VITALI'S THEOREM FOR INVARIANT MEASURES

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1. Introduction. If \( \mathcal{U} \) is a family of closed sets each with positive \( k \)-dimensional Lebesgue measure, and if the subset \( A \) of the Euclidean space \( X = \mathbb{R}^k \) has the property that for each neighborhood \( V \) of each point \( x \in A \) there exists \( U \in \mathcal{U} \) with \( x \in U \subseteq V \), then \( \mathcal{U} \) is said to be a Vitali cover for \( A \). The classical Vitali theorem for the case \( k = 1 \) asserts that if \( \mathcal{U} \) is a Vitali cover for the set \( A \subseteq X = \mathbb{R} \), then there is in \( \mathcal{U} \) a sequence of pairwise disjoint elements whose union exhausts all of \( A \) but a Lebesgue null set. The conclusion follows also if \( k > 1 \), provided one assumes that the Vitali cover \( \mathcal{U} \) of \( A \) is regular in the following sense: For each \( x \in A \) there is a number \( \alpha > 0 \) and a sequence \( U_n \) of elements of \( \mathcal{U} \) and a sequence \( S_n \) of spheres (i.e., "balls") for which \( x \in U_n \subseteq S_n \), \( \lim_{n \to \infty} \lambda \alpha S_n = 0 \), and \( \lambda \alpha U_n / \lambda \alpha S_n \geq \alpha \). (\( \lambda \alpha \) denotes Lebesgue measure in the space \( X = \mathbb{R}^k \). The number \( \alpha \) is called a parameter of regularity at \( x \).)

The invariance under translation of the set-function \( \lambda \alpha \) suggests the point of view adopted in the present generalization of Vitali's theorem. We consider a group \( G \) acting transitively on a Hausdorff space \( X \), the latter endowed with a measure \( \mu \) for which \( \mu(gB) = \mu B \) whenever \( g \in G \) and \( B \) is a \( \mu \)-measurable subset of \( X \). A Vitali covering for a subset \( A \) of \( X \) is defined in the obvious way. Noticing that if \( S_\alpha \) denotes the sphere of radius \( \alpha \) centered at the origin \( \phi \) in \( \mathbb{R}^k \), then \( S_\alpha + S_\alpha \subseteq S_{2\alpha} \) and \( \lambda \alpha S_\alpha / \lambda \alpha S_{2\alpha} = 1/2^k \), we define regularity of a Vitali cover in terms of a set-theoretic "multiplication" defined between subsets of \( X \). We replace the spheres \( S_\alpha \) by what we call quasispheres. As a conceptual aid, the reader may regard the quasisphere \( S_\alpha \), of Definition 2.7, as corresponding to a sphere centered at \( \phi \) of radius \( 1/2^\alpha \). (See §4 for a precise treatment of this matter.)

The upshot of these considerations is our main result, Theorem 3.5, which may be informally stated as follows: If \( \mu \) is \( G \)-invariant on \( X \), and if \( \mathcal{U} \) is a regular Vitali cover for \( A \subseteq X \), then \( \mathcal{U} \) admits a sequence of pairwise disjoint elements whose union exhausts all of \( A \) but a \( \mu \)-null set. Our theorem includes the classical Vitali theorem in \( \mathbb{R}^k \), but the first example of §4 shows that even in \( \mathbb{R}^2 \) the conventional regularity restrictions imposed upon the cover \( \mathcal{U} \) are unnecessarily stringent. This example shows that Vitali's conclusion may hold, in nontrivial circumstances, even when \( \mathcal{U} \) admits no positive parameter of regularity at any point \( x \in A \).

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The question as to whether regular Vitali covers can exist other than in the case \( X = \mathbb{R}^k \) is settled affirmatively in §4. There we assert their existence whenever \( G \) satisfies at some point a "local equi-Lipschitz condition" (see §4) and \( X \) is locally Euclidean. A simple instance in which these hypotheses are satisfied is the case in which \( G = O_3 \) is the orthogonal group of rotations acting on the 2-sphere \( X = S^2 \subset \mathbb{R}^3 \), \( \mu \) being taken as 2-dimensional Lebesgue measure in \( X \); many other examples will occur to the reader.

