EXTENSION OF A RESULT OF BEURLING ON INVARIANT SUBSPACES(1)

BY

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1. Introduction. In a fundamental paper [1], A. Beurling has characterized the invariant subspaces of the shift operator \( S: (x_0, x_1, x_2, \ldots) \rightarrow (x_1, x_2, x_3, \ldots) \) in the Hilbert space \( l^2 \) of all complex square-summable sequences. Transforming to an equivalent problem in the analytic function space \( H^2 \), Beurling made use of a factorization theorem(2) to derive results implying that the lattice of invariant subspaces of \( S \) is isomorphic to the lattice of "inner functions."

It is the aim of the present paper to extend Beurling's result to the more general "tridiagonal" operator \( T: (x_0, x_1, \ldots) \rightarrow (y_0, y_1, \ldots) \) for which

\[
\begin{align*}
 y_0 &= \alpha x_0 + \beta x_1 \\
y_n &= \gamma x_{n-1} + \alpha x_n + \beta x_{n+1}, \quad n = 1, 2, \ldots.
\end{align*}
\]

Here \( \alpha, \beta, \gamma \) are complex numbers such that \( \beta \neq 0 \) and \( |\beta| \neq |\gamma| \). The term tridiagonal is suggested by the form of the infinite matrix generating \( T \). Since the value of \( \alpha \) does not affect the invariant subspaces of \( T \), it will be assumed \( \alpha = 0 \). The adjoint of such an operator \( T \) is given by \( T^* x = y \), where \( y_0 = \gamma x_1 \) and \( y_n = \tilde{\beta} x_{n-1} + \tilde{\gamma} x_{n+1} \), \( n = 1, 2, \ldots \).

2. Eigenvalues of \( T \). The vector equation \( T x = \lambda x \) is equivalent to the recurrence relation

\[
\begin{align*}
 -\lambda x_0 + \beta x_1 &= 0; \\
\gamma x_{n-1} - \lambda x_n + \beta x_{n+1} &= 0, \quad n = 1, 2, \ldots.
\end{align*}
\]

Let \( x_n = \rho_n(\lambda) \), \( n \geq 0 \), denote the solution to (1) normalized by \( x_0 = 1 \). Note that \( \rho_n(\lambda) \) is a polynomial of degree \( n \). The number \( \lambda \) is an eigenvalue of \( T \) if and only if \( \rho(\lambda) = (\rho_0(\lambda), \rho_1(\lambda), \ldots) \subseteq l^2 \).

Let \( z_1, z_2 \) denote the roots of \( f_T(z; \lambda) = \gamma - \lambda z + \beta z^2 \), the characteristic polynomial of (1). Then \( \rho(\lambda) \subseteq l^2 \) if and only if \( |z_k| = |z_k(\lambda)| < 1, \quad k = 1, 2 \). Hence it is important to study the functions \( z_k(\lambda) \), which are the inverses of \( \lambda = \lambda(z) = \beta z + \gamma/z \).

To this end, set \( \gamma/\beta = re^{i\theta} \) \((0 \leq \theta < 2\pi; \theta = 0 \text{ if } \gamma = 0)\), \( z = re^{i\theta} \), and \( w = u + iv \).

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(2) A more detailed proof of Beurling's factorization theorem has been given by Rudin [6].

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Thus the two circles \(|z| = \rho, |z| = r/p\) correspond to the ellipse \(u^2(\rho + r/p)^{-2} + v^2(\rho - r/p)^{-2} = 1\) in the \(w\)-plane. When \(\rho = r^{1/2}\), the two circles coincide, and the ellipse degenerates into the line segment \(-2r^{1/2} \leq u \leq 2r^{1/2}, v = 0\) joining the foci. Since \(\lambda = \beta e^{i\theta} w\) is a magnification followed by a rotation, the two circles correspond also to an ellipse \(E_\rho = E_{r/\rho}\) in the \(\lambda\)-plane. Each \(E_\rho\) has as foci the two points \(\lambda = \pm 2(\beta \gamma)^{1/2}\) for which \(f_\tau(z; \lambda)\) has a multiple root. The ellipse \(E_\rho\) will be denoted \((3) E_\tau\), and will be called the spectral ellipse of \(T\). Int \(E_\tau\) will denote the open set bounded by \(E_\tau\).

Lemma 1. Let \(\lambda \in \text{Int } E_\tau\). Then \(\lambda\) is a simple eigenvalue of \(T\) if \(r < 1\), while \(\lambda\) is a simple eigenvalue of \(T^*\) if \(r > 1\). Furthermore, for \(r < 1\) the convergence of \(\sum |p_n(\lambda)|^2\) is uniform in each closed subset of \(\text{Int } E_\tau\).

3. Generalized \(H_2\) spaces. The polynomials \(p_n(\lambda)\) possess a remarkable orthogonality property, first discovered by Szegö [7].

