CARTAN DECOMPOSITIONS FOR $L^*$ ALGEBRAS

BY
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1. Introduction. In the discussion of $L^*$ algebras given in [5] a classification theory was obtained for the separable simple algebras under the assumption of the existence of a Cartan decomposition relative to some Cartan subalgebra. The main result of this paper is a proof that any semi-simple $L^*$ algebra of arbitrary dimension has such a decomposition relative to any Cartan subalgebra.

In the process of proving this several additional results of interest in themselves are obtained, among them one concerning representations of finite-dimensional semi-simple Lie algebras which seems to be new. This is stated in detail in the second corollary of 4.5. The conclusion obtained in 4.5 also adds a new result to the theory of commutators of operators on a Hilbert space.

2. Continuous decompositions.

Definitions and notation. An $L^*$ algebra is defined as a Lie algebra $L$ over the complex numbers whose underlying vector space is a Hilbert space and such that for each $x$ in $L$ there exists an element $x^*$ with $([x, y], z) = (y, [x^*, z])$ for all $y$ and $z$. For an $x$ in $L$, $X$ (occasionally $D_x$) will denote the linear operator defined by $Xy = [x, y]$ for all $y$ and we will assume that the norm on $L$ is chosen such that $\|X\| \leq \|x\|$. An $L^*$ algebra is semi-simple if and only if the mapping $x \rightarrow x^*$ is one-one. For the remainder of this paper $L$ will denote an arbitrary (but fixed) semi-simple $L^*$ algebra unless further restrictions are explicitly stated. As shown in [5] this implies the mapping $x \rightarrow x^*$ is a Hilbert space conjugation and anti-multiplicative, $D_x^*$ is the adjoint of $D_x$, and that $L$ is a direct sum of simple $L^*$ ideals. A Cartan subalgebra of $L$ is defined as a maximal abelian self-adjoint subalgebra of $L$. For subsets $M, N$ of $L$, $\text{Sp}(M)$ will represent the smallest closed linear subspace of $L$ containing $M$ and $[M, N] = \text{Sp}\{ [m, n] : m \in M, n \in N \}$. For subspaces $S_1, S_2$ the notation $S_1 + S_2$ will be used only when $S_1$ is orthogonal to $S_2$.

Suppose $A$ is a bounded self-adjoint operator on $L$. For $\lambda$ real and $\varepsilon > 0$ let $V(\lambda, \varepsilon) = \{ x : \|(A - \lambda)^n x\| \leq \varepsilon^n \|x\|, n = 1, 2, \cdots \}$. For a Borel set $M$ of the real numbers let $V(M, \varepsilon) = \text{Sp}\{ V(\lambda, \varepsilon) : \lambda \in M \}$ and $V(M) = \bigcap_{\varepsilon > 0} V(M, \varepsilon)$. It is proved in [1, pp. 66–69] that $V(\lambda, \varepsilon)$ is a closed subspace and equal to the set of $x$ such that the sequence $\{(e^{-\varepsilon}(A - \lambda))^n x\}$ is bounded. Furthermore, if $E$ is the real spectral measure such that $A = \int dE$ then the range of $E(M)$ is equal to $V(M)$ for $M$ compact. For any Borel set $M$ the range of $E(M)$ will be denoted by $S(M)$. Finally, for Borel sets $M$ and $N$ let $M + N$
\[= \{m+n: m \in M, n \in N\} \text{ and } -M = \{-m: m \in M\}. \text{ Then } M+N \text{ and } -M \text{ are also Borel sets.}\]

2.1. Suppose \(A\) is a bounded self-adjoint derivation on \(L\) and \(M, N\) are Borel sets of the real line. Then \([S(M), S(N)] \subseteq S(M+N)\) and \(S(M)^* = S(-M)\).

**Proof.** Suppose first that \(M\) and \(N\) are compact and \(\epsilon > 0\). Let \(M = \bigcup M_k\) and \(N = \bigcup N_k\) where \(\{M_k\}, \{N_k\}\) are sequences of disjoint Borel sets, each of diameter less than \((1/2)\epsilon\). Let \(x \in S(M)\) and \(y \in S(N)\). Then \(x = \sum x_j, y = \sum y_k\) where \(x_j \in S(M_j), y_k \in S(N_k)\) and \([x, y] = \sum [x_j, y_k]\). Suppose \(\lambda_j \in M_j, \mu_k \in N_k\). Then it follows from the spectral theorem that \(\|(A - \lambda_j)^* x_j\| \leq (2^{-n}) \|x_j\|\) and \(\|(A - \mu_k)^* y_k\| \leq (2^{-n}) \|y_k\|\) for each positive integer \(n\).

Since \(A\) is a derivation it follows by induction on \(n\) that \(\|(A - (\lambda_j + \mu_k))^* [x_j, y_k]\| = \|\sum \delta_n C_m (A - \lambda_j)^* x_j, (A - \mu_k)^* y_k]\| \leq \sum \delta_n C_m \| (A - \lambda_j)^* x_j\| \| (A - \mu_k)^* y_k\| \leq \sum \delta_n C_m 2^{-n} \|x_j\| \|y_k\| \epsilon \|x_j\| \|y_k\|\).

Hence the sequence \(\{(e^{-1}(A - (\lambda_j + \mu_k))^*[x_j, y_k]\) is bounded and this implies \([x_j, y_k] \in V(\lambda_j + \mu_k, \epsilon)\). Thus \([x, y] \in V(M+N, \epsilon)\).

Since \(\epsilon\) was arbitrary, \([x, y] \in V(M+N)\). The compactness of \(M\) and \(N\) implies \(M+N\) is also and hence \([x, y] \in S(M+N)\). Thus \([S(M), S(N)]\) is a subset of \(S(M+N)\) for \(M\) and \(N\) compact.

It is proved in [1] that \(E\) is regular, i.e., for any Borel sets \(M\) and \(N\), \(E(M) = \sup\{E(C): C \subseteq M, C \text{ compact}\}\) and similarly for \(E(N)\). For \(C \subseteq M\) and \(D \subseteq N\), we have \([S(C), S(D)] \subseteq S(C+D) \subseteq S(M+N)\). Letting \(C\) vary gives \([S(M), S(D)] \subseteq S(M+N)\). Letting \(D\) vary gives \([S(M), S(N)] \subseteq S(M+N)\).

Since \(A\) is a derivation, \([A, D_x] = D_{Ax}\) for all \(x\) and hence \(D_{(Ax)^*} = [A, D_x]^* = [D_x^*, A^*] = -[A, D_x^*] = D_{-Ax}\) so that \((Ax)^* = -Ax^*\) for all \(x\) in \(L\). Using this, a proof like that above can be constructed to prove the second assertion.

**Notation.** Suppose \(\mathcal{C}\) is a closed self-adjoint abelian subalgebra of \(L\). Let \(\mathcal{A} = \mathcal{A}(\mathcal{C})\) be the commutative \(C^*\) algebra of bounded operators generated by \(\{H: h \in \mathcal{C}\}\). Since each \(H\) is zero on \(\mathcal{C}\) then the identity operator is not in \(\mathcal{A}\). Let \(\Delta = \Delta(\mathcal{C})\) be the set of all homomorphisms of \(\mathcal{A}\) into the complex numbers. For \(A \in \Delta\) let \(\hat{A}\) be the function on \(\Delta\) defined by \(\hat{A}(\alpha) = \alpha(A)\). If \(\Delta\) is given the weakest topology making these functions continuous then \(\Delta\) is a compact Hausdorff space and the theory of \(C^*\) algebras shows that the mapping \(A \to \hat{A}\) is an algebraic isomorphism mapping \(\mathcal{A}\) isometrically onto the set of all continuous functions on \(\Delta\) vanishing at the zero homomorphism with \(A^*\) corresponding to the complex conjugate of \(\hat{A}\). The set \(\Delta\) with its topology will be called the spectrum of \(\mathcal{C}\). It is also known that the spectrum of an operator \(A \in \mathcal{A}\) is the range of the function \(\hat{A}\).

2.2. For each \(\alpha \in \Delta\) there is a unique \(x_\alpha \in \mathcal{C}\) such that \(\alpha(H) = (h, x_\alpha}\) for all \(h \in \mathcal{C}\). Also \(\|x_\alpha\| \leq 1\) and \(x_\alpha^* = x_\alpha\). If \(\{x_\alpha\}\) is given the induced weak topology of \(\mathcal{C}\) then \(\Delta\) is homeomorphic to \(\{x_\alpha\}\) under the mapping \(\alpha \to x_\alpha\).
Proof. All except the last statement were proved in [5]. The family \( \{ \hat{H} : h \in \mathcal{C} \} \) separates points of \( \Delta \) and all members vanish at infinity (zero). By Theorem 5G of [3], the topology of \( \Delta \) is that generated by the family. But this is clearly equivalent to the weak topology on \( \{ x_\alpha \} \), finishing the proof.

