CURRENTS AND AREA(1)

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1. Introduction. When a continuous map $f$ of a compact $k$ dimensional manifold $X$ into the Euclidean space $\mathbb{R}^n$ is uniformly approximated by smooth maps $f_i$, the areas of $f_i$ need of course not converge. This is the simple reason for the complexity of the theory of area. Many geometric properties of $f$ have been studied intensively in a search for useful and intuitively appealing concepts which suffice to determine the area of $f$ and which behave properly under uniform approximation. The author believes that this paper makes a decisive contribution toward the natural solution of this problem.

To each of the smooth maps $f_i$ corresponds a measure over $X$ whose values are $k$ dimensional currents in $\mathbb{R}^n$. This measure associates with any continuous real-valued function $\psi$ on $X$ the current $f_i\psi(X \wedge \psi)$ given by the formula (2)

$$f_i\psi(X \wedge \psi)(\phi) = \int_X \psi \wedge f_i^*(\phi)$$

whenever $\phi$ is a differential $k$ form of class $\infty$ on $\mathbb{R}^n$. The values of this measure are currents of finite mass; its total variation, using mass as norm, equals the area of $f_i$.

Applying to the limit map $f$ the monotone-light factorization

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(2) Of course this formula remains meaningful in case $\psi$ is a $k-j$ form on $X$ and $\phi$ is a $j$ form on $\mathbb{R}^n$, with $j \leq k$. The supremum of the integral, taking $\psi$ and $\phi$ with masses not exceeding 1, may be called the $j$ dimensional area of $f_i$. If $k > n$, when the $k$ dimensional area of $f_i$ equals 0, one can let $j = n$ to obtain the coarea of $f_i$ studied in [F9, §3].

204
whenever $\chi$ is a continuous real-valued function on $M_f$ and $\phi$ is a differential $k$ form of class $\infty$ on $R^n$.

The principal results of this paper may now be summarized as follows:

If $f$ has finite Lebesgue area and either $k=2$ or the range of $f$ has $k+1$ dimensional Hausdorff measure 0, then:

1. There exists a unique current-valued measure $\mu$ over $M_f$ such that for every sequence of smooth maps $f_i$, which converge uniformly to $f$ and whose areas are bounded, the measures $\mu_i$ converge weakly to $\mu$.

2. The total variation of $\mu$ is equal to the Lebesgue area of $f$, and also equal to the integralgeometric $M$ area of $f$ introduced in [F5].

3. $\mu$ is the indefinite integral, with respect to $k$ dimensional Hausdorff measure over $M_f$, of a density function whose values are simple $k$ vectors in $R^n$ with integer norms; these $k$ vectors describe the tangential properties of $f$, and the multiplicities with which $f$ assumes its values in $R^n$.

Much of the present work with currents depends on the recent joint paper [FF] by W. H. Fleming and the author. Hence the terminology of [FF] is readopted here without change. For those facts from the previous theory of Lebesgue area which are used here the reader may consult [F4; F5; F6; F8] and [DF]. It should be noted that this paper eliminates from geometric area theory the need for Morrey's representation theorem, cyclic element theory and the Moore-Roberts-Steenrod characterization of monotone images of 2 dimensional manifolds.

2. A representation theorem. The purpose of this section is to establish density properties of certain current-valued measures. Where classical differentiation theory is not applicable, arguments using the relative isoperimetric inequality fill the gap.

2.1. Theorem. Suppose:

1. $Z$ is a locally connected compact metric space, $g: Z \rightarrow R^n$ is a continuous light mapping, and $\Delta(z, r)$ is the component of $z$ in $g^{-1}\{w: |w-g(z)| < r\}$ whenever $z \in Z, r > 0$.

2. $\mu$ is a countably additive function whose domain is the class of all Borel subsets of $Z$, and whose range is a class of $k$ dimensional rectifiable currents in $R^n$; thus

\[
\mu \left( \bigcup_{i=1}^{\infty} A_i \right) (\phi) = \sum_{i=1}^{\infty} \mu(A_i) (\phi)
\]

whenever $\phi \in E^k(R^n)$ and $A_1, A_2, A_3, \cdots$ are disjoint Borel subsets of $Z$.

3. The total $M$ variation of $\mu$ is finite; hence a finite Borel measure $\|\mu\|$ over $Z$ is defined by the formula

\[
\|\mu\|(A) = \sup \left\{ \sum_{i=1}^{\infty} M[\mu(B_i)]: B_1, B_2, B_3, \cdots \text{ are disjoint Borel subsets of } A \right\}.
\]
206 HERBERT FEDERER [February

(4) For each Borel subset $A$ of $\mathbb{Z}$,

$$ \text{spt } \mu(A) \subseteq g(\text{Clos } A), \quad \text{spt } \partial \mu(A) \subseteq g(\text{Bdry } A). $$

Then there exists a Baire function $v: \mathbb{Z} \to \mathbb{A}_k(\mathbb{R}^n)$ with the following properties:

(5) For $\|\mu\|$ almost all $z$ in $\mathbb{Z}$, $v(z)$ is a simple $k$-vector, $|v(z)|$ is an integer, and

$$ \phi_1[g(z)]v(z) = \lim_{r \to 0^+} \alpha(k)^{-1}r^{-k} \mu(\Delta(z, r))(\phi) $$

whenever $\phi \in E^k(\mathbb{R}^n)$.

(6) If $A$ is a Borel subset of $\mathbb{Z}$ and $\phi \in E^k(\mathbb{R}^n)$, then

$$ \mu(A)(\phi) = \int_{\mathbb{R}^n} \phi(y) \left[ \sum_{z \in A \cap |v|} v(z) \right] dH^k y. $$

**Proof.** Where possible define $v(z) \in \mathbb{A}_k(\mathbb{R}^n)$ so that

$$ f[v(z)] = \lim_{r \to 0^+} \alpha(k)^{-1}r^{-k} \mu(\Delta(z, r))(\phi) $$

whenever $f \in \mathbb{A}_k(\mathbb{R}^n)$, $\phi(x) = f$ for $x \in \mathbb{R}^n$. In case the limit fails to exist for some $f$, let $v(z) = 0$. Applying classical arguments to the real valued measures $\mu(\cdot)(\phi)$ one sees that the components of $v$ are Baire functions.

We let $\gamma = g_t(\|\mu\|)$, so that $\gamma(Y) = \|\mu\| \|g_t-1(Y)\|$ for every Borel subset $Y$ of $\mathbb{R}^n$, and divide the remainder of the argument into eleven parts.

**Part 1.** If $A$ and $Y$ are Borel subsets of $\mathbb{Z}$ and $\mathbb{R}^n$, then

$$ \mu(A) \cap Y = \mu[A \cap g^{-1}(Y)]. $$

**Proof.** For a fixed set $A$, both members of the above equation are countably additive with respect to $Y$. Hence it suffices to verify the equation in the special case when $Y$ is closed.

Let $K_1, K_2, K_3 \ldots$ be closed sets whose union is $\mathbb{R}^n - Y$ and observe that

$$ \text{spt } \mu[A \cap g^{-1}(Y)] \subseteq Y, \quad \text{spt } \mu[A \cap g^{-1}(K_i)] \subseteq K_i, $$

$$ \mu[A \cap g^{-1}(Y)] = \mu[A \cap g^{-1}(Y)] + \mu[A \cap g^{-1}(K_i)] \cap Y, $$

$$ \mu(A) \cap Y - \mu[A \cap g^{-1}(Y)] = \mu[A \cap g^{-1}(Y \cup K_i)] \cap Y, $$

$$ M(\mu(A) \cap Y - \mu[A \cap g^{-1}(Y)]) \leq \|\mu\| \|A - g^{-1}(Y \cup K_i)\| \to 0 \text{ as } i \to \infty. $$

**Part 2.** If $A \subseteq B$ are Borel subsets of $\mathbb{Z}$, then

$$ \|\mu\|(A) - M[\mu(A)] \leq \|\mu\|(B) - M[\mu(B)]. $$

**Proof.** $\mu(B) = \mu(A) + \mu(B - A)$, hence

$$ M[\mu(B)] - M[\mu(A)] \leq M[\mu(B - A)] \leq \|\mu\|(B - A) = \|\mu\|(B) - \|\mu\|(A). $$
PART 3. There exists a countable family $F$ of $k$ dimensional proper regular submanifolds of class 1 of $\mathbb{R}^n$ such that

$$\gamma(\mathbb{R}^n - \bigcup F) = 0.$$ 

Proof. Suppose $\epsilon > 0$.

Choose disjoint Borel sets $B_1, B_2, B_3, \ldots$ for which

$$\bigcup_{i=1}^{\infty} B_i = \mathbb{R}^n, \quad \sum_{i=1}^{\infty} M[\mu(B_i)] > \|\mu\|(Z) - \epsilon,$$

then apply [FF, 8.16] to secure countable families $G_1, G_2, G_3, \ldots$ of $k$ dimensional proper regular submanifolds of class 1 of $\mathbb{R}^n$ such that

$$\|\mu(B_i)\|(\mathbb{R}^n - \bigcup G_i) = 0 \quad \text{for } i = 1, 2, 3, \ldots,$$

and consider the family

$$G = \bigcup_{i=1}^{\infty} G_i.$$ 

Letting $A_i = B_i \cap g^{-1}(\mathbb{R}^n - \bigcup G_i)$ one sees from Part 1 that

$$\mu(A_i) = \mu(B_i) \cap (\mathbb{R}^n - \bigcup G_i) = 0 \quad \text{for } i = 1, 2, 3, \ldots,$$

and uses Part 2 to obtain

$$\epsilon > \sum_{i=1}^{\infty} (\|\mu\|(B_i) - M[\mu(B_i)]) \geq \sum_{i=1}^{\infty} \|\mu\|(A_i)$$

$$\geq \sum_{i=1}^{\infty} \|\mu\|[B_i \cap g^{-1}(\mathbb{R}^n - \bigcup G)] = \gamma(\mathbb{R}^n - \bigcup G).$$

PART 4. If $Y$ is a Borel subset of $\mathbb{R}^n$ for which $\mathcal{H}^k(Y) = 0$, then $\gamma(Y) = 0$.

