BANACH ALGEBRAS WITH SCATTERED STRUCTURE SPACES(1)

BY

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1. For a commutative semisimple Banach algebra B, let $\mathfrak{M}_B$ denote its space of nonzero multiplicative linear functionals, and $x \rightarrow \hat{x}$ its Gelfand representation. A well-known theorem of Šilov [1; 15] shows that for any compact-open subset U of $\mathfrak{M}_B$ there is an x in B with $\hat{x}$ the characteristic function ϕ_U of U. An immediate consequence is the fact that B is regular if $\mathfrak{M}_B$ is totally disconnected. The present note is devoted to a similar application of Šilov's result which has apparently escaped notice.

Normally when A is a subalgebra of B (closed or not), $\mathfrak{M}_A$ may contain functionals other than those provided by the restrictions of the elements of $\mathfrak{M}_B$. But at least when A is closed the Šilov boundary ∂_A of A is produced by the elements of ∂_B. In any case, if we assume ∂_A is produced by ∂_B, and that ∂_B is *scattered* (i.e., *contains no nonvoid perfect subset*), then Šilov's theorem shows not only that B is regular but indeed that A is regular as well so that all of $\mathfrak{M}_A = \partial_A$ arises from $\partial_B = \mathfrak{M}_B$. When ∂_B is discrete the same is true of ∂_A, and we can trivially identify the smallest hull-less ideal $j_A(\infty)$ of A; it is precisely the span of the idempotents in A. Consequently A is tauberian if and only if it is the closed span of its idempotents.

As an application we can easily determine all closed *tauberian* subalgebras of $L_1(G)$ and $L_2(G)$ when G is a compact abelian group. In the L_2 case every closed subalgebra A is tauberian, and is determined by just the sets of constancy of the Fourier transforms $A^\frown$; the same prescription applies to the tauberian closed subalgebras A of $L_1(G)$, but whether nontauberian subalgebras exist seems to be a difficult problem. Borrowing from the L_2 case we can easily see that A is tauberian if (and of course only if) $A \cap L_2$ is dense in A. (Some application can also be made to closed *commutative* semisimple subalgebras of $L_1(G)$ and $L_2(G)$ when G is compact nonabelian, cf. §4.)

The notation used below is essentially standard, as in [9], with the exception of our use of "scattered," as defined above. We denote the hull of an ideal I by hI, and by kF the kernel of a subset F of $\mathfrak{M}_A$, while $j_A(\infty)$ is the set of a in A for which $\hat{a}$ has compact support. A is called tauberian when every hull-less ideal is dense in A; when A is regular this amounts to the density of $j_A(\infty)$ [9]. All algebras will be assumed *commutative semisimple.* Although $^\frown$ may be used for any Gelfand representation which arises, it will always be clear from the context which is intended. Finally, A_a will be used to

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denote the closed subalgebra of A generated by an element a, $\sigma_A(a)$ the spectrum of a in A, and bndry F the ordinary boundary of a plane set F.

It is a pleasure to record my indebtedness to Edwin Hewitt for raising a question which led to this investigation, and to Walter Rudin for suggesting the use of scattered sets.

2. **The main result.** Let A be a Banach algebra which forms, algebraically, a subalgebra of a commutative semisimple Banach algebra B. If the functionals in ∂_A are included in the restrictions to A of the elements of ∂_B we shall simply say ∂_A arises from, or is produced by, ∂_B. As is well known this occurs(2) if A is a closed subalgebra of B, and as a consequence, bndry $\sigma_{A_a}(a) \subset \hat{a}(\partial_A)$. Indeed, $\sigma_{A_a}(a)$ can be identified with $\mathfrak{M}_{A_a}$ in such a way that bndry $\sigma_{A_a}(a)$ is precisely ∂_{A_a} (which arises from ∂_A).

