1. For a commutative semisimple Banach algebra $B$, let $\mathcal{M}_B$ denote its space of nonzero multiplicative linear functionals, and $x \mapsto \hat{x}$ its Gelfand representation. A well-known theorem of Šilov [15] shows that for any compact-open subset $U$ of $\mathcal{M}_B$ there is an $x$ in $B$ with $\hat{x}$ the characteristic function $\phi_U$ of $U$. An immediate consequence is the fact that $B$ is regular if $\mathcal{M}_B$ is totally disconnected. The present note is devoted to a similar application of Šilov’s result which has apparently escaped notice.

Normally when $A$ is a subalgebra of $B$ (closed or not), $\mathcal{M}_A$ may contain functionals other than those provided by the restrictions of the elements of $\mathcal{M}_B$. But at least when $A$ is closed the Šilov boundary $\partial_A$ of $A$ is produced by the elements of $\partial_B$. In any case, if we assume $\partial_A$ is produced by $\partial_B$, and that $\partial_B$ is scattered (i.e., contains no nonvoid perfect subset), then Šilov’s theorem shows not only that $B$ is regular but indeed that $A$ is regular as well so that all of $\mathcal{M}_A = \partial_A$ arises from $\partial_B = \mathcal{M}_B$. When $\partial_B$ is discrete the same is true of $\partial_A$, and we can trivially identify the smallest hull-less ideal $j_A(\infty)$ of $A$; it is precisely the span of the idempotents in $A$. Consequently $A$ is tauberian if and only if it is the closed span of its idempotents.

As an application we can easily determine all closed tauberian subalgebras of $L_1(G)$ and $L_2(G)$ when $G$ is a compact abelian group. In the $L_2$ case every closed subalgebra $A$ is tauberian, and is determined by just the sets of constancy of the Fourier transforms $\hat{A}$; the same prescription applies to the tauberian closed subalgebras $A$ of $L_1(G)$, but whether nontauberian subalgebras exist seems to be a difficult problem. Borrowing from the $L_2$ case we can easily see that $A$ is tauberian if (and of course only if) $A \cap L_2$ is dense in $A$. (Some application can also be made to closed commutative semisimple subalgebras of $L_1(G)$ and $L_2(G)$ when $G$ is compact nonabelian, cf. §4.)

The notation used below is essentially standard, as in [9], with the exception of our use of “scattered,” as defined above. We denote the hull of an ideal $I$ by $hI$, and by $kF$ the kernel of a subset $F$ of $\mathcal{M}_A$, while $j_A(\infty)$ is the set of $a$ in $A$ for which $\hat{a}$ has compact support. $A$ is called tauberian when every hull-less ideal is dense in $A$; when $A$ is regular this amounts to the density of $j_A(\infty)$ [9]. All algebras will be assumed commutative semisimple. Although $\sim$ may be used for any Gelfand representation which arises, it will always be clear from the context which is intended. Finally, $A_a$ will be used to
denote the closed subalgebra of $A$ generated by an element $a$, $\sigma_A(a)$ the spectrum of $a$ in $A$, and $\text{bdry } F$ the ordinary boundary of a plane set $F$.

It is a pleasure to record my indebtedness to Edwin Hewitt for raising a question which led to this investigation, and to Walter Rudin for suggesting the use of scattered sets.

2. The main result. Let $A$ be a Banach algebra which forms, algebraically, a subalgebra of a commutative semisimple Banach algebra $B$. If the functionals in $\partial_A$ are included in the restrictions to $A$ of the elements of $\partial_B$ we shall simply say $\partial_A$ arises from, or is produced by, $\partial_B$. As is well known this occurs\(^{(2)}\) if $A$ is a closed subalgebra of $B$, and as a consequence, $\text{bdry } \sigma_{A_A}(a) \subseteq \partial(\partial_A)$. Indeed, $\sigma_{A_A}(a)$ can be identified with $\mathcal{M}_{A_a}$ in such a way that $\text{bdry } \sigma_{A_A}(a)$ is precisely $\partial_{A_A}$ (which arises from $\partial_A$).

**Theorem 2.1.** Let $A$ be a Banach algebra which is algebraically a subalgebra of a commutative semisimple Banach algebra $B$. Suppose $\sigma_A$ arises from $\partial_B$, while $\partial_B$ is scattered. Then $A$ is regular, and consequently $\mathcal{M}_A = \partial_A$ arises from $\partial_B = \mathcal{M}_B$.