Proof. For each Borel set $B \subset g^{-1}(Y)$ one sees from Part 1 and [FF, 8.16] that

$$\mu(B) = \mu(B \cap g^{-1}(Y)) = \mu(B) \cap Y = 0.$$ 

PART 5. For $\gamma$ almost every $y$ in $\mathbb{R}^n$ there exists an $M \subset F$ such that $y \in M$ and $\Theta^k(\gamma, R^n - M, y) = 0$.

Proof. For each $M \subset F$ it follows from [F3, 3.2] and Part 4 that $\Theta^k(\gamma, R^n - M, y) = 0$ for $\mathcal{H}^k$ and $\gamma$ almost all $y$ in $M$.

PART 6. $\Theta^k(\gamma, y) < \infty$ for $\mathcal{H}^k$ and $\gamma$ almost all $y$ in $\mathbb{R}^n$.

Proof. If $M \subset F$, then

$$\Theta^k(\mathcal{H}^k, M, y) = 1, \quad \Theta^k(\gamma, y) = \frac{d\gamma}{d(\mathcal{H}^k \cap M)} (y) < \infty$$

for $\mathcal{H}^k$ and $\gamma$ almost all $y$ in $M$. Moreover [F3, 3.2] implies that
for $H^k$ and $\gamma$ almost all $y$ in $R^n - UF$. 

**Part 7.** Suppose $y \in M \subset F$, 
\[ \Theta^k(\gamma, y) < \infty, \quad \Theta^k(\gamma, R^n - M, y) = 0, \]

$P$ is an oriented $k$ dimensional plane through $y$ tangent to $M$, and
\[ W_r = R^n \cap \{ w : |w - y| < r \} \quad \text{for } r > 0. \]

Then to each open subset $V$ of $Z$, such that $y \in g(\partial V)$, corresponds a unique integer $m(V)$ for which
\[ \lim_{r \to 0^+} r^{-k}F[\mu(V) \cap W_r - m(V) \cdot (P \cap W_r)] = 0. \]

In fact if $\xi > 0$ and $\nu \geq 1$ are as in [FF, 8.18], and if
\[ 0 < \epsilon < \inf\{\xi, \alpha(k)/3\}, \quad 0 < t < 1, \]
\[ \nu\alpha(k)\Theta^k(\gamma, y)(t^k - 1) < \epsilon < \alpha(k)t^k/3, \]
then there exists a $\rho > 0$ such that
\[ F[\mu(V) \cap W_r - m(V) \cdot (P \cap W_r)] \leq \nu t^{-k}\|\mu\|(V \cap g^{-1}[W_r - (M \cap W_r)]) + 2rt^{-k}\|\mu\|(V \cap g^{-1}(W_r)) \leq \epsilon r^k \]
whenever $0 < r < \rho$. $V$ is an open subset of $Z$ and $g(\partial V) \subset R^n - W_r$. 

**Proof.** Observing that
\[ \lim_{r \to 0^+} r^{-k}\gamma(W_r) = \alpha(k)\Theta^k(\gamma, y), \]
\[ \lim_{r \to 0^+} r^{-k}\gamma(W_r - W_0) = \alpha(k)\Theta^k(\gamma, y)(1 - t^k), \]
\[ \lim_{r \to 0^+} r^{-k}\gamma(W_r - M) = 0, \]
choose $\rho > 0$ so that
\[ \nu t^{-k}r^{-k}\gamma(W_r - (M \cap W_r)) + 2rt^{-k}r^{-k}\gamma(W_r) < \epsilon \]
and the conclusion of [FF, 8.19] holds whenever $0 < r \leq \rho$.

Now suppose $V$ is an open subset of $Z$, $y \in g(\partial V)$, let
\[ G = \{ r : 0 < r \leq \inf\{\rho, \text{distance } [y, g(\partial V)]\} \}, \]
and let $H$ be the subset of $G$ consisting of those points where $\gamma(W_r)$ is differentiable with respect to $r$; clearly $L_1(G - H) = 0$.

If $r \in H$, then $spt \partial \mu(V) \subset R^n - W_r$ and the proof of [FF, 3.9], with $T = \mu(V)$ and omitting all references to $\partial T$, shows (see also [FF, 8.14]) that
\[
\mu(V) \cap W_r \subset I_{k}(\text{Clos } W_r),
\]

\[
\text{spt } \partial[\mu(V) \cap W_r] \subset \text{Bdry } W_r.
\]

Choosing \(f\) according to [FF, 8.19] one obtains

\[
X = f_{\ast}[\mu(V) \cap W_r] \subset I_{k}(\text{Clos } W_r)
\]

with \(\partial X = \partial[\mu(V) \cap W_r]\) and

\[
r^{-k}\|X\|_{(R^n - P)} \leq r^{-k}\|\mu(V)\|_{[W_r - (M \cap W_\nu)]}
\]

\[
\leq r^{-k}\gamma(W_r - (M \cap W_\nu)) < \epsilon < \zeta.
\]

Accordingly [FF, 8.18] yields an integer \(m_r(V)\) for which

\[
M[X - m_r(V) \cdot (P \cap W_r)] \leq \nu\|X\|_{(R^n - P)}
\]

\[
\leq \nu r^{-k}\|\mu\|_{(V \cap g^{-1}[W_r - (M \cap W_\nu)])}.
\]

Letting \(h\) be the linear homotopy from \(f\) to the identity map of \(R^n\), one also finds that

\[
\mu(V) \cap W_r - X = \partial h(I \times [\mu(V) \cap W_r]),
\]

\[
M[h(I \times [\mu(V) \cap W_r])] \leq 2r^{-k}\|\mu(V)\|(W_r),
\]

hence

\[
F[\mu(V) \cap W_r - m_r(V) \cdot (P \cap W_r)] \leq \nu r^{-k}\|\mu\|_{(V \cap g^{-1}[W_r - (M \cap W_\nu)])}
\]

\[
+ 2r^{-k}\|\mu\|_{[V \cap g^{-1}[W_r])} < \epsilon r^k.
\]

Moreover, since \(er^k < \alpha(k)r^k/3 = M(P \cap W_r)/3\), the integer \(m_r(V)\) is uniquely characterized by the preceding inequality.

Next it will be shown that

\[
m_r(V) = m_s(V) \quad \text{whenever } r \in H, s \in H, tr < s < r.
\]

In fact

\[
M[\mu(V) \cap W_r - m_r(V) \cap W_s] \leq \gamma(W_r - W_\nu) < \epsilon r^k,
\]

\[
F[m_r(V) \cdot (P \cap W_r) - m_s(V) \cdot (P \cap W_s)] < 3\epsilon r^k,
\]

and the assumption \(m_r(V) \neq m_s(V)\) would imply that

\[
\alpha(k)s^k = M(P \cap W_s) < 3\epsilon r^k, \quad \alpha(k)t^k/3 < \epsilon.
\]

It is now obvious that \(m_r(V)\) has the same value, say \(m(V)\), for all \(r \in H\). Hence the desired inequality has been proved in case \(r \in H\). By left continuity, it remains valid for all \(r \in G\).

**Part 8.** If the conditions of Part 7 hold, \(V\) is an open subset of \(Z\) and \(C(V, r)\) is the family of components of \(V \cap g^{-1}(W_r)\), then
whenever $0 < r < p$ and $g(\text{Bdry } V) \subset \mathbb{R}^n - W_r$. Furthermore

$$\limsup_{r \to 0^+} \sum_{U \in C(V, r)} |m(U)| \leq \Theta^k(V, \gamma).$$

**Proof.** Since

$$\mu(V) \cap W_r = \mu[V \cap g^{-1}(W_r)] = \sum_{U \in C(V, r)} \mu(U),$$

one finds that

$$F\left[\mu(V) \cap W_r - \sum_{U \in C(V, r)} m(U) \cdot (P \cap W_r)\right]$$

$$\leq \sum_{U \in C(V, r)} (vt^{-k})||M||[U - g^{-1}(M \cap W_r)] + 2rt^{-k}||M|| (U)$$

$$\leq vt^{-k}||M||[V \cap g^{-1}[W_r - (M \cap W_r)] + 2rt^{-k}||M||[V \cap g^{-1}(W_r)] \leq \epsilon r^k.$$

Similarly one obtains

$$\alpha(k) r^k \sum_{U \in C(Z, r)} |m(U)| \leq \sum_{U \in C(Z, r)} (F[\mu(U)] + F[\mu(U) - m(U) \cdot (P \cap W_r)])$$

$$\leq \gamma(W_r) + \epsilon r^k.$$

**Part 9.** If the conditions of Part 7 hold and $P$ is oriented by the simple $k$-vector $\xi$ with $|\xi| = 1$, then for each $z \in g^{-1}\{y\}$ the conclusion of (5) holds with

$$v(z) = \lim_{r \to 0^+} m[\Delta(z, r)] \cdot \xi.$$

Furthermore

$$\Theta^k[\mu(V) \land \phi, y] = \phi(y) \left[\sum_{x \in V \cap g^{-1}\{y\}} v(z)\right]$$

whenever $V$ is an open subset of $Z$, $y \in \text{Bdry}(V)$, $\phi \in E^k(\mathbb{R}^n)$.