There is a unique spectral measure \( E \) on the Borel sets of \( \Delta \) such that \( (Ax, y) = \int \hat{A}(\alpha) d(E(\alpha)x, y) \) for all \( x, y \) in \( L \). Also it is easily seen that the range of \( E(0) \) is \( \{ x : [3C, x] = 0 \} \) where \( \{ 0 \} \) denotes the Borel set consisting of the zero homomorphism. For an arbitrary Borel set \( M \) of \( \Delta \) let \( S(M) \) denote the range of \( E(M) \).

Suppose \( \alpha, \beta \in \Delta \). If \( x_\alpha + x_\beta = x_\gamma \) for some \( \gamma \in \Delta \), let \( \gamma \) be denoted by \( \alpha + \beta \). If \( x_\alpha = -x_\gamma \) for some \( \gamma \in \Delta \), let \( \gamma \) be denoted by \( -\alpha \). Using this notation we have the following theorem which is the continuous version of the desired composition for \( L \) relative to \( \mathcal{C} \).

2.3. Suppose \( M, N \) are Borel sets of \( \Delta \) and \( M + N = \{ m + n : m \in M, n \in N \} \). Then \( M + N \) is a Borel set and \( [S(M), S(N)] \subset S(M + N) \). If \( -M = \{ -m : m \in M \} \) then \( S(-M) = S(M)^* \).

Proof. Choose a set \( \{ x_i : i \in I \} \) of elements of \( \mathcal{C} \) such that the set spans \( \mathcal{C} \) and \( x_i^* = x_i \) for each \( i \). Let \( \sigma_i \) be the spectrum of \( x_i \). Then \( \sigma_i \) is compact so that \( P = \prod x_i \) is compact. For \( \alpha \in \Delta \) let \( f(\alpha) \) be the element of \( P \) whose \( i \)-th coordinate is \( (x_i, x_\alpha) \). Then \( f \) is a homeomorphism of \( \Delta \) onto a compact subset of \( P \). If addition is defined in \( P \) (whenever possible) in the obvious coordinate-wise fashion then \( f \) preserves the algebraic structure of \( \Delta \) as well as the topology. The spectral measure \( E \) can be defined directly on \( P \) (hence on \( f(\Delta) \) and \( \Delta \)) by constructing the product measure obtained from the \( E_i \)'s where \( X_i = \int x_i dE_i \). The measure-theoretic details will be omitted here but a discussion of this type of problem may be found in [2].

A subset \( M \) of \( P \) will be called a rectangle if and only if \( M = \prod M_i \) where \( M_i \) is a Borel set of \( \sigma_i \) and \( M_i = \sigma_i \) for all but a finite number of indices. Then the Borel sets of \( P \) will coincide with the \( \sigma \)-algebra generated by the rectangles. In fact, Chapter 7 of [2] shows that this \( \sigma \)-algebra is obtained as the smallest monotone class containing all finite unions of disjoint rectangles. 2.1 can be used to prove 2.3 for the case when \( M, N \) are finite unions of disjoint rectangles. The collection of all sets for which 2.3 holds is clearly a monotone class, hence contains all Borel sets.

2.4. Corollary. Suppose \( L \) is separable and \( \mathcal{C} \) is a closed self-adjoint abelian subalgebra of \( L \) such that \( \{ ||x_\alpha|| : \alpha \in \Delta, x_\alpha \neq 0 \} \) is bounded away from zero. Then there exist disjoint Borel sets \( M_k \) of \( \Delta \), \( k = 0, 1, \ldots \) such that \( M_0 = \{ 0 \} \), \( L = \sum S(M_k) \), and \( [S(M_k), S(M_k)^*] \subset \{ x : [3C, x] = 0 \} \).

Proof. \( L \) separable implies \( \{ x_\alpha : \alpha \in \Delta \} \) is a separable metric space in the norm topology, hence contains a countable dense subset. Suppose \( ||x_\alpha|| > c > 0 \) for all \( \alpha \neq 0 \). Let \( \alpha_0 = 0 \) and let \( \{ x_{\alpha_k} : k \geq 1 \} \) be a countable dense subset of
\{x_0\} - \{0\}. If \( N_k = \{ \beta : \|x_0 - x_\alpha\| \leq 3^{-1}c \} \) then \( N_0 = \{0\} \) and \( \Delta = \bigcup N_k \). Furthermore, since \( \{x_0\} \) is weakly compact, \( N_k \) is weakly closed, hence a Borel set. If \( \alpha, \beta \in N_k \) the triangle inequality implies \( \|x_\alpha - x_\beta\| < c \) so that either \( \alpha = \beta \) or \( \alpha - \beta \in \Delta \). Hence \( N_k + (-N_k) = \{0\} \). In the usual way it is possible to choose Borel sets \( M_k \) such that \( M_k \subseteq N_k \) and \( \Delta \) is the disjoint union of the \( M_k \)'s. Then \( [S(M_k), (S(M_k))^*] \subseteq S(M_k + (-M_k)) \subseteq S(\{0\}) \) and the remaining statements follow easily from the spectral theory.


3.1. There exists a nonzero element \( a \in L \) such that \( (a, a^*) = 0 \) and \( A^3 = 0 \).

Proof. Let \( x \) be a self-adjoint element of \( L \) with \( \|X\| = 1 \) and let \( X = \bigwedge \delta E \). If \( M \) is the real interval \( (2/3, 1] \) and \( V \) is the range of \( E(M) \) then \( V \neq 0 \) and, since \( V^* \) is the range of \( E(-M) \), \( (V, V^*) = 0 \). Using 2.1, \( [V, [V, [F, L]]] = 0 \). Thus \( a \) may be chosen as any element of \( V \) different from zero.

Notation. For this section we choose a fixed \( a \) having the properties listed in 3.1 and let \( b = a^* \), \( c = [a, b] \). Then \( C = C^* \) and \( A^3 = B^3 = 0 \). This section is devoted to an analysis of the \( L^* \) subalgebra generated by \( a \), culminating in 3.8. This turns out to be a key result in the general existence proof for Cartan decompositions.

3.2. For any \( x \) in \( L \), \( A^2XA = AXA^2 \) and \( B^2XB = BXB^2 \).

Proof. \( [A, [A, [A, X]]] = 0 \) and \( A^3 = 0 \) together imply \( -3A^2XA + 3AXA^2 = 0 \). The second equation follows from the first by taking adjoints.

3.3. Suppose \( x \) is any element in the closure of the range of \( A^2 \). Then \( X^3 = 0 \).

Proof. By continuity of the adjoint representation it is sufficient to prove this for the case \( x = A^2z \) for some \( z \) in \( L \). Then \( X = [A, [A, Z]] = A^2Z - 2AZA^2 \). Using 3.2, \( A^2ZA = AZA^2 \) so that \( A^2ZA^2 = 0 \). A direct computation then shows that \( X^3 = A^2Z^2A^2ZA^2 = 0 \).

3.4. Suppose \( n \) is any positive integer. Then

(a) \( C^n a = -A^2(BA)^{n-1}b = (AB)^n a \),
(b) \( C^n b = (-1)^{n-1}B^2(AB)^{n-1}a = (-1)^n(BA)^n b \).

Proof. Since \( C = C^* \), \( (C^nx)^* = (-1)^nC^nx \) for any \( x \). Thus (b) follows from (a) by using adjoints.

The first equation of (a) is proved by induction. For \( n = 1 \), \( C = [\{a, b\}, a] = -A^2b \). Assuming the result for \( n \), \( C^{n+1} a = C(-A^2(BA)^{n-1}b) = -(AB - BA)A^2(BA)^{n-1}b = -ABA^2(BA)^{n-1}b = -A^3(BA)^n b \) after using 3.2.

For the second, repeated application of 3.2 gives \( C^n a = -A^2(BA)^{n-1}A^2b = (AB)^n a \).

Corollary. For all integers \( n \geq 0 \),

(a) \( AC^n a = BC^n b = 0 \),
(b) \( ABC^n a = C^{n+1} a, BAC^n b = -C^{n+1} b \),
(c) \( B^2C^n a = (-1)^nC^{n+1} b, A^2C^n b = (-1)^{n+1}C^{n+1} a \).