Our theorem is readily applicable to the instance in which a locally compact Hausdorff group acts by left-translation on itself, \( \mu \) being taken as Haar measure. We show that any Lie group \( G \) acting upon itself satisfies the "local equi-Lipschitz condition", and thus that every subset of \( G \) admits numerous Vitali covers which are regular relative to a naturally defined collection of quasispheres.

There exist many generalizations of the Vitali theorem; the interested reader is referred to \[2, pp. 267-268\] for a bibliography on this topic. None of these generalizations seems to be relevant to our present undertaking.

2. Preliminary discussion and definitions.

2.1. Standing conventions. \( X \) will denote a Hausdorff space, \( G \) a group which acts transitively on \( X \). By this we mean that \( G \) is a subgroup of the group of all one-to-one mappings of \( X \) onto itself such that, whenever \( x \in X \) and \( y \in X \), there is a \( g \in G \) for which \( gx = y \). We consider on \( X \) a countably additive, non-negative measure \( \mu \) which is \( G \)-invariant, in the sense that \( gU \) is \( \mu \)-measurable and \( \mu gU = \mu U \) whenever \( U \) is a \( \mu \)-measurable subset of \( X \) and \( g \in G \). We suppose also that for each \( x \in X \) and each \( \epsilon > 0 \) there exists a \( \mu \)-measurable neighborhood \( V \) of \( x \) for which \( \mu V < \epsilon \). The outer measure determined by \( \mu \) will be denoted \( \mu^* \).

2.2. Remark. The requirements of 2.1 are satisfied, for example, if \( \mu \) is Haar measure on a locally compact nondiscrete Hausdorff group \( G = X \) which acts on itself by left-translation. We do not specifically suppose \( \mu \) to be a regular Borel measure which assigns measure zero to each point of \( X \), since such an assumption does not shorten our discussion.

2.3. Definition. \( \phi \) will denote a point of \( X \), arbitrary but henceforth fixed.

2.4. Definition. For \( B \subset X \) and \( C \subset X \), let \( B^{-1}C = \bigcup \{ gC \mid g \in G, \phi \in gB \} \).

2.5. Remark. If \( \phi' \) had replaced \( \phi \), and if \( h \) is any element of \( G \) for which \( h\phi = \phi' \), then \( B^{-1}C \) is replaced by the set \( D = \bigcup \{ gC \mid g \in G, \phi \in gB \} = \bigcup \{ hgC \mid g \in G, \phi \in gB \} = \bigcup \{ hgC \mid g \in G, \phi \in gB \} = hB^{-1}C \). Thus a change in the choice of \( \phi \) results simply in a translation of the set \( B^{-1}C \).

In case \( X = G \) is a group acting by translation upon itself, \( B^{-1}C \) has its usual meaning provided \( \phi \) is chosen to be the identity element; that is, \( B^{-1}C \) represents the set of all elements of the form \( g^{-1}h \) with \( g \in B \) and \( h \in C \). In the discussion which follows we shall be concerned chiefly with the situation where \( B = C \); \( B^{-1}B \) may be described as the union of all translates of \( B \) which contain \( \phi \).
Purely as a notational convenience, we introduce the following definition.

2.6. Definition. For \( B \subset X \) and \( C \subset X \), let

\[
BC = \bigcup \{ gC \mid g \in G, g\varphi \in B \}.
\]

The following two definitions describe the covers to which our theorem applies.

2.7. Definition. A sequence \( S_n \) of subsets of \( X \) will be called a sequence of quasispheres if, for some \( \varepsilon > 0 \) and each positive integer \( n \):

1. \( S_{n+1}^{-1} S_{n+1} \subset S_n \);
2. \( \mu^* S_{n+1} > \varepsilon \mu^* S_n \).