**Proof.** One sees from Part 8 that for $0 < r < p$ the number of elements of the set

$$D(r) = C(Z, r) \cap \{U : m(U) \neq 0\}$$

does not decrease as $r$ decreases, and is bounded, hence constant for small $r$, say for $0 < r < \delta$. Moreover for $0 < s < r < \delta$ the relation

$$\{(U, V) : U \subset D(r), V \subset D(s), U \supset V\}$$

is an $m$ preserving univalent map of $D(r)$ onto $D(s)$.

Since $Z$ is compact and $g$ is light, the points of $g^{-1}\{y\}$ constitute the components of
If \( 0 < r < \delta \), each \( U \in D(r) \) contains a unique \( z \in g^{-1}\{y\} \) such that \( m[\Delta(z, s)] = m(U) \) whenever \( 0 < s \leq r \). All other points \( z \in g^{-1}\{y\} \) have the property that \( m[\Delta(z, s)] = 0 \) for all sufficiently small \( s > 0 \).

For \( z \in g^{-1}\{y\} \) and \( \phi \in E^k(\mathbb{R}^n) \) one infers, with the help of Part 7, that

\[
\phi(y) \left( \lim_{r \to 0^+} m[\Delta(z, r)] \right) = \lim_{r \to 0^+} \alpha(k)^{-1} r^{-k} m[\Delta(z, r)] \int_{P \cap W} \phi
\]

Furthermore, if \( V \) is an open subset of \( Z \) and \( y \in g(\text{Bdry } V) \), then

\[
\Theta^k[\mu(V) \wedge \phi, y] = \lim_{r \to 0^+} \alpha(k)^{-1} r^{-k} m(V) \int_{P \cap W} \phi
\]

\[
= \lim_{r \to 0^+} \sum_{U \in C(V, r)} m(U) \alpha(k)^{-1} r^{-k} \int_{P \cap W} \phi
\]

\[
= \sum_{z \in V \cap g^{-1}\{y\}} \lim_{r \to 0^+} m[\Delta(z, r)] \phi(y)(\xi).
\]

**Part 10. If \( Y \) is a Borel subset of \( \mathbb{R}^n \), then**

\[
\int_Y \Theta^k(\gamma, y) dH^k y \leq \gamma(Y).
\]

**Proof.** Apply \([F3, 3.1, 3.2]\) and Parts 4, 6.

**Part 11. Proof of (6).**

Fix \( \phi \in E^k(\mathbb{R}^n) \), let \( C \) be the class of those Borel subsets \( A \) of \( Z \) for which the conclusion of (6) holds, and let \( D \) be the class of those open subsets \( V \) of \( Z \) for which \( \gamma[g(\text{Bdry } V)] = 0 \). Since

\[
\sum_{x \in g^{-1}\{y\}} | v(z) | \leq \Theta^k(\gamma, y)
\]

whenever \( y \in \mathbb{R}^n \), one readily verifies with the help of Part 10 (with \( Y = \mathbb{R}^n \)) that \( C \) is closed to monotone convergence.

For every \( k \) dimensional rectifiable current \( T \) in \( \mathbb{R}^n \) one knows from \([FF, 8.16 (2)]\) that

\[
T(\phi) = \int_{\mathbb{R}^n} \frac{d(T \wedge \phi)}{d||T||} (y) d||T|| y
\]

\[
= \int_{\mathbb{R}^n} \frac{d(T \wedge \phi)}{d||T||} (y) \Theta^k(||T||, y) dH^k y = \int_{\mathbb{R}^n} \Theta^k(T \wedge \phi, y) dH^k y.
\]
Hence Parts 5, 6, 9, 10 (with $\gamma(Y) = 0$) imply that $D \subseteq C$.

If $Y$ is an open subset of $R^n$ for which $\gamma(\text{Bdry } Y) = 0$, then every component of $g^{-1}(Y)$ belongs to $D$. Since $g$ is a light mapping it follows that $D$ contains a base for the topology of $Z$. Moreover $D$ is closed to finite union and intersection.

Accordingly $C$ is the class of all Borel subsets of $Z$.

2.2. Corollary. In case $g$ has the Lipschitz constant $1$ and the diameter of $\Delta(z, r)$ never exceeds $2r$, then the following additional statements hold:

(7) For $H^k$ almost all $z$ in $Z$,
$$|v(z)| = \lim_{r \to 0^+} \alpha(k)^{-1}r^{-k}||\mu||[\Delta(z, r)].$$

(8) If $A$ is a Borel subset of $Z$, then
$$||\mu||(A) = \int _A |v(z)| dH^kz.$$

(9) If $\psi$ is a Baire function on $Z$ such that $\psi(z) = 0$ whenever $v(z) = 0$, then
$$\int _Z \psi(z) dH^kz = \int _{R^n} \sum _{x \in g^{-1}(y)} \psi(z) dH^k y.$$

(10) If $A$ is a Borel subset of $Z$ and $\phi \in E^k(R^n)$, then
$$\mu(A)(\phi) = \int _A \phi[g(z)][v(z)] dH^kz.$$

Proof. For $A \subseteq Z$ and $y \in R^n$ let $N(g, A, y)$ be the number (possibly $\infty$) of elements in $A \cap g^{-1}\{y\}$. Since
$$H^k(B) \geq H^k[g^{-1}(B)] \quad \text{for } B \subseteq Z,$$

it follows from [F1, 4.1] that
$$H^k(A) \geq \int _{R^n} N(g, A, y) dH^k y$$

for every Borel subset $A$ of $Z$, and consequently
$$\int _Z \psi(z) dH^kz \geq \int _{R^n} \sum _{x \in g^{-1}(y)} \psi(z) dH^k y$$

for every nonnegative Baire function $\psi$ on $Z$.

Noting that [F3, 3.1, 3.2] are easily adapted to $Z$, with the spherical balls of $R^n$ replaced by the neighborhoods $\Delta(z, r)$, one sees with the help of (6) that, for every Borel subset $A$ of $Z$,
\[ \|\mu\|(A) \geq \int_A \limsup_{t \to 0^+} \alpha(k)^{-1}r^{-k}\|\mu\|[\Delta(z, r)]dH^p z \]

\[ \geq \int_A \liminf_{t \to 0^+} \alpha(k)^{-1}r^{-k}\|\mu\|[\Delta(z, r)]dH^p z \]

\[ = \int_A |v(z)| dH^p z \]

\[ = \int_{\mathbb{R}^n} \sum_{z \in A \cap \mathbb{R}^n} |v(z)| dH^p y \]

\[ \geq \|\mu\|(A). \]

Therefore (7) and (8) are proved, and the equation

\[ H^k(A) = \int_{\mathbb{R}^n} N(g, A, y) dH^k y \]

holds in case \(|v(z)|\) equals a fixed positive integer for all \(z \in A\). Then (9) follows readily, and (10) is a consequence of (6) and (9).

3. The convergence property. Suppose \(X\) is a compact oriented \(k\) dimensional manifold of class \(\infty\).

To each map \(f: X \to \mathbb{R}^n\) of class \(\infty\) corresponds a countably additive function which associates with each Borel subset \(B\) of \(X\) the \(k\) dimensional rectifiable current \(f_*(B)\); here

\[ f_*(B)(\phi) = \int_B f^*(\phi) \quad \text{for} \quad \phi \in E^k(\mathbb{R}^n). \]

One readily verifies that the \(M\) variation of this countably additive function over \(B\) equals the classical area integral of \(f\mid B\).

It will be shown how these concepts can be extended to continuous maps \(f: X \to \mathbb{R}^n\) of finite Lebesgue area, at least in case \(H^{k+1}[f(X)] = 0\) or \(k = 2\).

3.1. Definition. Every continuous map \(f: X \to \mathbb{R}^n\) has a monotone-light factorization

\[ f = l_f \circ m_f, \quad m_f: X \to M_f, \quad l_f: M_f \to \mathbb{R}^n \]

whose middle space \(M_f\) consists of the maximal continua of constancy of \(f\); the distance between two points \(\xi\) and \(\eta\) of \(M_f\) is

\[ d_f(\xi, \eta) = \inf \left\{ \text{diam} f(C) : C \text{ is a continuum containing } \xi \cup \eta \right\}; \]

this metric \(d_f\) and the identification map \(m_f\) induce the same topology.

Assuming that \(f\) has finite Lebesgue area, consider any sequence of maps \(f_i: X \to \mathbb{R}^n\) of class \(\infty\) which converge uniformly to \(f\) and whose areas are bounded; these areas need not converge to the Lebesgue area of \(f\). Let \(\mu_i\) be
the countably additive function associating with each Borel subset $A$ of $M_f$ the rectifiable current

$$f_d[m_f^{-1}(A)] \in E_k(R^n).$$

We say that $f$ has the convergence property if and only if for every such choice of approximating maps $f_i$, the corresponding sequence of measures $\mu_i$ over $M_f$ is weakly convergent; this means that the sequence of numbers

$$\mu_i(\chi)(\phi) = \int_X (\chi \circ m_i) \wedge f_i(\phi)$$

is convergent whenever $\phi \in E^k(R^n)$ and $\chi$ is a continuous real-valued function on $M_f$; the limit then equals

$$\mu(\chi)(\phi)$$

where $\mu$ is a Borel measure over $M_f$ with values in $E_k(R^n)$.

In case $f$ has the convergence property, the weak limit is clearly independent of the choice of approximating maps $f_i$, and we shall refer to $\mu$ as the limit measure corresponding to $f$.