3.5. Let \( S_0 = \text{Sp}\{a, b\} \), \( S_n = \text{Sp}\{D_1 \cdots D_n : s = a, b; D_i = A, B\} \) for \( n = 1, 2, \cdots \). Let \( S = \text{Sp}\{S_n : n = 0, 1, \cdots \} \).
(a) $S$ is the $L^*$ subalgebra generated by $a$.
(b) $S_{2n} = Sp \{ C^na, C^nb \}, n = 0, 1, \ldots$
(c) $S_{2n+1} = Sp \{ BC^na, AC^nb \}, n = 0, 1, \ldots$
(d) $(S_{2n}, S_{2n+1}) = 0, m, n = 0, 1, \ldots$

**Proof.** (a) It is clear that $S_n = S^*_n$ and hence $S = S^*$. Since $AS_n \subseteq S_{n+1}$,
$BS_n \subseteq S_{n+1}$, then $S$ is invariant under $A$ and $B$. Hence $S$ is invariant under $X$
for any $x$ in the $L^*$ subalgebra $S'$ generated by $a$, i.e., $[S', S] \subseteq S$. But clearly
$S \subseteq S'$. Hence $[S, S] \subseteq S$ and therefore $S = S'$.
(b) and (c) are true for $n = 0$. Suppose they hold for some $n$. Then

$$S_{2n+2} = Sp \{ AS_{2n+1}, BS_{2n+1} \} = Sp \{ ABC^n a, A^* C^n b, B^* C^n a, BAC^n b \}$$
using the corollary of 3.4. Hence

$$S_{2n+2} = Sp \{ C^{n+1} a, C^{n+1} b \},$$

since $AC^{n+1} a = BC^{n+1} b = 0$. Thus, by induction on $n$, (b) and (c) are true for
all $n$.
(d) $(C^na, BC^n a) = (AC^n a, C^n a) = 0$ and

$$(C^na, AC^n b) = (-1)^n (C^na, A(BA)^n b) = (-1)^n (C^na, (AB)^n a b)$$

$$= (-1)^{n+1} (C^{n+1} a, Ba) = (-1)^{n+1} (AC^{n+1} a, a) = 0.$$

Similarly $(C^nb, AC^n b) = (C^nb, BC^n a) = 0$, completing the proof of (d).

3.6. Letting $n$ range over the non-negative integers, let $V_0 = Sp \{ S_{2n+1} \}$,
$V_1 = Sp \{ C^n a \}, V_2 = Sp \{ C^n b \}$.

(a) $S = V_0 + V_1 + V_2$
(b) $V_0 = V_0^*, V_1^* = V_2, V_2^* = V_1$
(c) $[V_1, V_1] = [V_2, V_2] = 0$
(d) $[V_1, V_1^*] = [V_2, V_2^*] = 0$

**Proof.** (a) It only remains to prove that $V_1$ is orthogonal to $V_2$. Now
$(a, b) = 0$ and, if either $n$ or $m$ is nonzero, then $(C^na, C^nb) = -(A^*(BA)^{n+m-1} b, b)$
$$= -((BA)^{n+m-1} b, B^* b) = 0.$$ Hence $(V_1, V_2) = 0$.
(b) $V_1^* = Sp \{ (C^n a)* \} = Sp \{ C^n b \} = V_2$. Similarly $V_2^* = V_1$. Since $S^* = S$,
$V_0^* = V_0$.
(c) It is sufficient to prove $[V_1, V_1] = 0$ or $[C^n a, C^n a] = 0$ for all $m$ and $n$.
This is done by induction on $n$. The case $n = 0$ is given by the corollary of 3.4.
Suppose $[C^{n-1} a, C^n a] = 0$ for all $m$. Then $C^n [a, C^n a] = C^n 0 = 0$. By Leibniz’s
rule, $0 = [C^n a, C^n a] + \text{terms of the form } [C^p a, C^q a]$ where $p < n$. Each of these
latter terms is zero by the induction hypothesis.
(d) $V_0 = Sp \{ [a, C^n b], [b, C^n a] \}$ implies $V_0 \subseteq [V_1, V_1^*]$. But $(V_1, [V_1, V_1^*])$
$$= ([V_1, V_1], V_1) = 0$$ implies $[V_1, V_1^*]$ is orthogonal to $V_1$. Similarly, since
$[V_1, V_1^*] = [V_2, V_2^*]$, $[V_1, V_1^*]$ is also orthogonal to $V_2$, hence must be a
subset of $V_0$.

3.7. $[c, V_0] = 0$. 

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Proof. By using adjoints it is sufficient to prove $CAC^nb = 0$ for all $n$. Since $C$ is self-adjoint it is sufficient to prove $C^2AC^nb = 0$. Using the appropriate cases of the corollary of 3.4,

$$CAC^nb = (AB - BA)AC^nb = A(BAC^nb) - B(A^2C^nb)$$

$$= -AC^{n+1}b + (-1)^nBC^{n+1}a.$$ 

Hence

$$C^2AC^nb = -A(BAC^{n+1}b) + B(A^2C^{n+1}b) + (-1)^nAB^2C^{n+1}a + (-1)^{n+1}BABC^{n+1}b$$

$$= AC^{n+2}b + (-1)^nBC^{n+2}a + (-1)^{n+1}AC^{n+1}b + (-1)^{n+1}BC^{n+1}a = 0.$$ 

**Corollary 1.** $[C^na, C^mb] = (-1)^n[a, C^{n+m}b]$ for all $n, m$.

**Proof.** $[C^na, C^mb]E V_0$ by 3.6. Also we may assume $n$ is positive. Then

$$0 = C^2[0, C^mb] = [C^na, C^mb] + [C^na, C^{m+1}b]$$

so that $[C^na, C^mb] = -[C^{n-1}a, C^{n+1}b]$. Applying this repeatedly gives the result in general.

**Corollary 2.** $[V_0, V_i] = V_i$ for $i = 1, 2$.

**Proof.** It is sufficient to prove this for $i = 1$. Now $[V_0, C^na] = C^n[V_0, a]$ and $V_1$ is invariant under $C$ so it is enough to prove $[V_0, a]E V_1$ in order to prove $[V_0, V_1]E V_1$. But $AV_0 = Sp\{A^nC^nb, A^2BC^ma\}E V_1$ by the corollary of 3.4.

For the reverse inclusion, suppose $xE V_1$ and $(x, [V_0, V_1]) = 0$. Then $([x, x^*], V_0) = 0$ so that $[x, x^*] = 0$ and $X$ is normal. To prove $x$ must be zero it is sufficient to show that $X$ is nilpotent. In fact, for future reference, we will prove that $X^2 = 0$ for all $xE V_1$. Since $V_1 = Sp\{C^na; nE N\}$, it is clear that $Sp\{C^na; nE 1\}$ is either all of $V_1$ or a hyperplane in $V_1$. In the first case $V_1$ is contained in the closure of the range of $A^2$ (see 3.4) and the assertion follows from 3.3. In the other case an element $x$ of $V_1$ must be of the form $x = u + y$ where $u$ is a scalar and $y$ is in the closure of the range of $A^2$. A proof like that of 3.3 then shows that $A^2Y = AY^2 = 0$. Since $A^2 = Y^2 = 0$, the binomial theorem gives $X^2 = 0$.

**Corollary 3.** $[V_0, V_0] = 0$.

**Proof.** Suppose $xE V_0$ and $(x, AC^nb) = 0$ for all $n$. Then $([b, x], C^nb) = 0$ for all $n$ implies $[b, x] = 0$ since $[b, x]E V_2$. But then $0 = C^nXb = XC^nb$ and this implies $x$ is zero on $V_0$ so that $(x, V_0) = (x, [V_0, V_0^*]) = 0$ and hence $x$ is zero.

Thus $V_0 = Sp\{AC^nb\}$. Now $0 = [C^nb, C^nb]$ implies $A^2[C^nb, C^nb] = 0$ so that $[A^2C^nb, C^nb] + 2[A^2C^nb, C^nb] + [C^nb, A^2C^nb] = 0$. The sum of the first and last terms on the left side is $(-1)^{n+1}[C^{n+1}a, C^nb] + (-1)^{n+1}[C^nb, C^{n+1}a]$ which is zero by Corollary 1. Thus $[AC^nb, AC^nb] = 0$ and, since $V_0 = Sp\{AC^nb\}$, $V_0$ is abelian.