2.8. Definition. Let \( A \subset X \), and let \( \mathcal{U} \) be a family of closed, \( \mu \)-measurable subsets of \( X \) each with positive \( \mu \)-measure. Then \( \mathcal{U} \) is said to be a regular Vitali cover for \( A \) if there exists a sequence \( S_n \) of quasispheres and a real-valued function \( M \) on \( A \) such that, for each \( x \in A \) and each neighborhood \( V \) of \( x \), there is a \( U \in \mathcal{U} \), a \( g \in G \) and an integer \( n \) for which \( x \in U \subset V \), \( gU \subset S_n \) and \( \mu^* S_n \leq M(x) \mu U \).

2.9. Remark. The concept of a regular Vitali cover has been defined in terms of the set-theoretic multiplication of Definition 2.4, which depends in turn upon the choice of \( \phi \in X \). But from Remark 2.5 and the fact that \( (hB)^{-1} (hC) = B^{-1} C \) for each \( h \in G \), it is easy to see that if a cover \( \mathcal{U} \) is a regular Vitali cover for \( A \) for a particular choice of \( \phi \), then it is a regular Vitali cover for every choice of \( \phi \).

The function \( M \) of Definition 2.8 corresponds to the parameter of regularity of the classical theory. This observation motivates the following definition.

2.10. Definition. A family \( \mathcal{U} \) of closed, \( \mu \)-measurable subsets of \( X \) is said to be a regular Vitali cover with constant parameter of regularity if it satisfies the conditions of the preceding definition with some constant function \( M \).

3. The main theorem.

3.1. Lemma. Let \( A \subset X \), and let \( \mathcal{U} \) be a collection of closed \( \mu \)-measurable subsets of \( X \) each of which has positive \( \mu \)-measure. Let \( K \) be an open subset of \( X \) for which \( A \subset K \) and \( \mu^* K < \infty \), and suppose that for each \( x \in A \) and each neighborhood \( V \) of \( x \) there exists \( U \in \mathcal{U} \) for which \( x \in U \subset V \). Suppose there is a real number \( P \) and a sequence \( C_n \) of subsets of \( X \) for which

1. \( N \mu^* (C_n C_n) < P \) for each \( n \); and
2. if \( U \in \mathcal{U} \) and \( g \in G \) with \( \phi \in gU \) and \( \mu U < 1/n \),

then \( gU \subset C_n \).

Then there is a (possibly finite) sequence \( U_k \) of pairwise disjoint elements of \( \mathcal{U} \) for which \( \mu^* (A \setminus \bigcup_{k=1}^{\infty} U_k) = 0 \). The sequence may be chosen so that \( \bigcup_{k=1}^{\infty} U_k \subset K \).

Proof. Insofar as the proof of Banach (see [1]) is applicable in our gener-
alized context, we present it here as reported in [3, §39] by Munroe.

We may as well suppose, discarding certain elements of $\mathcal{U}$ if necessary, that $U \subseteq K$ whenever $U \subseteq \mathcal{U}$.

The sequence $U_k$ is defined recursively. Choose any $U_1 \subseteq \mathcal{U}$, and suppose that $U_k$ has been chosen for $1 \leq k \leq n$. If $A \subseteq \bigcap_{i=1}^{n+1} U_k$, the construction terminates. Otherwise, since the elements of $\mathcal{U}$ are closed, there are a point $x \in A \setminus \bigcap_{i=1}^{n+1} U_k$ and a $U \subseteq \mathcal{U}$ for which $x \in U$ and $U \cap \bigcap_{i=1}^{n+1} U_k = \emptyset$. Then, with $\delta_n = \text{sup} \{ \mu U \mid U \subseteq \mathcal{U}, U \cap \bigcap_{i=1}^{n+1} U_k = \emptyset \}$, we have $\delta_n > 0$. We select $U_{n+1} \subseteq \mathcal{U}$ with $U_{n+1} \cap \bigcap_{i=1}^{n+1} U_k = \emptyset$ and $\mu U_{n+1} > \delta_n / 2$.

