THE ŠILOV BOUNDARY INDUCED BY A CERTAIN BANACH ALGEBRA(1)

BY

W. W. COMFORT(2)

1. **Introduction: preliminary definitions and discussion.** The l_1-algebra of a commutative semi-group G consists of all complex-valued functions α on G satisfying the condition $\infty > \sum_{z\in G} |\alpha(z)| = \|\alpha\|$, addition and scalar multiplication being defined point-wise and multiplication (convolution) being defined by the relation

$$(\alpha * \beta)(z) = \sum_{u,v\in G;\, uv=z} \alpha(u)\beta(v).$$

A multiplicative function f on G is a complex-valued function on G which satisfies the relation $f(uv) = f(u)\cdot f(v)$ for each pair of elements u, v in G. A semi-character on G is a bounded multiplicative function on G that is not identically zero. The set of semi-characters on the commutative semi-group G will be denoted by $\hat{G}$.

Many of the results of the present paper depend on the joint work of Hewitt and Zuckerman. They show in [3] that the set $\mathfrak{M}$ of nontrivial multiplicative linear functionals on $l_1(G)$, known to be in (1-1) correspondence with the space of regular maximal ideals in $l_1(G)$, may be identified with $\hat{G}$. Indeed, Theorem 2.7 of [3] shows that if with the function $\chi\in\hat{G}$ we associate the function $\eta(\chi) = \tau\in\mathfrak{M}$ defined by $\tau(\alpha) = \sum_{z\in G} \alpha(z)\chi(z)$, then η is a (1-1) mapping of $\hat{G}$ onto $\mathfrak{M}$.

We give the set $\hat{G}$ the topology it inherits when considered as a subset of the dual space $l_1^*(G)$ of $l_1(G)$. We recognize this relative topology on $\hat{G}$ as the product, Gel'fand, or w^* topology. For each $\chi\in\hat{G}$, the set $U = \{\psi\in\hat{G} \mid |\psi(z) - \chi(z)| < \epsilon \text{ for each } z\in F\}$, where $\epsilon > 0$ and F is a finite subset of G, is an open neighborhood of χ, and the set $\mathcal{B}(\chi)$ of such neighborhoods is a fundamental neighborhood system at χ. If 0 is isolated from $\hat{G}$ in $l_1^*(G)$, then $\hat{G}$ is compact [3, Theorem 8.5]. In any case $\hat{G}$ is locally compact, and

Presented to the Society, September 1, 1960, under the title *Šilov boundaries induced by certain Banach algebras*, April 19, 1958; received by the editors April 22, 1960.

(1) The present paper is based upon the author's doctoral dissertation (University of Washington, May, 1958). Research on that dissertation was supported in part by the National Science Foundation under Grant G-2053. The extensions of that dissertation here incorporated were obtained at Harvard University while the author was supported in part by Office of Ordnance Research, U. S. Army, under contract DA-19-020-ORD-3778.

(2) The author is indebted to Professor Edwin Hewitt for his encouragement and generous assistance during the preparation of this paper.

it is easy to see that if with the function $\alpha \in l_1(G)$ we associate the function $\hat{\alpha}$ defined on $\hat{G}$ by the relation $\hat{\alpha}(\chi) = \sum_{z \in G} \alpha(z)\chi(z)$, then $\hat{\alpha}$ is continuous on $\hat{G}$. The topology on $\hat{G}$ is, in fact, the smallest topology relative to which each $\hat{\alpha}$ is continuous.

If χ_1 and χ_2 are distinct elements of $\hat{G}$, so that there exists $x \in G$ for which $\chi_1(x) \neq \chi_2(x)$, then the function $\alpha \in l_1(G)$ defined on G by $\alpha(x) = 1$, $\alpha(z) = 0$ if $z \neq x$, has the property that $\hat{\alpha}(\chi_1) \neq \hat{\alpha}(\chi_2)$.

Straightforward computation verifies the assertion that $\hat{\alpha} \cdot \hat{\beta} = (\alpha * \beta)\hat{}$ for each α, β in $l_1(G)$, so that the set $(l_1(G))\hat{w} = \{\hat{\alpha} \mid \alpha \in l_1(G)\}$ is closed under point-wise multiplication. Thus $(l_1(G))\hat{}$ is a separating algebra of functions vanishing at ∞ and continuous on the locally compact Hausdorff space $\hat{G}$.

All hypotheses of Šilov's theorem (see [5, p. 80, Theorem 24E] or [2, §24]) being satisfied, we have also the conclusion: there is a unique closed subset ∂ of $\hat{G}$ with the following properties:

(a) if $\alpha \in l_1(G)$, then $|\hat{\alpha}|$ assumes its maximum on ∂;

(b) no proper closed subset of ∂ satisfies (a).

The set ∂ will be called the Šilov boundary induced in $\hat{G}$ by $(l_1(G))\hat{}$ or, less precisely, the boundary (see 9.1). All proofs known to the author of the existence of the boundary have been effected by transfinite arguments.

The present paper studies the nature of the boundary. Theorem 6.4 identifies those semi-groups G for which $\partial = \hat{G}$, and in our main result, Theorem 8.7, we offer, for each $\chi \in \hat{G}$, a collection $\mathcal{G}(\chi)$ of sets of semi-characters, characterized in purely algebraic terms, with the property that $\chi \in \partial$ if and only if $\chi \in \overline{A}$ for each $A \in \mathcal{G}(\chi)$.

2. **Two initial simplifications.** In 24.5 of [2], directly after defining and proving the existence of the boundary, Šilov made a remark which in our notation reads as follows: for each $\chi \in \hat{G}$, we have $\chi \in \partial$ if and only if there is, for each neighborhood U of χ, an $\alpha \in l_1(G)$ for which $|\hat{\alpha}|$ assumes its maximum only on U. Theorem 2.3 below shows that in the present context α may be chosen to vanish off of a finite subset of G.

The other result of this section is a theorem whose content justifies the assumption, made in all later sections of this paper, that the set $\hat{G}$ of semi-characters on G distinguishes points of G. It is easy to construct semi-groups not satisfying this property (see 9.2); this property has been shown in [3, Theorem 3.5] to be equivalent to the semi-simplicity of the algebra $l_1(G)$. In §§4 and 5.7 of [3], Hewitt and Zuckerman impose upon the commutative semi-group G an equivalence relation $\sim$ defined as follows: $u \sim v$ if and only if $\chi(u) = \chi(v)$ for each $\chi \in \hat{G}$. They show that if $T_u = \{v \in G \mid v \sim u\}$ and $G' = \{T_u \mid u \in G\}$, then, under multiplication defined by the relation $T_u \cdot T_v = T_{uv}$, G' becomes a commutative semi-group. The mapping $u \to T_u$ is a homomorphism of G onto G'.

2.1. THEOREM. *Let G be a commutative semi-group, and define G' as above.*

For each $\chi\in\hat{G}$ define $\eta(\chi)=\chi'$ on G' by the relation $\chi'(T_u)=\chi(u)$. Let ∂' denote the Šilov boundary induced in $(G')^\wedge$ by $(l_1(G'))^\wedge$. Then

(a) *η is a homeomorphism from $\hat{G}$ onto $(G')^\wedge$;*

(b) *$\eta(\partial)=\partial'$.*

Proof. (a) It is easy to see that η is (1-1). That $\chi'=\eta(\chi)$ is multiplicative, bounded and not identically zero follows from the corresponding facts for χ. If $U\in\mathcal{B}(\chi)$, say $U=\{\psi\in(G')^\wedge \mid |\chi(z)-\psi(z)|<\epsilon \text{ for each } z\in F\}$ where $\epsilon>0$ and F is a finite subset of G, then $\eta(U)=\{\psi'\in G' \mid |\chi'(T_z)-\psi'(T_z)|<\epsilon$ for each $T_z\in T_F\}$, where $T_F=\{T_z \mid z\in F\}$. Similarly $\eta^{-1}(U')\in\mathcal{B}(\chi)$ whenever $U'\in\mathcal{B}(\chi')$. Hence η is a homeomorphism.

