bdry } V)]=0$. Since

$$\sum_{z\in g^{-1}\{y\}} |v(z)| \leqq \Theta_*^k(\gamma, y)$$

whenever $y\in R^n$, one readily verifies with the help of Part 10 (with $Y=R^n$) that C is closed to monotone convergence.

For every k dimensional rectifiable current T in R^n one knows from [FF, 8.16 (2)] that

$$\begin{aligned}T(\phi) &= \int_{R^n} \frac{d(T\wedge\phi)}{d\|T\|}(y)d\|T\|y\\ &= \int_{R^n} \frac{d(T\wedge\phi)}{d\|T\|}(y)\Theta^k(\|T\|, y)\, dH^k y = \int_{R^n}\Theta^k(T\wedge\phi, y)\, dH^k y.\end{aligned}$$

Hence Parts 5, 6, 9, 10 (with $\gamma(Y)=0$) imply that $D\subset C$.

If Y is an open subset of R^n for which $\gamma(\text{Bdry } Y)=0$, then every component of $g^{-1}(Y)$ belongs to D. Since g is a light mapping it follows that D contains a base for the topology of Z. Moreover D is closed to finite union and intersection.

Accordingly C is the class of all Borel subsets of Z.

2.2. COROLLARY. *In case g has the Lipschitz constant* 1 *and the diameter of* $\Delta(z, r)$ *never exceeds* $2r$, *then the following additional statements hold*:

(7) *For* H^k *almost all* z *in* Z,

$$|v(z)| = \lim_{r\to 0+} \alpha(k)^{-1}r^{-k}\|\mu\|[\Delta(z, r)].$$

(8) *If* A *is a Borel subset of* Z, *then*

$$\|\mu\|(A) = \int_A |v(z)|\, dH^k z.$$

(9) *If* ψ *is a Baire function on* Z *such that* $\psi(z)=0$ *whenever* $v(z)=0$, *then*

$$\int_Z \psi(z)\, dH^k z = \int_{R^n} \sum_{z\in g^{-1}\{y\}} \psi(z)\, dH^k y.$$

(10) *If* A *is a Borel subset of* Z *and* $\phi\in E^k(R^n)$, *then*

$$\mu(A)(\phi) = \int_A \phi[g(z)][v(z)]\, dH^k z.$$

Proof. For $A\subset Z$ and $y\in R^n$ let $N(g, A, y)$ be the number (possibly ∞) of elements in $A\cap g^{-1}\{y\}$. Since

$$H^k(B) \geqq H^k[g(B)] \qquad \text{for } B\subset Z,$$

it follows from [F1, 4.1] that

$$H^k(A) \geqq \int_{R^n} N(g, A, y)\, dH^k y$$

for every Borel subset A of Z, and consequently

$$\int_Z \psi(z)\, dH^k z \geqq \int_{R^n} \sum_{z\in g^{-1}\{y\}} \psi(z)\, dH^k y$$

for every nonnegative Baire function ψ on Z.

Noting that [F3, 3.1, 3.2] are easily adapted to Z, with the spherical balls of R^n replaced by the neighborhoods $\Delta(z, r)$, one sees with the help of (6) that, for every Borel subset A of Z,

$$\begin{aligned}\|\mu\|(A) &\geqq \int_A \limsup_{r\to 0+} \alpha(k)^{-1}r^{-k}\|\mu\|[\Delta(z, r)]dH^k z \\ &\geqq \int_A \liminf_{r\to 0+} \alpha(k)^{-1}r^{-k}\|\mu\|[\Delta(z, r)]\, dH^k z \\ &\geqq \int_A |v(z)|\, dH^k z \\ &\geqq \int_{R^n} \sum_{z\in A\cap g^{-1}\{y\}} |v(z)|\, dH^k y \\ &\geqq \|\mu\|(A).\end{aligned}$$

Therefore (7) and (8) are proved, and the equation

$$H^k(A) = \int_{R^n} N(g, A, y)\, dH^k y$$

holds in case $|v(z)|$ equals a fixed positive integer for all $z\in A$. Then (9) follows readily, and (10) is a consequence of (6) and (9).

3. **The convergence property.** *Suppose X is a compact oriented k dimensional manifold of class ∞.*

To each map $f: X\to R^n$ of class ∞ corresponds a countably additive function which associates with each Borel subset B of X the k dimensional rectifiable current $f_\#(B)$; here

$$f_\#(B)(\phi) = \int_B f^\#(\phi) \qquad \text{for } \phi \in E^k(R^n).$$

One readily verifies that the M variation of this countably additive function over B equals the classical area integral of $f|B$.

It will be shown how these concepts can be extended to continuous maps $f: X\to R^n$ of finite Lebesgue area, at least in case $H^{k+1}[f(X)]=0$ or $k=2$.

3.1. Definition. Every continuous map $f: X\to R^n$ has a monotone-light factorization

$$f = l_f \circ m_f, \qquad m_f: X \to M_f, \qquad l_f: M_f \to R^n$$

whose middle space M_f consists of the maximal continua of constancy of f; the distance between two points ξ and η of M_f is

$$d_f(\xi, \eta) = \inf\{\operatorname{diam} f(C) : C \text{ is a continuum containing } \xi \cup \eta\};$$

this metric d_f and the identification map m_f induce the same topology.

Assuming that f has finite Lebesgue area, consider any sequence of maps $f_i: X\to R^n$ of class ∞ which converge uniformly to f and whose areas are bounded; these areas need not converge to the Lebesgue area of f. Let μ_i be the countably additive function associating with each Borel subset A of M_f the rectifiable current

$$f_{i\#}[m_f^{-1}(A)] \in \boldsymbol{E}_k(R^n).$$

We say that f has the *convergence property* if and only if for every such choice of approximating maps f_i the corresponding sequence of measures μ_i over M_f is weakly convergent; this means that the sequence of numbers

$$\mu_i(\chi)(\phi) = \int_X (\chi \circ m_f) \wedge f_i^{\#}(\phi)$$

is convergent whenever $\phi \in \boldsymbol{E}^k(R^n)$ and χ is a continuous real-valued function on M_f; the limit then equals

$$\mu(\chi)(\phi)$$

where μ is a Borel measure over M_f with values in $\boldsymbol{E}_k(R^n)$.

In case f has the convergence property, the weak limit is clearly independent of the choice of approximating maps f_i, and we shall refer to μ as the *limit measure corresponding to* f.