To show the sequence $U_k$ is as required, we set $T = \limsup T_n = \bigcap_{i=1}^{n+1} \bigcup_{i=1}^{n+1} T_n$ where, for each positive integer $n$, we define

$$T_n = U \{ U \subseteq \mathcal{U} \mid \mu U \leq 2\mu U_n, U \cap \bigcup_{i=1}^{n+1} U_{n+1} = \emptyset \}.$$  

We will show (1) $A \setminus \bigcup_{i=1}^{n+1} U_k \subseteq T$ and (2) $\mu^* T = 0$.

Let $x \in A \setminus \bigcup_{i=1}^{n+1} U_k$, and choose a positive integer $m$. Then $x \in \bigcup_{i=1}^{n+1} U_k$, so there is a $U \subseteq \mathcal{U}$ with $x \in U$ and $U \cap \bigcup_{i=1}^{n+1} U_k = \emptyset$. Since the measurable sets $U_k$ are pairwise disjoint subsets of $K$, it follows from our hypothesis $\mu^* K < \infty$ that $\mu U_k = 0$. If $U \cap \bigcap_{i=1}^{n+1} U_k = \emptyset$ for each $k > m$, then $\mu U \leq \delta_k < 2\mu U_{k+1}$ for each $k > m$, so that $\mu U = 0$. From this contradiction it follows that there is a smallest integer $N$ for which $U \cap \bigcap_{i=1}^{n+1} U_k = \emptyset$. Then $\mu U \leq \delta_N < 2\mu U_N$, so that $x \in U \subseteq T_N$. Since $N > m$, we have $x \in \bigcup_{n=m}^{n-1} T_n$. Thus (1) is established.

To prove (2), we first find an integer $n_0$ for which $2\mu U_n < 1$ whenever $n \geq n_0$. Let $n \geq n_0$ be fixed, and let $z$ be a fixed point in $U_n$. We will show that if $N$ is the largest integer for which $2\mu U_n < 1/N$, and if $h$ is any element of $G$ for which $hz = \phi$, then $T_n \subseteq h^{-1}C_N C_N$. For each $x \in T_n$ there exist $U \subseteq \mathcal{U}$ and $y \in U \cap \bigcap_{i=1}^{n+1} U_k$ for which $x \in U$ and $\mu U < 2\mu U_n$. Choosing a point $g \in G$ for which $gy = \phi$, and $Z \subseteq h^{-1}C_N C_N$, we see from hypothesis (2) of the present lemma that $gU \subseteq C_N$ and $hU_n \subseteq C_N$, so that $gx \subseteq C_N$ and $hy \subseteq C_N$. Then $\phi = (gh^{-1})hy \subseteq gh^{-1}C_N$, so that $gh^{-1}C_N \subseteq C_N$ by 2.6. Hence $hx \subseteq C_N C_N$ for each $x \in T_n$. Now since $2\mu U_n \geq 1/(2N)$, we have $\mu^* T_n \leq \mu^* C_N C_N < P/N \leq 4P\mu U_n$ for each $n \geq n_0$, so that $\sum_{n=n_0}^{n+1} \mu^* T_n \leq 4P \sum_{n=n_0}^{n+1} \mu U_n \leq 4P \mu^* K < \infty$. That $\mu^* T = 0$ now follows from the relations $\mu^* T \leq \mu^* \bigcup_{i=1}^{n+1} T_n \leq \sum_{i=1}^{n+1} \mu^* T_n$, valid for each $k$. This concludes the proof of the lemma.

The next two lemmas concern the properties of the product $B^{-1}C$.