THEOREM 2.1. *Let A be a Banach algebra which is algebraically a subalgebra of a commutative semisimple Banach algebra B. Suppose ∂_A arises from ∂_B, while ∂_B is scattered. Then A is regular, and consequently $\mathfrak{M}_A = \partial_A$ arises from $\partial_B = \mathfrak{M}_B$.*

Proof. Rudin [12, Theorem 1] has shown that a continuous image of a compact scattered space is scattered. Since the one-point compactification $\partial_B \cup \{0\}$ of ∂_B is clearly scattered, and bndry $\sigma_{A_a}(a) \subset \hat{a}(\partial_A) \subset \hat{a}(\partial_B)$, we conclude that bndry $\sigma_{A_a}(a)$ is scattered. But nontrivial closed connected plane sets cannot be scattered, so bndry $\sigma_{A_a}(a)$ is totally disconnected; consequently $\sigma_{A_a}(a)$, and its subset $\sigma_A(a)$, are all boundary. Thus $\hat{a}(\mathfrak{M}_A) = \sigma_A(a)$ is totally disconnected, for each a in A, so $\mathfrak{M}_A$ is totally disconnected.

By Šilov's theorem [1; 15] for each compact-open U in $\mathfrak{M}_A$ there is an a in A with $\hat{a} = \phi_U$, so that A is regular, and of course $\mathfrak{M}_A = \partial_A$. Since in particular we may take $A = B$, B is regular and $\mathfrak{M}_B = \partial_B$, completing the proof.

Clearly $A^\frown$ contains sufficiently many real valued functions to apply Stone-Weierstrass, yielding a result of Rudin [12].

COROLLARY 2.2. *$A^\frown$ is dense in $C_0(\mathfrak{M}_A)$.*

COROLLARY 2.3. *Let A and B be as in* 2.1, *while ∂_B is discrete. Then $\mathfrak{M}_A$ is discrete and, for each $M \in \mathfrak{M}_A$, $\phi_{\{M\}} \in A^\frown$.*

We need only verify discreteness. But since $\mathfrak{M}_A$ arises from ∂_B, and $\hat{a}$, considered as a function on ∂_B, has $|\hat{a}(M)| \geqq \epsilon > 0$ for only finitely many M in ∂_B, $|\hat{a}(M)| \geqq \epsilon$ for only finitely many M in $\mathfrak{M}_A$. Clearly then the compact subsets of $\mathfrak{M}_A$ are finite, and thus $\mathfrak{M}_A$ is discrete.

COROLLARY 2.4. *Let A be a commutative semisimple Banach algebra, and E*

(2) By the Beurling-Gelfand formula, which shows $a \in A$ has the same spectral norm in A and in B. Thus under the dual to the injection $A \to B$, ∂_B maps onto a closed set on which $|\hat{a}|$ maximizes.

a scattered hull-kernel closed subset of $\mathfrak{M}_A$. Then the relative topology on E is the hull-kernel topology.

For the semisimple algebra $B = A/kE$ has E in its relative topology as $\mathfrak{M}_B$, whether the usual (w^*) or hull-kernel topologies are used [9, 20G]; since B is regular by 2.1, these two topologies coincide on $\mathfrak{M}_B$, so that the relative topology on E is the same whether $\mathfrak{M}_A$ is taken in its usual topology or the hull-kernel topology.

There is, more or less, a converse to the fact (in 2.1) that ∂_A is scattered if ∂_B is. Indeed

Corollary 2.5. *Let A and B be commutative semisimple Banach algebras and τ a homomorphism of A onto a dense subalgebra of B. Then if ∂_A is scattered, so is ∂_B.*

Proof. By semisimplicity τ is continuous, with closed kernel I. Let C be the semisimple algebra A/I, and $\bar{\tau}$ the induced isomorphism of C into B. Since we can identify $\mathfrak{M}_C$ with the scattered set hI, and C is regular, by a result of Rickart [11, Theorem 1] the dual map to $\bar{\tau}$ takes $\mathfrak{M}_B$ onto $\partial_C = \mathfrak{M}_C$, and, since $\bar{\tau}C$ is dense, is a homeomorphism. Consequently $\mathfrak{M}_B$ is scattered, yielding the result.