3.2. LEMMA. *If $f: X \to R^n$ is of class ∞, $u: R^n \to R$ has Lipschitz constant λ, and $C(s)$ is the set of components of*

$$\{x: (u \circ f)(x) < s\}$$

whenever $s \in R$, then

$$\int_{-\infty}^{\infty} \sum_{V \in C(s)} \boldsymbol{M}[\partial f_\#(V)]\, ds \leqq \lambda \text{ area } (f).$$

Proof. Let

$$\gamma(s) = \text{area}(f \mid \{x: (u \circ f)(x) < s\}) \qquad \text{for } s \in R$$

note that

$$\int_{-\infty}^{\infty} \gamma'(s)\, ds \leqq \text{area}(f),$$

and consider a real number s for which $\gamma'(s) < \infty$. For each $V \in C(s)$ one may apply [FF, 3.9] with

$$T = f_\#(V) = f_\#(V) \cap \{y: u(y) < s\},$$
$$\text{spt } \partial T \subset f(\text{Bdry } V) \subset \{y: u(y) = s\}$$

to obtain

$$\boldsymbol{M}[\partial f_\#(V)] \leqq \lambda \liminf_{h \to 0+} h^{-1} \|f_\#(V)\| (\{y: s - h \leqq y < s\}).$$

It follows that

$$\sum_{V\in C(s)} \boldsymbol{M}[\partial f_\#(V)] \leqq \lambda \liminf_{h\to 0+} h^{-1} \sum_{V\in C(s)} \|f_\#(V)\|(\{y: s-h \leqq u(y) < s\})$$

$$\leqq \lambda \liminf_{h\to 0+} h^{-1} \operatorname{area}[f \mid \{x: s-h \leqq (u\circ f)(x) < s\}]$$

$$= \lambda\gamma'(s).$$

3.3. REMARK. If ψ is a continuous real-valued function on X, then all but countably many real numbers s have the following property:

No component of $\{x:\psi(x)\leqq s\}$ contains two distinct components of $\{x:\psi(x)<s\}$.

In fact, choosing b in a countable dense subset of X, let $B(s)$ and $D(s)$ be the components of b in $\{x:\psi(x)<s\}$ and $\{x:\psi(x)\leqq s\}$ whenever $\psi(b)<s$. Since the sets $D(s)-B(s)$ form a disjointed family, all but countably many have an empty interior, in which case $B(s)$ is the only component of $\{x:\psi(x)<s\}$ contained in $D(s)$.

3.4. THEOREM. *If $f: X\to R^n$ is a continuous map and $f_1, f_2, f_3, \cdots$ are maps of class ∞ which converge uniformly to f and whose areas are bounded, then the corresponding sequence $\mu_1, \mu_2, \mu_3, \cdots$ of measures over M_f has a weakly convergent subsequence, whose limit measure μ satisfies the conditions of* 2.1 *and* 2.2 *with $Z=M_f$ and $g=l_f$.*

Proof. By Cantor's diagonal process the given sequence may be replaced by a subsequence such that

$$\lim_{i\to\infty} \mu_i(\chi)(\phi)$$

exists for countable $\boldsymbol{M}$ dense sets of forms $\phi\in \boldsymbol{E}^k(R^n)$ and continuous real-valued functions χ on M_f. Then weak convergence follows because the total $\boldsymbol{M}$ variations of μ_i do not exceed the areas of f_i, which are bounded.

Consider also the measures γ_i over X defined by

$$\gamma_i(B) = \operatorname{area}(f_i \mid B)$$

for every Borel subset B of X. After passing once more to a subsequence, one may assume that the sequence $\gamma_1, \gamma_2, \gamma_3, \cdots$ is weakly convergent to a Borel measure γ over X.

Inasmuch as

$$\begin{aligned} \operatorname{spt} \mu_i(A) &\subset f_i[\operatorname{spt} m_f^{-1}(A)] \\ &\subset f_i[\operatorname{Clos} m_f^{-1}(A)] \subset f_i[m_f^{-1}(\operatorname{Clos} A)], \\ \operatorname{spt} \partial\mu_i(A) &\subset f_i[\operatorname{spt} \partial m_f^{-1}(A)] \\ &\subset f_i[\operatorname{Bdry} m_f^{-1}(A)] \subset f_i[m_f^{-1}(\operatorname{Bdry} A)] \end{aligned}$$

for all i and every Borel subset A of M_f, one readily verifies all but one of the conditions of 2.1 and 2.2; the only real problem is to show that $\mu(A)$ is a rectifiable current.

Let F be the family of those Borel subsets A of M_f for which the current $\mu(A)$ is rectifiable. Obviously F is closed to countable disjoint union, and to proper subtraction.

It will be shown that if $u: R^n \to R$ is Lipschitzian, then for L_1 almost all real numbers s each component of

$$Z(s) = M_f \cap \{z: (u \circ l_f)(z) < s\}$$

belongs to F. Letting $C_i(s)$ be the set of components of

$$X \cap \{x: (u \circ f_i)(x) < s\}$$

one sees from 3.2 that

$$\sup_i \int_{-\infty}^{\infty} \sum_{V \in C_i(s)} M[\partial f_{i\#}(V)]ds < \infty$$

and infers from Fatou's lemma that

$$\liminf_{i \to \infty} \sum_{V \in C_i(s)} M[\partial f_{i\#}(V)] < \infty$$

for L_i almost all s. For all but countably many real numbers s it is also true that

$$\gamma(\{x: (u \circ f)(x) = s\}) = 0$$

and that the property of 3.3 holds with $\psi = u \circ f$.

Now suppose s is a real number satisfying these three conditions, A is a component of $Z(s)$, $b \in B = m_f^{-1}(A)$, D is the component of b in $\{x: (u \circ f)(x) \leq s\}$, and $b \in V_i \in C_i(s)$ for $i = 1, 2, 3, \cdots$. Then

$$m_f^{-1}(\text{Bdry } A) \subset \{x: (u \circ f)(x) = s\},$$

$$\|\mu\|(\text{Bdry } A) \leq \gamma[m_f^{-1}(\text{Bdry } A)] = 0,$$

$$\mu(A) = \lim_{i \to \infty} f_{i\#}(B),$$

$$B \subset \bigcup_{j=1}^{\infty} \text{Int} \bigcap_{i=j}^{\infty} V_i,$$

$$\bigcap_{j=1}^{\infty} \text{Clos} \bigcup_{i=j}^{\infty} V_i \subset D \subset B \cup \{x: (u \circ f)(x) = s\},$$

hence $\epsilon > 0$ implies that

$$(B - V_i) \cup (V_i - B) \subset \{x: |(u \circ f)(x) - s| < \epsilon\}$$

for large i, and one obtains

$$\limsup_{i\to\infty} \mathbf{M}[f_{i\#}(V_i) - f_{i\#}(B)] \leqq \limsup_{i\to\infty} \gamma_i[(V_i - B) \cup (B - V_i)] = 0,$$

$$\liminf_{i\to\infty} \mathbf{N}[f_{i\#}(V_i)] < \infty,$$

$$\mu(A) = \lim_{i\to\infty} f_{i\#}(V_i) \in \mathbf{I}_k(R^n)$$

by [FF, 8.14, 8.13], hence $A \in F$.