(b) To show that $|(\alpha')^\wedge|$ assumes its maximum on $\eta(\partial)$ whenever $\alpha'\in l_1(G')$, we associate with each $T_z\in G'$ a point $f(T_z)\in T_z$. Now define α on G as follows:

if $x\in T_z$ and $x=f(T_z)$, then $\alpha(x)=\alpha'(T_z)$;

if $x\in T_z$ and $x\neq f(T_z)$, then $\alpha(x)=0$.

Let $|\hat{\alpha}|$ attain its maximum (over $\hat{G}$) at $\chi\in\partial$, and suppose that there exists a point $\psi'=\eta(\psi)\in(G')^\wedge$ for which $|(\alpha')^\wedge(\psi')|>|(\alpha')^\wedge(\chi')|$. Then $|\hat{\alpha}(\psi)|=|(\alpha')^\wedge(\psi')|>|(\alpha')^\wedge(\chi')|=|\hat{\alpha}(\chi')|$, a contradiction establishing the fact that $|(\alpha')^\wedge|$ assumes its maximum (over $(G')^\wedge$) at the point $\chi'=\eta(\chi)\in\eta(\partial)$.

Now let D' be a proper closed subset of $\eta(\partial)$, so that $D=\eta^{-1}(D')$ is a proper closed subset of ∂ and there is a point $\alpha\in l_1(G)$ for which $|\hat{\alpha}|$ assumes its maximum only on $\partial\backslash D$. Defining $\alpha'(T_z)=\sum_{x\in T_z}\alpha(x)$ and letting $|\hat{\alpha}|$ assume its maximum at $\chi\in\partial\backslash D$, we have $|(\alpha')^\wedge(\chi')|=|\hat{\alpha}(\chi)|>|\hat{\alpha}(\psi)|=|(\alpha')^\wedge(\psi')|$ for each $\psi'\in(G')^\wedge\backslash[\eta(\partial)\backslash D']$.

$\eta(\partial)$ being a closed subset of $(G')^\wedge$ that satisfies conditions (a) and (b) of §1, we conclude that $\eta(\partial)=\partial'$.

2.2. Remark. The theorem above shows that we may reasonably restrict our investigations into the nature of ∂ to the case in which $\hat{G}$ separates points of G. For suppose the boundary were known for each such commutative semi-group, and let G be any commutative semi-group. Construct G' and η as above and let the boundary induced in $(G')^\wedge$ by $(l_1(G'))^\wedge$ be ∂'. Then, by 2.1, ∂ is given by the formula $\partial=\eta^{-1}(\partial')$.

We shall henceforth assume, then, that $G=G'$, a condition shown in Theorems 2.8 and 3.5 of [3] to be equivalent to the condition that the homomorphism of $l_1(G)$ onto $(l_1(G))^\wedge$ given by $\alpha\to\hat{\alpha}$ is actually an isomorphism.

2.3. Theorem. *Let $\chi\in\hat{G}$. Then the following assertions are equivalent:*

(a) $\chi\in\partial$;

(b) *for each $U\in\mathcal{B}(\chi)$ there are an $\alpha\in l_1(G)$ and a $\chi'\in U$ for which α vanishes off of a finite subset of G and $|\hat{\alpha}(\psi)|<|\hat{\alpha}(\chi')|$ whenever $\psi\notin U$.*

Proof. The implication (b)$\to$(a) is a special case of Šilov's observation [2, 28.5]. If (a) holds and if $U\in\mathcal{B}(\chi)$, then there are $\beta\in l_1(G)$ and $\chi'\in U$ for

which $|\hat{\beta}(\chi')|>M$, where $M=\sup_{\psi\in U}|\hat{\beta}(\psi)|$. There is a finite subset F of G for which $\sum_{z\notin F}|\hat{\beta}(z)|<[|\hat{\beta}(\chi')|-M]/4$, and we may obtain (b) by defining $\alpha=\beta\cdot f$, where f denotes the characteristic function of the set F.

3. **Two important inclusions.** We show in this section that ∂ lies between two closed subsets of $\hat{G}$, each of which is easily identified. Our first result, 3.3, shows that ∂ contains each semi-character whose multiplicative inverse exists in $\hat{G}$.

3.1. LEMMA. *Let γ be a complex number such that $\gamma\neq 1$ and $|\gamma|\leqq 1$. Then $|\gamma+\gamma^2|<2$.*

3.2. LEMMA. *If χ is a semi-character on G and $x\in G$, then $|\chi(x)|\leqq 1$.*

3.3. THEOREM. *Let $|\chi|=1$. Then $\chi\in\partial$.*

Proof. We need only show that for each $U\in\mathcal{B}(\chi)$ there is an $\alpha\in l_1(G)$ for which $|\hat{\alpha}|$ assumes its maximum only on U. Let $U=\{\psi\in\hat{G}\,|\,|\psi(x)-\chi(x)|<\epsilon$ for each $x\in F\}$, where F is a finite subset of G and $\epsilon>0$. We may obviously suppose that $\epsilon<1/2$. Define $J=F\cup\{x^2\,|\,x\in F\}$, let $m=\text{card } J$, and define α on G by the following relations:

if $z\in J$, then $\alpha(z)=\bar{\chi}(z)$;

if $z\notin J$, then $\alpha(z)=0$.

Then $\alpha\in l_1(G)$, and we have

$$\hat{\alpha}(\chi)=\sum_{z\in G}\alpha(z)\chi(z)=\sum_{z\in J}\alpha(z)\chi(z)=m.$$

Now let $\psi\notin U$. Then there exists an $x\in F$ for which $\psi(x)\neq\chi(x)$. If $\psi(x)=0$, then

$$|\hat{\alpha}(\psi)|=\left|\sum_{z\in J}\alpha(z)\psi(z)\right|=\left|\sum_{z\in J;\,z\neq x}\alpha(z)\psi(z)\right|\leqq m-1<m=|\hat{\alpha}(\chi)|.$$

If $\psi(x)\neq 0$, then $x\neq x^2$. For if $x=x^2$, then every semi-character assumes at x either the value 0 or the value 1, and in the present case we would have $\psi(x)=1=\chi(x)$. Then, since $x\neq x^2$, we have

$$\begin{aligned}|\hat{\alpha}(\psi)| &= \left|\sum_{z\in G}\alpha(z)\psi(z)\right|=\left|\sum_{z\in J}\alpha(z)\psi(z)\right|\\ &\leqq\left|\sum_{z\in J;\,z\neq x,\,z\neq x^2}\alpha(z)\psi(z)\right|+|\alpha(x)\psi(x)+\alpha(x^2)\psi(x^2)|\\ &\leqq m-2+|\gamma+\gamma^2|,\end{aligned}$$

where $\gamma=\alpha(x)\psi(x)=\bar{\chi}(x)\psi(x)\neq 1$. Since $|\gamma|\leqq 1$, we have from 3.1 the relation $|\gamma+\gamma^2|<2$, so that $|\hat{\alpha}(\psi)|<m-2+2=|\hat{\alpha}(\chi)|$. Thus $|\hat{\alpha}(\psi)|<|\hat{\alpha}(\chi)|$ for each $\psi\notin U$, so that $|\hat{\alpha}|$ assumes its maximum only on U.

3.4. COROLLARY. *Let G be a group. Then $\partial=\hat{G}$.*

Proof. If there were $\chi\in\hat{G}$ and $z\in G$ for which $|\chi(z)|\neq 1$, then either $|\chi(z)|>1$ or $|\chi(z^{-1})|>1$, contrary to 3.2. Hence 3.3 applies.

3.5. REMARK. Theorem 3.3 shows that the boundary contains the group of semi-characters mapping G into the group $\{w \mid w$ is a complex number and $|w|=1\}$. Our next result bounds ∂ from above.