3.2. Lemma. If $g \in G$ and $U \subseteq X$, then $g \phi \in U^{-1}U$ if and only if $g^{-1}\phi \in U^{-1}U$.

Proof. If $g \phi \in U^{-1}U$, then there is an $h \in G$ for which $g \phi = hU$ and $\phi = hU$. But then $\phi = g^{-1}hU$ and $g^{-1}\phi = g^{-1}hU$, so that $g^{-1}\phi \in U^{-1}U$ by 2.4. The replacement of $g$ by $g^{-1}$ in this argument proves the converse.

3.3. Lemma. Let $U_i \subseteq X$ for each $i \in I$, and let $C = \bigcup_{i \in I} U_i^{-1}U_i$. Then $CD = C^{-1}D$ for each $D \subseteq X$.

Proof. We have, using 3.2,
We now prove as a lemma a special case of our theorem.

3.4. Lemma. Let $\mathcal{U}$ be a regular Vitali cover for $A$ with constant parameter of regularity $M$. Let $K$ be an open subset of $X$ for which $A \subset K$ and $\mu^K < \infty$. Then there exists a (possibly finite) sequence $U_k$ of pairwise disjoint elements of $\mathcal{U}$ for which $\mu^*(A \setminus \bigcup_{k=1}^{\infty} U_k) = 0$. The sequence may be chosen so that $\bigcup_{k=1}^{\infty} U_k \subset K$.

Let $\varepsilon$ and $s_n$ be as in Definition 2.8. We may clearly suppose, discarding certain elements from $\mathcal{U}$ if necessary, that for each $U \in \mathcal{U}$ there are a $g \in G$ and an $S_k$ with $gU \subset S_k$ and $\mu^*S_k \leq M\mu U$.

Since $\phi \in B^{-1}B$ for each nonempty subset $B$ of $X$, we have $\phi \in S_{k+1}^{-1}S_{k+1} \subset S_k$ for each $k$. Denoting by $e$ the identity element of the group $G$, we have $S_{k+1} = eS_{k+1} \subset \bigcup \{gS_{k+1} | g \in G, \phi = g\phi\} = \{\phi\}^{-1}S_{k+1} \subset S_{k+1}^{-1}S_{k+1} \subset S_k$, so that the real sequence $\mu^*S_k$ is monotone decreasing. Since $\mathcal{U}$ is a Vitali cover for the set $A$, each point of which admits neighborhoods of arbitrarily small $\mu$-measure, we see from the conclusion of the preceding paragraph that $\mu^*S_k \to 0$.

Now for each positive integer $n$ we set $C_n = \bigcup \{U^{-1}U | U \in \mathcal{U}, \mu U < 1/n\}$. The sequence $C_n$ clearly satisfies hypothesis (2) of 3.1, so by 3.1 the proof of the present lemma may be completed by showing that the real sequence $n\mu^*C_nC_n$ is bounded. To this end we restrict our attention to those positive integers $n$ for which $n > M/\mu S_2$, and we associate with each such $n$ the smallest integer $k_n$ for which $\mu^*S_{k_n} \leq M/n$. Then always $k_n \geq 3$ and $S_k \subset S_{k_n}$ whenever $\mu^*S_k \leq M/n$. Thus for each $U \in \mathcal{U}$ with $\mu U < 1/n$ there is a $g \in G$ for which $gU \subset S_k$. Hence $U^{-1}U = (gU)^{-1}(gU) \subset S_k^{-1}S_k \subset S_{k_n}$ whenever $U \in \mathcal{U}$ and $\mu U < 1/n$. Thus $C_n \subset S_{k_n-1}$, and from 3.3 we have $C_nC_n = C_n^{-1}C_n \subset S_{k_n-1}^{-1}S_{k_n-1} \subset S_{k_n-2}$. Hence $\mu^*C_nC_n \leq \mu^*S_{k_n-2} \leq \mu^*S_{k_n-1}/\varepsilon^2 \leq M/(\varepsilon^2n)$ whenever $n > M/\mu S_2$, and the proof is complete.