For each $a \in R^n$ one may consider the function u defined by

$$u(y) = \sup\{\,|\,y_i - a_i\,| : i = 1, 2, \cdots, n\}$$

for $y \in R^n$. One finds that, for almost every n dimensional cube W in R^n, each component of $l_f^{-1}(W)$ belongs to F. Since l_f is light, approximation by finite sums of such components shows that every open subset of M_f belongs to F. One concludes that F is in the class of all Borel subsets of M_f.

3.5. Corollary. *If $f: X \to R^n$ is a continuous map of finite Lebesgue area with the convergence property, then the limit measure μ corresponding to f satisfies the conditions of* 2.1 *and* 2.2 *with $Z = M_f$ and $g = l_f$.*

3.6. Corollary. *If $f: X \to R^k$ is a continuous map of finite Lebesgue area, then f has the convergence property.*

Proof. It is sufficient to prove that if

$$f_1, f_2, f_3, \cdots \quad \text{and} \quad g_1, g_2, g_3, \cdots$$

are two sequences of maps of class ∞ which converge uniformly to f and whose areas are bounded, and if the corresponding sequences

$$\mu_1, \mu_2, \mu_3, \cdots, \nu_1, \nu_2, \nu_3, \cdots$$

of measures over M_f are weakly convergent to μ and ν respectively, then $\mu = \nu$.

Almost every k dimensional cube W in R^k has the property that

$$\|\mu\|[l_f^{-1}(\mathrm{Bdry}\, W)] = 0 = \|\nu\|[l_f^{-1}(\mathrm{Bdry}\, W)].$$

If A is a component of $l_f^{-1}(W)$ and $B = m_f^{-1}(A)$, then

$$\mu(A) = \lim_{i\to\infty} f_{i\#}(B), \qquad \nu(A) = \lim_{j\to\infty} g_{j\#}(B).$$

Letting $h_{i,j}$ be the linear homotopy from f_i to g_j one obtains

$$g_{j\#}(B) - f_{i\#}(B) = h_{i,j\#}(I \times \partial B)$$

because $\mathbf{E}_{k+1}(R^k) = \{0\}$. It follows that

$$\nu(A) - \mu(A) = \lim_{i,j\to\infty} h_{i,j\#}(I \times \partial B)$$

is a k dimensional rectifiable current with support in Bdry W, hence equals 0.

3.7. LEMMA. *Suppose* $f: X \to R^n$ *is a continuous map of finite Lebesgue area and either* $H^{k+1}[f(X)]=0$ *or* $k=2$. *If* $a \in R^n$, *then almost all orthogonal projections* p *of* R^n *onto* R^k *have the following property*:

X contains no continuum C such that $a \in f(C)$, *f is not constant on C,* $p \circ f$ *is constant on C.*

Proof. Assume $a=0$ and consider three cases:

CASE 1. $H^{k+1}[f(X)]=0$.

For $r>0$, let $A(r)=f(X) \cap \{y: |y|=r\}$. The Eilenberg inequality ([E] or [F7, 3.2]) allows one to choose a sequence of numbers $r_1 > r_2 > r_3 > \cdots$ with limit 0 such that $H^k[A(r_i)]=0$ for $i=1, 2, 3, \cdots$. Letting S_i be the set of those $n-k$ dimensional planes in R^n which meet $A(r_i)$, one sees from [F3, 7.5] that S_i has Haar measure 0. Moreover, if p is an orthogonal projection of R^n onto R^k such that X contains a continuum C with $0 \in f(C)$, $f(C) \neq \{0\}$, $(p \circ f)(C) = \{0\}$, then the kernel of p belongs to S_i for large i.

CASE 2. $k=2$ *and* $n=3$. Let

$$S = R^3 \cap \{w: |w| = 1\},$$

$$Q: R^3 \to R^2, \qquad Q(y) = (y_1, y_2) \qquad \text{for } y \in R^3,$$

$$g: R^3 - \{0\} \to R^3,$$

$$g(y) = (y_1/|y|, y_2/|y|, |y|) \qquad \text{for } y \in R^3,$$

choose finitely triangulable sets $X_1 \subset X_2 \subset X_3 \subset \cdots$ such that

$$\bigcup_{=1}^{\infty} X_j = X - f^{-1}\{0\},$$

and let U_j be the set of those $u \in R^2$ for which X_j contains a continuum D such that $(g \circ f)(D)$ is nondegenerate and $(Q \circ g \circ f)(D) = \{u\}$. Replacing [F6, 8.10] by [DF, 4.1] in the proof of [F6, 8.11] one sees that $L_2(U_j)=0$, hence

$$H^2[S \cap Q^{-1}(U_j)] = 0.$$

Now observe that if p is an orthogonal projection of R^3 onto R^2 for which the conclusion of the lemma fails, and if

$$w \in S \cap \text{kernel } p,$$

then X contains a continuum C such that

$$0 \in f(C), \qquad \{0\} \neq f(C) \subset \{tw: t \in R\}.$$

Therefore

$$m_f(C) = m_f(C \cap f^{-1}\{0\}) \cup \bigcup_{j=1}^{\infty} m_f(C \cap X_j)$$

is a nondegenerate continuum, while

$$m_f(C \cap f^{-1}\{0\}) \subset l_f^{-1}\{0\}$$

is totally disconnected, hence [HW, Theorem 2.2] yields a positive integer j for which

$$\dim[m_f(C \cap X_j)] > 0.$$

Choosing a continuum $D \subset C \cap X_j$ such that f is not constant on D, one finds that

$$(g \circ f)(D) \text{ is nondegenerate,}$$

$$(Q \circ g \circ f)(D) = \{\pm Q(w)\},$$

$$w \in S \cap Q^{-1}(U_j) \text{ or } -w \in S \cap Q^{-1}(U_j).$$

Accordingly the set of all such points w has H^2 measure 0.

CASE 3. *$k=2$ and $n>3$.*

Let G_n and G_3 be the orthogonal groups of R^n and R^3, consider the orthogonal projections

$$\begin{aligned} &P: R^n \to R^2, && P(y) = (y_1, y_2) && \text{for } y \in R^n, \\ &Q: R^3 \to R^2, && Q(w) = (w_1, w_2) && \text{for } w \in R^3, \\ &S_i: R^n \to R^3, && S_i(y) = (y_1, y_2, y_i) && \text{for } y \in R^n, \end{aligned}$$

corresponding to $i=3, \cdots, n$ and let K_i be the set of all those $g \in G_n$ for which there exists a continuum $C \subset X$ such that

$$0 \in f(C), \qquad (S_i \circ g \circ f)(C) \neq \{0\}, \qquad (P \circ g \circ f)(C) = \{0\}.$$

Inasmuch as

$$\bigcup_{i=3}^{n} \{P \circ g: g \in K_i\}$$

is the class of those orthogonal projections p of R^n onto R^2 for which the conclusion of the lemma fails, it is sufficient to prove that each K_i has Haar measure 0.