3.6. THEOREM. *Let* $\Gamma=\{\chi\in\hat{G} \mid |\chi|=0$ *or* $1\}$. *Then* $\partial\subset\Gamma$.

Proof. Let $\chi\notin\Gamma$, so that there exists an $x\in G$ and an $\epsilon>0$ for which $0<|\chi(x)|-2\epsilon<|\chi(x)|+2\epsilon<1$. Let $U=\{\psi\in\hat{G} \mid |\psi(x)-\chi(x)|<\epsilon\}$. By 2.3 it will suffice, in order to show that $\chi\notin\partial$, to prove that if α is an element of $l_1(G)$ that vanishes off a finite subset of G, then $|\hat{\alpha}|$ assumes its maximum somewhere on $\hat{G}\backslash U$. Suppose that $|\hat{\alpha}|$ assumes its maximum at $\psi\in\hat{G}$. Only the case $\psi\in U$ requires our attention. If $|\hat{\alpha}(\psi)|=0$, then $|\hat{\alpha}|$ assumes its maximum, 0, also at the semi-character $1\notin U$. Otherwise we let $x_1, x_2, \cdots, x_n$ be exactly those points z of G for which $\alpha(z)\psi(z)\neq 0$, and for each complex number w we define the function ψ_w on G as follows:

if $\psi(z)=0$, then $\psi_w(z)=0$;

if $\psi(z)\neq 0$, then $\psi_w(z)=\psi(z)\cdot e^{w \log|\psi(z)|}$,

where log denotes the elementary logarithm.

Now let u and v be arbitrary points of G. If $\psi(u)=0$ or $\psi(v)=0$, then $\psi(uv)=0$ and $\psi_w(uv)=0=\psi(u)\psi(v)=\psi_w(u)\psi_w(v)$. If $\psi(u)\neq 0$ and $\psi(v)\neq 0$, then $\psi(uv)\neq 0$ and $\psi_w(uv)=\psi(uv)e^{w \log|\psi(uv)|}=\psi(u)e^{w \log|\psi(u)|}\cdot\psi(v)e^{w \log|\psi(v)|}=\psi_w(u)\cdot\psi_w(v)$. If $\Re(w)>-1$ and $z\in G$ with $\psi(z)\neq 0$, then $|\psi_w(z)|=|\psi(z)e^{w \log|\psi(z)|}|=|e^{(w+1) \log|\psi(z)|}|=|e^{\log|\psi(z)|\Re(w+1)}|\leqq 1$, the final weak inequality being derived from the fact that $-\infty<\log|\psi(z)|\leqq 0$. These computations show that if $\Re(w)>-1$, then $\psi_w\in\hat{G}$.

Now find a number $\delta>0$ with the property that if $0<\epsilon\leqq r\leqq 1$ and $0<s<\delta$, then $r^s>1-\epsilon$. For each complex number w we write

$$f(w) = \sum_{i=1}^{n} \alpha(x_i)e^{w \log|\psi(x_i)|}\cdot\psi(x_i).$$

Then f is an entire function. We now distinguish two cases.

CASE I. f is constant. Let w_0 be a nonzero real number for which $0<w_0+1<\delta$. The relation $\psi\in U$ implies that $\epsilon<|\psi(x)|<1$, so that $|\psi_{w_0}(x)|=|\psi(x)|^{w_0+1}>1-\epsilon$, and $\psi_{w_0}\notin U$. Also

$$\begin{aligned}|\hat{\alpha}(\psi_{w_0})| &= \left|\sum_{z\in G}\alpha(z)\psi_{w_0}(z)\right| = \left|\sum_{z\in G;\psi(z)\neq 0}\alpha(z)e^{w_0 \log|\psi(z)|}\cdot\psi(z)\right| \\ &= \left|\sum_{i=1}^{n}\alpha(x_i)e^{w_0 \log|\psi(x_i)|}\cdot\psi(x_i)\right| = |f(w_0)| = |f(0)| \\ &= \left|\sum_{i=1}^{n}\alpha(x_i)\psi(x_i)\right| = |\hat{\alpha}(\psi)|,\end{aligned}$$

and $|\hat{\alpha}|$ assumes its maximum at the point $\psi_{w_0}\notin U$.

CASE II. f is nonconstant. Then there exists a complex number $w_0 \neq 0$ for which $1-\epsilon < |w_0| < 1$ and satisfying $|f(w_0)| > |f(0)|$. The relation $\mathcal{R}(w_0) > -1$ implies that $\psi_{w_0} \in \hat{G}$, and

$$\begin{aligned} |\hat{\alpha}(\psi_{w_0})| &= \left| \sum_{z \in G} \alpha(z)\psi_{w_0}(z) \right| = \left| \sum_{z \in G; \psi(z) \neq 0} \alpha(z) e^{w_0 \log |\psi(z)|} \cdot \psi(z) \right| \\ &= \left| \sum_{i=1}^{n} \alpha(x_i) e^{w_0 \log |\psi(x_i)|} \cdot \psi(x_i) \right| = |f(w_0)| \\ &> |f(0)| = |\hat{\alpha}(\psi)|. \end{aligned}$$

This contradiction returns us to Case I and completes the proof.

3.7. REMARK. Let B denote the set of semi-characters χ for which $x \in G$ implies $|\chi(x)| = 1$. Then Theorems 3.3 and 3.6 assert simply that $B \subset \partial \subset \Gamma$. The semi-character 1 is always in B, and the case in which G is a group is an instance in which $B = \partial = \Gamma = \hat{G}$. That the sets under consideration do not always collapse so conveniently is shown by Example 9.3, where each of the inclusions $B \subset \partial \subset \Gamma \subset \hat{G}$ is proper.

4. **An equivalence relation on $\hat{G}$.** We introduce in this section an equivalence relation on $\hat{G}$ with the property that if one of two equivalent semi-characters is in the boundary, then the other is also. We show also that the boundary is closed under the formation of conjugates.

4.1. THEOREM. *Let B be defined as in 3.7, and let $\chi \in B$ and $\chi_1 \in \partial$. Then $\chi\chi_1 \in \partial$.*

Proof. We need only show that for each $U \in \mathcal{B}(\chi\chi_1)$ there is an α in $l_1(G)$ for which $|\hat{\alpha}|$ assumes its maximum only on U. Let $U = \{\psi \in \hat{G} \mid |\psi(x) - \chi\chi_1(x)| < \epsilon$ for each $x \in F\}$, where F is a finite subset of G and $\epsilon > 0$, and let $V = \{\psi \in \hat{G} \mid |\psi(x) - \chi_1(x)| < \epsilon$ for each $x \in F\}$, so that $V \in \mathcal{B}(\chi_1)$. Since $\chi_1 \in \partial$, there is by 2.3 a β in $l_1(G)$ and a χ' in V for which $|\hat{\beta}(\psi')| < |\hat{\beta}(\chi')|$ whenever $\psi' \notin V$. Let $\alpha = \beta\bar{\chi}$ and let $\chi'' = \chi'\chi$. Then $|\chi''(x) - \chi\chi_1(x)| = |\chi'(x) - \chi_1(x)| < \epsilon$ for each $x \in F$, so that $\chi'' \in U$. If $\psi \notin U$, then $\psi\bar{\chi} \notin V$ and $|\hat{\alpha}(\psi)| = |\hat{\beta}(\bar{\chi}\psi)| < |\hat{\beta}(\chi')| = |\sum_{x \in G} \alpha(x)\chi''(x)| = |\hat{\alpha}(\chi'')|$.

4.2. DEFINITION. If χ_1 and χ_2 are semi-characters, we shall write $\chi_1 \sim \chi_2$ if there exists a $\chi \in B$ for which $\chi\chi_1 = \chi_2$.

4.3. THEOREM. *The relation $\sim$ is an equivalence relation on $\hat{G}$.*

4.4. THEOREM. *For each $\chi \in \hat{G}$, let $C_\chi = \{\psi \mid \psi \sim \chi\}$. Then the boundary is the union of those equivalence classes C_χ which it intersects.*

Proof. The present theorem is a restatement of 4.1.

4.5. REMARK. If $\sim$ is extended to the set $\hat{G} \cup \{0\}$ of all bounded complex-valued multiplicative functions on G by defining $0 \sim 0$ and $0 \nsim \chi$ for each $\chi \in \hat{G}$, then $\sim$ becomes a congruence relation on $\hat{G} \cup \{0\}$. It follows that if

the space $\hat{G}$ is a semi-group under point-wise multiplication, then the equivalence relation $\sim$ of 4.2 is actually a congruence relation on $\hat{G}$.