**Proof.** Rudin [12, Theorem 1] has shown that a continuous image of a compact scattered space is scattered. Since the one-point compactification $\partial_B \cup \{0\}$ of $\partial_B$ is clearly scattered, and $\text{bdry } \sigma_{A_A}(a) \subseteq \partial(\partial_A) \subseteq \partial(\partial_B)$, we conclude that $\text{bdry } \sigma_{A_A}(a)$ is scattered. But nontrivial closed connected plane sets cannot be scattered, so $\text{bdry } \sigma_{A_A}(a)$ is totally disconnected; consequently $\sigma_{A_A}(a)$, and its subset $\sigma_A(a)$, are all boundary. Thus $\partial(\mathcal{M}_A) = \sigma_A(a)$ is totally disconnected, for each $a$ in $A$, so $\mathcal{M}_A$ is totally disconnected.

By Šilov's theorem [1; 15] for each compact-open $U$ in $\mathcal{M}_A$ there is an $a$ in $A$ with $\partial = \phi_U$, so that $A$ is regular, and of course $\mathcal{M}_A = \partial_A$. Since in particular we may take $A = B$, $B$ is regular and $\mathcal{M}_B = \partial_B$, completing the proof.

Clearly $A^\sim$ contains sufficiently many real valued functions to apply Stone-Weierstrass, yielding a result of Rudin [12].

**Corollary 2.2.** $A^\sim$ is dense in $C_0(\mathcal{M}_A)$.

**Corollary 2.3.** Let $A$ and $B$ be as in 2.1, while $\partial_B$ is discrete. Then $\mathcal{M}_A$ is discrete and, for each $M \in \mathcal{M}_A$, $\phi_{|M|} \in A^\sim$.

We need only verify discreteness. But since $\mathcal{M}_A$ arises from $\partial_B$, and $\partial$, considered as a function on $\partial_B$, has $|\partial(M)| \geq \epsilon > 0$ for only finitely many $M$ in $\partial_B$, $|\partial(M)| \geq \epsilon$ for only finitely many $M$ in $\mathcal{M}_A$. Clearly then the compact subsets of $\mathcal{M}_A$ are finite, and thus $\mathcal{M}_A$ is discrete.

**Corollary 2.4.** Let $A$ be a commutative semisimple Banach algebra, and $E$

\(^{(2)}\) By the Beurling-Gelfand formula, which shows $a \in A$ has the same spectral norm in $A$ and in $B$. Thus under the dual to the injection $A \rightarrow B$, $\partial_B$ maps onto a closed set on which $|\partial|$ maximizes.
a scattered hull-kernel closed subset of $\mathcal{M}_A$. Then the relative topology on $E$ is the hull-kernel topology.

For the semisimple algebra $B = A/kE$ has $E$ in its relative topology as $\mathcal{M}_B$, whether the usual ($w^*$) or hull-kernel topologies are used [9, 20G]; since $B$ is regular by 2.1, these two topologies coincide on $\mathcal{M}_B$, so that the relative topology on $E$ is the same whether $\mathcal{M}_A$ is taken in its usual topology or the hull-kernel topology.

There is, more or less, a converse to the fact (in 2.1) that $\partial_A$ is scattered if $\partial_B$ is. Indeed

**Corollary 2.5.** Let $A$ and $B$ be commutative semisimple Banach algebras and $\tau$ a homomorphism of $A$ onto a dense subalgebra of $B$. Then if $\partial_A$ is scattered, so is $\partial_B$.

**Proof.** By semisimplicity $\tau$ is continuous, with closed kernel $I$. Let $C$ be the semisimple algebra $A/I$, and $\tilde{\tau}$ the induced isomorphism of $C$ into $B$. Since we can identify $\mathcal{M}_C$ with the scattered set $hI$, and $C$ is regular, by a result of Rickart [11, Theorem 1] the dual map to $\tilde{\tau}$ takes $\mathcal{M}_B$ onto $\partial_C = \mathcal{M}_C$, and, since $\tilde{\tau}C$ is dense, is a homeomorphism. Consequently $\mathcal{M}_B$ is scattered, yielding the result.