Fix i, let u be the characteristic function of K_i, and with each $h \in G_3$ associate $\rho(h) \in G_n$ so that

$$S_i \circ \rho(h) = h \circ S_i, \qquad \rho(h)(y) = y \qquad \text{for } y \in \text{kernel } S_i.$$

Integrating with respect to Haar measures over G_n and G_3 one obtains

$$\int_{G_n} u(g)\,dg = \int_{G_3}\int_{G_n} u[\rho(h)\circ g]\,dgdh$$
$$= \int_{G_n}\int_{G_3} u[\rho(h)\circ g]\,dhdg = 0$$

because for each $g \in G_n$ one may apply Case 2 to the map $S_i \circ g \circ f$, taking account of the fact that

$$(Q\circ h)\circ(S_i\circ g\circ f) = P\circ[\rho(h)\circ g]\circ f \qquad \text{for } h\in G_3.$$

3.8. COROLLARY. *If the conditions of* 3.7 *hold and* γ *is a Radon measure over* M_f, *then*

$$\gamma(M_f - M_{p\circ f}) = 0$$

for almost all orthogonal projections p *of* R^n *onto* R^k.

Proof. Note that

$$S = \{(z, p): z \in M_f - M_{p\circ f}\}$$

is a Borel set of type F_σ. Since, for each $z \in M_f$,

$$\{p: (z, p) \in S\} \text{ has Haar measure } 0,$$

by 3.7, the Fubini theorem implies that, for almost all p,

$$\gamma(\{z: (z, p) \in S\}) = 0.$$

3.9. THEOREM. *If* $f: X \to R^n$ *is a continuous map of finite Lebesgue area and either* $H^{k+1}[f(X)] = 0$ *or* $k=2$, *then* f *has the convergence property.*

Proof. In view of 3.4 it suffices to prove that if

$$f_1, f_2, f_3, \cdots \quad \text{and} \quad g_1, g_2, g_3, \cdots$$

are two sequences of maps of class ∞ which converge uniformly to f and whose areas are bounded, and if the corresponding sequences

$$\mu_1, \mu_2, \mu_3, \cdots \quad \text{and} \quad \nu_1, \nu_2, \nu_3, \cdots$$

of measures over M_f are weakly convergent to μ and ν respectively, then $\mu = \nu$.

According to 3.8 almost every orthogonal projection p of R^k onto R^n has the property that

$$\|\mu\|(M_f - M_{p\circ f}) = 0 = \|\nu\|(M_f - M_{p\circ f}).$$

Factoring $m_{p\circ f} = h\circ m_f$, where

$$h: M_f \to M_{p\circ f}, \qquad z \subset h(z) \in M_{p\circ f} \qquad \text{for } z \in M_f,$$

one infers from 3.6 (applied to $p\circ f$) that if $\omega \in E^k(R^k)$ and ζ is a real valued continuous function on $M_{p\circ f}$, then

$$
\begin{aligned}
\mu(\zeta\circ h)[p^{\#}(\omega)] &= \lim_{i\to\infty} \mu_i(\zeta\circ h)[p^{\#}(\omega)] \\
&= \lim_{i\to\infty} \int_X (\zeta\circ h\circ m_f)\wedge f_i^{\#}[p^{\#}(\omega)] \\
&= \lim_{i\to\infty} \int_X (\zeta\circ m_{p\circ f})\wedge (p\circ f_i)^{\#}(\omega) \\
&= \lim_{j\to\infty} \int_X (\zeta\circ m_{p\circ f})\wedge (p\circ g_j)^{\#}(\omega) \\
&= \lim_{j\to\infty} \int_X (\zeta\circ h\circ m_f)\wedge g_j^{\#}[p^{\#}(\omega)] \\
&= \lim_{j\to\infty} \nu_j(\zeta\circ h)[p^{\#}(\omega)] = \nu(\zeta\circ h)[p^{\#}(\omega)].
\end{aligned}
$$

It follows that the equation

$$\mu(\zeta\circ h)[p^{\#}(\omega)] = \nu(\zeta\circ h)[p^{\#}(\omega)]$$

holds also in case ζ is a real valued bounded Baire function on $M_{p\circ f}$, and in particular

$$\mu[h^{-1}(B)][p^{\#}(\omega)] = \nu[h^{-1}(B)][p^{\#}(\omega)]$$

for every Borel subset B of $M_{p\circ f}$. Now, if A is any closed subset of M_f, then

$$(h^{-1}\circ h)(A) - A \subset M_f - M_{p\circ f},$$

$$\mu(A)[p^{\#}(\omega)] = \mu(h^{-1}[h(A)])[p^{\#}(\omega)] = \nu(h^{-1}[h(A)])[p^{\#}(\omega)] = \nu(A)[p^{\#}(\omega)],$$

and consequently

$$\mu(\chi)[p^{\#}(\omega)] = \nu(\chi)[p^{\#}(\omega)]$$

for every real valued bounded Baire function χ on M_f. Furthermore one sees from 2.2 (10) that if $\psi\in \boldsymbol{E}^0(R^n)$, then

$$\mu(\chi)[\psi\wedge p^{\#}(\omega)] = \mu[\chi\cdot(\psi\circ l_f)][p^{\#}(\omega)] = \nu[\chi\cdot(\psi\circ l_f)][p^{\#}(\omega)] = \nu(\chi)[\psi\wedge p^{\#}(\omega)].$$

Observing that $\boldsymbol{E}^k(R^n)$ consists of finite sums of such forms $\psi\wedge p^{\#}(\omega)$, one finally obtains

$$\mu(\chi)(\phi) = \nu(\chi)(\phi) \qquad \text{for } \phi\in \boldsymbol{E}^k(R^n).$$

3.10. Remark. The preceding theorem remains true without the assumption that X is compact, provided f is proper $[f^{-1}(Y)$ is compact for every compact $Y\subset R^n]$. If the maps f_i converge to f, uniformly on each compact subset of X, then

$$\lim_{i\to\infty} \mu_i(\chi)$$

exists for every continuous $\chi: M_f \to R$ with compact support.

To prove this choose $u: X \to R$ of class ∞ so that

$$\operatorname{spt}(\chi \circ m_f) \subset \{x: u(x) > 0\},$$
$$\operatorname{Clos}\{x: u(x) > 0\} \text{ is compact},$$
$$du(x) \neq 0 \quad \text{whenever} \quad u(x) = 0.$$

By doubling $\{x: u(x) > 0\}$ with respect to $\{x: u(x) = 0\}$ one obtains the compact manifold

$$Q = (X \times R) \cap \{(x, y): u(x) = y^2\}$$

of class ∞, and the maps

$$\xi: Q \to X, \qquad \xi(x, y) = x,$$
$$\eta: \{x: u(x) > 0\} \to Q, \qquad \eta(x) = (x, u(x)^{1/2}),$$
$$g = f \circ \xi \quad \text{and} \quad g_i = f_i \circ \xi: Q \to R^n.$$

Moreover there exists a continuous $\psi: M_g \to R$ such that

$$\operatorname{spt}(\psi \circ m_g) \subset \operatorname{range} \eta,$$
$$\psi \circ m_g \circ \eta = \chi \circ m_f \,|\, \{x: u(x) > 0\},$$

and 3.9 implies for each $\phi \in E^k(R^n)$ the existence of

$$\lim_{i \to \infty} \int_Q (\psi \circ m_g) \wedge \overset{\#}{g_i}(\phi) = \lim_{i \to \infty} \int_{\{x:\, u(x) > 0\}} \overset{\#}{\eta}[(\psi \circ m_g) \wedge \overset{\#}{g_i}(\phi)]$$
$$= \lim_{i \to \infty} \int_X (\chi \circ m_f) \wedge \overset{\#}{f_i}(\phi).$$

The restriction that χ have compact support is essential, as seen from the example where f maps an open circular disc conformally onto a plane region bounded by a simple closed curve with positive L_2 measure.