4.6. REMARK. Theorems 4.1 and 4.4 suggest that if $\chi\in\partial$ and $|\psi|=|\chi|$, then $\psi\in\partial$. Example 9.4 shows that, even in the case that $\hat{G}$ is a semi-group, this need not be the case. Indeed, 9.4 gives an example of a function $\chi\in\partial$ for which $\chi^2\notin\partial$.

4.7. THEOREM. *Let $\chi\in\partial$. Then $\bar{\chi}\in\partial$.*

Proof. Let U be any neighborhood of $\bar{\chi}$ and let $V=\{\bar{\psi}|\psi\in U\}$. Then V is a neighborhood of χ and, since $\chi\in\partial$, there is a β in $l_1(G)$ and a $\chi'\in V$ for which $\psi\notin V$ implies $|\hat{\beta}(\psi)|<|\hat{\beta}(\chi')|$. We observe that $\overline{\chi'}\in U$ and define $\alpha=\bar{\beta}$. Then if ψ is any semi-character not in U, in which case $\bar{\psi}\notin V$, we have

$$\begin{aligned}|\hat{\alpha}(\psi)| &= \left|\sum_{x\in G}\alpha(x)\psi(x)\right| = \left|\sum_{x\in G}\bar{\alpha}(x)\bar{\psi}(x)\right| = \left|\sum_{x\in G}\beta(x)\bar{\psi}(x)\right| \\ &= |\hat{\beta}(\bar{\psi})| < |\hat{\beta}(\chi')| = \left|\sum_{x\in G}\beta(x)\chi'(x)\right| = \left|\sum_{x\in G}\bar{\beta}(x)\overline{\chi'}(x)\right| \\ &= \left|\sum_{x\in G}\alpha(x)\overline{\chi'}(x)\right| = |\hat{\alpha}(\overline{\chi'})|.\end{aligned}$$

5. **The Hewitt-Zuckerman decomposition of** G. In this section we review from [3] a number of results to be used later. Though this section is self-contained, the reader is referred for additional background and discussion, to [3, especially §§4 and 5]. Our Theorem 5.7, though not to be found in [3], is nevertheless in the Hewitt-Zuckerman spirit.

5.1. DEFINITION. For $u, v\in G$, write $u\sim v$ if, for each $\chi\in\hat{G}$, $\chi(u)=0$ if and only if $\chi(v)=0$. Let $H_u=\{v\in G|v\sim u\}$.

5.2. THEOREM. *The relation $\sim$ is an equivalence relation on G. The mapping $u\to H_u$ is a homomorphism of G onto $\{H_u|u\in G\}$, multiplication in the latter space being given by $H_uH_v=H_{uv}$.*

5.3. DEFINITION. If $u, v\in G$, we write $H_u\leqq H_v$ if and only if $H_uH_v=H_u$. For each $u\in G$, we write $A_u=\{v\in G|H_u\leqq H_v\}$.

5.4. THEOREM. *The partial ordering given by 5.3 makes the idempotent semi-group $\{H_u|u\in G\}$ a semi-lattice.*

5.5. THEOREM. *Let $u\in G$. Then H_u and A_u are sub-semi-groups of G. If $v\in H_u$ and $z\in H_u$ and $uz=vz$, then $u=v$.*

Proof. The first assertions are obvious, and the third follows from the fact that $\hat{G}$ separates points of G.

5.6. THEOREM. *Let S be a commutative semi-group with cancellation. Then S may be embedded isomorphically in a group.*

Proof. S may be embedded in the group of equivalence classes of formal quotients of elements of S, the two quotients u_1/v_1 and u_2/v_2 being considered equivalent if $u_1v_2=u_2v_1$.

5.7. THEOREM. *Let S be a commutative semi-group with cancellation, and suppose that whenever a complex-valued multiplicative function χ on S assumes a nonzero value at some $s\in S$, then χ vanishes nowhere on S. If S has an idempotent element, then S is a group.*

Proof. Let $e^2=e\in S$, and let

$$S_1 = \{u \in S \mid \text{ there is a } v \in S \text{ for which } uv = e\}.$$

The cancellation law in S easily yields the fact that $es=s$ for each $s\in S$. To show that S is a group, we need then only show that $S_1=S$. Let $\chi(s)=0$ for each $s\in S\backslash S_1$, and let $\chi(s)=1$ for each $s\in S_1$. Since $S_1\cdot S_1\subset S_1$, χ fails to be multiplicative only if there are points $s\in S\backslash S_1$, $s_1\in S_1$ and $t\in S$ with $st=s_1$. But this is impossible since if s_2 satisfies the condition $s_1s_2=e$, then $s(ts_2)=e$, so that $s\in S_1$.

5.8. COROLLARY. *If $u\in G$ and H_u contains an idempotent, then H_u is a group.*

6. Characterization of those semi-groups G for which $\partial=\hat{G}$.

6.1. LEMMA. *If $z\in G$, then there is at most one element (e, x) of $G\times G$ for which the following relations hold:*

(a) $ez=z$;

(b) $ex=x$;

(c) $xz=e$.

Proof. Suppose that (e_1, x_1) and (e_2, x_2) are elements of $G\times G$ such that $e_iz=z$, $ex_i=x_i$ and $x_iz=e_i$ $(i=1, 2)$. Then $e_1=x_1z=x_1ze_2=e_1e_2=e_1zx_2=zx_2=e_2$, and $x_1=e_1x_1=e_2x_1=x_2zx_1=x_2e_1=x_2e_2=x_2$.

6.2. COROLLARY. *Let G_1 and G_2 be nondisjoint subgroups of G, and let e_i denote the identity of G_i. Then $e_1=e_2$.*

Proof. Let $z\in G_1\cap G_2$ and let x_i be the inverse in G_i of z. Then $x_1=x_2$, whence $e_1=e_2$.

6.3. LEMMA. *Suppose that there exists a function $'$ on G to G satisfying the following conditions:*

(i) *$z''=z$ for each $z\in G$;*

(ii) *$(z_1z_2)'=z_1'z_2'$ for each z_1, z_2 in G;*

(iii) *$\chi(z)=\bar{\chi}(z')$ for each $z\in G$ and each $\chi\in\hat{G}$.*

Then $\partial=\hat{G}$.

Proof. We define a function $\tilde{}$ from $l_1(G)$ to $l_1(G)$ as follows: if $\alpha \in l_1(G)$, then $\tilde{\alpha}$ is that function on G whose value at $z \in G$ is $\tilde{\alpha}(z) = \bar{\alpha}(z')$. Then it is obvious that $(\tilde{\alpha})^{\sim} = \alpha$, that $\tilde{\alpha} + \tilde{\beta} = (\alpha + \beta)^{\sim}$, and that $\bar{\lambda}\tilde{\alpha} = (\lambda\alpha)^{\sim}$. The relation $\tilde{\alpha} * \tilde{\beta} = (\alpha * \beta)^{\sim}$ also holds; we omit the proof. The mapping $\tilde{}$ is continuous because $\|\tilde{\alpha}\| = \|\alpha\|$, and

$$\begin{aligned}\tilde{\alpha}^{\wedge}(\chi) &= \sum_{z \in G} \tilde{\alpha}(z)\chi(z) = \sum_{z \in G} \bar{\alpha}(z')\chi(z) = \sum_{z \in G} \bar{\alpha}(z')\bar{\chi}(z') \\ &= \sum_{z' \in G} \bar{\alpha}(z')\bar{\chi}(z') = (\hat{\alpha}(\chi))^{-}\end{aligned}$$

for each $\chi \in \hat{G}$. Thus $l_1(G)$ is a self-adjoint Banach algebra in the sense of [5, §26, pp. 87 and 88], and it follows that $\partial = \hat{G}$ (see, for example, [5, p. 88, Lemma 1 and corollary]).

6.4. THEOREM. *Let G be a commutative semi-group whose dual space $\hat{G}$ separates points. Then the following assertions are equivalent*:

(a) *G is a union of groups*;

(b) $\partial = \hat{G}$;

(c) $\Gamma = \hat{G}$.

Proof. (a)$\Rightarrow$(b). With each $z \in G$ we associate that element x of G for which simultaneously $ez = z$, $ex = x$ and $xz = e$ for some $e \in G$. This x is unique by 6.1. Writing $x = z'$, we see from 6.1 and 6.2 that the mapping $z \to z'$ satisfies the hypotheses of 6.3. Hence $\partial = \hat{G}$.