3.2. Lemma. If $f : X \rightarrow R^n$ is of class $\infty$, $u : R^n \rightarrow R$ has Lipschitz constant $\lambda$, and $C(s)$ is the set of components of

$$\{x : (u \circ f)(x) < s\}$$

whenever $s \in R$, then

$$\int_{-\infty}^{\infty} \sum_{V \in C(s)} M[\partial f_i(V)] \, ds \leq \lambda \text{ area } (f).$$

Proof. Let

$$\gamma(s) = \text{area}(f | \{x : (u \circ f)(x) < s\})$$

for $s \in R$

note that

$$\int_{-\infty}^{\infty} \gamma'(s) \, ds \leq \text{area}(f),$$

and consider a real number $s$ for which $\gamma'(s) < \infty$. For each $V \in C(s)$ one may apply [FF, 3.9] with

$$T = f_i(V) = f_i(V) \cap \{y : u(y) < s\},$$

$$\text{spt } \partial T \subset f(Bdry V) \subset \{y : u(y) = s\}$$

to obtain

$$M[\partial f_i(V)] \leq \lambda \liminf_{h \rightarrow 0+} h^{-1} ||f_i(V)|| (\{y : s - h \leq y < s\}).$$
It follows that
\[
\sum_{V \in \mathcal{C}(x)} M[\partial f_t(V)] \leq \lambda \liminf_{h \to 0^+} h^{-1} \sum_{V \in \mathcal{C}(x)} \left[ f_t(V) \left( \{ y : s - h \leq u(y) < s \} \right) \right] \\
\leq \lambda \liminf_{h \to 0^+} h^{-1} \text{area}[\{ x : s - h \leq (u \circ f)(x) < s \}] \\
= \lambda \gamma'(s).
\]

3.3. Remark. If \( \psi \) is a continuous real-valued function on \( X \), then all but countably many real numbers \( s \) have the following property:

No component of \( \{ x : \psi(x) \leq s \} \) contains two distinct components of \( \{ x : \psi(x) < s \} \).

In fact, choosing \( b \) in a countable dense subset of \( X \), let \( B(s) \) and \( D(s) \) be the components of \( b \) in \( \{ x : \psi(x) < s \} \) and \( \{ x : \psi(x) \leq s \} \) whenever \( \psi(b) < s \). Since the sets \( D(s) - B(s) \) form a disjointed family, all but countably many have an empty interior, in which case \( B(s) \) is the only component of \( \{ x : \psi(x) < s \} \) contained in \( D(s) \).

3.4. Theorem. If \( f : X \to \mathbb{R}^n \) is a continuous map and \( f_1, f_2, f_3, \ldots \) are maps of class \( \infty \) which converge uniformly to \( f \) and whose areas are bounded, then the corresponding sequence \( \mu_1, \mu_2, \mu_3, \ldots \) of measures over \( M_f \) has a weakly convergent subsequence, whose limit measure \( \mu \) satisfies the conditions of 2.1 and 2.2 with \( Z = M_f \) and \( g = \lambda f \).

Proof. By Cantor's diagonal process the given sequence may be replaced by a subsequence such that
\[
\lim_{i \to \infty} \mu_i(\phi)
\]
exists for countable \( M \) dense sets of forms \( \phi \in \mathcal{B}(\mathbb{R}^n) \) and continuous real-valued functions \( \chi \) on \( M_f \). Then weak convergence follows because the total \( M \) variations of \( \mu_i \) do not exceed the areas of \( f_i \), which are bounded.

Consider also the measures \( \gamma_i \) over \( X \) defined by
\[
\gamma_i(B) = \text{area}(f_i | B)
\]
for every Borel subset \( B \) of \( X \). After passing once more to a subsequence, one may assume that the sequence \( \gamma_1, \gamma_2, \gamma_3, \ldots \) is weakly convergent to a Borel measure \( \gamma \) over \( X \).

Inasmuch as
\[
\text{spt} \mu_i(A) \subset f_i[\text{spt} m^{-1}_f(A)] \\
\subset f_i[\text{Clos} m^{-1}_f(A)] \subset f_i[m^{-1}_f(\text{Clos} A)], \\
\text{spt} \partial \mu_i(A) \subset f_i[\text{spt} \partial m^{-1}_f(A)] \\
\subset f_i[\text{Bdry} m^{-1}_f(A)] \subset f_i[m^{-1}_f(\text{Bdry} A)],
\]

for all \( i \) and every Borel subset \( A \) of \( M_f \), one readily verifies all but one of the conditions of 2.1 and 2.2; the only real problem is to show that \( \mu(A) \) is a rectifiable current.

Let \( F \) be the family of those Borel subsets \( A \) of \( M_f \) for which the current \( \mu(A) \) is rectifiable. Obviously \( F \) is closed to countable disjoint union, and to proper subtraction.

It will be shown that if \( u : \mathbb{R}^n \to \mathbb{R} \) is Lipschitzian, then for \( L_1 \) almost all real numbers \( s \), each component of

\[
Z(s) = M_f \cap \{ z : (u \circ l_i)(z) < s \}
\]

belongs to \( F \). Letting \( C_i(s) \) be the set of components of

\[
X \cap \{ x : (u \circ f_i)(x) < s \}
\]

one sees from 3.2 that

\[
\sup_i \int_{-\infty}^{\infty} \sum_{V \in C_i(s)} M[\partial f_s(V)] ds < \infty
\]

and infers from Fatou’s lemma that

\[
\liminf_{i \to \infty} \sum_{V \in C_i(s)} M[\partial f_s(V)] < \infty
\]

for \( L_1 \) almost all \( s \). For all but countably many real numbers \( s \) it is also true that

\[
\gamma(\{ x : (u \circ f)(x) = s \}) = 0
\]

and that the property of 3.3 holds with \( \psi = u \circ f \).

Now suppose \( s \) is a real number satisfying these three conditions, \( A \) is a component of \( Z(s) \), \( b \in B = m_f^{-1}(A) \), \( D \) is the component of \( b \) in \( \{ x : (u \circ f)(x) \leq s \} \), and \( b \in V_i \in C_i(s) \) for \( i \) = 1, 2, 3, \ldots . Then

\[
m_f^{-1}(\text{Bdry } A) \subset \{ x : (u \circ f)(x) = s \},
\]

\[
\| \mu \|(\text{Bdry } A) \leq \gamma[m_f^{-1}(\text{Bdry } A)] = 0,
\]

\[
\mu(A) = \lim_{i \to \infty} f_s(B),
\]

\[
B \subset \bigcup_{i=1}^{\infty} \text{Int} \cap_{i=1}^{\infty} V_i,
\]

\[
\bigcap_{j=1}^{\infty} \text{Clos} \bigcup_{i=j}^{\infty} V_i \subset D \subset B \bigcup \{ x : (u \circ f)(x) = s \},
\]

hence \( \epsilon > 0 \) implies that

\[
(B - V_i) \cup (V_i - B) \subset \{ x : |(u \circ f)(x) - s| < \epsilon \}
\]

for large \( i \), and one obtains
\[ \limsup_{t \to \infty} \mathcal{M}[f_d(V_t) - f_d(B)] \leq \limsup_{t \to \infty} \gamma_t[(V_t - B) \cup (B - V_t)] = 0, \]

\[ \liminf_{t \to \infty} \mathcal{N}[f_d(V_t)] < \infty, \]

\[ \mu(A) = \lim_{t \to \infty} f_d(V_t) \in I_b(R^n) \]

by [FF, 8.14, 8.13], hence \( A \subseteq F \).

For each \( a \in R^n \) one may consider the function \( u \) defined by

\[ u(y) = \sup\{|y_i - a_i| : i = 1, 2, \ldots, n\} \]

for \( y \in R^n \). One finds that, for almost every \( n \) dimensional cube \( W \) in \( R^n \), each component of \( l^\perp(W) \) belongs to \( F \). Since \( l^\perp \) is light, approximation by finite sums of such components shows that every open subset of \( M_f \) belongs to \( F \). One concludes that \( F \) is in the class of all Borel subsets of \( M_f \).

3.5. **Corollary.** If \( f : X \to R^n \) is a continuous map of finite Lebesgue area with the convergence property, then the limit measure \( \mu \) corresponding to \( f \) satisfies the conditions of 2.1 and 2.2 with \( Z = M_f \) and \( g = l_f \).

3.6. **Corollary.** If \( f : X \to R^k \) is a continuous map of finite Lebesgue area, then \( f \) has the convergence property.

**Proof.** It is sufficient to prove that

\[ f_1, f_2, f_3, \ldots \quad \text{and} \quad g_1, g_2, g_3, \ldots \]

are two sequences of maps of class \( \infty \) which converge uniformly to \( f \) and whose areas are bounded, and if the corresponding sequences

\[ \mu_1, \mu_2, \mu_3, \ldots, \nu_1, \nu_2, \nu_3, \ldots \]

of measures over \( M_f \) are weakly convergent to \( \mu \) and \( \nu \) respectively, then \( \mu = \nu \).

Almost every \( k \) dimensional cube \( W \) in \( R^k \) has the property that

\[ \|\mu\|[l^{-1}_f(\text{Bdry } W)] = 0 = \|\nu\|[l^{-1}_g(\text{Bdry } W)]. \]

If \( A \) is a component of \( l^{-1}_f(W) \) and \( B = m^{-1}_f(A) \), then

\[ \mu(A) = \lim_{t \to \infty} f_d(B), \quad \nu(A) = \lim_{t \to \infty} g_d(B). \]

Letting \( h_{i,j} \) be the linear homotopy from \( f_i \) to \( g_j \) one obtains

\[ g_d(B) - f_d(B) = h_{i,j}(I \times \partial B) \]

because \( E_{k+1}(R^k) = \{0\} \). It follows that

\[ \nu(A) - \mu(A) = \lim_{i,j \to \infty} h_{i,j}(I \times \partial B) \]
is a $k$ dimensional rectifiable current with support in $\text{Bdry } W$, hence equals 0.