The next result, our main theorem, is obtained by removing some of the hypotheses of the preceding lemma.

3.5. Theorem. Let $\mathcal{U}$ be a regular Vitali cover for $A$, and suppose that there is a sequence $V_k$ of open subsets of $X$ for which $A \subset \bigcup_{k=1}^{\infty} V_k$ and always $\mu^*V_k < \infty$. Then there is a (possibly finite) sequence $U_k$ of pairwise disjoint elements of $\mathcal{U}$ for which $\mu^*(A \setminus \bigcup_{k=1}^{\infty} U_k) = 0$.

Proof. Let $S_n$ be the sequence of quasispheres, $M$ the (not necessarily constant) function given by 2.8. We suffer no loss in generality in supposing that $V_k \subset V_{k+1}$ for each positive integer $k$. For each positive integer $k$, we define

$$A_k = A \cap M^{-1}(0, k] \cap V_k$$
and  $\mathcal{U}_k = \{ U \in \mathcal{U} \mid g U \subseteq S_n \text{ for some } g \in G \text{ and some integer } n \text{ with } \mu^* S_n < k \mu U \}$. Then  $\mathcal{U}_k$ is a regular Vitali cover for  $A_k$ with constant parameter of regularity  $k$, so by 3.4 we can find pairwise disjoint elements  $U_1, U_2, \ldots, U_n$, of  $\mathcal{U}_1$ for which  $\mu^*(A_1 \setminus \bigcup_{k=1}^{n-1} U_k) < 1$. Proceeding inductively on the assumption that pairwise disjoint elements  $U_1, \ldots, U_n$ of  $\mathcal{U}_n$ have been chosen so that  $\mu^*(A_n \setminus \bigcup_{k=1}^{n} U_k) < 1/n$, we notice that  $W_n = V_n \setminus \bigcup_{k=1}^{n-1} U_k$ is an open subset of  $X$ for which  $A_n \setminus \bigcup_{k=1}^{n} U_k \subset W_n$ and  $\mu^* W_n < \infty$. Thus there are by Lemma 3.4 pairwise disjoint elements  $U_{k_{n+1}}, \ldots, U_{k_{n+1}}$ of  $\mathcal{U}_{n+1}$ for which

$$\mu^* \left[ \left( A_n \setminus \bigcup_{k=1}^{k_{n+1}} U_k \right) \setminus \bigcup_{k=k_{n+1}}^{k_{n+1}} U_k \right] < \frac{1}{n+1}$$

and

$$\bigcup_{k=k_{n+1}}^{k_{n+1}} U_k \subset W_n = V_n \setminus \bigcup_{k=1}^{n} U_k.$$

From these computations it follows that

$$\mu^* \left( A_n \setminus \bigcup_{k=1}^{n} U_k \right) \leq \mu^* \left( A_n \setminus \bigcup_{k=1}^{n} U_k \right) < 1/n \text{ for each positive integer } n,$$

which together with the relations  $A = \bigcup_{n=1}^{\infty} A_n$ and  $A_n \subset A_{n+1}$  $(n \geq 1)$ yields the conclusion  $\mu^* (A \setminus \bigcup_{n=1}^{\infty} U_k) = 0$.

### 4. Examples

The examples presented here are designed to indicate the applicability of our theorem to numerous instances in which a group acts transitively on a measure space. For the sake of simplicity we consider cases where each  $U$ in the cover  $\mathcal{U}$ is a translate of a quasisphere. It will be clear how more complicated examples can be constructed. We begin by stating a corollary to our theorem.

#### 4.1. Corollary

Let  $S_n$ be a sequence of closed, measurable quasispheres, and let  $A$ be contained in a countable union of open sets each of which has finite outer measure. Suppose that for each neighborhood  $V$ of each point  $x$ in  $A$ there exists an element  $g \in G$ and a positive integer  $n$ for which  $x \in g S_n \subset V$. Then there are a sequence  $g_k$ of elements of  $G$ and a sequence  $n_k$ of positive integers for which the sets  $g_k S_{n_k}$ are pairwise disjoint and  $\mu^*(A \setminus \bigcup_{n=1}^{\infty} g_k S_{n_k}) = 0$.