(b)$\Rightarrow$(c). From 3.6, we have $\partial \subset \Gamma \subset \hat{G}$.

(c)$\Rightarrow$(a). We use the following theorem of Ross, proved in [6]: if χ is a semi-character on the sub-semi-group T of the commutative semi-group S, then there is a semi-character χ' of S that agrees with χ on T if and only if, for each t_1, t_2 in T, the relation $|\chi(t_2)| \leqq |\chi(t_1)|$ holds whenever $st_1 = t_2$ for some $s \in S$. If there is a $z \in G$ for which H_z is a nongroup, then we apply Ross' theorem with $S = H_z$ and $T = \{z^n \mid n \text{ is an integer and } n > 0\}$. Choosing a complex number γ for which $0 < |\gamma| < 1$, we define $\chi(z^n) = \gamma^n$. Ross' hypotheses hold since if $xz^{n+k} = z^n$ for some positive integers n and k and $x \in G$, then H_z would contain the idempotent element xz^k, hence by 5.7 would be a group. Thus χ admits an extension $\chi' \in \hat{G}$, and since $0 < |\chi'(z)| < 1$ we have $\chi' \in \hat{G} \backslash \Gamma$.

6.5. COROLLARY. *Let G be an idempotent semi-group, in the sense that $z^2 = z$ for each $z \in G$. Then $\partial = \hat{G}$.*

7. Additive functions on torsion-free semi-groups.

7.1. DEFINITION. In what follows, we denote by R the rational field.

7.2. DEFINITION. If V is a linear space over the field K, a function f on V to K will be called a linear functional on V if it satisfies the relation $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$ for every two elements α and β of K and every two elements u and v of V.

7.3. LEMMA. *Let P be a linear space over R, and let F be a finite subset of P with the property that*

$$\text{if } \sum_{x\in F} \alpha(x)x = \varnothing, \text{ where each } \alpha(x) \geqq 0, \text{ then each } \alpha(x) = 0.$$

Then there is a linear functional f on P satisfying the following two conditions:

(a) *if $x\in F$, then $f(x)\geqq 0$*;

(b) *there is an x in F for which $f(x)\neq 0$.*

Proof. The proof presented in [8], in which the rational field R is replaced by the real line, carries over directly to the present context.

7.4. DEFINITION. A real-valued function p on G is said to be additive if $p(xy)=p(x)+p(y)$ for every two elements x and y of G.

7.5. THEOREM(3). *Let G be a commutative semi-group with cancellation in which each element has infinite order, and let F be a finite subset of G. Then there is an additive, rational-valued function p on G with the following two properties*:

(a) *$p\geqq 0$ on F*;

(b) *there is an $x\in F$ for which $p(x)\neq 0$.*

Proof. We have observed in 5.6 that G may be embedded in a commutative group G^0, which itself may be embedded in a divisible group D (see, for example, [4, p. 165]). Since D is the direct sum of a torsion-free group P and a torsion group K, we have $F\subset G^0\subset D=P\oplus K$, where P is an algebraic direct sum of rational fields R. The natural homomorphism $\phi: D\to D/K=P$ carries no $x\in G$ into the neutral element $\varnothing$ of P: for if $\phi(x)=\varnothing$ for an element x of G, then x has finite order in G.

Now let $\bar{G}=\phi(G)$, and suppose that $\sum_{i=1}^{n} \alpha_i\bar{x}_i=\varnothing$, where each of the rational numbers α_i is non-negative and each $\bar{x}_i$ has the form $\bar{x}_i=\phi(x_i)$ for some $x_i\in G$. If some $\alpha_i\neq 0$, then there are integers $\beta_j>0$ and elements $\bar{y}_j$ of $\bar{G}$ for which $\sum\beta_j\bar{y}_j=\varnothing$. But then if $\phi(y_j)=\bar{y}_j$ and $y=\prod y_j^{\beta_j}\in G$, it follows that $\phi(y)=\varnothing$. This contradiction shows that P and $\bar{F}=\phi(F)$ satisfy all hypotheses of 7.3. Then there is a linear functional f on P satisfying conditions (a) and (b) of 7.3. Now define $p=f\circ\phi$. Then p is additive because f is linear and ϕ is a homomorphism; $p(x)=f(\bar{x})\geqq 0$ for each $x\in F$; and if $\bar{x}=\phi(x)$ is an element of $\bar{F}$ for which $f(\bar{x})\neq 0$, then $p(x)=f(\bar{x})\neq 0$.

7.6. THEOREM. *Let G be a commutative semi-group with cancellation in which each element has infinite order, and let F be a finite subset of G. Then there is an additive, rational-valued function p on G for which $p(y)>0$ whenever $y\in F$.*

Proof. We prove, for each positive integer n, the statement $S(n)$: if k is a positive integer not less than n, and if F is a k-membered subset of G, then

(3) An outline of the proof here presented, together with reference [8], was suggested to the author by R. S. Pierce.

there is a rational-valued additive function p_n on G which is non-negative on F and positive on some n-membered subset of F.

$S(1)$ is a restatement of Theorem 7.5. If $S(n)$ holds for each $n<m$, and if all hypotheses of $S(m)$ are satisfied, then there is an additive, rational-valued function p_{m-1} on G which is non-negative on F and strictly positive on some $(m-1)$-membered subset Q of F. If $\overline{F}=F\backslash Q\neq\Lambda$, then there is by $S(1)$ an additive, rational-valued function $\bar{p}_1$ on G which is non-negative on $\overline{F}$ and positive at some point $y\in\overline{F}$. Let M be a positive integer for which $\bar{p}_1(z)+Mp_{m-1}(z)>0$ for each $z\in Q\cup\{y\}$. Then $p_m=\bar{p}_1+Mp_{m-1}$ is a function whose existence yields the truth of $S(m)$.

7.7. THEOREM. *Let S be a commutative semi-group, and let T be a sub-semi-group of S whose complement T' is an ideal, in the sense that $xy\in T'$ whenever $x\in S$ and $y\in T'$. Let $y_1, y_2, \cdots, y_m$ be elements of T', and suppose that there are no points x_1, x_2 in T, z in T' and u in T' for which $x_1zu=x_2u$. Then there exists an additive, rational-valued function p on S such that $p\equiv 0$ on T and $p(y_j)>0$ for $1\leqq j\leqq m$.*

Proof. For z_1, z_2 in T', we set $z_1\sim z_2$ if $x_1z_1u=x_2z_2u$ for some points $x_i\in T$ and $u\in T'$. Then $\sim$ is an equivalence relation on T', and for each $z\in T'$ we set $\bar{z}=\{w\in T'\mid w\sim z\}$. Relative to the multiplication $\bar{z}_1\bar{z}_2=(z_1z_2)^-$, the set $\overline{Z}=\{\bar{z}\mid z\in T'\}$ is a commutative semi-group. If $\bar{z}^n=\bar{z}$ for some $z\in T'$ and some integer $n>0$, then $x_2zu=x_1z^nu=x_1z^{n-1}(zu)$ for some points $x_i\in T$ and $u\in T'$; since $z^{n-1}\in T'$ and $zu\in T'$, this is impossible. Also if $\bar{z}\bar{z}_1=\bar{z}\bar{z}_2$ for some points z, z_i in T', so that $x_1zz_1u=x_2zz_2u$, then $x_1z_1(zu)=x_2z_2(zu)$, so that $z_1\sim z_2$ and $\bar{z}_1=\bar{z}_2$. Thus $\overline{Z}$ is a commutative semi-group with cancellation in which each element has infinite order. By 7.6 there is an additive, rational-valued function $\bar{p}$ on $\overline{Z}$ for which $\bar{p}(\bar{y}_j)>0$ whenever $1\leqq j\leqq m$. We define $p(z)=\bar{p}(\bar{z})$ for each $z\in T'$, and we extend p to the rest of S by defining $p(z)=0$ for each $z\in T$. It is easy to verify that p has the required properties.

8. Characterization of ∂.

8.1. DEFINITION. If $\chi\in\hat{G}$, then the support of χ, denoted $S(\chi)$, is the set $S(\chi)=\{z\in G\mid\chi(z)\neq 0\}$.