3.7. **Lemma.** Suppose $f: X \to \mathbb{R}^n$ is a continuous map of finite Lebesgue area and either $\mathcal{H}^{k+1}[f(X)] = 0$ or $k = 2$. If $a \in \mathbb{R}^n$, then almost all orthogonal projections $p$ of $\mathbb{R}^n$ onto $\mathbb{R}^k$ have the following property:

$X$ contains no continuum $C$ such that $a \in f(C)$, $f$ is not constant on $C$, $p \circ f$ is constant on $C$.

**Proof.** Assume $a = 0$ and consider three cases:

**Case 1.** $\mathcal{H}^{k+1}[f(X)] = 0$.

For $r > 0$, let $A(r) = f(X) \cap \{y: |y| = r\}$. The Eilenberg inequality ([E] or [F, 3.2]) allows one to choose a sequence of numbers $r_1 > r_2 > r_3 > \cdots$ with limit 0 such that $\mathcal{H}^k[A(r_i)] = 0$ for $i = 1, 2, 3, \cdots$. Letting $S_i$ be the set of those $n - k$ dimensional planes in $\mathbb{R}^n$ which meet $A(r_i)$, one sees from [F3, 7.5] that $S_i$ has Haar measure 0. Moreover, if $p$ is an orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^k$ such that $X$ contains a continuum $C$ with $0 \in f(C)$, $f(C) \neq \{0\}$, then the kernel of $p$ belongs to $S_i$ for large $i$.

**Case 2.** $k = 2$ and $n = 3$. Let

$$S = \mathbb{R}^3 \cap \{w: |w| = 1\},$$

$$Q: \mathbb{R}^3 \to \mathbb{R}^2, \quad Q(y) = (y_1, y_2) \quad \text{for } y \in \mathbb{R}^3,$$

$$g: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3,$$

$$g(y) = (y_1/|y|, y_2/|y|, y_3/|y|) \quad \text{for } y \in \mathbb{R}^3,$$

choose finitely triangulable sets $X_1 \subset X_2 \subset X_3 \subset \cdots$ such that

$$\bigcup_{i=1}^{\infty} X_i = \mathbb{R}^3 - f^{-1}\{0\},$$

and let $U_j$ be the set of those $u \in \mathbb{R}^3$ for which $X_i$ contains a continuum $D$ such that $(g \circ f)(D)$ is nondegenerate and $(Q \circ g \circ f)(D) = \{u\}$. Replacing [F6, 8.10] by [DF, 4.1] in the proof of [F6, 8.11] one sees that $L_3(U_j) = 0$, hence

$$\mathcal{H}^2[S \cap Q^{-1}(U_j)] = 0.$$ 

Now observe that if $p$ is an orthogonal projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$ for which the conclusion of the lemma fails, and if

$$w \in S \cap \text{kernel } p,$$

then $X$ contains a continuum $C$ such that

$$0 \in f(C), \quad \{0\} \neq f(C) \subset \{tw: t \in \mathbb{R}\}.$$ 

Therefore
\[ m_f(C) = m_f(C \cap f^{-1}\{0\}) \cup \bigcup_{j=1}^{\infty} m_f(C \cap X_j) \]

is a nondegenerate continuum, while
\[ m_f(C \cap f^{-1}\{0\}) \subset f^{-1}\{0\} \]

is totally disconnected, hence [HW, Theorem 2.2] yields a positive integer \( j \) for which
\[ \dim[m_f(C \cap X_j)] > 0. \]

Choosing a continuum \( D \subseteq C \cap X_j \) such that \( f \) is not constant on \( D \), one finds that
\[ (g \circ f)(D) \]

is nondegenerate,
\[ (Q \circ g \circ f)(D) = \{ \pm Q(w) \}, \]
\[ w \in S \cap Q^{-1}(U_j) \text{ or } -w \in S \cap Q^{-1}(U_j). \]

Accordingly the set of all such points \( w \) has \( H^2 \) measure 0.

**Case 3.** \( k = 2 \) and \( n > 3 \).

Let \( G_n \) and \( G_3 \) be the orthogonal groups of \( R^n \) and \( R^3 \), consider the orthogonal projections
\[ P: R^n \to R^2, \quad P(y) = (y_1, y_2) \text{ for } y \in R^n, \]
\[ Q: R^3 \to R^2, \quad Q(w) = (w_1, w_2) \text{ for } w \in R^3, \]
\[ S_i: R^n \to R^3, \quad S_i(y) = (y_1, y_2, y_i) \text{ for } y \in R^n, \]

corresponding to \( i = 3, \ldots, n \) and let \( K_i \) be the set of all those \( g \in G_n \) for which there exists a continuum \( C \subseteq X \) such that
\[ 0 \in f(C), \quad (S_i \circ g \circ f)(C) \neq \{0\}, \quad (P \circ g \circ f)(C) = \{0\}. \]

Inasmuch as
\[ \bigcup_{i=3}^{n} \{ P \circ g : g \in K_i \} \]

is the class of those orthogonal projections \( P \) of \( R^n \) onto \( R^2 \) for which the conclusion of the lemma fails, it is sufficient to prove that each \( K_i \) has Haar measure 0.

Fix \( i \), let \( u \) be the characteristic function of \( K_i \), and with each \( h \in G_3 \) associate \( \rho(h) \in G_n \) so that
\[ S_i \circ \rho(h) = h \circ S_i, \quad \rho(h)(y) = y \text{ for } y \in \text{kernel } S_i. \]

Integrating with respect to Haar measures over \( G_n \) and \( G_3 \) one obtains
\[ \int_{a_n} u(g) \, dg = \int_{a_n} \int_{a_n} u[\rho(h) \circ g] \, dg \, dh = \int_{a_n} \int_{a_n} u[\rho(h) \circ g] \, dh \, dg = 0 \]

because for each \( g \in G_n \) one may apply Case 2 to the map \( S_i \circ g \circ f \), taking account of the fact that

\[
(Q \circ h) \circ (S_i \circ g \circ f) = P \circ [\rho(h) \circ g] \circ f \quad \text{for} \ h \in G_2.
\]

### 3.8. Corollary

If the conditions of 3.7 hold and \( \gamma \) is a Radon measure over \( M_f \), then

\[
\gamma(M_f - M_{p \circ f}) = 0
\]

for almost all orthogonal projections \( p \) of \( R^n \) onto \( R^k \).

**Proof.** Note that

\[
S = \{(z, p) : z \in M_f - M_{p \circ f}\}
\]

is a Borel set of type \( F \). Since, for each \( z \in M_f \),

\[
\{p : (z, p) \in S\} \text{ has Haar measure } 0,
\]

by 3.7, the Fubini theorem implies that, for almost all \( p \),

\[
\gamma(\{z : (z, p) \in S\}) = 0.
\]

### 3.9. Theorem

If \( f : X \to R^n \) is a continuous map of finite Lebesgue area and either \( H^{k+1}(f(X)) = 0 \) or \( k = 2 \), then \( f \) has the convergence property.

**Proof.** In view of 3.4 it suffices to prove that if

\[
f_1, f_2, f_3, \ldots \quad \text{and} \quad g_1, g_2, g_3, \ldots
\]

are two sequences of maps of class \( \infty \) which converge uniformly to \( f \) and whose areas are bounded, and if the corresponding sequences

\[
\mu_1, \mu_2, \mu_3, \ldots \quad \text{and} \quad \nu_1, \nu_2, \nu_3, \ldots
\]

of measures over \( M_f \) are weakly convergent to \( \mu \) and \( \nu \) respectively, then \( \mu = \nu \).

According to 3.8 almost every orthogonal projection \( p \) of \( R^k \) onto \( R^k \) has the property that

\[
||\mu||(M_f - M_{p \circ f}) = 0 = ||\nu||(M_f - M_{p \circ f}).
\]

Factoring \( m_{p \circ f} = h \circ m_f \), where

\[
h : M_f \to M_{p \circ f}, \quad z \subset h(z) \in M_{p \circ f} \quad \text{for} \ z \in M_f,
\]

one infers from 3.6 (applied to \( p \circ f \)) that if \( \omega \in E^k(R^k) \) and \( \zeta \) is a real valued continuous function on \( M_{p \circ f} \), then

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\[
\mu(\xi \circ h) [\psi(\omega)] = \lim_{i \to \infty} \mu_i(\xi \circ h) [\psi(\omega)]
\]
\[
= \lim_{i \to \infty} \int_X (\xi \circ h \circ m_i) \wedge f_i [\psi(\omega)]
\]
\[
= \lim_{i \to \infty} \int_X (\xi \circ m_{p \circ f_i}) \wedge (p \circ f_i) [\psi(\omega)]
\]
\[
= \lim_{j \to \infty} \int_X (\xi \circ m_{p \circ j}) \wedge (p \circ g_j) [\psi(\omega)]
\]
\[
= \lim_{j \to \infty} \nu_j(\xi \circ h) [\psi(\omega)] = \nu(\xi \circ h) [\psi(\omega)].
\]

It follows that the equation
\[
\mu(\xi \circ h) [\psi(\omega)] = \nu(\xi \circ h) [\psi(\omega)]
\]
holds also in case \( \xi \) is a real valued bounded Baire function on \( M_{p \circ f} \), and in particular
\[
\mu[\nu^{-1}(B)] [\psi(\omega)] = \nu[\nu^{-1}(B)] [\psi(\omega)]
\]
for every Borel subset \( B \) of \( M_{p \circ f} \). Now, if \( A \) is any closed subset of \( M_f \), then
\[
(\nu^{-1}(A)) A \subset M_f - M_{p \circ f},
\]
\[
\mu(A) [\psi(\omega)] = \mu(\nu^{-1}(A)) [\psi(\omega)] = \nu(\nu^{-1}(A)) [\psi(\omega)] = \nu(A) [\psi(\omega),
\]
and consequently
\[
\mu(\chi) [\psi(\omega)] = \nu(\chi) [\psi(\omega)]
\]
for every real valued bounded Baire function \( \chi \) on \( M_f \). Furthermore one sees from 2.2 (10) that if \( \psi \in E^0(R^a) \), then
\[
\mu(\chi) [\psi \wedge \psi(\omega)] = \mu(\chi \circ (\psi \circ l_i)) [\psi(\omega)] = \nu(\chi \circ (\psi \circ l_i)) [\psi(\omega)] = \nu(\chi) [\psi \wedge \psi(\omega)].
\]
Observing that \( E^*(R^a) \) consists of finite sums of such forms \( \psi \wedge \psi(\omega) \), one finally obtains
\[
\mu(\chi)(\phi) = \nu(\chi)(\phi) \quad \text{for } \phi \in E^*(R^a).
\]

3.10. Remark. The preceding theorem remains true without the assumption that \( X \) is compact, provided \( f \) is proper \([f^{-1}(Y) \text{ is compact for every compact } Y \subset R^a] \). If the maps \( f_i \) converge to \( f \), uniformly on each compact subset of \( X \), then
\[
\lim_{i \to \infty} \mu_i(\chi)
\]
exists for every continuous \( \chi: M_f \to \mathbb{R} \) with compact support.