Actually fuller use of Rickart's result can be made if τ is an algebraic isomorphism, since then we do not require the continuity of τ produced by the semisimplicity of B.

Corollary 2.6. *Let τ be an algebraic isomorphism of the commutative semisimple Banach algebra A onto a dense subalgebra of the non-semisimple Banach algebra B, and suppose $\partial_A = \mathfrak{M}_A$ is scattered. Then $\mathfrak{M}_B$ is scattered.*

For the dual to τ maps $\mathfrak{M}_B$ onto $\partial_A = \mathfrak{M}_A$, and again is a homeomorphism.

2.7. Remarks. Since countable locally compact spaces are scattered the results of [2; 3] can be obtained from ours. As a more novel application, we note the following. Let $M_0(T^1)$ denote the subalgebra of the algebra of measures on the circle group T^1 consisting of those μ with Fourier-Stieltjes transforms in $C_0(Z)$ (Z the integers). It is known that $\mathfrak{M}_{M_0(T^1)}$ contains Z properly [14, 2.5 (f)]; as a consequence we conclude that *Z does not contain $\partial_{M_0(T^1)}$* (by 2.1). (And in general, a scattered closed proper subset of $\mathfrak{M}_A$ cannot contain ∂_A.) In passing, we may as well note that the Fourier-Stieltjes transformation, as a map of $M_0(T^1)$ into $C_0(Z)$, provides us with an example of a nontopological isomorphism $A \rightarrow B$ with $\mathfrak{M}_B$ scattered and $\mathfrak{M}_A$ not scattered. Indeed with $A = M_0(T^1)$, Z is a closed subset of $\mathfrak{M}_A$, as is easily seen, and thus a closed proper subset on which no nonzero representative function can vanish (by the 1-1 nature of the Fourier-Stieltjes transformation); consequently A cannot be regular.

Finally, in 2.1, if ∂_B is not scattered but each $\hat{a}(\partial_B)$ is, then A is regular and $\mathfrak{M}_A = \partial_A$ arises from ∂_B by the same argument.

3. **Tauberian algebras with discrete boundaries.** Let ∂_A be discrete. Since A is regular it contains a smallest hull-less ideal, $j_A(\infty)$, consisting of all a for which $\hat{a}$ vanishes off a compact (i.e., finite) subset of the discrete space $\mathfrak{M}_A = \partial_A$. Moreover since we know $A^\frown$ contains $\phi_{\{M\}}$ for each M in $\mathfrak{M}_A$, $j_A(\infty)$ is just the span of the corresponding idempotent elements of A. Recalling that a semisimple algebra is called tauberian if it has no proper closed hull-less ideals (which, for regular algebras, says exactly that $j_A(\infty)$ is dense), we have

THEOREM 3.1. *Let ∂_A be discrete. Then A is tauberian if and only if it is the closed span of its idempotent elements.*

As a consequence we obtain a result due to Rudin [13](3).

THEOREM 3.2. *Let A be a closed subalgebra of a tauberian algebra B, and suppose $\mathfrak{M}_B$ is discrete. Then if $A^\frown$ separates the elements of $\mathfrak{M}_B \cup \{0\}$, $A = B$.*

Proof. By 2.1 and 2.3 and our hypothesis of separation, we have $\mathfrak{M}_A = \mathfrak{M}_B$ and $\phi_{\{M\}} \in A^\frown$ for every M in $\mathfrak{M}_B$. Thus $j_B(\infty) \subset A$, and $A = B$.

COROLLARY 3.3. *Let A be tauberian and $\mathfrak{M}_A$ discrete. The following are equivalent:*

1°. *A is separable.*
2°. *$\mathfrak{M}_A$ is countable.*
3°. *A is singly generated.*

Proof. If 1° holds then $C_0(\mathfrak{M}_A)$ is separable by 2.2, and $\mathfrak{M}_A$, being discrete, is then clearly countable. But if $\mathfrak{M}_A = \{M_1, M_2, \cdots\}$, let $\hat{a}_n = \phi_{\{M_n\}}$, and choose a sequence $\lambda_1, \lambda_2, \cdots$ of distinct nonzero numbers satisfying $\sum |\lambda_n| \cdot \|a_n\| < \infty$. Then $a = \sum \lambda_n a_n$ is an element of A for which $\hat{a}$ separates $\mathfrak{M}_A \cup \{0\}$, so that $A_a = A$ by 3.2. Finally that 3° implies 1° is clear.