3.11. Remark. It is an open question whether Theorem 3.9 remains true without the assumption that $H^{k+1}[f(X)] = 0$ or $k = 2$; certainly Lemma 3.7 becomes false, as seen from the following simple example: Let u be a continuous map of $I = \{t: 0 \leq t \leq 1\}$ onto $R^4 \cap \{y: |y| \leq 2\}$, and define

$$f: I \times I \times I \to R^4, \qquad f(x_1, x_2, x_3) = x_2 u(x_1).$$

Then $L_3(f) = 0$, but $f(\{t\} \times I \times I)$ is the line segment from 0 to $u(t)$.

A slightly more complicated example shows that in case $k > 2$ the sets $(p \circ l_f)(M_f - M_{p \circ f})$ can have interior points for (almost) all orthogonal projections of R^n onto R^k. Let u be as above, choose $c \in R^4$ with $|c| = 1$, and define

$$f: I \times I \times I \to R^4. \qquad f(x_1, x_2, x_3) = u(x_1) + x_2 u[c \bullet u(x_1)].$$

Again $L_3(f)=0$, yet if $|c\bullet u(s)|>0$, then

$$W_s = R^4 \cap \left\{ w: \left| w + \frac{s - c\bullet w}{c\bullet u(s)} u(s) \right| < 2 \right\}$$

is open and nonempty, because $sc \in W_s$, and for each $w\in W_s$ there exists a $t\in I$ such that

$$u(t) = w + \frac{s - c\bullet w}{c\bullet u(s)} u(s),$$

hence $c\bullet u(t)=s$ and $f(\{t\}\times I\times I)$ is a segment of length $|u(s)|$ on the straight line through w in the direction of $u(s)$.

4. **The additivity of Lebesgue area.** *Suppose $f: X\to R^n$ is a continuous map of finite Lebesgue area for which either $\boldsymbol{H}^{k+1}[f(X)]=0$ or $k=2$, μ is the limit measure corresponding to f, and v is as in* 2.2 *and* 2.3 (*with $Z=M_f$, $g=l_f$*).

For each finitely triangulable subset T of X, let $L_k(f|T)$ be the Lebesgue area of $f|T$. For each open subset U of X, let $L_k(f|U)$ be the supremum of $L_k(f|T)$ over all finitely triangulable subsets T of U.

The purpose of this section is to establish the precise relation (4.3, 4.9) between μ and L_k.

4.1. LEMMA. *If $n=k$ and A is an open subset of M_f, then*

$$\|\mu\|(A) = L_k[f|m_f^{-1}(A)].$$

Proof. If W is a k dimensional open cube in R^k,

$$\|\mu\|[l_f^{-1}(\text{Bdry } W)] = 0,$$

V is a component of $l_f^{-1}(W)$, $U=m_f^{-1}(V)$, then obviously

$$\mu(V) = \text{degree}(f|U)\cdot W,$$

where the degree is obtained from the induced homomorphism

$$f^*: H^k(R^k, R^k - W) \to H^k(X, X-U)$$

of integral Čech cohomology groups. This formula implies that

$$\boldsymbol{M}[\mu(V)] = |\text{degree}(f|U)|\cdot L_k(W),$$

and the proof of the lemma may be completed by reference to [F4, §4] and [F8, 7.3].

4.2. LEMMA. *Almost all orthogonal projections p of R^n onto R^k have the property that*

$$\int_A |p[v(z)]|\, d\boldsymbol{H}^k z = L_k[p\circ f|m_f^{-1}(A)]$$

whenever A is an open subset of M_f.

Proof. Recalling 3.8 assume $\|\mu\|(M_f - M_{p \circ f}) = 0$, factor $m_{p \circ f} = h \circ m_f$ as in 3.9, and note that the limit measure corresponding to $p \circ f$ is $h_\sharp(p_\sharp \circ \mu)$. Since h is univalent except on $M_f - M_{p \circ f}$, it follows from 4.1 that

$$L_k[p \circ f \mid m_{p \circ f}^{-1}(B)] = \|h_\sharp(p_\sharp \circ \mu)\|(B)$$
$$= \|p_\sharp \circ \mu\|[h^{-1}(B)] = \int_{h^{-1}(B)} |p \circ v| \, dH^k$$

for every open subset B of $M_{p \circ f}$.

First taking $B = M_{p \circ f} - h(M_f - A)$ one obtains

$$h^{-1}(B) \subset A, \qquad A - h^{-1}(B) \subset M_f - M_{p \circ f},$$

$$\int_A |p \circ v| \, dH^k = \int_{h^{-1}(B)} |p \circ v| \, dH^k$$
$$= L_k[p \circ f \mid m_{p \circ f}^{-1}(B)] \leqq L_k[p \circ f \mid m_f^{-1}(A)].$$

Next suppose T is a finitely triangulable subset of $m_f^{-1}(A)$ and choose open subsets $B_1 \supset B_2 \supset B_3 \supset \cdots$ of $M_{p \circ f}$ such that

$$\bigcap_{i=1}^{\infty} B_i = m_{p \circ f}(T).$$

Then

$$\bigcap_{i=1}^{\infty} h^{-1}(B_i) = (h^{-1} \circ m_{p \circ f})(T) \subset (h^{-1} \circ h)(A) \subset A \cup (M_f - M_{p \circ f}),$$

$$L_k[p \circ f \mid T] \leqq \lim_{i \to \infty} L_k[p \circ f \mid m_{p \circ f}^{-1}(B_i)]$$
$$= \lim_{i \to \infty} \int_{h^{-1}(B_i)} |p \circ v| \, dH^k \leqq \int_A |p \circ v| \, dH^k.$$

4.3. THEOREM. *If A is an open subset of M_f, then*

$$\|\mu\|(A) = \beta(n, k)^{-1} \int_{G_n} L_k[P \circ \rho \circ f \mid m_f^{-1}(A)] \, d\phi_n \rho,$$

where P is an orthogonal projection of R^n onto R^k, G_n is the orthogonal group of R^n with the Haar measure ϕ_n such that $\phi_n(G_n) = 1$, and

$$\beta(n, k) = \alpha(k)\alpha(n - k)\alpha(n)^{-1}\binom{n}{k}^{-1}.$$

Proof. Computing the above integral by means of 4.2, Fubini's theorem, [F2, 4.4, 5.4] and 2.2 one obtains

$$\int_{G_n}\int_A |(P\circ\rho\circ v)(z)|\, dH^k z\, d\phi_n\rho = \int_A\int_{G_n} |(P\circ\rho\circ v)(z)|\, d\phi_n\rho\, dH^k z$$

$$= \int_A \beta(n,k)\,|v(z)|\, dH^k z = \beta(n,k)\|\mu\|(A).$$

4.4. Corollary. $\|\mu\|(A) \leqq L_k[f|m_f^{-1}(A)]$.

Proof. It is known from [F5, §6] or [F8, §6, §7] that the right member of the first equation in 4.3 does not exceed $L_k[f|m_f^{-1}(A)]$.