To prove this choose \( u: X \to \mathbb{R} \) of class \( \infty \) so that
\[
\text{spt}(\chi \circ m_f) \subset \{ x: u(x) > 0 \},
\]
\[
\text{Clos} \{ x: u(x) > 0 \} \text{ is compact,}
\]
\[
du(x) = 0 \text{ whenever } u(x) = 0.
\]
By doubling \( \{ x: u(x) > 0 \} \) with respect to \( \{ x: u(x) = 0 \} \) one obtains the compact manifold
\[
Q = (X \times \mathbb{R}) \cap \{ (x, y): u(x) = y^3 \}
\]
of class \( \infty \), and the maps
\[
\xi: Q \to X, \quad \xi(x, y) = x,
\]
\[
\eta: \{ x: u(x) > 0 \} \to Q, \quad \eta(x) = (x, u(x)^{1/2}),
\]
\[
g = f \circ \xi \quad \text{and} \quad g_i = f_i \circ \xi: Q \to \mathbb{R}^n.
\]
Moreover there exists a continuous \( \psi: M_f \to \mathbb{R} \) such that
\[
\text{spt}(\psi \circ m_f) \subset \text{range } \eta,
\]
\[
\psi \circ m_f \circ \eta = \chi \circ m_f \mid \{ x: u(x) > 0 \},
\]
and 3.9 implies for each \( \phi \in E^k(\mathbb{R}^n) \) the existence of
\[
\lim_{t \to +} \int_Q (\psi \circ m_f) \wedge \tilde{g}_i(\phi) = \lim_{t \to +} \int_{\{ x: u(x) > 0 \}} \tilde{g}_i(\psi \circ m_f) \wedge \tilde{g}_i(\phi)
\]
\[
= \lim_{t \to +} \int_X (\chi \circ m_f) \wedge f_i(\phi).
\]

The restriction that \( \chi \) have compact support is essential, as seen from the example where \( f \) maps an open circular disc conformally onto a plane region bounded by a simple closed curve with positive \( L_2 \) measure.

3.11. Remark. It is an open question whether Theorem 3.9 remains true without the assumption that \( H^{k+1}[f(X)] = 0 \) or \( k = 2 \); certainly Lemma 3.7 becomes false, as seen from the following simple example: Let \( u \) be a continuous map of \( I = \{ t: 0 \leq t \leq 1 \} \) onto \( R^4 \cap \{ y: |y| \leq 2 \} \), and define
\[
f: I \times I \times I \to R^4, \quad f(x_1, x_2, x_3) = x_2u(x_1).
\]
Then \( L_2(f) = 0 \), but \( f(\{ t \} \times I \times I) \) is the line segment from 0 to \( u(t) \).

A slightly more complicated example shows that in case \( k > 2 \) the sets \( (\psi \circ l_f)(M_f - M_f \circ f) \) can have interior points for (almost) all orthogonal projections of \( R^4 \) onto \( R^k \). Let \( u \) be as above, choose \( c \in R^4 \) with \( |c| = 1 \), and define
\[
f: I \times I \times I \to R^4, \quad f(x_1, x_2, x_3) = u(x_1) + x_2u[c \bullet u(x_1)].
\]
Again $L_A(f) = 0$, yet if $|c \bullet u(s)| > 0$, then

$$W_* = \mathbb{R}^4 \cap \left\{ w : \left| w + \frac{s - c \bullet w}{c \bullet u(s)} u(s) \right| < 2 \right\}$$

is open and nonempty, because $s c \in W_*$, and for each $w \in W_*$ there exists a $t \in I$ such that

$$u(t) = w + \frac{s - c \bullet w}{c \bullet u(s)} u(s),$$

hence $c \bullet u(t) = s$ and $f(\{t\} \times I \times I)$ is a segment of length $|u(s)|$ on the straight line through $w$ in the direction of $u(s)$.

4. The additivity of Lebesgue area. Suppose $f : \mathbb{X} \to \mathbb{R}^n$ is a continuous map of finite Lebesgue area for which either $H^{k+1}[f(\mathbb{X})] = 0$ or $k = 2$, $\mu$ is the limit measure corresponding to $f$, and $v$ is as in 2.2 and 2.3 (with $Z = M_f$, $g = l_2$).

For each finitely triangulable subset $T$ of $\mathbb{X}$, let $L_k(f|T)$ be the Lebesgue area of $f|T$. For each open subset $U$ of $\mathbb{X}$, let $L_k(f|U)$ be the supremum of $L_k(f|T)$ over all finitely triangulable subsets $T$ of $U$.

The purpose of this section is to establish the precise relation (4.3, 4.9) between $\mu$ and $L_k$.

4.1. Lemma. If $n = k$ and $A$ is an open subset of $M_f$, then

$$\|\mu\|(A) = L_k[f|_f^{-1}(A)].$$

Proof. If $W$ is a $k$ dimensional open cube in $\mathbb{R}^k$,

$$\|\mu\|[f^{-1}(\text{Bdry } W)] = 0,$$

$V$ is a component of $f^{-1}(W)$, $U = f^{-1}(V)$, then obviously

$$\mu(V) = \text{degree} (f|U) \cdot W,$$

where the degree is obtained from the induced homomorphism

$$f^*: H^k(\mathbb{R}^k, \mathbb{R}^k - W) \to H^k(\mathbb{X}, \mathbb{X} - U)$$

of integral Čech cohomology groups. This formula implies that

$$M[\mu(V)] = |\text{degree}(f|U)| \cdot L_k(W),$$

and the proof of the lemma may be completed by reference to [F4, §4] and [F8, 7.3].

4.2. Lemma. Almost all orthogonal projections $p$ of $\mathbb{R}^n$ onto $\mathbb{R}^k$ have the property that

$$\int_A |p[v(z)]| d\mu^k z = L_k[p \circ f|_f^{-1}(A)].$$
whenever \( A \) is an open subset of \( M_f \).

**Proof.** Recalling 3.8 assume \( \|\mu\|((M_f - M_{p \circ f})) = 0 \), factor \( m_{p \circ f} = h \circ m_f \) as in 3.9, and note that the limit measure corresponding to \( p \circ f \) is \( h_\#(p \circ \mu) \). Since \( h \) is univalent except on \( M_f - M_{p \circ f} \), it follows from 4.1 that

\[
L_h[\rho \circ f \mid m_{p \circ f}^{-1}(B)] = \|h_\#(p \circ \mu)\|(B) = \|\rho \circ \mu\|_{h^{-1}(B)} = \int_{h^{-1}(B)} |p \circ v| \, dH^k
\]

for every open subset \( B \) of \( M_{p \circ f} \).

First taking \( B = M_{p \circ f} - h(M_f - A) \) one obtains

\[
h^{-1}(B) \subset A, \quad A - h^{-1}(B) \subset M_f - M_{p \circ f},
\]

\[
\int_A |p \circ v| \, dH^k = \int_{h^{-1}(B)} |p \circ v| \, dH^k = L_h[\rho \circ f \mid m_{p \circ f}^{-1}(B)] \leq L_h[\rho \circ f \mid m_f^{-1}(A)].
\]

Next suppose \( T \) is a finitely triangulable subset of \( m_f^{-1}(A) \) and choose open subsets \( B_1 \supset B_2 \supset B_3 \supset \cdots \) of \( M_{p \circ f} \) such that

\[
\bigcap_{i=1}^\infty B_i = m_{p \circ f}(T).
\]

Then

\[
\bigcap_{i=1}^\infty h^{-1}(B_i) = (h^{-1} \circ m_{p \circ f})(T) \subset (h^{-1} \circ h)(A) \subset A \cup (M_f - M_{p \circ f}),
\]

\[
L_h[\rho \circ f \mid T] \leq \lim_{i \to \infty} L_h[\rho \circ f \mid m_{p \circ f}^{-1}(B_i)] = \lim_{i \to \infty} \int_{h^{-1}(B_i)} |p \circ v| \, dH^k \leq \int_A |p \circ v| \, dH^k.
\]

**4.3. Theorem.** If \( A \) is an open subset of \( M_f \), then

\[
\|\mu\|(A) = \beta(n, k)^{-1} \int_{G_n} L_h[P \circ \rho \circ f \mid m_f^{-1}(A)] \, d\phi_n,\]

where \( P \) is an orthogonal projection of \( R^n \) onto \( R^k \), \( G_n \) is the orthogonal group of \( R^n \) with the Haar measure \( \phi_n \) such that \( \phi_n(G_n) = 1 \), and

\[
\beta(n, k) = \alpha(k)\alpha(n - k)^{-1}\binom{n}{k}^{-1}.
\]

**Proof.** Computing the above integral by means of 4.2, Fubini's theorem,
[F2, 4.4, 5.4] and 2.2 one obtains
\[
\int_{\partial A} \int_{A} |(P \circ \rho \circ v)(z)| \, dH^p = \int_{A} \int_{\partial A} |(P \circ \rho \circ v)(z)| \, d\phi_{n}dH^p \rho
\]
\[
= \int_{A} \beta(n, k) |v(z)| \, dH^p = \beta(n, k) ||\mu|| (A).
\]

4.4. Corollary. \(|\mu||A) \leq L_k[f \,| \, m_f^{-1}(A)\).

Proof. It is known from [F5, §6] or [F8, §6, §7] that the right member of the first equation in 4.3 does not exceed \(L_k[f \,| \, m_f^{-1}(A)\).