Actually fuller use of Rickart’s result can be made if $\tau$ is an algebraic isomorphism, since then we do not require the continuity of $\tau$ produced by the semisimplicity of $B$.

**Corollary 2.6.** Let $\tau$ be an algebraic isomorphism of the commutative semisimple Banach algebra $A$ onto a dense subalgebra of the non-semisimple Banach algebra $B$, and suppose $\partial_A = \mathcal{M}_A$ is scattered. Then $\mathcal{M}_B$ is scattered.

For the dual to $\tau$ maps $\mathcal{M}_B$ onto $\partial_A = \mathcal{M}_A$, and again is a homeomorphism.

2.7. Remarks. Since countable locally compact spaces are scattered the results of [2; 3] can be obtained from ours. As a more novel application, we note the following. Let $M_0(T^1)$ denote the subalgebra of the algebra of measures on the circle group $T^1$ consisting of those $\mu$ with Fourier-Stieltjes transforms in $C_0(Z)$ ($Z$ the integers). It is known that $\mathcal{M}_{M_0(T^1)}$ contains $Z$ properly [14, 2.5 (f)]; as a consequence we conclude that $Z$ does not contain $\partial_{M_0(T^1)}$ (by 2.1). (And in general, a scattered closed proper subset of $\mathcal{M}_A$ cannot contain $\partial_A$.) In passing, we may as well note that the Fourier-Stieltjes transformation, as a map of $M_0(T^1)$ into $C_0(Z)$, provides us with an example of a nontopological isomorphism $A \rightarrow B$ with $\mathcal{M}_B$ scattered and $\mathcal{M}_A$ not scattered. Indeed with $A = M_0(T^1)$, $Z$ is a closed subset of $\mathcal{M}_A$, as is easily seen, and thus a closed proper subset on which no nonzero representative function can vanish (by the 1-1 nature of the Fourier-Stieltjes transformation); consequently $A$ cannot be regular.

Finally, in 2.1, if $\partial_B$ is not scattered but each $\partial(\partial_B)$ is, then $A$ is regular and $\mathcal{M}_A = \partial_A$ arises from $\partial_B$ by the same argument.
3. Tauberian algebras with discrete boundaries. Let \( \partial_A \) be discrete. Since \( A \) is regular it contains a smallest hull-less ideal, \( j_A(\infty) \), consisting of all \( a \) for which \( a \) vanishes off a compact (i.e., finite) subset of the discrete space \( \mathcal{M}_A = \partial_A \). Moreover since we know \( A^\sim \) contains \( \phi_M \) for each \( M \) in \( \mathcal{M}_A \), \( j_A(\infty) \) is just the span of the corresponding idempotent elements of \( A \). Recalling that a semisimple algebra is called tauberian if it has no proper closed hull-less ideals (which, for regular algebras, says exactly that \( j_A(\infty) \) is dense), we have

**Theorem 3.1.** Let \( \partial_A \) be discrete. Then \( A \) is tauberian if and only if it is the closed span of its idempotent elements.

As a consequence we obtain a result due to Rudin [13](3).

**Theorem 3.2.** Let \( A \) be a closed subalgebra of a tauberian algebra \( B \), and suppose \( \mathcal{M}_B \) is discrete. Then if \( A^\sim \) separates the elements of \( \mathcal{M}_B \cup \{0\} \), \( A = B \).

**Proof.** By 2.1 and 2.3 and our hypothesis of separation, we have \( \mathcal{M}_A = \mathcal{M}_B \) and \( \phi_M \in A^\sim \) for every \( M \) in \( \mathcal{M}_B \). Thus \( j_B(\infty) \subset A \), and \( A = B \).

**Corollary 3.3.** Let \( A \) be tauberian and \( \mathcal{M}_A \) discrete. The following are equivalent:

1°. \( A \) is separable.
2°. \( \mathcal{M}_A \) is countable.
3°. \( A \) is singly generated.