**Corollary 4.** $S$ is semi-simple with $V_0$ as a Cartan subalgebra.
Proof. Suppose $x \in S$ and $[x, V_0] = 0$. Then $0 = ([x, V_0], V_i) = (x, V_i)$ for $i = 1, 2$ implies $x \in V_0$ and $V_0$ is maximal abelian. To show that $S$ is semi-simple suppose that $x \in S$ and $[x, S] = 0$. Then $x \in V_0$ and $[x, b] = 0$. The proof of Corollary 3 shows that $x$ must be zero.

3.8. Suppose $S$ is a semi-simple $L^*$ algebra and $S = V_0 + V_1 + V_2$ with $V_0$ as a Cartan subalgebra and that the relations of 3.6 hold. Then $S$ is a direct sum of three-dimensional ideals $I_j$ where $I_j = \text{Sp} \{ e_j, e_j^*, [e_j, e_j^*] \}$ for some nonzero $e_j \in V_1$.

Proof. The decomposition theorem of [5] for semi-simple algebras shows that $S$ can be written as the direct sum of simple ideals $I_j$ where $I_j = H_j + [H_j, S]$ for some closed self-adjoint subspace $H_j$ of $V_0$. We choose a fixed $I_j$ and let $U_0 = H_j$, $U_1 = [H_j, V_1]$, $U_2 = [H_j, V_2]$. Then $I_j = U_0 + U_1 + U_2$ and it is clear that $[U_i, U_i] = 0$ for each $i$, $[U_0, U_i] = U_i$ for $i = 1, 2$, while $[U_i, U_i^*] = U_0$ for $i = 1, 2$.

Suppose $U_1 = P + Q$ where $P$ and $Q$ are closed subspaces invariant under $U_0$. Then $([U_1, U_1^*], P, Q) = 0$ implies $([U_1, U_1^*], Q, P^*) = 0$ so that $[Q, P^*] = 0$. Since $[Q, P]$ is also zero, it follows that $U_0 = [P, Q] + [P, Q^*]$ and, furthermore, that $XY = 0$ (on $S$) for all $x \in [P, P^*]$ and $y \in [Q, Q^*]$. Referring to the proof in [5] of the decomposition theorem and using the simplicity of $I_j$ we must have $[P, P^*] = 0$ or $[Q, Q^*] = 0$. But every element of $U_1$ is nilpotent on $S$ so that necessarily either $P = 0$ or $Q = 0$. Hence $U_1$ contains no nontrivial closed subspaces invariant under $U_0$. By the spectral theorem $U_1$ must be one-dimensional and this completes the proof.

Corollary 1. The $L^*$ algebra generated by $a$ is a direct sum of three-dimensional ideals.

Corollary 2. There exists a nonzero element $x \in L$ such that $X^3 = 0$ and $[[x, x^*], x] = \lambda x$ with $\lambda$ positive. In fact $\lambda \|x\|^2 = \|[[x, x^*], x]\|^2$.

Proof. Let $I_j$ be a simple ideal of $S$ as above and let $x = e_j$. Then $[[x, x^*], x] = \lambda x$ for some $\lambda$. Hence $\lambda \|x\|^2 = ([[[x, x^*], x], x], x) = \|[[x, x^*], x]\|^2$. Since $x \in V_1$, the proof of Corollary 2 of 3.7 shows that $X^3 = 0$. Thus $[x, x^*] \neq 0$ and $\lambda$ must be different from zero.

4. A commutator equation. For this section, $A$ will denote a fixed nonzero bounded operator on a Hilbert space such that $[[A, A^*], A] = \lambda A$ for some $\lambda \neq 0$. From this considerable information about the spectra of $AA^*, A^*A$, and $[A, A^*]$ can be obtained. Also there are some interesting consequences for representations of Lie algebras as bounded operators on a Hilbert space. By assuming 4.1 and Corollary 1 of 4.2 much of what is done here is valid for elements $A, A^*$ of an arbitrary algebra with identity over a field of characteristic zero.

4.1. $A$ is nilpotent.

Proof. The mapping $B \mapsto [[A, A^*], B]$ is a derivation on the set of all
bounded operators and has norm not exceeding \(|A, A^*|| = n\lambda A^n \text{ for } n \text{ a positive integer. Then } n\lambda A^n = [[A, A^*], A^n] \implies \frac{1}{n} \left| [A^* A - \frac{\lambda}{2} (n-1) A^n] \right| \leq 2\| [A, A^*]\| \|A^n\| \text{ so that } A^* \text{ is zero for some } n. \\

4.2. \([A^*, A^n] = n A^n - (\lambda/2) (n-1) A^{n-1}\) for \(n=1, 2, \ldots\) \\

Proof. The case \(n=1\) is trivial. Assuming the equation for \(n\) gives \\
\[
[A^*, A^{n+1}] = A^n [A^*, A] + [A^*, A^n] A \\
= A^n [A^*, A] + n A^n [A^*, A] - n\lambda A^n - (\lambda/2) (n-1) A^n \\
= (n+1) A^n [A^*, A] - (\lambda/2) (n-1) A^n.
\]

Corollary 1. \(\lambda\) is real and positive and \([[A^*, A], A^*] = \lambda A^*\). \\

Proof. Choose \(n\) such that \(A^n \neq 0\) but \(A^{n+1} = 0\). Then \(0 = [A^*, A^{n+1}] = (n+1) A^n [A^*, A] - (\lambda/2) (n-1) A^n \text{ implies } A^n (A^* A - n (\lambda/2)) = 0.\) Since \(A^n \neq 0\), \(A^* A - n (\lambda/2)\) does not have a bounded inverse, hence \(n (\lambda/2)\) is in the spectrum of the positive operator \(A^* A\) and \(\lambda\) must be positive. Since \(\lambda\) is real, taking adjoints of both sides of the equation \([[A, A^*], A] = \lambda A\) gives the second assertion.

Corollary 2. \([A^n, A] = n A^{n-1} [A^*, A] + (\lambda/2) (n-1) A^{n-1}\). \\

Proof. Taking adjoints of both sides of the equation in 4.2 gives \\
\[
[A^n, A] = n [A^*, A] A^{n-1} - (\lambda/2) (n-1) A^{n-1} \\
= n A^{n-1}[A^*, A] + \lambda n (n-1) A^{n-1} - (\lambda/2) (n-1) A^{n-1}.
\]

Corollary 3. \(AA^*\) commutes with \(A^* A\). \\

Proof. Since \([A, A^*] = AA^* - A^* A\), it is sufficient to prove \(AA^*\) commutes with \([A, A^*]\). But \\
\[\begin{align*}
\end{align*}\]

4.3. For each non-negative integer \(n\) let \(B_n = A^n A^* A^n\) and \(D_n = A^n A A^n\). \\
(a) \(B_n A A^* = (1/n + 1) B^n + (n/n + 1) B_n^* A A^* + n (\lambda/2) B_n^*\). \\
(b) \(D_n A A^* = (1/n + 1) D^n + (n/n + 1) D^n A A^* + n (\lambda/2) D_n^*\). \\
(c) For all \(n, m \geq 0\), \(B_n\) and \(D_n\) commute with \(B_m\) and \(D_m\). \\

Proof. (a) \\
\[
= B_{n+1} + A^n (n A^{n-1} [A^*, A] + (\lambda/2) (n-1) A^{n-1}) A^* \\
= B_{n+1} + (\lambda/2) (n-1) B_n + n A^n A A^* \\
= B_{n+1} + (\lambda/2) (n-1) B_n + n\lambda A A^* \\
= B_{n+1} + (\lambda/2) (n+1) B_n + n B_n A A^* - n B_n A A^*.
\]

Solving for \(B_n A A^*\) gives the assertion in (a).
(b) Because of the symmetry between $A$ and $A^*$ the proof is like that for (a).

(c) If either $n$ or $m$ is zero the result is immediate. By Corollary 3 of 4.2, $B_1$ commutes with $D_1$. Using equations (a) and (b) an induction on $n$ shows that $B_n$ and $D_n$ are polynomials in $B_1$ and $D_1$ and this gives (c).