From this corollary it follows, speaking loosely, that Theorem 3.5 will be applicable to any situation in which quasispheres can be introduced. Euclidean  $k$-space  $R^k$ is one setting in which this can be done in such a way that the cover which results from a translation of the quasispheres to the points of a nonempty subset  $A$ of  $R^k$ is nowhere regular in the classical sense. By using a sequence of solid ellipses with increasing eccentricity, we give in Example 4.2 a possible construction for the case  $k = 2$.

#### 4.2. Example

Let the topological group  $G = X = R^k$ act on itself by trans-
lation, and let \( \mu \) denote 2-dimensional Lebesgue measure. Choosing \( \phi \) to be the identity element in \( \mathbb{R}^2 \) we define
\[
S_n = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } (2^n x_1)^2 + (2^n x_2)^2 \leq 1\},
\]
for each positive integer \( n \). To see that the closed sets \( S_n \) form a sequence of quasispheres in the sense of Definition 2.7, we observe that \( S_{n+1} \cap S_{n+1} = 2S_{n+1} = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } (2^n x_1)^2 + (2^n x_2)^2 \leq 1\} \subset S_n \), and that \( \mu S_{n+1} = \pi / 2^{n+2} > \epsilon \pi / 2^n = \epsilon \mu S_n \) if \( \epsilon < 1/8 \). The sequence \( S_n \) being fundamental at \( \phi \), we see that the hypotheses of Corollary 4.1 are satisfied for an arbitrary subset \( A \) of \( \mathbb{R}^2 \). Since the smallest disk \( D_n \) in \( \mathbb{R}^2 \) containing \( S_n \) has area \( \pi / 2^n \), however, the sequence \( \mu S_n / \mu D_n \) has limit zero and the classical Vitali theorem is not applicable.

We next show how to find quasispheres in a more general situation.

4.3. Example. Suppose that the space \( X \) admits a point, which we may as well choose for \( \phi \), some neighborhood \( V \) of which admits a homeomorphism onto a neighborhood of the origin in \( \mathbb{R}^k \). If \( \eta \) is such a homeomorphism, arbitrary but henceforth fixed, then \( \eta^{-1}(\eta x + \eta y) \), whenever it is defined, will be denoted by \( x+y; x-y \) is defined similarly. When \( x \in V \), we write \( |x| \) for the distance from \( \eta x \) to the origin. Then quasispheres can be defined as the image under \( \eta^{-1} \) of spheres in \( \mathbb{R}^k \), provided that the elements of \( G \) satisfy the following condition E-L, a sort of local equi-Lipschitz condition.

Condition E-L. There is an open subset \( W \) of \( A \) and a real number \( \epsilon \) for which \( \eta \in C^2(W) \) and for which \( \eta x \in V \) and \( |\eta x - \eta y| < \epsilon |x - y| \) whenever \( x \in W \) and \( y \in W \) and \( \eta x = \phi \).

Indeed, let \( r \) be a positive number for which \( x \in W \) whenever \( |x| < r \), and for which the sphere of radius \( r \) about the origin in \( \mathbb{R}^k \) is contained in \( \eta V \). Now define
\[
S_n = \{x \mid |x| \leq r/(2P)^n\},
\]
for each positive integer \( n \). Our discussion will be complete if we can show that the sequence \( S_n \) satisfies conditions (1) and (2) of Definition 2.7.