Lemma 2. For each \(\rho > 0\),
\[
\frac{1}{2\pi} \int_{E_\rho} p_n(\lambda)\overline{p_m(\lambda)} \omega(\lambda) \, d\lambda = \begin{cases} 0, & n \neq m, \\ \frac{2\pi^2(n+1)}{\rho^2(n+1)} + \frac{(r/\rho)^2(n+1)}{2}, & n = m, \end{cases}
\]
where \((1) \, \omega(\lambda) = |\beta|^{-2}|\lambda^2 - 4\beta\gamma|^{1/2} = 1/2\).

The proof is essentially the same as that of Szegö for the Chebyshev polynomials of the second kind, of which the \(p_n(\lambda)\) are a generalization; the mapping \(\lambda = \beta z + \gamma z\) is used to transform the path of integration to the unit circle in the \(z\)-plane. The details are worked out in [3]. As an illustration, let \(\beta = 1, \gamma = 0, \rho = 1\). Then \(p_n(\lambda) = \lambda^n, E_\rho\) is the circle \(|\lambda| = 1, \omega(\lambda) = 1\) on \(E_\rho\); and Lemma 2 reduces to the familiar relation
\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i\sigma} e^{-i\sigma} d\phi = \delta_{nm}.
\]

The tridiagonal operator with \(\beta = 1, \gamma = 0\) is precisely the shift operator \(S\).

For \(r < 1\), let \(\mathcal{F}\) denote the set of all functions \(f(\lambda) = \sum_{n=0}^\infty a_n p_n(\lambda)\) for which the complex sequence \(\{a_n\}\) is square-summable. In view of Lemma 1 and Schwarz’ inequality, each such \(f(\lambda)\) is analytic in \(\text{Int } E_\tau\). Indeed, \(\mathcal{F}\) is the generalized Hardy space \([5] H_2(E_\tau)\) consisting of all functions \(f(\lambda)\) analytic in \(\text{Int } E_\tau\) with the property that \(|f(\lambda)|^2\) has a harmonic majorant there.

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Lemma 3. $\mathcal{E} = H_2(\mathcal{E}_T)$. Moreover, there exist positive constants $A$ and $B$ such that

$$A u_T(0) \leq \sum_{n=0}^{\infty} |a_n|^2 \leq B u_T(0),$$

where $u_T(\lambda)$ is the least harmonic majorant of $|f(\lambda)|^2 = \sum a_n \rho_n(\lambda)^2$.

Proof. It follows from Lemma 2 that

$$\lim_{\rho \to 1-0} \frac{1}{2\pi} \int_{\mathcal{E}_p} |f(\lambda)|^2 \omega(\lambda) \, d\lambda = \sum_{n=0}^{\infty} |a_n|^2 \left[ 1 + r^{2(n+1)} \right],$$

where $f(\lambda) = \sum a_n \rho_n(\lambda)$. On the other hand, it is known [5] that

$$\lim_{p \to 1-0} \frac{1}{2\pi} \int_{\mathcal{E}_p} \left| f(\lambda) \right|^2 \frac{\partial g_p(\lambda, 0)}{\partial n} \, d\lambda = u_T(0),$$

provided $f \in H_2(\mathcal{E}_T)$. Here $g_p(\lambda, \mu)$ is Green's function for Int $\mathcal{E}_p$ with pole at $\lambda = \mu$, and the derivative is taken along the inner normal. Let $g(\lambda, \mu) = g_1(\lambda, \mu)$ denote Green's function for Int $\mathcal{E}_T$.

To complete the proof, therefore, it suffices to establish the existence of positive constants $a$ and $b$, independent of $\rho$, such that $a \leq \left| \frac{\partial g_p(\lambda, 0)}{\partial n} \right| \leq b$, $\lambda \in \mathcal{E}_p$, for all sufficiently large $\rho < 1$. Since [5] $\frac{\partial g(\lambda, 0)}{\partial n} > 0$ on $\mathcal{E}_T$, this follows from

$$\lim_{\rho \to 1-0} \left\{ \frac{\partial g_p(\lambda, 0)}{\partial n} - \frac{\partial g(\lambda, 0)}{\partial n} \right\} = 0$$

uniformly in $\lambda \in \mathcal{E}_p$.

It remains to prove (2). Application of the inversion principle [4; 2, pp. 87–88] to the algebraic curves $\mathcal{E}_p$ shows that the functions $g_p(\lambda, 0)$ have, for all $\rho > \rho_0$, harmonic extensions to a fixed domain Int $\mathcal{E}_{p_1}$ ($p_1 > 1$) with $\lambda = 0$ deleted. Let $\kappa = 1/2(1+\rho_1)$. Because [4, pp. 42 ff.] the sequence of harmonic functions $\{g_p(\lambda, 0) - g(\lambda, 0)\} \to 0$ as $\rho \to 1-0$ uniformly in each closed subdomain of Int $\mathcal{E}_T$, it follows from Vitali's theorem that the same is true uniformly on $\mathcal{E}_p$. Thus differentiation of the formula

$$g_p(\lambda, 0) - g(\lambda, 0) = \frac{1}{2\pi} \int_{\mathcal{E}_p} \left\{ g_p(v, 0) - g(v, 0) \right\} \frac{\partial g_p(v, \lambda)}{\partial n} \, dv$$

under the integral sign yields (2).