4.4. Let $p$, $q$ be non-negative integers and $n = p + q$. Then

$$B_p D_q A A^* = (q + 1/n + 1)B_{p+1} D_q + (q/n + 1)B_{p+2} D_{q+1} + (λ/2)(q + 1)B_p D_q.$$

**Proof.** The case $q = 0$ is given by equation (a) of 4.3 and the case $p = 0$ reduces to $D_n A A^* = A A^* D_n$. Thus we may assume both $p$ and $q$ are positive. Now $B_p D_q A A^* = (B_p A A^*) D_q = (1/p + 1)B_{p+1} + (p/p + 1)B_p A A^* + p(λ/2)B_p D_q$ implies

$$B_p D_q A A^* = (1/p + 1)B_{p+1} D_q + p(λ/2)B_p D_q + (q/p + 1)B_p D_q A^* A.$$

But, using equation (b) of 4.3,

$$B_p(D_q A A^*) = B_p((q + 1/q)D_q A^* A - (q/q)D_{q+1} - (λ/2)(q + 1)D_q)$$

which gives

$$B_p D_q A A^* = (1/p + 1)B_{p+1} D_q + (q/q)D_{q+1} + (λ/2)(q + 1)B_p D_q A^* A.$$

Using (1) and (2) to eliminate the term $B_p D_q A^* A$ gives the conclusion.

**Corollary.** For $n$ a positive integer $(A A^*)^n$ is a linear combination of the $B_p D_q$ where $1 ≤ p ≤ n$, $0 ≤ q ≤ n$.

**Proof.** For $n = 1$, $A A^* = B_1 D_0$. 4.4 and an induction on $n$ gives the result in general.

4.5. Let $n$ be the greatest integer such that $A^n ≠ 0$. Then

$$(A A^*)^n \prod (A A^* - (λ/2)(q + 1)) = 0$$

where the product is taken over all pairs $p$, $q$ with $1 ≤ p ≤ n$, $0 ≤ q ≤ n$.

**Proof.** $(A A^*)^n$ is a linear combination of terms of the form $B_p D_q$ with $1 ≤ p ≤ n$, $0 ≤ q ≤ n$. Thus it is sufficient to show that for each such pair $p$, $q$, $B_p D_q \prod (A A^* - (λ/2)(k)(m + 1)) = 0$ where the product is taken over all pairs $k$, $m$ with $p ≤ k ≤ n$, $q ≤ m ≤ n$. If we define the degree of $B_p D_q$ as $p + q$ then $B_p D_q (A A^* - (λ/2)(q + 1)) = (q + 1/p + q + 1)B_{p+1} D_q + (p/p + q + 1)B_p D_{q+1}$ (by 4.4) and hence is a sum of terms of degree greater than that of $B_p D_q$. If the degree of $B_p D_q$ is $n$ each of the terms on the right is zero since $A^{n+1} = A^{n+1} = 0$. In general an induction on the terms of higher degree will yield the conclusion.

**Corollary.** $A A^*$ and $A^* A$ have finite spectra contained in the set \{ $k(λ/2)$: $k = 0, 1, \cdots, n(n+1)$ \}. $[A, A^*]$ has spectrum contained in this set and its negatives.

**Proof.** Let $α$ be the commutative $C^*$ algebra generated by $A A^*$ and $A^* A$. 4.5
For any homomorphism $\alpha$ of $\mathfrak{a}$ onto the complex numbers the value of $\alpha$ at $AA^*$ must satisfy the same polynomial relation as $AA^*$, proving the first part. Because of the symmetry between $A$ and $A^*$, $A^*A$ must also satisfy a polynomial identity like that of 4.5. Since $[A, A^*] = AA^* - A^*A$, a similar argument applies here.

**Corollary.** Suppose $L$ is a finite-dimensional semi-simple complex Lie algebra with $\mathfrak{c}$ as a Cartan subalgebra and $\{h_\alpha, e_\alpha : \alpha$ a root\} is a Weyl basis of $L$ relative to $\mathfrak{c}$. Let $\sigma$ be the associated involution and $x^* = -\sigma(x)$ for all $x$ in $L$. Suppose $\phi$ is a representation of $L$ as bounded operators on a Hilbert space with $\phi(x^*) = \phi(x)^*$ for all $x$.

(a) If $[x, x^*] = 0$ then $\phi(x)$ is diagonalizable with finite spectrum.

(b) The eigenvalues of $\phi(h_\alpha)$ are integer multiples of $(1/2)\alpha(h_\alpha)$.

(c) $\phi(\mathfrak{c})$ is diagonalizable.

(d) $\phi(e_\alpha)$ is nilpotent.

**Proof.** For each $\alpha$, $h_\alpha = [e_\alpha, e_\alpha^*]$ and $[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha$ together with the first corollary give (b). (d) is a consequence of this and 4.1. (c) is true since $\phi(\mathfrak{c})$ is spanned by the finitely many diagonalizable operators $\phi(h_\alpha)$ which are mutually commutative. If $[x, x^*] = 0$ then $x$ is contained in some Cartan subalgebra of the $L^*$ algebra $L$ and this subalgebra is spanned by elements of the form $[f_\beta, f_\beta^*]$ where $f_\beta$ is a root vector relative to it so that the arguments used in (a) and (b) can be used to prove $\phi(x)$ is diagonalizable.

A slightly improved version of 4.5 for a special case will be needed later and this is proved below.

4.6. Suppose $A$ is equal to $D_a$ for some $a$ in $L$ and $A^3 = 0$. Then the spectrum of $[A, A^*]$ lies in the set $\{k(\sqrt{2}) : k = 0, 1, 2, -1, -2\}$. 

**Proof.** By 3.2, $A^2A^*A = AA^*A^2$ and, using the argument in the proof of the first corollary of 4.2, each of these is equal to $\lambda A^2$. Because of symmetry similar relations hold with $A$ and $A^*$ interchanged.

Now $(AA^*)^2 = A^2A^*^2 + A[A^*, A]A^* = A^2A^*^2 + AA^*[A^*, A] + \lambda AA^*$ which implies $2(\lambda A^2)^2 = A^2A^*^2 + AA^*A + \lambda (AA^*)^2$. From these two relations a direct computation shows that $2(AA^*)^3 - 3(\lambda A^2)^2 + \lambda^2 AA^* = 0$ and (by symmetry) that the same relation holds for $A^*A$. Then an argument like that used in proving the corollary of 4.5 will finish the proof.

5. Reduction to the separable case.

**Definition.** Let $\mathfrak{c}$ be a Cartan subalgebra of $L$. If $L^\prime$ is a semi-simple subalgebra of $L$, $L^\prime$ will be called regular (with respect to $\mathfrak{c}$) if and only if $L^\prime$ is separable and $\mathfrak{c} = \mathfrak{c} \cap L^\prime$ is a Cartan subalgebra of $L^\prime$. It will be proved here that if each regular $L^\prime$ has a Cartan decomposition with respect to the accompanying $\mathfrak{c}^\prime$ then $L$ has a decomposition with respect to $\mathfrak{c}$. For $x$ in $L$ let $M(x)$ denote the smallest closed subspace of $L$ containing $x$ and invariant under $\mathfrak{c}$. Then $M(x) = \text{Sp} \{ V_n : n = 0, 1, \cdots \}$ where $V_0 = \text{Sp} \{ x \}$, $V_n = [\mathfrak{c}, V_{n-1}]$ for $n \geq 1$.

5.1. Let $x$ be fixed and let $B$ be the bounded operator on $\mathfrak{c}$ defined by
(Bh, h') = (Hx, H'x) for h, h' in C. Then B is self-adjoint and completely continuous.

**Proof.** \((Bh, h') \geq 0\) implies \(B = B^*\). Let \(E\) be the spectral measure on the spectrum of \(C\) such that \((Hy, z) = \int(h, x_\alpha)d(E(\alpha)y, z)\) for \(h\) in \(C\) and \(y, z\) in \(L\). Then \((Bh, h') = \int(h, x_\alpha)(h', x_\alpha)d(E(\alpha)x, x)\). If \(\{h_n\}\) converges weakly to \(h\) and \(\{x_n\}\) to \(h'\) then both sequences are bounded and the Lebesgue dominated convergence theorem implies \((Bh_n, h')\) converges to \((Bh, h')\). By [4, Definition 2, p. 206], \(B\) is completely continuous.

5.2. For \(x\) in \(L\) let \(C'(x) = \{h : h \in C, [h, x] = 0\}\) and let \(C(x)\) be the orthogonal complement of \(C'(x)\) in \(C\). If \(x\) is self-adjoint then so are \(C(x)\) and \(M(x)\) and both are separable.

**Proof.** It is clear that \(C'(x)\) is self-adjoint and hence the same is true of \(C(x)\). Since \(V_0\) is self-adjoint, induction on \(n\) proves that each \(V_n\) is also and hence \(M(x)\) is.