**Proof.** If 1° holds then \( C^0(\mathcal{M}_A) \) is separable by 2.2, and \( \mathcal{M}_A \), being discrete, is then clearly countable. But if \( \mathcal{M}_A \) = \{ \( M_1, M_2, \cdots \) \}, let \( \partial_n = \phi_{\{M_n\}} \), and choose a sequence \( \lambda_1, \lambda_2, \cdots \) of distinct nonzero numbers satisfying \( \sum |\lambda_n| \cdot ||a_n|| < \infty \). Then \( a = \sum \lambda_n a_n \) is an element of \( A \) for which \( \partial \) separates \( \mathcal{M}_A \cup \{0\} \), so that \( A_\partial = A \) by 3.2. Finally that 3° implies 1° is clear.

It is trivial to identify the maximal closed subalgebras of a tauberian \( B \) with \( \mathcal{M}_B \) discrete, by virtue of 3.2.

**Theorem 3.4.** Let \( B \) be a tauberian algebra with \( \mathcal{M}_B \) discrete, and let \( A \) be a maximal closed subalgebra which is not a maximal regular ideal. Then(4) \( A = \{ x : x \in B, \hat{x}(M_1) = \hat{x}(M_2) \} \) for some \( M_1 \neq M_2 \) in \( \mathcal{M}_B \).

3.5. If \( \mathcal{M}_A \) is discrete one can easily determine all closed tauberian subalgebras of \( A \) (an example is given in §4). But closed subalgebras need not

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(3) Rudin's proof (which makes no use of Šilov's theorem) appears in mimeographed notes of a Symposium on Harmonic Analysis and Related Integral Transforms held in summer 1956 at Cornell University. (It was applied there only to the special case \( B = L_1(G) \), \( G \) a compact abelian group.) We might note that in 3.2 we could alternatively assume \( \partial_B \) is scattered and \( B \) spanned by idempotents, obtaining a result of Katznelson and Rudin [8, Theorem 3] (for once \( \mathcal{M}_A = \mathcal{M}_B \) we have \( \hat{\epsilon} \in A^\sim \) for any \( \epsilon \in \mathcal{E} \) by Šilov's theorem).

(4) The special case \( B = L_1(G) \), \( G \) a compact abelian group, strengthens the final remark of [3].
be tauberian even if $A$ is; Mirkil [10] gives an example of such a tauberian $A$
containing a closed ideal $I \neq khI$, so that spectral synthesis fails, and $khI$
provides a nontauberian subalgebra since $I$ is always a hull-less ideal in $khI$.
Indeed for just this reason it is apparent that for a given algebra $A$, all closed
subalgebras are tauberian if and only if all admit spectral synthesis. (For a
tauberian $A$ with $\mathcal{M}_A$ discrete, spectral synthesis is equivalent to $x \in (Ax)^{-}$,
all $x$ in $A$ [10].)

3.6. Coddington [4] has given an example of a tauberian $A$ with $\mathcal{M}_A$ dis-
crete which is not self-adjoint.

3.7. Our next results yield a class of algebras $A$ with $\mathcal{M}_A$ discrete. Let
$a \rightarrow L_a$ denote the regular representation of $A : L_ax = ax$, $x$ in $A$. Let $\mathcal{B}(X)$
denote the algebra of all bounded linear maps of a Banach space $X$ into itself;
we shall say $T \in \mathcal{B}(X)$ has an essentially simple spectrum if $\sigma_{\mathcal{B}(X)}(T) \setminus \{0\}$
consists only of eigenvalues of finite multiplicity, having only 0 as a point of
accumulation.

**Theorem 3.8.** Let $A$ be a commutative semisimple Banach algebra. If each
$L_a$ has an essentially simple spectrum, $\mathcal{M}_A$ is discrete.

When each $L_a$ is actually a compact operator this is a special case of a
result of Kaplansky [7, 5.1]; our tauberian $A$'s with $\mathcal{M}_A$ discrete all fall into
this category since $j_\alpha(\infty)$ provides a uniformly dense set of compact $L_a$. But
even with all $L_a$ compact $A$ need not be tauberian(6).

For the proof of 3.8 we require the following well-known fact.

**Lemma 3.9.** Let $A$ be a commutative Banach algebra, and let $A^-$ be the uni-
form closure of $\{L_a : a \in A\}$ in $\mathcal{B}(A)$. Then we can identify the spaces $\mathcal{M}_A^-$ and
$\mathcal{M}_A$ in such a way that $L_a = \hat{a}$.