4.5. Lemma. If \(A\) is an open subset of \(M_f\), \(B\) is a Borel subset of \(A \cap \{z: v(z) \neq 0\}\), and \(\gamma\) is a simple \(k\)-vector of \(R^n\), then
\[
L_k[f \,| \, m_f^{-1}(A)\]
\]
\[
\leq ||\mu|| (A) + \left[\left(\begin{array}{c} n \\ k \end{array}\right) - 1\right]\left(||\mu|| (A - B) + \int_{B} \frac{v(z)}{|v(z)|} - \gamma |d||\mu|| z\right).
\]

Proof. Recalling the notation of [FF, 8.1] one infers from [DF, 3.16, 5.7] and 4.2 that, for almost all orthogonal transformations \(g\) of \(R^n\),
\[
L_k[f \,| \, m_f^{-1}(A)\] = \sum_{\lambda \in \Lambda(k,n)} L_k[\varphi_{\lambda} \circ (g)(m_f^{-1}(A))]
\]
\[
= \sum_{\lambda \in \Lambda(k,n)} \int_{A} |(\varphi_{\lambda} \circ g)(v(z))| \, dH^p.
\]
The resulting inequality
\[
L_k[f \,| \, m_f^{-1}(A)\]
\]
\[
\leq \sum_{\lambda \in \Lambda(k,n)} \int_{A} |(\varphi_{\lambda} \circ g)(v(z))| \, dH^p
\]
holds, by continuity, for every orthogonal transformation \(g\) of \(R^n\). Choosing \(g\) so that
\[
\varphi_{\lambda} [g(\gamma)] = 0 \quad \text{whenever} \quad \lambda \in \Lambda(k,n) - \{(1, \cdots, k)\},
\]
one completes the proof by observing that
\[
\int_{A} |(\varphi_{(1,\cdots,k)} \circ g)(v(z))| \, dH^p \leq \int_{A} |v(z)| \, dH^p = ||\mu|| (A)
\]
and that, for \(\lambda \in \Lambda(k,n) - \{(1, \cdots, k)\},
\[
\int_{A-B} |(\varphi_{\lambda} \circ g)(v(z))| \, dH^p \leq \int_{A-B} |v(z)| \, dH^p = ||\mu|| (A - B),
\]
\[
\int_B |(p^\circ g)[v(z)]| \, dH^2z = \int_B \left| (p^\circ g) \left[ \frac{v(z)}{|v(z)|} - \gamma \right] \right| \cdot |v(z)| \, dH^2z \\
\leq \int_B \left| \frac{v(z)}{|v(z)|} - \gamma \right| \cdot d|\mu||z.
\]

4.6. **Lemma.** For every \(\delta > 0\) there is a closed subset \(Y\) of \(R^n\) such that
\[
|\mu| \left[ l_f^{-1}(Y) \right] = 0
\]
and, if \(\mathcal{Z}\) is the set of components of \(M_f - l_f^{-1}(Y)\), then \(\xi(U) \subset U\) may be associated with \(U \in \mathcal{Z}\) so that \(v[\xi(U)]\) is simple and
\[
\sum_{U \in \mathcal{Z}} \int_{U \cap \{z: v(z) \text{ is simple and } \neq 0\}} \left| \frac{v(z)}{|v(z)|} - \frac{v[\xi(U)]}{|v[\xi(U)]|} \right| d|\mu||z < \delta.
\]

**Proof.** Let \(P = \{z: v(z) \text{ is simple and } \neq 0\}\) and \(j = v/|v| : P \rightarrow \mathbf{A}_k(R^n)\). Since \(j\) is \(|\mu|\) summable over \(P\), there exists a continuous \(w: M_f \rightarrow \mathbf{A}_k(R^n)\) for which
\[
\int_P |j - w| \, d|\mu| < \delta/3.
\]
Using the lightness of \(l_f\) one may then construct \(Y\), as the union of finitely many \(n-1\) dimensional planes in \(R^n\), so that \(|\mu| \left[ l_f^{-1}(Y) \right] = 0\) and the oscillation of \(w\) on each member of \(\mathcal{Z}\) is less than \(\delta/3|\mu|(P)\).

For each \(U \in \mathcal{Z}\) select \(\xi(U) \subset U \cap P\) so that
\[
|w[\xi(U)] - j[\xi(U)]| \cdot |\mu||(U \cap P) \leq \int_{U \cap P} |w - j| \, d|\mu|,
\]
and observe that
\[
\int_{U \cap P} |j(z) - j[\xi(U)]| \, d|\mu||z
\]
\[
\leq \int_{U \cap P} |j - w| \, d|\mu| + \int_{U \cap P} |w(z) - w[\xi(U)]| \, d|\mu||z
\]
\[
+ \int_{U \cap P} |w[\xi(U)] - j[\xi(U)]| \, d|\mu||z
\]
\[
\leq 2 \int_{U \cap P} |j - w| \, d|\mu| + \delta|\mu||(U \cap P)/3|\mu|(P).
\]

4.7. **Remark.** If \(\psi: M_f \rightarrow R^n\) is continuous and
\[
F = \psi \circ m_f,
\]
then there exists a unique monotone \( h : M_f \to M_F \) such that
\[
m_F = h \circ m_f \quad \text{and} \quad \psi = l_F \circ h.
\]

Assuming the convergence property for \( F \) as well as for \( f \), let \( \mu_f \) and \( \mu_F \) be the limit measures corresponding to \( f \) and \( F \), with the associated densities \( v_f \) and \( v_F \).

If \( W \) is an open subset of \( M_f \) such that \( \psi|_W = l_f|_W \), then \( W \) is an open subset of \( M_F \), \( h(z) = z \) for \( z \in W \), \( h|_W \) is a homeomorphism, and 3.10 implies that
\[
\mu_f(B) = \mu_F(B) \quad \text{for every Borel set} \ B \subset W, \ v_f|_W = v_F|_W.
\]

The following two special cases occur in the sequel:

1. There exist a neighborhood \( H \) of \( l_f(M_f - W) \) in \( R^n \) and a Lipschitzian \( \Gamma : H \to R^n \) such that \( \psi \) agrees with \( \Gamma \circ l_f \) in some neighborhood of \( M_f - W \). Then
\[
\mu_f(B) = \Gamma_f(\mu_f[h^{-1}(B)])
\]
for every Borel set \( B \subset M_F - W \); moreover
\[
\|\mu_f\|(B) \leq \int_{k^{-1}(B)} (\lambda \circ l_f) \, d\|\mu_f\|
\]
if \( \lambda : H \to R \) is continuous with \( |D\Gamma(y)| \leq \lambda(y) \) for \( L_n \) almost all \( y \) in \( Y \).

2. If \( k = 2 \) and \( \psi(M_f - W) \) is a polygon. Then
\[
\mu_f(B) = 0 \quad \text{for every Borel set} \ B \subset M_F - W.
\]

4.8. Lemma. \( \|\mu\|(M_f) \geq L_k(f) \).

Proof. Suppose \( \delta > 0 \), and again write \( \mu = \mu_f \).

Choose \( Y \) according to 4.6, let \( V \) be a neighborhood of \( Y \) in \( R^n \) for which
\[
\|\mu_f\|[l_f^{-1}(V)] < \delta / \binom{n}{k},
\]
suppose
\[
0 < \epsilon < \text{distance}(Y, R^n - V)/(7n),
\]
and consider the maps \( \omega, \tau_a : R^n \to R^n \) defined by
\[
\omega(y) = \epsilon y, \quad \tau_a(y) = a + y \quad \text{for} \ y, \quad a \in R^n.
\]

Recalling \([FF, 5.1, 5.2]\) and abbreviating
\[
B = R^n \cap \{ b : |b_i| < 1 \text{ for } i = 1, \ldots, n \}
\]
one finds that
\[
\int_{\omega(B)} \int_{I_r^{-1}(V)} \left( u_k \circ \omega^{-1} \circ \tau_{-a} \circ I_r \right)^{-k} d\|\mu_r\| dL_n a
\]

\[
= \varepsilon^n \int_{\Omega} \int_{I_r^{-1}(V)} \left( u_k \circ \tau_{-b} \circ \omega^{-1} \circ I_r \right)^{-k} d\|\mu_r\| dL_n b
\]

\[
= \varepsilon^n \int_{\Omega} \int_{\omega^{-1}(V)} \left( u_k \circ \tau_{-b} \right)^{-k} d(\omega^{-1} \circ I_r)(\|\mu_r\|) dL_n b
\]

\[
= \varepsilon^n L_n(B) \left( \begin{pmatrix} n \\ k \end{pmatrix} \right) (\omega^{-1} \circ I_r)(\|\mu_r\|)[\omega^{-1}(V)]
\]

\[
= L_n[\omega(B)] \left( \begin{pmatrix} n \\ k \end{pmatrix} \right) \|\mu_r\| [I_r^{-1}(V)] < L_n[\omega(B)] \delta.
\]

Hence the points \( a \) satisfying the condition

\[
\int_{I_r^{-1}(V)} \left( u_k \circ \omega^{-1} \circ \tau_{-a} \circ I_r \right)^{-k} d\|\mu_r\| < \delta
\]

form a set of positive \( L_n \) measure. In this set a point \( a \) will be selected, subject to an additional requirement, as follows:

\textbf{In case} \( H^{k+1}[f(X)] = 0 \), one may choose \( a \) so that

\[
f(X) \subset R^n - (\tau_a \circ \omega)(C_{n-k-1}),
\]

because obviously this requirement holds for \( L_n \) almost all \( a \) (see the proof of [F8, 7.8]). In this case let \( \psi = I_r \), \( F = f \).