Let \(h' \in C\). By the definition of the operator \(B\) in 5.1, \((Bh, h') = 0\) for all \(h\) if and only if \(H'x = 0\). Hence \(C'(x)\) is the null-space of \(B\) and, since \(B\) is self-adjoint, \(C(x)\) is the closure of the range of \(B\). Since \(B\) is completely continuous, the reference in [4] shows that \(C(x)\) must be separable. Then an induction on \(n\) proves that \(V_n = [H(x), V_{n-1}]\) and that each \(V_n\) is separable so that \(M(x)\) is separable.

5.3. Suppose \(x\) is self-adjoint, nonzero, and orthogonal to \(C\). Let \(L'\) be the \(L^*\) algebra generated by \(C(x) + M(x)\). Then \(C(x) = \sigma(C)\cap L'\), \(L'\) is regular, \([C'(x), L'] = 0\), and \((C'(x), L') = 0\).

**Proof.** Since the orthogonal complement of \(C\) is invariant under \(C\) and contains \(x\) it also contains \(M(x)\) so that the indicated sum is actually direct. Since \(C(x)\) and \(M(x)\) are separable and self-adjoint it is possible to choose a countable (or finite) orthogonal basis of the space \(C(x) + M(x)\), say \(\{e_n\}\), such that each \(e_n\) is self-adjoint. Then a proof like that for 3.5 (a) shows that \(L'\) is spanned by products of the form \(E_{i_1} \cdots E_{i_n}e_n\) and, since the set of these is countable, \(L'\) is separable.

For \(h' \in C'(x)\), \(H'\) is zero on \(C(x) + M(x)\), hence on a set of generators of \(L'\). Since \(H'\) is a derivation, \(H'\) is zero on \(L'\), proving \([C'(x), L'] = 0\). Now \((C'(x), e_n) = 0\) for each \(n\) and, since \([C'(x), L'] = 0\), it follows readily that \(C'(x)\) is orthogonal to each of the finite products of generators, hence is orthogonal to \(L'\). This implies \(C(x) = \sigma(C)\cap L'\).

Finally, if \(y \in L'\) and \([y, C(x)] = 0\) then \([y, C(x)] = 0\) so that \(y \in C\) and hence \(y \in C(x)\). Thus \(C(x)\) is a maximal abelian subalgebra of \(L'\). If \(y \in L'\) and \([y, L'] = 0\) then \(y \in C(x)\cap C'(x)\) implies \(y\) is zero. Thus \(L'\) is semi-simple and \(C(x)\) is a Cartan subalgebra of \(L'\).

5.4. Suppose every regular subalgebra \(L'\) of \(L\) has a Cartan decomposition with respect to \(C' = C\cap L'\). Then \(L\) has a decomposition with respect to \(C\).

**Proof.** Let \(K\) be the \(L^*\) subalgebra of \(L\) obtained by letting \(K = C + V\) where \(V\) is the span of all the nonzero root vectors of \(L\) relative to \(C\). It is sufficient to prove \(K = L\). Now \(K\) is invariant under \(C\) so that \(K'\), the orthog-
onal complement of $K$ in $L$, is also invariant. Furthermore, $K'^* = K'$ and if $K' \neq 0$ there is a nonzero self-adjoint element $x \in K'$. Then $M(x) \subset K'$. Let $L'$ be the $L^*$ subalgebra generated by $M(x)$ and $M(x)$. By 5.3 $L'$ is regular so that the hypothesis here implies $L'$ has a Cartan decomposition with respect to $M(x)$. Since $M(x)$ is invariant under $M(x)$ it will be spanned by root vectors of $M(x)$ and hence there is a nonzero $\psi$ in $M(x)$ which is a common eigenvector for all $H, h \in M(x)$. But if $h' \in M(x)$ then $h'\psi = 0$ so that it follows immediately that $\psi$ is a common eigenvector for $M(x)$. Since $\psi \in M(x) \subset K'$ this gives the desired contradiction.


Remark. It will be proved here that if $L$ is simple and separable there is a Cartan subalgebra $\mathfrak{A}$ of $L$ such that $L$ as a decomposition with respect to $\mathfrak{A}$. Hence $L$ must be one of the five types $A, A', B, C, D$ obtained in [5]. Since each of these is a Lie subalgebra of an $H^*$ algebra, Theorem 2 of [5] shows that $L$ has a decomposition with respect to any Cartan subalgebra. From this it is clear that any separable semi-simple $L^*$ algebra has a decomposition with respect to any Cartan subalgebra. Finally, 5.4 shows that this is true with no restriction on the dimension of $L$.

6.1. Suppose $a_1, a_2$ are self-adjoint elements of $L$ and $A_1 A_2 = 0$. Then either $a_1 = 0$ or $a_2 = 0$.

Proof. Since $A_i$ is self-adjoint, $A_2 A_1$ is also zero. Let $C_i$ be the null-space of $A_i$ and $R_i$ the closure of the range of $A_i$, $i = 1, 2$. Then $L = C_i + R_i$ and both $C_i$ and $R_i$ are self-adjoint. Let $I_i = \text{Sp} \{ R_i, [R_i, R_i] \}$. Since $A_1 A_2 = 0$ then $(R_1, R_2) = 0$. Also $[[a_1, L], [a_2, L]] = [[[[a_1, L], a_2], L] + [a_2, [[a_1, L], L]] = [a_2, [[a_1, L], L]] \subset R_2$. Similarly $[[a_1, L], [a_2, L]] \subset R_1$. From this we have $[R_1, R_2] \subset R_1 \cap R_2 = 0$. Hence the Jacobi identity gives $[I_1, I_2] = 0$. From the above it is easy to see that $I_1$ is orthogonal to $I_2$. Let $W$ be the orthogonal complement of $I_1 + I_2$. Then $(W, R_i) = 0$ implies $W \subset C_i \cap C_i$. This in turn implies that $R_i$ is invariant under $W$. But $(W, R_i) = (W, [R_i, R_i]) = 0$ so that $[W, R_i] = 0$ and hence $[W, I_i] = 0$. Since $L = W + I_1 + I_2$ it follows immediately that $I_i$ is an ideal of $L$. By the simplicity of $L$ either $I_1$ or $I_2$ must be zero. Now $A_i L \subset R_i \subset I_i$ so that either $A_1$ or $A_2$ is zero.

Notation. By Corollary 2 of 3.8 there exists an element $a$ of $L$ such that $A^2 = 0, \|a\| = 1$, and $[[a, a^*], a] = \lambda a$ where $\lambda = \|[[a, a^*], a]\|^2 \neq 0$. Thus $[[A, A^*], A] = \lambda A$ and 4.6 implies $L = V_0 + V_{A/2} + V_{-A/2} + V_A + V_{-A}$ where $V_\mu$ is the eigenspace for $[A, A^*]$ with the indicated subscript as eigenvalue. The usual relations hold between these subspaces, i.e., $[V_\mu, V_\nu] \subset V_{\mu + \nu}$, and $V_\mu^* = V_{-\mu}$ for each $\mu$. In particular, $X^* = 0$ for all $x \in V_\lambda$.

6.2. Let $S = [V_\lambda, V_\lambda^*] + V_\lambda + V_\lambda^*$. Then $S$ is a semi-simple $L^*$ algebra.

Proof. Clearly $S^* = S$ and $[V_\lambda, V_\lambda] = [V_\lambda^*, V_\lambda^*] = 0$. Since $V_\lambda, V_\lambda^*$ are both invariant under $[V_\lambda, V_\lambda^*]$, so $[V_\lambda, V_\lambda^*] = 0$. From this it follows that $S$ is a subalgebra. If $x \in S$ and $[x, S] = 0$ then $[A, A^*]x = 0$ implies $x \in [V_\lambda, V_\lambda^*]$. But $(x, [V_\lambda, V_\lambda^*]) = 0$ since $[x, V_\lambda] = 0$. Thus $x$ is zero and $S$ is semi-simple.

Definition. By Zorn's Lemma it is possible to choose a subset $\mathcal{F}$ of $V$
which is maximal with respect to the following properties:

(i) $b \in \mathcal{F}$ implies $|b| = 1$ and $[(b, b^*), b] = \lambda_0 b \ (\lambda_0 = \|b, b^*\|^2)$.

(ii) $b, c \in \mathcal{F}$ implies $[(b, b^*), [c, c^*]] = 0$.

Necessarily $a$ is in $\mathcal{F}$ since $[V_\lambda, V_\lambda^*] \subseteq V_\lambda$. Let $M = \text{Sp} \{ [b, b^*] : b \in \mathcal{F} \}$. Then $M$ is self-adjoint and abelian. Let $C(M) = \{ x : x \in S, [x, M] = 0 \}$.