**Proof.** Each $M$ in $\mathcal{M}_A^-$ of course produces a nonzero multiplicative linear
functional on $A$ since $a \rightarrow L_a$ is multiplicative and $\{L_a : a \in A\}$ is dense. Con-
versely if $M \subseteq \mathcal{M}_A$ choose a $u_M$ in $A$ with $M(u_M) = 1$, and set $M^*(T) = M(Tu_M)$,
$T \in A^-$. Clearly $M^*$ is linear; if $L_{a_n} \rightarrow T$ and $L_{a_n} \rightarrow S$ in $A^-$ then $M^*(TS)$
$= M(TS u_M) = \lim M(L_{a_n}L_{a_n} u_M) = \lim M(a_n u_M)M(b_n u_M) = M(Tu_M)M(Su_M)$
$= M^*(T)M^*(S)$, and $M^* \subseteq \mathcal{M}_A^-$. Finally since $M^*(L_a) = M(a)$ and $\{L_a : a \in A\}$
is dense in $A^-$ our correspondence clearly preserves topology and yields
$L_a = \hat{a}$ on the identified space.

**Proof of 3.8.** Since the spectrum $\sigma_{\mathcal{B}(A)}(L_a)$ of the operator $L_a$ has at most
0 as a point of accumulation, the same is true of bdny $\sigma_{A^-}(L_a) \subseteq \sigma_{\mathcal{B}(A)}(L_a)$.
Thus $\sigma_{A^-}(L_a) = \hat{L_a} = \hat{\mathcal{M}_A} = \hat{\sigma}_{\mathcal{B}(A)}(L_a)$ has the same property, and $\mathcal{M}_A$ is totally
disconnected and $A$ regular, as in 2.1.

Now if $\mathcal{M}_A$ is not discrete it contains some compact infinite subset, and

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(6) For example Mirkil's algebra [10], and its closed subalgebras, satisfy the hypothesis
of 3.8.
thus for some $a$ in $A$, $\delta > 0$, and sequence $\{ M_n \}$ of distinct elements of $\mathcal{M}_A$ we have $|\delta(M_n)| \geq \delta$. In view of the nature of $\delta(M_n)$ we may as well assume $\delta(M_n) = 1$ for all $n$. But again for the same reason $\delta = 1$ on a neighborhood $V_n$ of $M_n$, which can of course be chosen so that $M_m \notin V_n$ for $m < n$. Let $a_n \in A$ be chosen so that $\delta_n(M_n) = 1$ while $\delta_n$ vanishes off $V_n$, so $\delta_n(M_m) = 0$ for $m < n$. Clearly the $a_n$ are linearly independent, while $\delta \delta_n = \delta_n$ implies $L_a a_n = a_n$.

Since the operator $L_a$ can only have finitely many linearly independent eigenvectors corresponding to a single eigenvalue, we have obtained the desired contradiction, completing the proof.

As a consequence of 3.8 we can say something about some commutative semisimple subalgebras of $L_1(G)$ when $G$ is noncommutative (below).

Neither 3.8 nor our next result contains the other, although both are variants of the same theme.

**Theorem 3.10.** Let $A$ be a commutative, semisimple, and uniformly closed algebra of operators with essentially simple spectra on a Banach space $X$. Then $\mathcal{M}_A$ is discrete.

**Proof.** The relation $\text{bdry } \sigma_A(a) \subset \sigma_{\text{bdry } X}(a)$ yields the fact that $\sigma_A(a) = \sigma(\mathcal{M}_A)$ has at most 0 as a point of accumulation; thus $\mathcal{M}_A$ is totally disconnected. Again if $\mathcal{M}_A$ is not discrete we may assume $\delta(M_n) = 1$ for a sequence of distinct $M_n$ in $\mathcal{M}_A$, with $\delta = 1$ on a neighborhood $V_n$ of $M_n$, and with the $V_n$ now chosen so that $V_n \cap V_m = \emptyset$, $n \neq m$. By Silov's theorem we have a nonzero idempotent $a_n$ in $A$ with $\delta_n$ vanishing off $V_n$, and thus $aa_n = a_n$, while $aa_m = 0$, $n \neq m$.

Now if $x_n$ is any nonzero element of the range of $a_n$ we have $x_n = a_n x_n$, so $ax_n = aa_n x_n = a_n x_n = x_n$, $ax_n = x_n$; on the other hand, for $n \neq m$, $a_n x_m = a_n a_m x_m = 0$, so the $x_n$ are surely linearly independent, contradicting the spectral property of $a$ and completing the proof.