It is trivial to identify the maximal closed subalgebras of a tauberian B with $\mathfrak{M}_B$ discrete, by virtue of 3.2.

THEOREM 3.4. *Let B be a tauberian algebra with $\mathfrak{M}_B$ discrete, and let A be a maximal closed subalgebra which is not a maximal regular ideal. Then*(4) *$A = \{x : x \text{ in } B,\ \hat{x}(M_1) = \hat{x}(M_2)\}$ for some $M_1 \neq M_2$ in $\mathfrak{M}_B$.*

3.5. If $\mathfrak{M}_A$ is discrete one can easily determine all closed tauberian subalgebras of A (an example is given in §4). But closed subalgebras need not

(3) Rudin's proof (which makes no use of Šilov's theorem) appears in mimeographed notes of a Symposium on Harmonic Analysis and Related Integral Transforms held in summer 1956 at Cornell University. (It was applied there only to the special case $B = L_1(G)$, G a compact abelian group.) We might note that in 3.2 we could alternatively assume ∂_B is scattered and B spanned by idempotents, obtaining a result of Katznelson and Rudin [8, Theorem 3] (for once $\mathfrak{M}_A = \mathfrak{M}_B$ we have $\hat{e} \in A^\frown$ for any $e = e^2 \in B$ by Šilov's theorem).

(4) The special case $B = L_1(G)$, G a compact abelian group, strengthens the final remark of [3].

be tauberian even if A is; Mirkil [10] gives an example of such a tauberian A containing a closed ideal $I \neq khI$, so that spectral synthesis fails, and khI provides a nontauberian subalgebra since I is always a hull-less ideal in khI. Indeed for just this reason it is apparent that for a given algebra A, all closed subalgebras are tauberian if and only if all admit spectral synthesis. (For a tauberian A with $\mathfrak{M}_A$ discrete, spectral synthesis is equivalent to $x \in (Ax)^-$, all x in A [10].)

3.6. Coddington [4] has given an example of a tauberian A with $\mathfrak{M}_A$ discrete which is not self-adjoint.

3.7. Our next results yield a class of algebras A with $\mathfrak{M}_A$ discrete. Let $a \to L_a$ denote the regular representation of $A: L_a x = ax$, x in A. Let $\mathfrak{B}(X)$ denote the algebra of all bounded linear maps of a Banach space X into itself; we shall say $T \in \mathfrak{B}(X)$ has an *essentially simple spectrum* if $\sigma_{\mathfrak{B}(X)}(T) \setminus \{0\}$ consists only of eigenvalues of finite multiplicity, having only 0 as a point of accumulation.

THEOREM 3.8. *Let A be a commutative semisimple Banach algebra. If each L_a has an essentially simple spectrum, $\mathfrak{M}_A$ is discrete.*

When each L_a is actually a compact operator this is a special case of a result of Kaplansky [7, 5.1]; our tauberian A's with $\mathfrak{M}_A$ discrete all fall into this category since $j_A(\infty)$ provides a uniformly dense set of compact L_a. But even with all L_a compact A need not be tauberian(5).

For the proof of 3.8 we require the following well-known fact.