8.2. THEOREM. *For each $\chi\in\hat{G}$, $S(\chi)$ is a sub-semi-group of G, and $G\backslash S(\chi)$ is an ideal in G.*

8.3. DEFINITION. For each finite subset F of G, we let $\hat{G}_F$ be the collection of semi-characters χ in Γ with the property that if $y\in F$, then either

(1) there exist points x_1, x_2 in $S(\chi)$ for which $x_1y=x_2y$ and $\chi(x_1)\neq\chi(x_2)$, or

(2) there exist points x_1, x_2 in $S(\chi)$ and $z\in G\backslash S(\chi)$ for which $x_1yz=x_2y$.

8.4. THEOREM. *If F is a finite subset of G and $\chi\in\hat{G}_F$, then $F\cap S(\chi)=\Lambda$.*

8.5. THEOREM. *The relation $\hat{G}_F=\bigcap_{y\in F}\hat{G}_{\{y\}}$ holds for each finite subset F of G.*

8.6. REMARK. The following characterization of the boundary would assume a simpler, more elegant form if the subsets $\hat{G}_F$ of $\hat{G}$ were closed. That the sets $\hat{G}_F$ need not be closed is shown in 9.6, where we give an example of a semi-character χ lying in ∂ but in no $\hat{G}_F$; by 8.7 we have $\chi\in(\hat{G}_F)^-$ whenever $F\cap S(\chi)=\Lambda$. Examples of this sort suggest that, while the characterization of ∂ offered by 8.7 may fail to be "best possible," it may nevertheless be difficult to improve upon it. That is, it may be difficult to obtain a short list of criteria P_i, each expressed solely in terms of the algebraic structure of G and the action of χ on G, with the property that for each $\chi\in\hat{G}$ one has $\chi\in\partial$ if and only if χ satisfies each P_i.

8.7. THEOREM. *Let $\chi\in\hat{G}$. Then the following assertions are equivalent*:
(a) $\chi\in(\hat{G}_F)^-$ *whenever F is a finite subset of G with $F\cap S(\chi)=\Lambda$*;
(b) $\chi\in\partial$.

Proof. (a)$\Rightarrow$(b). If (b) fails, then there is a neighborhood $U\in\mathcal{B}(\chi)$ of χ whose complement in $\hat{G}$ contains ∂. We let $U=\{\psi\in\hat{G}\mid|\psi(z)-\chi(z)|<\epsilon$ for each $z\in F\}$ where $\epsilon>0$ and F is a finite subset of G. We define $F_1=F\cap S(\chi)$, $F_0=F\backslash S(\chi)$. Since $B\subset\partial$ by 3.3, we may suppose that $F_0\neq\Lambda$. Under the hypotheses of the present theorem there is a point $\chi'\in U\cap\hat{G}_{F_0}$. By 8.4, χ' vanishes on F_0. We write $F_0=F_0'\cup F_0''$, where $F_0'=\{y\in F_0\mid$ there exist x_1, x_2 in $S(\chi')$ for which $x_1y=x_2y$ and $\chi'(x_1)\neq\chi'(x_2)\}$ and $F_0''=F_0\backslash F_0'$. With each $y\in F_0'$ we associate appropriate points $x_1(y)$, $x_2(y)$ in $S(\chi')$, and with each $y\in F_0''$ we associate points $x_1(y)$, $x_2(y)$ in $S(\chi')$ and $z(y)$ in $G\backslash S(\chi')$ for which $x_1yz=x_2y$. We may suppose that $x_1(y)$ and $x_2(y)$ are in F_1 for each $y\in F_0$, and we set $J=F_1\cup\{x^2\mid x\in F_1\}$, $n=\text{card } J$. We may clearly suppose ϵ to be so small that $\epsilon<1/(4n)$, that $\epsilon<|\chi'(x_1(y))-\chi'(x_2(y))|/2$ for each $y\in F_0'$, and that $\mathcal{R}w>\cos(\pi/9)$ whenever w is a complex number for which $|w-1|<2\epsilon$. We now choose positive numbers η and k which satisfy the following conditions: $|1+w|<2-\eta$ whenever w is a complex number and $|w|=1$ and $|w-1|\geqq 2\epsilon$; $k>2(\text{card } F_0)/\eta$. We define $\alpha_0(x)=\overline{\chi'}(x)$ if $x\in J$, $\alpha_0(x)=0$ if $x\in G\backslash J$. For each $y\in F_0''$, we define $\alpha_y(z(y))=-\chi'(x_1(y))\overline{\chi'}(x_2(y))/k$, $\alpha_y(x)=0$ if $x\in G$ and $x\neq z(y)$. We define $\alpha=\alpha_0+\sum_{y\in F_0''}\alpha_y$, so that $\alpha\in l_1(G)$. Since $\chi'\in\Gamma$, we have $\hat{\alpha}(\chi')=(\alpha_0\hat{\ }(\chi')=\text{card } J=n$.

Now choose $\psi\in\hat{G}\backslash U$. The desired contradiction will be obtained by showing that $|\hat{\alpha}(\psi)|<|\hat{\alpha}(\chi')|$. It suffices to consider only the case $\psi\in\partial$.

CASE I. There is a point $u\in F_1$ for which $|\psi(u)-\chi'(u)|\geqq\epsilon$. If $u=u^2$ then, since $\psi(u)\neq\chi'(u)=1$, we must have $\psi(u)=0$. Then

$$\begin{aligned}|\hat{\alpha}(\psi)| &= \left|\sum_{x\in G}\alpha(x)\psi(x)\right| \leqq \left|\sum_{x\in S(\chi')}\alpha(x)\psi(x)\right| + \left|\sum_{x\notin S(\chi')}\alpha(x)\psi(x)\right| \\ &= \left|\sum_{x\in J;\,x\neq u}\alpha(x)\psi(x)\right| + \left|\sum_{x\notin S(\chi')}\alpha(x)\psi(x)\right| \leqq (n-1)+(\text{card } F_0)/k \\ &< (n-1)+\eta/2 < n = |\hat{\alpha}(\chi')|.\end{aligned}$$

If $u \neq u^2$, then

$$\begin{aligned} |\hat{\alpha}(\psi)| &= \left| \sum_{x \in G} \alpha(x)\psi(x) \right| \leqq \left| \sum_{x \in J;\, x \neq u,\, x \neq u^2} \alpha(x)\psi(x) \right| \\ &\quad + |\alpha(u)\psi(u) + \alpha(u^2)\psi(u^2)| + \left| \sum_{x \notin S(\chi')} \alpha(x)\psi(x) \right| \\ &\leqq (n-2) + (2-\eta) + \eta/2 < n = |\hat{\alpha}(\chi')|. \end{aligned}$$

CASE II. Case I fails. Then $|\psi(u) - \chi'(u)| < \epsilon$ for each $u \in F_1$, so that $|\psi(u) - \chi'(u)| < 2\epsilon$ for each $u \in J$. If $\psi(y) \neq 0$ for some $y \in F_0'$, then from the fact that $x_1(y) \in S(\psi)$ and $x_2(y) \in S(\psi)$ it would follow that $\psi(x_1(y)) = \psi(x_2(y))$, contradicting the relations

$$|\psi(x_i(y)) - \chi'(x_i(y))| < \epsilon < \left| \frac{\chi'(x_1(y)) - \chi'(x_2(y))}{2} \right|, \qquad (i = 1, 2).$$

This Case II can arise only when $\psi(y) = 0$ for each $y \in F_0'$. Let

$$Y = \{y \in F_0'' \mid \psi(y) \neq 0\}, \qquad m = \text{card } Y.$$

We know card $F_0'' \geqq m > 0$. Define

$$\begin{aligned} r_1 &= \sum_{x \in S(\chi')} \alpha(x)\psi(x), \\ r_2 &= \sum_{y \in Y} \alpha(z(y))\psi(z(y)) \\ &= \sum_{y \in Y} \chi'(x_1(y))\overline{\chi'}(x_2(y))\bar{\psi}(x_1(y))\psi(x_2(y))/k, \end{aligned}$$

so that $|\hat{\alpha}(\psi)| = |r_1 - r_2|$. We observe that $|r_1| \leqq n$ and $\Re r_1 > \cos(\pi/9)$, so that $|\Im r_1| < n \sin(\pi/9)$; also $|r_2| \leqq m/k$ and $\Re r_2 > (m/k)\cos(2\pi/9)$, so that $|\Im r_2| < (m/k)\sin(2\pi/9)$. Hence, setting $s_1 = ne^{i\pi/9}$ and $s_2 = (m/k)e^{-2i\pi/9}$, we have $|r_1 - r_2|^2 \leqq |s_1 - s_2|^2 = n^2 + (m/k)^2 - (2mn/k)\cos(\pi/3) = n^2 + (m/k)^2 - mn/k < n^2$, so that $|\hat{\alpha}(\psi)| < n = |\hat{\alpha}(\chi')|$.