The norm of $f \in H_2(\mathcal{E}_T)$ will be defined as $\|f\| = [u_T(0)]^{1/2}$.

4. Invariant subspaces. Assuming $r < 1$, let $\tau$ denote the one-one antilinear(\footnote{\(\tau(ax+by) = \overline{a}r(x) + \overline{b}r(y)\) for all $x, y \in l_2$ and complex $a, b$.}) transformation of $l_2$ onto $H_2(\mathcal{E}_T)$ defined by

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Since \( T\rho(x) = \lambda \rho(x) \) by construction, it is clear that \( \tau(T^* x) = \lambda f(x) \), where \( f(x) = \tau(x) \). This observation, together with Lemma 3, shows that the lattice of invariant subspaces of \( T^* \) in \( l_2 \) is isomorphic to that of the operator \( M : f(\lambda) \rightarrow \lambda f(\lambda) \) in \( H_2(\mathbb{C}_T) \). It suffices, therefore, to consider the invariant subspaces of \( M \).

Let \( z = \xi(\lambda) \), \( \lambda = \eta(z) \) be a fixed conformal mapping of \( \text{Int} \mathbb{C}_T \) onto \( |z| < 1 \) which carries 0 into 0: \( \eta(0) = \xi(0) = 0 \). Then [5] the transformation \( \pi : f(\lambda) \rightarrow F(z) = f(\eta(z)) \) is an isometric isomorphism of \( H_2(\mathbb{C}_T) \) onto the classical space \( H_2 \).

For \( f, g \in H_2(\mathbb{C}_T) \) let \( \mathcal{V}[f, g] \) denote the (closed) subspace spanned by the functions \( \lambda^n f(\lambda) \) and \( \lambda^n g(\lambda) \), \( n = 0, 1, \ldots \). Let \( \mathcal{V}[f] = \mathcal{V}[f, 0] \).

**Lemma 4.** A subspace of \( H_2 \) is invariant under multiplication by \( z \) if and only if it is invariant under multiplication by \( \eta(z) \).

**Proof.** Suppose first that \( zU \subseteq U \), and let \( F(z) \in U \). Since \( \eta(z) \) is analytic in \( |z| \leq 1 \), there exists a polynomial \( Q(z) \) such that \( |\eta(z) - Q(z)| < \varepsilon \) for \( |z| \leq 1 \). Thus \( ||\eta F - QF|| \leq \varepsilon ||F|| \). But \( Qf \in U \), by hypothesis. The converse is proved similarly.

**Corollary.** \( \pi(\mathcal{V}[f, g]) = \mathcal{V}[\pi(f), \pi(g)] \).

Lemma 4 and its corollary, together with Beurling’s Theorems I, III, and IV, now yield the following extensions of those theorems.

**Theorem 1.** Let \( f, g \in H_2(\mathbb{C}_T) \) be \( \neq 0 \), and let \( F = \pi(f) \), \( G = \pi(g) \). Then \( g \in \mathcal{V}[f] \) if and only if the inner factor (*) \( F_0 \) of \( F \) divides the inner factor \( G_0 \) of \( G \).

**Theorem 2.** Let \( f, g, F, G \) be as in Theorem 1. Then \( \mathcal{V}[f, g] = \mathcal{V}[\pi^{-1}(F_0)] \), where \( H_0 \) is the greatest common divisor of \( F_0 \) and \( G_0 \).

**Theorem 3.** Let \( U \subseteq H_2(\mathbb{C}_T) \) be a non-null invariant subspace of \( M \). Then there exists a unique inner function \( G_0(z) \) such that \( \mathcal{V}[\pi^{-1}(G_0)] = U \).

From Theorems 1, 2, 3 one deduces that the lattice of invariant subspaces of \( T^* \) is isomorphic to the lattice of inner functions. Implicit is the assumption \( r = |\gamma/\beta| < 1 \). When \( r > 1 \), one derives the same result for \( (T^*)^* = T \). But the invariant subspaces of \( T^* \) are the orthocomplements of the invariant subspaces of \( T \); indeed, interchanging “unions” and intersections, the lattice of invariant subspaces of \( T^* \) is isomorphic to the lattice of invariant subspaces of \( T \). Consequently:

**Theorem 4.** The lattice of invariant subspaces of a tridiagonal operator \( \lambda \neq 1 \) is isomorphic to the lattice of inner functions.

(*) The reader is referred to Beurling’s paper for definitions of terms.
References


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