To show \( S_{n+1} \cap S_{n+1} \subset S_n \), which is (1), we notice that the constant \( P \) of condition E-L exceeds 1, so that always \( S_n \subset W \). If \( x \in S_{n+1} \cap S_{n+1} \), so that there exist points \( y \) and \( z \) in \( S_{n+1} \) and \( g \in G \) for which \( \eta y = x \) and \( \eta z = \phi \), then we have
\[
|x| = |\eta y - \eta z| < P |y - z| \leq P (|y| + |z|) \leq 2Pr/(2P)^{n+1} = r/(2P)^n;
\]
thus \( x \in S_n \) and (1) of Definition 2.7 is established.

To prove (2), we notice that for each positive integer \( n \) the sets \( \eta S_n, \eta S_{n+2} \) are spheres in \( \mathbb{R}^k \), the radius of the former being \( (2P)^2 \) times the radius of the latter. Thus there is an integer \( m \), dependent only on \( k \) and \( P \), for which \( \eta S_n \subset \bigcup_{i=1}^{m} \eta x_i + \eta S_{n+2} \) for appropriately chosen points \( x_i \in S_n \). Now if \( y \in \eta S_{n+2} \), then the distance in \( \mathbb{R}^k \) from the origin to \( \eta x_i + \eta y \) does not exceed \( r/(2P)^n + r/(2P)^{n+2} < r \), so that \( \eta^{-1}(\eta x_i + \eta y) = x_i + y \) exists and lies in \( V \). Choosing \( g_i \in G \) so that \( g_i x_i = \phi \), we have
Thus \( g_i(x_i + S_{n+2}) \subseteq S_{n+1} \), and from the relation \( S_n \subseteq \bigcup_{i=1}^{m} x_i + S_{n+2} \) it follows that

\[
\mu^* S_n \leq \sum_{i=1}^{m} \mu^* (x_i + S_{n+2}) \leq \sum_{i=1}^{m} \mu^* S_{n+1} = m \mu^* S_{n+1}.
\]

4.4. Example. We consider finally the case in which a Lie group \( G = X \) acts upon itself by translation. Retaining the notation introduced in 4.3, we recall from Smith [4] that the homeomorphism \( \eta \) may be chosen to introduce on \( X \) a coordinate system which is right regular, in the sense that there is a function \( F \) on \( V \times V \), jointly continuous at \((\phi, \phi)\), for which

\[
u v = u + v + \left| v \right| F(u,v),
\]

whenever each of the points, \( u, v, uv, F(u,v) \) and \( \left| v \right| F(u,v) \) lies in \( V \). (By \( \left| v \right| F(u,v) \), we mean \( \eta^{-1}(\left| v \right| \eta F(u,v)) \).) The fact that our theorem is applicable in this situation will be proved by showing that if \( \eta \) is so chosen, then condition E-L is satisfied. The condition now reads as follows: There is an open subset \( W \) of \( X \) and a real number \( P \) for which \( \phi \in W \subseteq V \) and for which \( x^{-1}y \in V \) and \( \left| x^{-1}y \right| < P \left| x-y \right| \) whenever \( x \in W \) and \( y \in W \).

Now if \( x \) and \( y \) are arbitrary points of \( V \) and if \( z \in X \), we could write

\[
y = xx^{-1}y = x + x^{-1}y + \left| x^{-1}y \right| z,
\]

whence

\[
y - x - \left| x^{-1}y \right| z = x^{-1}y,
\]

provided that each of the points appearing in equations (1) and (2) is well-defined (i.e., lies in \( V \)). In the present case, a real number \( P > 1 \) being given, we choose \( W \) so small that (1) and (2) are meaningful for \( z = F(x, x^{-1}y) \), and so small that \( \left| F(x, x^{-1}y) \right| < (P-1)/P \) whenever \( x \in W \) and \( y \in W \). With this choice of \( W \) we have \((1/P)\left| x^{-1}y \right| < (1 - \left| z \right|) \left| x^{-1}y \right| < \left| y-x \right| = \left| x-y \right| \) whenever \( x \in W \) and \( y \in W \), so that condition E-L is satisfied.

References


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