(b)$\Rightarrow$(a). We shall suppose that (a) fails. Then there is a finite subset F_0 of $G \backslash S(\chi)$ for which $\chi \notin (\hat{G}_{F_0})^-$, so that for an appropriate finite subset F of G and an $\epsilon > 0$ we have $U \cap \hat{G}_{F_0} = \Lambda$, where $U = \{\psi \in \hat{G} \mid |\psi(z) - \chi(z)| < \epsilon$ for each $z \in F\}$. We may as well suppose that $F_0 \subset F$. To show that then $\chi \notin \partial$, it will suffice to show that if $\alpha \in l_1(G)$ and α vanishes off a finite subset of G, then $|\hat{\alpha}|$ assumes its maximum off U. Indeed, let $|\hat{\alpha}|$ be maximal at the point $\psi \in U \cap \partial$. Then $\psi \notin \hat{G}_{F_0}$ so by 8.5 there is a point $y \in F_0$ for which $\psi \notin \hat{G}_{\{y\}}$. We may suppose that $\epsilon < 1/2$, hence that $\Lambda \neq F \cap S(\chi) \subset S(\psi)$. We set $x = \prod\{x_i \in S(\psi) \mid x_i \in F$ or $\alpha(x_i) \neq 0\}$, so that $x \in S(\psi)$. We define $\psi' = \psi$ on A_x, $\psi' = 0$ elsewhere. Then $S(\psi') = A_x$ and $\psi' \in U$. Since $\psi \in \partial \subset \Gamma$, we see that $\psi' \in \Gamma$. Also $\hat{\alpha}(\psi) = \sum_{z \in S(\psi)} \alpha(z)\psi(z) = \sum_{z \in A_x} \alpha(z)\psi(z) = \hat{\alpha}(\psi')$. Since $S(\psi') \subset S(\psi)$, there can exist no points x_1, x_2 in $S(\psi')$ with $x_1 y = x_2 y$ and $\psi'(x_1)$

$\neq\psi'(x_2)$. Hence by [1, Theorem 4.2], there is a point $\psi''\in\Gamma$ for which $\psi''=\psi'$ on $S(\psi')$ and $\psi''(y)\neq 0$. Then $\psi''\notin U$. We may clearly suppose that ψ'' vanishes off A_{xy}, so that $S(\psi'')=A_{xy}$.

CASE I. $\alpha\equiv 0$ on $A_{xy}\backslash S(\psi')$. Then

$$\begin{aligned}\hat{\alpha}(\psi'') &= \sum_{z\in S(\psi')}\alpha(z)\psi''(z)\\ &= \sum_{z\in S(\psi')}\alpha(z)\psi'(z) = \hat{\alpha}(\psi') = \hat{\alpha}(\psi),\end{aligned}$$

so that $|\hat{\alpha}|$ assumes its maximum also at the point $\psi''\notin U$.

CASE II. Case I fails. We then let $y_1, y_2, \cdots, y_n$ be an enumeration of those points $z\in A_{xy}\backslash S(\psi')$ for which $\alpha(z)\neq 0$. None of the n points y_j can lie in $S(\psi)$, since if $z\in S(\psi)$ and $\alpha(z)\neq 0$, then $z\in A_x=S(\psi')$. We observe that A_{xy} is a commutative semi-group, $A_{xy}\cap S(\psi)$ a sub-semi-group of A_{xy} whose complement in A_{xy} is an ideal in A_{xy}. We claim further that there do not exist points x_1, x_2 in $A_{xy}\cap S(\psi)$ and z, u in $A_{xy}\backslash S(\psi)$ for which $x_1zu=x_2u$. For if such points did exist, then the relation $xx_1zy(xyu)=xx_2y(xyu)$ would hold; since each of the three points xx_1zy, xx_2y, xyu lies in the cancellation semi-group H_{xy}, it would follow that $(xx_1)zy=(xx_2)y$ which, since $xx_i\in S(\psi)$ and $z\in A_{xy}\backslash S(\psi)\subset G\backslash S(\psi)$, would contradict the fact that $\psi\notin\hat{G}_{\{y\}}$. Thus all hypotheses of 7.7 are satisfied, and there is an additive, rational-valued function p on A_{xy} for which $p\equiv 0$ on $A_{xy}\cap S(\psi)$ and $p(y_j)>0$ for $1\leqq j\leqq n$. We may take p to be integral at each y_j. Let θ be an irrational multiple of 2π, and define $\omega(z)=e^{i\theta p(z)}$ for each $z\in A_{xy}$. Then $\omega\equiv 1$ on $S(\psi)\cap A_{xy}$. Let $v=e^{i\theta}$, so that $\omega(y_j)=v^{p(y_j)}$, and define

$$f(w) = \sum_{j=1}^{n}\alpha(y_j)\psi''(y_j)w^{p(y_j)} + \sum_{z\in S(\psi')}\alpha(z)\psi'(z),$$

so that f is an entire function. If f is constant, then

$$\begin{aligned}\hat{\alpha}(\psi'') &= \sum_{j=1}^{n}\alpha(y_j)\psi''(y_j) + \sum_{z\in S(\psi')}\alpha(z)\psi''(z) = f(1)\\ &= f(0) = \sum_{z\in S(\psi')}\alpha(z)\psi'(z) = \hat{\alpha}(\psi') = \hat{\alpha}(\psi),\end{aligned}$$

so that $|\hat{\alpha}|$ assumes its maximum also at the point $\psi''\notin U$.

If f is nonconstant, then there is by the maximum modulus principle a complex number w_0 for which $|w_0|=1$ and $|f(w_0)|>|f(0)|$. Let W be a neighborhood of w_0 for which $|f(w)|>|f(0)|$ whenever $w\in W$. Since θ is an irrational multiple of 2π, there is an integer $m>0$ for which $v^m\in W$. The function $\psi''\omega^m$ is in $\hat{G}$, and we have $S(\psi''\omega^m)=A_{xy}$. We observe that if $z\in A_x=S(\psi')$, then $z\in S(\psi)\cap A_{xy}$, so that $\omega(z)=1$. Hence

$$\begin{aligned}
|\hat{\alpha}(\psi''\omega^m)| &= \left| \sum_{z\in A_{xy}\setminus S(\psi')} \alpha(z)\psi''(z)\omega^m(z) + \sum_{z\in S(\psi')} \alpha(z)\psi''(z)\omega^m(z) \right| \\
&= \left| \sum_{j=1}^{n} \alpha(y_j)\psi''(y_j)v^{mp(y_j)} + \sum_{z\in S(\psi')} \alpha(z)\psi'(z) \right| \\
&= |f(v^m)| > |f(0)| = |\hat{\alpha}(\psi')| = |\hat{\alpha}(\psi)|,
\end{aligned}$$

contradicting the fact that $|\hat{\alpha}|$ assumes its maximum at ψ.

8.8. Remark. Šilov showed in [7, §5.4] that in the maximal ideal space of a commutative Banach algebra, a point χ is an isolated point of ∂ if and only if χ is isolated in the entire maximal ideal space. The author in [1] has identified the isolated points of $\hat{G}$, and has shown in particular that each such point must lie in Γ. It follows that a semi-character χ is isolated in $\hat{G}$ if and only if $\chi\in\partial$ and χ is an isolated point of Γ. Using Theorem 5.2 of [1], the preceding sentence yields Theorem 8.11 below.