LEMMA 3.9. *Let A be a commutative Banach algebra, and let A^- be the uniform closure of $\{L_a : a \in A\}$ in $\mathfrak{B}(A)$. Then we can identify the spaces $\mathfrak{M}_{A^-}$ and $\mathfrak{M}_A$ in such a way that $\hat{L}_a = \hat{a}$.*

Proof. Each M in $\mathfrak{M}_{A^-}$ of course produces a nonzero multiplicative linear functional on A since $a \to L_a$ is multiplicative and $\{L_a : a \in A\}$ is dense. Conversely if $M \in \mathfrak{M}_A$ choose a u_M in A with $M(u_M) = 1$, and set $M^e(T) = M(Tu_M)$, $T \in A^-$. Clearly M^e is linear; if $L_{a_n} \to T$ and $L_{b_n} \to S$ in A^- then $M^e(TS) = M(TSu_M) = \lim M(L_{a_n} L_{b_n} u_M) = \lim M(a_n u_M) M(b_n u_M) = M(Tu_M) M(Su_M) = M^e(T) M^e(S)$, and $M^e \in \mathfrak{M}_{A^-}$. Finally since $M^e(L_a) = M(a)$ and $\{L_a : a \in A\}$ is dense in A^- our correspondence clearly preserves topology and yields $\hat{L}_a = \hat{a}$ on the identified space.

Proof of 3.8. Since the spectrum $\sigma_{\mathfrak{B}(A)}(L_a)$ of the operator L_a has at most 0 as a point of accumulation, the same is true of bndry $\sigma_{A^-}(L_a) \subset \sigma_{\mathfrak{B}(A)}(L_a)$. Thus $\sigma_{A^-}(L_a) = \hat{L}_a(\mathfrak{M}_{A^-}) = \hat{a}(\mathfrak{M}_A)$ has the same property, and $\mathfrak{M}_A$ is totally disconnected and A regular, as in 2.1.

Now if $\mathfrak{M}_A$ is not discrete it contains some compact infinite subset, and

(5) For example Mirkil's algebra [10], and its closed subalgebras, satisfy the hypothesis of 3.8.

thus for some a in A, $\delta>0$, and sequence $\{M_n\}$ of distinct elements of $\mathfrak{M}_A$ we have $|\hat{a}(M_n)|\geqq\delta$. In view of the nature of $\hat{a}(\mathfrak{M}_A)$ we may as well assume $\hat{a}(M_n)=1$ for all n. But again for the same reason $\hat{a}=1$ on a neighborhood V_n of M_n, which can of course be chosen so that $M_m\notin V_n$ for $m<n$. Let $a_n\in A$ be chosen so that $\hat{a}_n(M_n)=1$ while $\hat{a}_n$ vanishes off V_n, so $\hat{a}_n(M_m)=0$ for $m<n$. Clearly the a_n are linearly independent, while $\hat{a}\hat{a}_n=\hat{a}_n$ implies $L_a a_n=a_n$. Since the operator L_a can only have finitely many linearly independent eigenvectors corresponding to a single eigenvalue, we have obtained the desired contradiction, completing the proof.

As a consequence of 3.8 we can say something about some commutative semisimple subalgebras of $L_1(G)$ when G is noncommutative (below).

Neither 3.8 nor our next result contains the other, although both are variants of the same theme.

THEOREM 3.10. *Let A be a commutative, semisimple, and uniformly closed algebra of operators with essentially simple spectra on a Banach space X. Then $\mathfrak{M}_A$ is discrete.*

Proof. The relation bndry $\sigma_A(a)\subset\sigma_{\mathfrak{B}(X)}(a)$ yields the fact that $\sigma_A(a)=\hat{a}(\mathfrak{M}_A)$ has at most 0 as a point of accumulation; thus $\mathfrak{M}_A$ is totally disconnected. Again if $\mathfrak{M}_A$ is not discrete we may assume $\hat{a}(\mathrm{M}_n)=1$ for a sequence of distinct M_n in $\mathfrak{M}_A$, with $\hat{a}=1$ on a neighborhood V_n of M_n, and with the V_n now chosen so that $V_n\cap V_m=\emptyset$, $n\neq m$. By Šilov's theorem we have a nonzero idempotent a_n in A with $\hat{a}_n$ vanishing off V_n, and thus $aa_n=a_n$, while $a_na_m=0$, $n\neq m$.

Now if x_n is any nonzero element of the range of a_n we have $x_n=a_nx_n$, so $ax_n=aa_nx_n=a_nx_n=x_n$, $ax_n=x_n$; on the other hand, for $n\neq m$, $a_nx_m=a_na_mx_m=0$, so the x_n are surely linearly independent, contradicting the spectral property of a and completing the proof.