4. **Applications to group algebras.** Let $G$ be a compact abelian group. Trivially every idempotent in $L_1(G)$ or $L_2(G)$ is a finite sum of characters, and thus if $A$ is a closed subalgebra of either of these, $j_A(\infty)$ is just the span of an appropriate set of such finite sums. Moreover it is quite trivial to identify the basic set of generating idempotents $\{ e_M: M \in \mathcal{M}_A \}$, where $\delta_M = \phi(M)$, since $\mathcal{M}_A$ arises from $G^\sim$: given $M$, for all $a$ in $A$ we have $\delta(M) = \delta(\hat{a})$, for $\hat{a}$ in a certain subset $F_M$ of $G^\sim$, and clearly such characters $\hat{a}$ are just those (finitely many) for which $\delta_M(\hat{a}) = 1 = \delta_M(M)$. Since the Fourier transform $\delta_M^* = \delta_M^*$ must vanish elsewhere, $e_M$ is precisely the sum of these characters.

Consequently $A$ (or rather, the map of $G^\sim \to \mathcal{M}_A \cup \{ 0 \}$) provides a subdivision of $G^\sim$ into certain finite "sets of constancy" $\{ F_M \}$, on each of which all the Fourier transforms in $A$ are constant (plus a possibly infinite set which we ignore for the moment, the "hull" of $A$, on which all these transforms vanish). Conversely the subdivision $\{ F_M \}$ determines $j_A(\infty)$ at least, and thus determines $A$ if $A$ is tauberian. In case $A$ is a closed subalgebra of
$L_2$, $A$ is tauberian; indeed it is simply a matter of rearranging terms in the 
in ($L_2$) unconditionally convergent Fourier series expansion of $a \in A$ to write 
a = \sum M \delta(g_M)e_M$, where $g_M$ is any element of $F_M$. In case $A$ is a closed sub-
alg of $L_1$ it is not at all clear that $A$ must be tauberian; but we can fall back upon 
the $L_2$ case at least to assert that $A$ is tauberian if (and obviously only if) $A \cap L_2$ is dense in $A$. Indeed approximation of $a \in A \cap L_2$ by an element 
of $j_A(\infty)$ in the $L_2$ norm improves when we pass to the $L_1$ norm so that 
$A \cap L_2 \subseteq j_A(\infty)$ and $A = j_A(\infty)$.

Whether all closed subalgebras of $L_1(G)$ are tauberian seems a rather diffi-
ult question. But of course under various assumptions about the “sets of 
constancy” an algebra must be tauberian; as the simplest example suppose 
each $F_M$ consists of a single character. Then $j_A(\infty)$ is clearly an ideal in $L_1(G)$ 
whose hull is $G(\mathcal{U}F_M)$, i.e., the “hull” of $A$. By spectral synthesis for 
$L_1(G)$ it coincides with the kernel of its hull, which of course contains $A$. 
(Thus the closed ideals of $L_1(G)$ are characterized as those closed subalgebras 
with degenerate (i.e., single element) sets of constancy.) As a second example, 
$A$ is tauberian if the union of the nondegenerate sets of constancy forms a 
lacunary subset $E$ of $G$ in the sense of [6, 9.2] (which includes the classical 
case when $G$ is the group $\mathbb{Z}$ of integers). For let $(f, h) = \int f\hat{h}dg, f^*(g) = f(g^{-1})$. 
By the Hahn-Banach Theorem it suffices to show $h \in L_2(G)$ with $(e_M, h) = 0$ 
for all $M$ satisfies $(a, h) = 0, a \in A$; alternatively that the continuous function 
a * $h^*$ vanishes at the identity $1$ of $G$. Since $0 = (e_M, h) = \sum_{\hat{\delta} \in F_M} \hat{h}^{-}(\hat{\delta})$ and $\hat{a}$ is constant on $F_M$, the formal series 

$$\sum (a * h^*)^{-}(\hat{g}) = \sum \hat{a}(\hat{g})\hat{h}^{-}(\hat{g})$$

for $a * h^*(1)$ can be rearranged and grouped in blocks with each block having 
sum zero. Thus $A$ will be tauberian if we can guarantee that (4.01), as re-
grouped, sums to $a * h^*(1)$. In our special case the fact that $\hat{h}$ vanishes on 
each degenerate $F_M$ shows $(a * h^*)^{-}$ vanishes off the lacunary set $E$; thus 
(4.01) is absolutely convergent by [6, 9.2, 8.5], hence converges uncondi-
tionally to $a * h^*(1) = 0$. (The approach through (4.01) yields many special 
cases when $G = \mathbb{Z}$.)