8.9. Definition. A subset K of G is called a face of G if, for each u and v of G, $uv\in K$ if and only if $u\in K$ and $v\in K$. If K is a face of G and L the union of all proper sub-faces of K, then the set $K\setminus L$ is denoted core K and is called the core of K.

8.10. Theorem. *If $\chi\in\hat{G}$, then $S(\chi)$ is a face of G.*

8.11. Definition. For each $\chi\in\hat{G}$, let $N(\chi)$ denote the set of points $y\in G$ for which there exist points x_1, x_2 in $S(\chi)$ for which $x_1y=x_2y$ and $\chi(x_1)\neq\chi(x_2)$.

8.12. Theorem. *Let χ be an isolated point of Γ. Then core $S(\chi)$ is a finite group, and the following assertions are equivalent:*

(a) *there is a finite subset I of $G\setminus S(\chi)$, I consisting only of idempotents, for which $A_{xy}\cap[I\cup N(\chi)]\neq\Lambda$ whenever $y\in G\setminus S(\chi)$ and $x\in$* core $S(\chi)$.

(b) $\chi\in\partial$.

9. **Examples.** In this section we give a number of examples designed to illustrate the theory already presented.

9.1. From our present point of view, one obtains a rationalization for applying the term "boundary" to the set ∂ by considering the set G of non-negative integers, "multiplication" being given by the usual integral addition. Each semi-character assumes the value 1 at $0\in G$. Since a semi-character is determined by its value at $1\in G$, which may be any complex number w with $|w|\leqq 1$, and since $\hat{G}$ carries the topology of uniform convergence on finite subsets of G, we see that, as a topological space, $\hat{G}$ coincides with the closed complex unit disk. By 8.7 or from 3.3, $\partial=\{\chi\in\hat{G}\,|\,|\chi(1)|=1\}=\{\text{complex } w\,|\,|w|=1\}$, the topological boundary of the unit disk in the plane.

9.2. Let G be a set whose cardinality exceeds 1, and let one of the elements of G be named 0. For every two elements x and y of G, define $xy=0$. Then $\hat{G}$

consists of the single semi-character 1 on G, so that $\hat{G}$ does not separate points.

9.3. Let S be the set consisting of all integral powers of the element $a \in S$, the two elements a^m and a^n of S being considered distinct if $m \neq n$. Let T be a two-element set with elements b and e, and let $S \cap T = \Lambda$. Let $G = S \cup T$ and define multiplication on G as follows:

$$a^m a^n = a^n a^m = a^{m+n}; \qquad eb = be = b;$$

$$b^2 = e^2 = e; \qquad ea^m = a^m e = ba^m = a^m b = a^m.$$

This associative multiplication makes G a commutative semi-group. If a multiplicative function on G assumes the value 0 at e, then it obviously assumes the value 0 at each $x \in G$. Thus each semi-character assumes the value 1 at e.

Define the function χ_0 on G as follows: $\chi_0(e) = 1$; $\chi_0(b) = -1$; $\chi_0(x) = 0$ if $x \in S$. Evidently $\chi_0 \in \hat{G}$ and $\chi_0 \notin B$. Now define $\alpha = \chi_0$; then $\alpha \in l_1(G)$, and we have $|\hat{\alpha}(\chi_0)| = 2$. Since $b^2 = e$, each semi-character assumes at b either the value 1 or the value -1. If $\chi \in \hat{G}$ and $\chi(a) \neq 0$, then from the relation $a = ab$ it follows that $\chi(b) = 1$. Now let χ be any semi-character not equal to χ_0. We have just shown that $\chi(b) = 1$. Then $\hat{\alpha}(\chi) = (1)(1) + (-1)(1) = 0$. Hence $|\hat{\alpha}|$ assumes its maximum only at χ_0, and $\chi_0 \in \partial$.

Let $\chi_1(e) = \chi_1(b) = 1$, $\chi_1(x) = 0$ if $x \in S$; clearly $\chi_1 \in \hat{G}$. Since $\hat{G}_F$ consists only of the single point χ_0 for each finite subset F of $S = G \backslash S(\chi_1)$, we see from 8.7 that $\chi_1 \notin \partial$.

Let $\chi_2(e) = \chi_2(b) = 1$, and let $\chi_2(a^n) = (1/2)^n$ for each positive integer n. Then $\chi_2 \in \hat{G}$ and $\chi_2 \notin \Gamma$.

The three semi-characters χ_0, χ_1, χ_2 on G show respectively that the three inclusions $B \subset \partial \subset \Gamma \subset \hat{G}$ are proper.

9.4. Let G be as in 9.3. Since $\chi(e) = 1$ for each $\chi \in \hat{G}$, we see that $\hat{G}$ is a semi-group. The two functions χ_0 and χ_1 defined in 9.3 are in Γ. $|\chi_0| = |\chi_1|$ and $\chi_0^2 = \chi_1$. But $\chi_0 \in \partial$ and $\chi_1 \notin \partial$. Theorem 5.1 guarantees the fact, here obvious, that there is no $\chi \in B$ for which $\chi\chi_0 = \chi_1$.

9.5. Let A be an index set, and let G_α be a commutative semi-group for each $\alpha \in A$, the sets G_α being taken pairwise disjoint. If $<$ is a linear ordering on A, then the set $G = \bigcup_{\alpha \in A} G_\alpha$ may be made into a commutative semi-group as follows: if $u \in G_\alpha$ and $v \in G_\alpha$, then the product uv in G is the product uv in G_α; if $u \in G_\alpha$ and $v \in G_\beta$ with $\alpha < \beta$, then $uv = vu = v$. If each G_α is a group, then from 6.4 we have $\partial = \hat{G}$.

9.6. Let $G = \{u, u^2, \cdots, u^n, \cdots\} \cup \{v, v^2, \cdots, v^n, \cdots\}$, where all powers of u and v are distinct. Define a multiplication in G as follows: $u^n u^m = u^{m+n}$; $v^n v^m = v^{n+m}$; $u^n v^m = v^m u^n = u^n$. Then G is a commutative semi-group. If $\chi(v^m) = 1$ and $\chi(u^n) = 0$ for each pair of positive integers m and n, then $\chi \in \hat{G}$. Since χ agrees at every two points of its support, and since there

exist no points x_1, x_2 in $S(\chi)$ and y, z in $G\backslash S(\chi)$ for which $yz = x_1yz = x_2y = y$, we see that $\chi \in \hat{G}_F$ for no finite subset F of $G\backslash S(\chi)$. But for each $\epsilon > 0$ there is a semi-character ψ for which $0 < |\psi(v) - \chi(v)| < \epsilon$, so that, for each finite subset F of $G\backslash S(\chi)$, each neighborhood of χ meets $\hat{G}_F$. Thus $\chi \in \partial$ by 8.7.

References

1. W. W. Comfort, *The isolated points in the dual of a commutative semi-group*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 227–233.

2. I. M. Gel'fand, D. A. Raĭkov and G. E. Šilov, *Commutative normed rings*, Amer. Math. Soc. Translations Series 2, vol. 15 (1957) pp. 115–220.

3. Edwin Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. vol. 83 (1956) pp. 70–97.

4. A. G. Kurosh, *Theory of groups*, Volume I, New York, Chelsea Publishing Company, 1955 (translated by K. A. Hirsch).

5. Lynn H. Loomis, *An introduction to abstract harmonic analysis*, New York, van Nostrand, 1953.

6. K. A. Ross, *A note on extending semicharacters on semigroups*, Proc. Amer. Math. Soc. vol. 10 (1959) pp. 579–583.

7. G. E. Šilov, *On the decomposition of a commutative normed ring into a direct sum of ideals*, Amer. Math. Soc. Translations Series 2, vol. 1 (1955) pp. 37–48.

8. Hermann Weyl, *The elementary theory of convex polyhedra*, Annals of Mathematics Studies no. 24, 1950, pp. 3–18.

9. André Weil, *L'intégration dans les groupes topologiques et ses applications*, Actualités Scientifiques et Industrielles no. 869, Paris, Hermann, 1940.

University of Washington,
Seattle, Washington
Harvard University,
Cambridge, Massachusetts