6.3. Let $\Delta$ be the spectrum of $M$ (acting on $S$) and suppose $\alpha \in \Delta$ with $\alpha$ nonzero. Then $\|x_\alpha\| \geq (1/2) \lambda_0^{1/2}$.

**Proof.** For any $b$ in $\mathcal{F}$, $[(b, b^*), [a, a^*]] = \lambda$ implies $\lambda \leq \|a, a^*\| \|b, b^*\| = \lambda^{1/2} \lambda_0^{1/2}$ so that $\lambda \leq \lambda_0$. Since $\alpha$ is not zero there is a $b \in \mathcal{F}$ such that $(x_\alpha, [b, b^*]) \neq 0$. By the first corollary of 4.5 the spectrum of $[B, B^*]$ consists of integer multiples of $(1/2) \lambda_0$. Thus we must have $\|x_\alpha\| \|b, b^*\| \geq (1/2) \lambda_0$ which gives $\|x_\alpha\| \geq (1/2) \lambda_0^{1/2}$.

**Corollary.** There exist subspaces $V_k$ of $S$, invariant under $C(M)$, such that $S = C(M) + \sum V_k$ and $[V_k, V_k^*] \subseteq C(M)$.

**Proof.** The existence of the $V_k$'s is implied by 2.4. Since they are spectral subspaces they are invariant under all operators commuting with $\{ X : x \in M \}$, hence invariant under $C(M)$.

6.4. $M = C(M)$ and $M$ is a Cartan subalgebra of $S$.

**Proof.** For $x$ in $C(M)$, $[A, A^*]x = 0$ implies $x \in V_\alpha$, hence $x \in [V_\lambda, V_\lambda^*]$. Thus $V_\lambda$ is invariant under $C(M)$. Then if $W_k = V_\lambda \cap V_k$ we have $W_k$ is invariant under $X$ for any $x$ commuting with $M$, $V_\lambda = \sum W_k$, and $[W_k, W_k^*] \subseteq C(M)$.

Suppose $c \in W_k$. Then $W_k$ is invariant under $[C, C^*]$. Hence $\text{Sp} \{ Xc : X = [C, C^*], n = 0, 1, \ldots \} \subseteq W_k$. Since $C^* = 0$, the proof of 3.8 shows that there exists an orthonormal set $\{ e_i \} \subseteq W_k$ such that $[e_i, e_i^*]$, $e_i = \lambda_i e_i$, $[e_i, e_j^*] = [e_i, e_j^*] = 0$ for $i \neq j$, and $c = \sum \lambda_i e_i$. Then $[c, c^*] = \sum \lambda_i [e_i, e_i^*]$. By the maximality of $\mathcal{F}$, each $e_i \in F$ since $[e_i, e_i^*] \subseteq C(M)$. Hence $[e_i, e_i^*] \subseteq M$ so that $[c, c^*] \subseteq M$.

For future reference we will now prove that if $x$ is any element of $L$ with $[x, M] = 0$ and $(x, M) = 0$ then $[x, V_\alpha] = 0$. To see this note first that $x \in V_\alpha$ (since $[a, a^*] \subseteq M$) and this implies $V_\lambda$ and $V_k$ are invariant under $X$ so that $V_k$ is also invariant. If $c \in W_k$ then $[c, c^*] \subseteq M$ so that $0 = (x, [c, c^*]) = (Xc, c)$. Since the operator $X$ on $W_k$ is completely determined by the quadratic form $(Xc, c)$ this gives $X$ is zero on $W_k$ for each $k$ so that $X$ is zero on $V_\lambda$.

The preceding paragraph shows that if $x \in C(M)$ and $(x, M) = 0$ then $[x, V_\lambda] = 0$. Now $M$ is self-adjoint which implies the same for $C(M)$ and thus $x^* \in C(M)$, $(x^*, M) = 0$ so that we also have $[x^*, V_\lambda] = 0$ and this gives $[x, V_\lambda^*] = 0$. But then $[x, [V_\lambda, V_\lambda^*]]$ is also zero and this implies $X$ is zero on $S$ so that $x = 0$. Hence $M = C(M)$ so that $M$ is maximal abelian in $S$, hence a Cartan subalgebra of $S$.

6.5. $S$ has a Cartan decomposition with respect to $M$.

**Proof.** Using the notation of 6.4 we now have $V_\lambda = \sum W_k$ with $[W_k, W_k^*] \subseteq C(M)$.
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Let $k$ be fixed and let $S_1 = [W_k, W_k^*] + W_k + W_k^*$. Then it is easily seen that $S_1$ is an $L^*$ subalgebra since $W_k$ is invariant under $M$. Let $P$ be the projection of $S_1$ onto $S_1$ and $a_0 = P[a, a^*]$. Then for $z \in S_1$, $[a_0, z] = P[a, a^*] = [a, a^*] = z$ since $Z$ and $Z^*$ leave $S_1$ invariant. Hence if $[z, S_1] = 0$ then $[z, [a, a^*]] = 0$ so that $z \in [W_k, W_k^*]$. But $([z, [W_k, W_k^*]]) = 0$ since $Z, W_k = 0$, hence $z$ is zero and $S_1$ is semi-simple. Now $[W_k, W_k^*] \subset M$ implies $[W_k, W_k^*]$ is abelian and the proof above shows that it is maximal abelian in $S_1$, thus a Cartan subalgebra. By 3.8, $S_1$ is a direct sum of simple ideals $I_j$ where $I_j = \text{Sp}\{e_i, e_i^*, [e_i, e_i^*]\}$ for some $e_i \in W_k$. If $x \in M$ then $[x, S_1] \subset S_1$ and $X$ is zero on $S_1$ if and only if it is zero on $W_k$ which is equivalent to $(x, [W_k, W_k^*]) = 0$. From this it follows readily that each $e_i$ is a common eigenvector for all $X, x \in M$. Thus each $W_k$, and therefore $V_1$, is spanned by root vectors for $M$. By symmetry the same is true of $V_k^*$. Since $S$ is generated by $V_1$ and $V_k^*$, and since each finite product of eigenvectors for $M$ is again an eigenvector for $M$, $S$ is spanned by eigenvectors and this completes the proof.

**Definition.** Let $\mathfrak{s}_1$ be a set in $V_{1,\lambda}$ which is maximal with respect to the following properties:

(i) $b \in \mathfrak{s}_1$ implies $\|b\| = 1, [b, b^*], b] = \lambda_b b$,

(ii) $b, c \in \mathfrak{s}_1$ implies $[b, b^*], [c, c^*], = 0$,

(iii) $b \in \mathfrak{s}_1$ implies $[b, b^*], M] = 0$.

Let $\mathfrak{s}_0 = \text{Sp}\{ [b, b^*]: b \in \mathfrak{s}_1 \}$. Then $M \subset \mathfrak{s}_0$ and $\mathfrak{s}_0$ is abelian and self-adjoint. Let $\mathfrak{s} = \{x: [x, \mathfrak{s}_0] = 0\}$. Since $[a, a^*] \in \mathfrak{s}_0, \mathfrak{s} \subset V_0$. We will show that $\mathfrak{s}$ is the desired Cartan subalgebra. Note that $V_\mu$ is invariant under $\mathfrak{s}$ for each eigenspace $V_\mu$ of $[A, A^*]$.

6.6. Let $\Delta$ be the spectrum of $\mathfrak{s}_0$ and suppose $\alpha$ is a nonzero element of $\Delta$. Then $\|x_\alpha\| \leq (1/4)\lambda^{1/2}$. Hence $L = \mathfrak{s} + \sum V_\lambda$ where $[V_\lambda, V_\lambda^*] \subset \mathfrak{s}$ and $V_\lambda$ is invariant under $\mathfrak{s}$.

**Proof.** For $b \in \mathfrak{s}_1$, $([b, b^*], [a, a^*]) = (1/2)\lambda$ implies $\lambda_b^{1/2} \lambda^{1/2} \leq (1/2)\lambda$ so that $\lambda_b \leq (1/4)\lambda$. The remainder of the argument is like that of 6.3 and the corollary.

6.7. $\mathfrak{s}$ is abelian and is a Cartan subalgebra of $L$.

**Proof.** If $\mathfrak{s}$ is abelian it is necessarily maximal abelian so that we need only prove the first assertion.