Finally we should note that if $\mathcal{U}F_M$ forms a lacunary subset of $G$ in the 
somewhat different sense of [5] or [6, 8.6], $A$ is tauberian since it is actually 
a subset of $L_2(G)$.

If $G$ is an arbitrary compact group, any commutative semisimple closed 
subalgebra $A$ of $L_1(G)$ or $L_2(G)$ satisfies the hypotheses of 3.8, and thus $\mathfrak{M}_A$ 
is discrete and $A$ regular. The form of idempotents in $L_1$ or $L_2$ is of course 
easily obtained, but now even in the $L_2$ case it is not at all clear that distinct 
idempotents $e_M, e_{M'}$, are orthogonal in $L_2$ if $G$ is nonabelian. Consequently

(*) The argument applies more generally; if $\mathfrak{M}_B$ is discrete and each $b$ in $B$ can be expressed 
by an unconditionally convergent series $\sum h_M e_M$, then all closed subalgebras are tauberian.
we shall restrict our attention to subalgebras $A$ consisting of normal elements $a: a * a^* = a^* * a$.

**Theorem 4.1.** Let $G$ be a compact group, $A$ a closed commutative subalgebra of $L_1(G)$ or $L_2(G)$ consisting of normal elements, and suppose $A \cap L_2$ is dense in $A$. Then $A$ is semisimple and self-adjoint, and is the closed span of ($L_2$- and ring theoretically-) orthogonal self-adjoint idempotent elements of $C(G)$ which provide, in a natural fashion, the discrete space $\mathcal{M}_A$.

**Proof.** $A$ is semisimple since the nonzero compact normal operator $f \mapsto a * f$ on $L_2$ must have a nonzero eigenvalue, which, as is easily seen, must lie in $\sigma_A(a)$. Each idempotent $e_M$ produces a compact normal idempotent operator on $L_2$ in the same way, so $f \mapsto e_M * f$ is an orthogonal projection onto a finite dimensional subspace of $L_2$. Since this projection must reduce to zero on all but finitely many minimal 2-sided ideals, we easily identify $e_M$ as an element of their span, and thus a continuous function. Of course the self-adjointness of $f \mapsto e_M * f$ yields $e_M = e_M^*$ (since $f \mapsto e_M^* * f$ is the adjoint); consequently if $M_1 \neq M_2$, $(e_{M_1}, e_{M_2}) = (e_{M_1} * e_{M_2}(1) = e_{M_1} * e_{M_2}(1) = 0$ (since $e_{M_1} * e_{M_2} = 0$), and the $e_M$ are orthogonal.

Since $(a, e_M) = a * e_M^* (1) = a * e_M(1) = \delta(M) e_M(1)$, and $e_M(1) = e_M * e_M^* (1) = (e_M, e_M) > 0$, any $a$ in $A \cap L_2$ which is orthogonal to $j_A(\infty)$ must be zero by semisimplicity. Thus $A \cap L_2$ lies in the closed $L_2$-span of the $e_M$, and each $a$ in $A \cap L_2$ can be appropriately approximated in $L_2$ norm; since the approximation improves in passing to the $L_1$ norm and $A \cap L_2$ is dense in $A$, even if $A$ is a subalgebra of $L_1$ the desired approximation is available. Finally since our involution is an isometry and maps a dense subset onto itself, $A = A^*$ clearly, completing the proof.

Much the same argument yields

**Theorem 4.2.** Let $H$ be an $H^*$ algebra which has only finite dimensional minimal 2-sided ideals. Let $A$ be a commutative closed subalgebra consisting of normal elements. Then $A$ is semisimple, self-adjoint, and spanned by a set of orthogonal self-adjoint idempotents; properly renormed, $A$ is an $H^*$ algebra.

(The first hypothesis of course guarantees that the operators $L_a$ are compact.)

**References**


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