4.5. Lemma. *If A is an open subset of M_f, B is a Borel subset of $A\cap\{z: v(z)\neq 0\}$, and γ is a simple k-vector of R^n, then*

$$L_k[f|m_f^{-1}(A)]$$

$$\leqq \|\mu\|(A) + \left[\binom{n}{k} - 1\right]\left(\|\mu\|(A-B) + \int_B \left|\frac{v(z)}{|v(z)|} - \gamma\right| d\|\mu\|z\right).$$

Proof. Recalling the notation of [FF, 8.1] one infers from [DF, 3.16, 5.7] and 4.2 that, for almost all orthogonal transformations g of R^n,

$$L_k[f|m_f^{-1}(A)] = L_k[g\circ f|m_f^{-1}(A)] \leqq \sum_{\lambda\in\Lambda(k,n)} L_k[p^\lambda\circ g\circ f|m_f^{-1}(A)]$$

$$= \sum_{\lambda\in\Lambda(k,n)} \int_A |(p^\lambda\circ g)[v(z)]|\, dH^k z.$$

The resulting inequality

$$L_k[f|m_f^{-1}(A)] \leqq \sum_{\lambda\in\Lambda(k,n)} \int_A |(p^\lambda\circ g)[v(z)]|\, dH^k z$$

holds, by continuity, for every orthogonal transformation g of R^n. Choosing g so that

$$p^\lambda[g(\gamma)] = 0 \quad \text{whenever} \quad \lambda\in\Lambda(k,n) - \{(1,\cdots,k)\},$$

one completes the proof by observing that

$$\int_A |(p^{(1,\cdots,k)}\circ g)[v(z)]|\, dH^k z \leqq \int_A |v(z)|\, dH^k z = \|\mu\|(A)$$

and that, for $\lambda\in\Lambda(k,n)-\{(1,\cdots,k)\}$,

$$\int_{A-B} |(p^\lambda\circ g)[v(z)]|\, dH^k z \leqq \int_{A-B} |v(z)|\, dH^k z = \|\mu\|(A-B),$$

$$\int_B |(p^\lambda \circ g)[v(z)]|\, dH^k z = \int_B \left|(p^\lambda \circ g)\left[\frac{v(z)}{|v(z)|} - \gamma\right]\right| \cdot |v(z)|\, dH^k z$$
$$\leqq \int_B \left|\frac{v(z)}{|v(z)|} - \gamma\right| d\|\mu\| z.$$

4.6. Lemma. *For every $\delta > 0$ there is a closed subset Y of R^n such that*

$$\|\mu\|[l_f^{-1}(Y)] = 0$$

and, if Ξ is the set of components of $M_f - l_f^{-1}(Y)$, then $\zeta(U) \in U$ may be associated with $U \in \Xi$ so that $v[\zeta(U)]$ is simple and

$$\sum_{U \in \Xi} \int_{U \cap \{z: v(z) \neq 0\}} \left| \frac{v(z)}{|v(z)|} - \frac{v[\zeta(U)]}{|v[\zeta(U)]|} \right| d\|\mu\| z < \delta.$$

Proof. Let $P = \{z: v(z) \text{ is simple and } \neq 0\}$ and $j = v/|v| : P \to \Lambda_k(R^n)$. Since j is $\|\mu\|$ summable over P, there exists a continuous w: $M_f \to \Lambda_k(R^n)$ for which

$$\int_P |j - w|\, d\|\mu\| < \delta/3.$$

Using the lightness of l_f one may then construct Y, as the union of finitely many $n-1$ dimensional planes in R^n, so that $\|\mu\|[l_f^{-1}(Y)] = 0$ and the oscillation of w on each member of Ξ is less than $\delta/3\|\mu\|(P)$.

For each $U \in \Xi$ select $\zeta(U) \in U \cap P$ so that

$$|w[\zeta(U)] - j[\zeta(U)]| \cdot \|\mu\|(U \cap P) \leqq \int_{U \cap P} |w - j|\, d\|\mu\|$$

and observe that

$$\int_{U \cap P} |j(z) - j[\zeta(U)]|\, d\|\mu\| z$$
$$\leqq \int_{U \cap P} |j - w|\, d\|\mu\| + \int_{U \cap P} |w(z) - w[\zeta(U)]|\, d\|\mu\| z$$
$$+ \int_{U \cap P} |w[\zeta(U)] - j[\zeta(U)]|\, d\|\mu\| z$$
$$\leqq 2\int_{U \cap P} |j - w|\, d\|\mu\| + \delta\|\mu\|(U \cap P)/3\|\mu\|(P).$$

4.7. Remark. If ψ: $M_f \to R^n$ is continuous and

$$F = \psi \circ m_f,$$

then there exists a unique monotone $h: M_f \to M_F$ such that

$$m_F = h \circ m_f \quad \text{and} \quad \psi = l_F \circ h.$$

Assuming the convergence property for F as well as for f, let μ_f and μ_F be the limit measures corresponding to f and F, with the associated densities v_f and v_F.

If W is an open subset of M_f such that $\psi | W = l_f | W$, then W is an open subset of M_F, $h(z) = z$ for $z \in W$, $h | W$ is a homeomorphism, and 3.10 implies that

$$\mu_f(B) = \mu_F(B) \text{ for every Borel set } B \subset W, \; v_f | W = v_F | W.$$

The following two special cases occur in the sequel:

(1) There exist a neighborhood H of $l_f(M_f - W)$ in R^n and a Lipschitzian $\Gamma: H \to R^n$ such that ψ agrees with $\Gamma \circ l_f$ in some neighborhood of $M_f - W$. Then

$$\mu_F(B) = \Gamma_\#(\mu_f[h^{-1}(B)])$$

for every Borel set $B \subset M_F - W$; moreover

$$\|\mu_F\|(B) \leqq \int_{h^{-1}(B)} (\lambda \circ l_f)^k d\|\mu_f\|$$

if $\lambda: H \to R$ is continuous with $|D\Gamma(y)| \leqq \lambda(y)$ for L_n almost all y in Y.