4. **Applications to group algebras.** Let G be a compact abelian group. Trivially every idempotent in $L_1(G)$ or $L_2(G)$ is a finite sum of characters, and thus if A is a closed subalgebra of either of these, $j_A(\infty)$ is just the span of an appropriate set of such finite sums. Moreover it is quite trivial to identify the basic set of generating idempotents $\{e_M\colon M\in\mathfrak{M}_A\}$, where $\hat{e}_M=\phi_{\{M\}}$, since $\mathfrak{M}_A$ arises from $G^\frown$: given M, for all a in A we have $\hat{a}(M)=\hat{a}(\hat{g})$, for $\hat{g}$ in a certain subset F_M of $G^\frown$, and clearly such characters $\hat{g}$ are just those (finitely many) for which $\hat{e}_M(\hat{g})=1=\hat{e}_M(M)$. Since the Fourier transform $\hat{e}_M(=\hat{e}_M^2)$ must vanish elsewhere, e_M is precisely the sum of these characters.

Consequently A (or rather, the map of $G^\frown\to\mathfrak{M}_A\cup\{0\}$) provides a subdivision of $G^\frown$ into certain finite "sets of constancy" $\{F_M\}$, on each of which all the Fourier transforms in $A^\frown$ are constant (plus a possibly infinite set which we ignore for the moment, the "hull" of A, on which all these transforms vanish). Conversely the subdivision $\{F_M\}$ determines $j_A(\infty)$ at least, and thus determines A if A is tauberian. In case A is a closed subalgebra of

L_2, A is tauberian; indeed it is simply a matter of rearranging terms in the (in L_2) unconditionally convergent Fourier series expansion of $a \in A$ to write $a = \sum_M \hat{a}(\hat{g}_M) e_M$, where $\hat{g}_M$ is any element of F_M(6). In case A is a closed subalgebra of L_1 it is not at all clear that A must be tauberian; but we can fall back upon the L_2 case at least to assert that A is tauberian if (and obviously only if) $A \cap L_2$ is dense in A. Indeed approximation of $a \in A \cap L_2$ by an element of $j_A(\infty)$ in the L_2 norm improves when we pass to the L_1 norm so that $A \cap L_2 \subset j_A(\infty)^-$ and $A = j_A(\infty)^-$.

Whether all closed subalgebras of $L_1(G)$ are tauberian seems a rather difficult question. But of course under various assumptions about the "sets of constancy" an algebra must be tauberian; as the simplest example suppose each F_M consists of a single character. Then $j_A(\infty)^-$ is clearly an ideal in $L_1(G)$ whose hull is $G^\frown \backslash (\cup F_M)$, i.e., the "hull" of A. By spectral synthesis for $L_1(G)$ it coincides with the kernel of its hull, which of course contains A. (Thus the closed ideals of $L_1(G)$ are characterized as those closed subalgebras with degenerate (i.e., single element) sets of constancy.) As a second example, A is tauberian if the union of the nondegenerate sets of constancy forms a lacunary subset E of $G^\frown$ in the sense of [6, 9.2] (which includes the classical case when $G^\frown$ is the group Z of integers). For let $(f, h) = \int_G f\bar{h}\, dg$, $f^*(g) = \overline{f(g^{-1})}$. By the Hahn-Banach Theorem it suffices to show $h \in L_\infty(G)$ with $(e_M, h) = 0$ for all M satisfies $(a, h) = 0$, $a \in A$; alternatively that the continuous function $a * h^*$ vanishes at the identity 1 of G. Since $0 = (e_M, h) = \sum_{\hat{g} \in F_M} \hat{h}^-(\hat{g})$ and $\hat{a}$ is constant on F_M, the formal series

$$\sum (a * h^*)^\frown(\hat{g}) = \sum \hat{a}(\hat{g})\hat{h}^-(\hat{g}) \tag{4.01}$$

for $a * h^*(1)$ can be rearranged and grouped in blocks with each block having sum zero. Thus A will be tauberian if we can guarantee that (4.01), as regrouped, sums to $a * h^*(1)$. In our special case the fact that $\hat{h}$ vanishes on each degenerate F_M shows $(a * h^*)^\frown$ vanishes off the lacunary set E; thus (4.01) is absolutely convergent by [6, 9.2, 8.5], hence converges unconditionally to $a * h^*(1) = 0$. (The approach through (4.01) yields many special cases when $G^\frown = Z$.)