Let $W_k = V_{1,\lambda} \cap W_k$. Then $W_k$ is invariant under $\mathfrak{s}$ and $V_{1,\lambda} = \sum W_k$. Also $[W_k, W_k^*] \subset \mathfrak{s}$. Suppose $\mathfrak{s}$ is not abelian. Then $[\mathfrak{s}, \mathfrak{s}]$ is a semi-simple $L^*$ subalgebra and $(\mathfrak{s}_0, [\mathfrak{s}, \mathfrak{s}]) = 0$. Thus there exists a $\mathfrak{w} \in [\mathfrak{s}, \mathfrak{s}^*]$ with $\|\mathfrak{w}\| = 1$ and $[\mathfrak{w}, [\mathfrak{w}, \mathfrak{w}^*], w] = \mu \mathfrak{w}$. Also $(\mathfrak{w}, \mathfrak{w}^*) = 0$.

Now $[W, W^*]$ has spectrum contained in the set $\{r\mu\}$ where $r$ is a half-integer (by the first corollary of 4.5). Let $T_{r\mu}$ be the eigenspace associated with the value $r\mu$. Choose a fixed $W_k$. Since $W_k$ is invariant under $[W, W^*], W_k = \sum Z_{r\mu}$ where $Z_{r\mu}$ is the intersection of $W_k$ with $T_{r\mu}$.

Suppose $r \neq 0$. Then $[Z_{r\mu}, Z_{r\mu}] \subset T_{2r\mu} \subset \text{Range} [W, W^*]$. But $[Z_{r\mu}, Z_{r\mu}]$
Since \([w, w^*] = 0\), the remark made in the proof of 6.4 shows that \([W, W^*]\) is zero on \(V_\lambda\) and thus \(V_\lambda\) is contained in the null-space of \([W, W^*]\). Hence \([Z_{r^*}, Z_{r^*}] = 0\).

Now suppose some \(Z_{r^*} \neq 0\) for \(r \neq 0\). For this \(r\) let \(S_1 = [Z_{r^*}, Z_{r^*}] + Z_{r^*} + Z_{r^*}^*\). Since \([Z_{r^*}, Z_{r^*}] \subset \mathcal{C} \cap \) null-space \([W, W^*]\) then \(Z_{r^*} = W_{r^*} \cap T_{r^*}\) is invariant under \([Z_{r^*}, Z_{r^*}]\) so it is easy to see that \(S_1\) is an \(L^*\) subalgebra. Using the technique of projecting \([w, w^*]\) onto \(S_1\), the proof of 6.5 can be used to show that \(S_1\) is semi-simple. Let \(c \in S_1\). Then \(c\) is a normalized root vector corresponding to some root \(\alpha\) then \(E_{\alpha} = 0\). There exists an \(e \in L'\) such that the eigenspace corresponding to the maximal eigenvalue of \([E_{\alpha}, E_{\alpha}^*]\) is finite-dimensional.

Proof. It is easily seen that \(\beta + 3\alpha\) is never a root for any root \(\alpha\) in the same root set as \(\alpha\) and this implies \(E_{3\alpha} = 0\). Hence the spectrum of \([E_{\alpha}, E_{\alpha}^*]\) is contained in the set \(\{\alpha, \beta : \alpha \in \mathcal{C} \}\). Now the eigenspace in question is spanned by root vectors \(e_{\beta}\) with \((\alpha, \beta) = (\alpha, \alpha)\). Using the notation of [5], the roots are obtained from the set \(\{\lambda_i - \lambda_j, 2\lambda_i, \lambda_i : i, j\}^\infty\) with \((\lambda_i, \lambda_j) = \delta_{i, j}\). Furthermore, \(L'\) must contain a root of the form \(\lambda_i - \lambda_j\). Taking \(\alpha = \lambda_i - \lambda_j\), \((\alpha, \beta) = (\alpha, \alpha)\) implies \(\beta\) is contained in the finite set \(\{\lambda_i - \lambda_j, 2\lambda_i\}\).

6.10. \(L\) has a Cartan decomposition with respect to \(\mathcal{C}\).

Proof. Let \(R\) be the (nonempty) set of all nonzero roots of \(L\) relative to \(\mathcal{C}\) and choose a normalized root vector \(e_{\alpha}\) for each \(\alpha \in R\). Let \(V' = \text{Sp}\{e_{\alpha} : \alpha \in R\}\), \(\mathcal{C}' = \text{Sp}\{e_{\alpha}^* : \alpha \in R\}\), and \(S' = \mathcal{C}' + V'\). Then \(S'\) is semi-simple, \(\mathcal{C}'\) is a Cartan subalgebra of \(S'\), and \(S\) has a Cartan decomposition with respect to \(\mathcal{C}'\). Then \(W\) is invariant under both \(S'\) and \(\mathcal{C}\). If \([S', W] = 0\) then it is immediate that \([S', L] \subset S'\) so that \(S'\) is a nonzero ideal, \(S' = L\), and the existence theorem is proved.
By the remarks above it is enough to prove \([S', W] = 0\). Now \(W\) is invariant under \(\mathcal{C}\) so that \(W = \sum W_k\) where \(W_k\) is invariant under \(\mathcal{C}\) and \([W_k, W_k^*] \subseteq \mathcal{C}\).

Let \(S''\) be any simple ideal of \(S\). By the classification theory of [5] and 6.9, there is an \(e_a\) in \(S''\) such that the eigenspace associated with the maximal eigenvalue of \([E_a, E_a^*]\) (restricted to \(S''\)) is finite-dimensional. We will show that \([E_a, E_a^*]\) is zero on \(W\).

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Since \([[e_a, e_a^*], e_a] = \mu e_a\), \(L\) is spanned by subspaces \(T_r\), where \(T_r\) is the eigenspace for \([E_a, E_a^*]\) associated with the value \(r\mu\), \(r\) being a half-integer. Suppose \(W_k \cap T_r\) contains a nonzero subspace \(U\) invariant under \(\mathcal{C}\) with \([U, U] = 0\) for some \(k\). If \(S_1 = [U, U^*] + U + U^*\) it is easy to see that \(S_1\) is a semi-simple \(L^*\) subalgebra (see the proof of 6.5) with \([U, U^*]\) as a Cartan subalgebra. By 3.8, \(S_1\) is a direct sum of three-dimensional ideals \(I_j\) with \(I_j = \text{Sp}\{e_j, e_j^*, [e_j, e_j^*]\}\) for some \(e_j\) in \(U\). Thus each \(e_j\) is a root vector for \([U, U^*] \subseteq \mathcal{C}\). Since \(S_1\) is invariant under \(\mathcal{C}\) it follows as in the proof of 6.8 that each \(e_j\) is a root vector for \(\mathcal{C}\), a contradiction of the definition of \(S''\).

Thus if \(r_{\alpha}\) is the maximal eigenvalue for \([E_a, E_a^*]\), since \(T_\alpha\) is abelian, we must have \(W \cap T_{r_{\alpha}} = 0\). But then \((T_{r_{\alpha}}, W) = 0\) and this implies \(T_{r_{\alpha}}\) is entirely contained in \(S''\). Since \(E^3 = 0\) on \(S''\), \(r_0\) must equal one. By the choice of \(\alpha\), \(T_\alpha\) is finite-dimensional, say of dimension \(m\).

Suppose \([E_a, E_a^*] \neq 0\) on \(W\). Since \(W\) is self-adjoint we must have \(Z = T_{1,2} \cap W_k \neq 0\) for some \(k\). Then \(Z\) is invariant under \(\mathcal{C}\). Suppose \(Z = Z_1 + \cdots + Z_{m+1}\) with each \(Z_i\) invariant under \(\mathcal{C}\). Then \([Z_i, Z_j^*], \mathcal{C}\) = \(0\) for \(i \neq j\). Hence \([Z_i, Z_j^*] = 0\) for \(i \neq j\) and this implies \([Z_i, Z_j] = 0\). Since \([Z_i, Z_i] \subseteq T_{1}\) for each \(i\), at least one \(Z_i\) must be abelian, so that the remarks above imply that \(Z\) contains no more than \(m\) mutually orthogonal subspaces invariant under \(\mathcal{C}\). From this it is a simple consequence of the spectral theorem that \(Z\) must contain a root vector for \(\mathcal{C}\); a contradiction. Hence \([E_a, E_a^*]\) must be zero on \(W\). Using the simplicity of \(S''\), this implies the representation of \(S''\) on \(W\) obtained by restricting the adjoint representation is trivial. Since \(S''\) was an arbitrary simple component of \(S\) this implies \([S', W] = 0\).

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