(2) $k = 2$ and $\psi(M_f - W)$ is a polygon. Then

$$\mu_F(B) = 0 \text{ for every Borel set } B \subset M_F - W.$$

4.8. Lemma. $\|\mu\|(M_f) \geqq L_k(f)$.

Proof. Suppose $\delta > 0$, and again write $\mu = \mu_f$.

Choose Y according to 4.6, let V be a neighborhood of Y in R^n for which

$$\|\mu_f\|[l_f^{-1}(V)] < \delta \Big/ \binom{n}{k},$$

suppose

$$0 < \epsilon < \text{distance}(Y, R^n - V)/(7n),$$

and consider the maps ω, $\tau_a: R^n \to R^n$ defined by

$$\omega(y) = \epsilon y, \qquad \tau_a(y) = a + y \text{ for } y, \qquad a \in R^n.$$

Recalling [FF, 5.1, 5.2] and abbreviating

$$B = R^n \cap \{b: |b_i| < 1 \text{ for } i = 1, \cdots, n\}$$

one finds that

$$\int_{\omega(B)}\int_{l_f^{-1}(V)}(u_k\circ\omega^{-1}\circ\tau_{-a}\circ l_f)^{-k}d\|\mu_f\|dL_na$$
$$= \epsilon^n\int_B\int_{l_f^{-1}(V)}(u_k\circ\tau_{-b}\circ\omega^{-1}\circ l_f)^{-k}d\|\mu_f\|dL_nb$$
$$= \epsilon^n\int_B\int_{\omega^{-1}(V)}(u_k\circ\tau_{-b})^{-k}d(\omega^{-1}\circ l_f)(\|\mu_f\|)dL_nb$$
$$= \epsilon^nL_n(B)\binom{n}{k}(\omega^{-1}\circ l_f)(\|\mu_f\|)[\omega^{-1}(V)]$$
$$= L_n[\omega(B)]\binom{n}{k}\|\mu_f\|[l_f^{-1}(V)] < L_n[\omega(B)]\delta.$$

Hence the points a satisfying the condition

$$\int_{l_f^{-1}(V)}(u_k\circ\omega^{-1}\circ\tau_{-a}\circ l_f)^{-k}d\|\mu_f\| < \delta$$

form a set of positive L_n measure. In this set a point a will be selected, subject to an additional requirement, as follows:

In case $H^{k+1}[f(X)]=0$, one may choose a so that

$$f(X)\subset R^n - (\tau_a\circ\omega)(C''_{n-k-1}),$$

because obviously this requirement holds for L_n almost all a (see the proof of [F8, 7.8]). In this case let $\psi=l_f$, $F=f$.

In case $k=2$, let $g_i\colon M_f\to R$ with

$$l_f(z) = (g_1(z),\cdots,g_n(z))\qquad\text{for } z\in M_f,$$

select T according to [DF, 5.3] with X replaced by M_f, and choose a so that

$$g_r^{-1}\{a_r+\epsilon j_r\}$$

is a subset of M_f-T and has dimension 0 at each point of

$$g_r^{-1}\{a_r+\epsilon j_r\}\cap g_s^{-1}\{a_s+\epsilon j_s\}\cap g_t^{-1}\{a_t+\epsilon j_t\}$$

whenever r, s, t are distinct elements of $\{1,\cdots,n\}$ and j_r, j_s, j_t are even integers; this choice is possible by [DF, 4.4]. Then use the construction of [DF, 5.6] to obtain continuous maps

$$\psi_i\colon M_f\to R,\qquad \psi\colon M_f\to R^n,\qquad F=\psi\circ m_f\colon X\to R^n,$$

such that, for $z\in M_f$,

$$\psi(z) = (\psi_1(z), \cdots, \psi_n(z)), \qquad |\psi(z) - l_f(z)| < \epsilon,$$
$$\psi(z) \notin (\tau_a \circ \omega)(C''_{n-3}) \cap \{y\colon \text{distance}(y, Y) \leqq 6n\epsilon\},$$
$$\psi(z) = l_f(z) \text{ whenever distance}[l_f(z), Y] \geqq 7n\epsilon,$$

and such that

$$\int_{l_F^{-1}(V)} (u_k \circ \omega^{-1} \circ \tau_{-a} \circ l_F)^{-k} d\|\mu_F\| < \delta.$$

The last requirements can be met because ψ may be constructed by finitely many successive modifications of the two types described in 4.7; those of type (1) involve orthogonal projections Γ of R^n onto $n-1$ dimensional planes and do not decrease the values of $u_k \circ \omega^{-1} \circ \tau_{-a}$.

In both cases choose $q\colon R^n \to \{t\colon 0 \leqq t \leqq 1\}$ with Lipschitz constant $(n\epsilon)^{-1}$ so that

$$q(y) = 1 \text{ whenever distance}(y, Y) \leqq 5n\epsilon,$$
$$q(y) = 0 \text{ whenever distance}(y, Y) \geqq 6n\epsilon,$$

and consider the continuous maps

$$\phi\colon R^n - (\tau_a \circ \omega)(C''_{n-k-1}) \to R^n,$$
$$\phi(y) = y + q(y) \cdot [(\tau_a \circ \omega \circ \sigma_k \circ \omega^{-1} \circ \tau_{-a})(y) - y],$$
$$\chi\colon M_F \to R^n,$$
$$\chi(z) = l_F(z) \text{ whenever distance}[l_F(z), Y] \geqq 6n\epsilon,$$
$$\chi(z) = (\phi \circ l_F)(z) \text{ whenever distance } [l_F(z), Y] \leqq 6n\epsilon,$$
$$G = \chi \circ m_F\colon X \to R^n,$$

as well as the monotone maps ξ, η completing the commutative diagram:

$$\begin{tikzcd}
 & X \arrow[dl, "m_f"'] \arrow[d, "m_F"] \arrow[dr, "m_G"] & \\
M_f \arrow[r, "\xi"] \arrow[d, "l_f"'] \arrow[dr, "\psi"] & M_F \arrow[r, "\eta"] \arrow[d, "l_F"] \arrow[dr, "\chi"] & M_G \arrow[d, "l_G"] \\
R^n & R^n & R^n
\end{tikzcd}$$

Clearly

$$\begin{aligned}
&|\phi(y) - y| < n\epsilon \qquad \text{for } y \in R^n - (\tau_a \circ \omega)(C''_{n-k-1}),\\
&|\chi(z) - l_F(z)| < n\epsilon \qquad \text{for } z \in M_F,\\
&|(\chi \circ \xi)(z) - l_f(z)| < (n+1)\epsilon \qquad \text{for } z \in M_f,\\
&|D\phi(y)| \leqq 1 + (n\epsilon)^{-1}(n\epsilon) + n(u_k \circ \omega^{-1} \circ \tau_{-a})(y)^{-1} + 1\\
&\qquad\qquad \leqq (n+3)(u_k \circ \omega^{-1} \circ \tau_{-a})(y)^{-1}
\end{aligned}$$

for L_n almost all y. Defining

$$W = M_f \cap \{z: \text{distance}[l_f(z), Y] > 6n\epsilon\},$$
$$P = W \cap \{z: v_f(z) \neq 0\},$$

one sees from 4.7 that W is also an open subset of M_F and M_G, with $\xi(z) = z = \eta(z)$ for $z \in W$, that $\|\mu_F\|$ and $\|\mu_G\|$ agree with $\|\mu_f\|$ on all Borel subsets of W, and that v_F and v_G agree with v_f on W; furthermore $\|\mu_f\|(W-P) = 0$.