Finally we should note that if $\cup F_M$ forms a lacunary subset of $G^\frown$ in the somewhat different sense of [5] or [6, 8.6], A is tauberian since it is actually a subset of $L_2(G)$.

If G is an arbitrary compact group, any commutative semisimple closed subalgebra A of $L_1(G)$ or $L_2(G)$ satisfies the hypotheses of 3.8, and thus $\mathfrak{M}_A$ is discrete and A regular. The form of idempotents in L_1 or L_2 is of course easily obtained, but now even in the L_2 case it is not at all clear that distinct idempotents e_{M_1}, e_{M_2}, are orthogonal in L_2 if G is nonabelian. Consequently

(6) The argument applies more generally: if $\mathfrak{M}_B$ is discrete and each b in B can be expressed by an unconditionally convergent series $\sum \lambda_M e_M$, then all closed subalgebras are tauberian.

we shall restrict our attention to subalgebras A consisting of normal elements $a: a * a^* = a^* * a$.

THEOREM 4.1. *Let G be a compact group, A a closed commutative subalgebra of $L_1(G)$ or $L_2(G)$ consisting of normal elements, and suppose $A \cap L_2$ is dense in A. Then A is semisimple and self-adjoint, and is the closed span of (L_2- and ring theoretically-) orthogonal self-adjoint idempotent elements of $C(G)$ which provide, in a natural fashion, the discrete space $\mathfrak{M}_A$.*

Proof. A is semisimple since the nonzero compact normal operator $f \to a * f$ on L_2 must have a nonzero eigenvalue, which, as is easily seen, must lie in $\sigma_A(a)$. Each idempotent e_M produces a compact normal idempotent operator on L_2 in the same way, so $f \to e_M * f$ is an orthogonal projection onto a finite dimensional subspace of L_2. Since this projection must reduce to zero on all but finitely many minimal 2-sided ideals, we easily identify e_M as an element of their span, and thus a continuous function. Of course the self-adjointness of $f \to e_M * f$ yields $e_M = e_M^*$ (since $f \to e_M^* * f$ is the adjoint); consequently if $M_1 \neq M_2$, $(e_{M_1}, e_{M_2}) = e_{M_1} * e_{M_2}^*(1) = e_{M_1} * e_{M_2}(1) = 0$ (since $e_{M_1} * e_{M_2} = 0$), and the e_M are orthogonal.

Since $(a, e_M) = a * e_M^*(1) = a * e_M(1) = \hat{a}(M) e_M(1)$, and $e_M(1) = e_M * e_M^*(1) = (e_M, e_M) > 0$, any a in $A \cap L_2$ which is orthogonal to $j_A(\infty)$ must be zero by semisimplicity. Thus $A \cap L_2$ lies in the closed L_2-span of the e_M, and each a in $A \cap L_2$ can be appropriately approximated in L_2 norm; since the approximation improves in passing to the L_1 norm and $A \cap L_2$ is dense in A, even if A is a subalgebra of L_1 the desired approximation is available. Finally since our involution is an isometry and maps a dense subset onto itself, $A = A^*$ clearly, completing the proof.

Much the same argument yields

THEOREM 4.2. *Let H be an H^* algebra which has only finite dimensional minimal 2-sided ideals. Let A be a commutative closed subalgebra consisting of normal elements. Then A is semisimple, self-adjoint, and spanned by a set of orthogonal self-adjoint idempotents; properly renormed, A is an H^* algebra.*

(The first hypothesis of course guarantees that the operators L_a are compact.)

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