\textbf{In case} \( k = 2 \), let \( g_i: M_f \rightarrow R \) with

\[
l_f(z) = (g_1(z), \ldots, g_n(z)) \quad \text{for} \quad z \in M_f,
\]

select \( T \) according to [DF, 5.3] with \( X \) replaced by \( M_f \), and choose \( a \) so that

\[
g_{r_i}^{-1}\{a_r + \varepsilon_j_1\}
\]

is a subset of \( M_f - T \) and has dimension 0 at each point of

\[
g_{r_i}^{-1}\{a_r + \varepsilon_j_1\} \cap g_{s_i}^{-1}\{a_s + \varepsilon_j_1\} \cap g_{t_i}^{-1}\{a_t + \varepsilon_j_1\}
\]

whenever \( r, s, t \) are distinct elements of \( \{1, \ldots, n\} \) and \( j_r, j_s, j_t \) are even integers; this choice is possible by [DF, 4.4]. Then use the construction of [DF, 5.6] to obtain continuous maps

\[
\psi_i: M_f \rightarrow R, \quad \psi: M_f \rightarrow R^n, \quad F = \psi \circ m_f: X \rightarrow R^n,
\]

such that, for \( z \in M_f \),
\[ \psi(z) = (\psi_1(z), \ldots, \psi_n(z)), \quad |\psi(z) - l_f(z)| < \epsilon, \]
\[ \psi(z) \in (\tau_\alpha \circ \omega)(C''_{n-2}) \cap \{ y: \text{distance}(y, Y) \leq 6ne \}, \]
\[ \psi(z) = l_f(z) \text{ whenever distance}[l_f(z), Y] \geq 7ne, \]

and such that

\[ \int_{l_f^{-1}(V)} (u_k \circ \omega^{-1} \circ \tau_{-\alpha} \circ l_f)^{-1} d\|\mu_f\| < \delta. \]

The last requirements can be met because \( \psi \) may be constructed by finitely many successive modifications of the two types described in 4.7; those of type (1) involve orthogonal projections \( \Gamma \) of \( R^n \) onto \( n-1 \) dimensional planes and do not decrease the values of \( u_k \circ \omega^{-1} \circ \tau_{-\alpha} \).

In both cases choose \( q : R^n \to \{ t: 0 \leq t \leq 1 \} \) with Lipschitz constant \( (ne)^{-1} \) so that

\[ q(y) = 1 \text{ whenever distance}(y, Y) \leq 5ne, \]
\[ q(y) = 0 \text{ whenever distance}(y, Y) \geq 6ne, \]

and consider the continuous maps

\[ \phi : R^n - (\tau_\alpha \circ \omega)(C''_{n-k-1}) \to R^n, \]
\[ \phi(y) = y + q(y) \cdot [(\tau_\alpha \circ \omega \circ \sigma_k \circ \omega^{-1} \circ \tau_{-\alpha})(y) - y], \]
\[ \chi : M_F \to R^n, \]
\[ \chi(z) = l_F(x) \text{ whenever distance}[l_F(x), Y] \geq 6ne, \]
\[ \chi(z) = (\phi \circ l_F)(z) \text{ whenever distance } [l_F(z), Y] \leq 6ne, \]
\[ G = \chi \circ m_F : X \to R^n, \]

as well as the monotone maps \( \xi, \eta \) completing the commutative diagram:

Clearly
for $L_n$ almost all $y$. Defining

$$W = M_f \cap \{z: \text{distance}[\ell_f(z), Y] > 6ne\},$$

$$P = W \cap \{z: v_f(z) \neq 0\},$$

one sees from 4.7 that $W$ is also an open subset of $M_f$ and $M_\partial$, with $\xi(z) = z = \eta(z)$ for $z \in W$, that $||\mu_f||$ and $||\mu_\partial||$ agree with $||\mu||$ on all Borel subsets of $W$, and that $v_f$ and $v_\partial$ agree with $v_f$ on $W$; furthermore $||\mu||(W - P) = 0$.

Now let $\Omega$ be the set of components of $M_\partial - l_\partial^{-1}(C_{k-1})$. Since $C_{k-1}$ is $k$-removable by [F8, 6.30],

$$L_k(G) = \sum_{Q \in \Omega} L_k[G | m_\partial^{-1}(Q)].$$

Also let

$$\Omega_1 = \Omega \cap \{Q: (\eta \circ \xi)^{-1}(Q) \text{ meets } l_\partial^{-1}(Y)\}, \quad \Omega_2 = \Omega - \Omega_1.$$

If $Q \in \Omega_1$, then $\eta^{-1}(Q) \subset l_\partial^{-1}(V)$. In fact, assuming $(\eta \circ \xi)(z) \in Q$ with $l_f(z) \in Y$, one finds that

$$\text{distance}[\psi(z), Y] < \epsilon, \quad \psi(z) = (l_f \circ \xi)(z),$$

$$\text{distance}[\chi \circ \xi)(z), Y] < (n + 1)\epsilon \leq 2ne,$$

and $(\chi \circ \xi)(z) = (l_\partial \circ \eta \circ \xi)(z)$ belongs to a component $E$ of $C_k - C_{k-1}$. Moreover $E$ is a $k$ dimensional cube with side length $2\epsilon$,

$$E \subset \{y: \text{distance}(y, Y) < 4ne\},$$

$E$ is open relative to $C_k \cap \{y: \text{distance}(y, Y) < 4ne\}$,

and inasmuch as

$$l_\partial(M_\partial) \cap \{y: \text{distance}(y, Y) < 4ne\} \subset C_k,$$

one infers that $l_\partial^{-1}(E)$ is open in $M_\partial$. Noting that

$$\text{Bdry } l_\partial^{-1}(E) \subset l_\partial^{-1}(C_{k-1}) \subset M_\partial - Q,$$

one concludes that $Q \cap l_\partial^{-1}(E)$ is nonempty, open and closed relative to $Q$, hence

$$Q \subset l_\partial^{-1}(E), \quad \eta^{-1}(Q) \subset \chi^{-1}(E) \subset l_\partial^{-1}(V).$$

Furthermore 4.5 yields the inequality
If $Q \in \Omega_2$, then $(\eta \circ \xi)^{-1}(Q) \subset U$ for a unique $U \in \Xi$, and 4.5 implies that

$$L_k[G \mid m^{-1}(Q)] \leq \left(\binom{n}{k}\right) \|\mu_\partial\|\|Q\|.$$

On the other hand, $\{\eta^{-1}(Q) : Q \in \Omega_2\}$ is a countable family of disjoint Borel subsets of $f^{-1}(V)$, it follows from 4.7 that

$$\sum_{Q \in \Omega_1} \|\mu_\partial\|\|Q\| + \sum_{Q \in \Omega_1} \|\mu_\partial\|\|Q \cap P\| \leq (n + 3)^3 \int_{f^{-1}(V)} (\mu_\partial \circ \omega^{-1} \circ \tau_{-\theta} \circ l_P)^{-1} \|\mu_P\| < (n + 3)^3 \delta.$$

Combining these estimates one concludes that

$$L_k(G) \leq \|\mu_k\|(M_f) + 2(n + 3)^3 \binom{n}{k} \delta,$$

with $|G(x) - f(x)| < (n + 1)\varepsilon$ for $x \in X$.

4.9. Theorem. If $A$ is an open subset of $M_f$, then

$$\|\mu\|(A) = L_k[f \mid m^{-1}(A)].$$

Proof. In view of 4.4 there would otherwise exist a finitely triangulable set $T \subset m^{-1}(A)$ such that

$$\|\mu\|(A) < L_k(f \mid T).$$

Letting $B = M_f - m_f(T)$ one would find that $T$ and $m_f^{-1}(B)$ are disjoint, $A \cup B = M_f$, and it would follow from 4.8 and 4.4 that
\[ \|\mu\|(M_f) \geq L_k(f) \geq L_k(f|T) + L_k[f|m^{-1}(B)] \]
\[ > \|\mu\|(A) + \|\mu\|(B) \geq \|\mu\|(M_f). \]

4.10. Remark. One readily verifies by the method of doubling, as in 3.10, that the results of this section remain true without the assumption that \( X \) is compact, provided \( f \) is proper.

Regarding the Lebesgue area densities introduced in [F6, §6] one infers from 4.9 and 2.2 that
\[ L^*_k(f, z) = L_{k+1}(f, z) = |\psi(z)| \]
for \( H^k \) almost all \( z \) in \( M_f \). Moreover one sees with the help of [DF] and an argument like the proof of [F6, 8.14 (7)] that the equation
\[ L_k(f) = \int_{M_f} L^*_k(f, z) \, dH^kz = \int_{\mathbb{R}^k} \sum_{v \in L_{k-1}(z)} L^*_k(f, z) \, dH^{k-1} \]
holds also in case \( L_k(f) = \infty \). The problems raised in [F6, pp. 326, 335] are thus solved provided \( H^{k+1}[f(X)] = 0 \) or \( k = 2 \).

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