Now let Ω be the set of components of $M_G - l_G^{-1}(C'_{k-1})$. Since C'_{k-1} is k-removable by [F8, 6.30],

$$L_k(G) = \sum_{Q\in\Omega} L_k[G \mid m_G^{-1}(Q)].$$

Also let

$$\Omega_1 = \Omega \cap \{Q: (\eta \circ \xi)^{-1}(Q) \text{ meets } l_f^{-1}(Y)\}, \qquad \Omega_2 = \Omega - \Omega_1.$$

If $Q \in \Omega_1$, then $\eta^{-1}(Q) \subset l_F^{-1}(V)$. In fact, assuming $(\eta \circ \xi)(z) \in Q$ with $l_f(z) \in Y$, one finds that

$$\text{distance}[\psi(z), Y] < \epsilon, \qquad \psi(z) = (l_F \circ \xi)(z),$$
$$\text{distance}[(\chi \circ \xi)(z), Y] < (n+1)\epsilon \leqq 2n\epsilon,$$

and $(\chi \circ \xi)(z) = (l_G \circ \eta \circ \xi)(z)$ belongs to a component E of $C'_k - C'_{k-1}$. Moreover E is a k dimensional cube with side length 2ϵ,

$$E \subset \{y: \text{distance}(y, Y) < 4n\epsilon\},$$

$$E \text{ is open relative to } C'_k \cap \{y: \text{distance } (y, Y) < 4n\epsilon\},$$

and inasmuch as

$$l_G(M_G) \cap \{y: \text{distance}(y, Y) < 4n\epsilon\} \subset C'_k,$$

one infers that $l_G^{-1}(E)$ is open in M_G. Noting that

$$\text{Bdry } l_G^{-1}(E) \subset l_G^{-1}(C'_{k-1}) \subset M_G - Q,$$

one concludes that $Q \cap l_G^{-1}(E)$ is nonempty, open and closed relative to Q, hence

$$Q \subset l_G^{-1}(E), \qquad \eta^{-1}(Q) \subset \chi^{-1}(E) \subset l_F^{-1}(V).$$

Furthermore 4.5 yields the inequality

$$L_k[G \mid m_G^{-1}(Q)] \leqq \binom{n}{k} \|\mu_G\|(Q).$$

If $Q \in \Omega_2$, then $(\eta \circ \xi)^{-1}(Q) \subset U$ for a unique $U \in \Xi$, and 4.5 implies that

$$L_k[G \mid m_G^{-1}(Q)] \leqq \|\mu_G\|(Q \cap P) + \binom{n}{k} \|\mu_G\|(Q - W)$$
$$+ \left[\binom{n}{k} - 1\right] \int_{Q \cap P} \left| \frac{v_G(z)}{|v_G(z)|} - \frac{v_f[\zeta(U)]}{|v_f[\zeta(U)]|} \right| d\|\mu_G\| z.$$

Since $\{\eta^{-1}(Q) : Q \in \Omega_1\} \cup \{\eta^{-1}(Q - W) : Q \in \Omega_2\}$ is a countable family of disjoint Borel subsets of $l_F^{-1}(V)$, it follows from 4.7 that

$$\sum_{Q \in \Omega_1} \|\mu_G\|(Q) + \sum_{Q \in \Omega_1} \|\mu_G\|(Q - W)$$
$$\leqq (n+3)^k \int_{l_F^{-}(V)} (u_k \circ \omega^{-1} \circ \tau_{-a} \circ l_F)^{-k} d\|\mu_F\| < (n+3)^k \delta.$$

On the other hand $\{Q \cap P : Q \in \Omega_2\}$ is a countable family of disjoint Borel subsets of P, and 4.6 implies that

$$\sum_{Q \in \Omega_2} \|\mu_G\|(Q \cap P) \leqq \|\mu_G\|(P) = \|\mu_f\|(P),$$

$$\sum_{U \in \Xi} \sum_{Q \in \Omega_2, (\eta \circ \xi)^{-1}(Q) \subset U} \int_{Q \cap P} \left| \frac{v_G(z)}{|v_G(z)|} - \frac{v_f[\zeta(U)]}{|v_f[\zeta(U)]|} \right| d\|\mu_G\| z$$
$$\leqq \sum_{U \in \Xi} \int_U \left| \frac{v_f(z)}{|v_f(z)|} - \frac{v_f[\zeta(U)]}{|v_f[\zeta(U)]|} \right| d\|\mu_f\| z < \delta.$$

Combining these estimates one concludes that

$$L_k(G) \leqq \|\mu_f\|(M_f) + 2(n+3)^k \binom{n}{k} \delta,$$

with $|G(x) - f(x)| < (n+1)\epsilon$ for $x \in X$.

4.9. THEOREM. *If A is an open subset of M_f, then*

$$\|\mu\|(A) = L_k[f \mid m_f^{-1}(A)].$$

Proof. In view of 4.4 there would otherwise exist a finitely triangulable set $T \subset m_f^{-1}(A)$ such that

$$\|\mu\|(A) < L_k(f \mid T).$$

Letting $B = M_f - m_f(T)$ one would find that T and $m_f^{-1}(B)$ are disjoint, $A \cup B = M_f$, and it would follow from 4.8 and 4.4 that

$$\|\mu\|(M_f) \geqq L_k(f) \geqq L_k(f \mid T) + L_k[f \mid m_f^{-1}(B)]$$
$$> \|\mu\|(A) + \|\mu\|(B) \geqq \|\mu\|(M_f).$$

4.10. REMARK. One readily verifies by the method of doubling, as in 3.10, that the results of this section remain true without the assumption that X is compact, provided f is proper.

Regarding the Lebesgue area densities introduced in [F6, §6] one infers from 4.9 and 2.2 that

$$\overset{*}{L_k}(f, z) = L_{*k}(f, z) = |v(z)|$$

for H^k almost all z in M_f. Moreover one sees with the help of [DF] and an argument like the proof of [F6, 8.14 (7)] that the equation

$$L_k(f) = \int_{M_f} \overset{*}{L_k}(f, z)\, dH^k z = \int_{R^n} \sum_{z \in l_f^{-1}\{y\}} \overset{*}{L_k}(f, z)\, dH^k y$$

holds also in case $L_k(f) = \infty$. The problems raised in [F6, pp. 326, 335] are thus solved provided $H^{k+1}[f(X)] = 0$ or $k = 2$.

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