EXTENSION OF A RESULT OF BEURLING ON INVARIANT SUBSPACES(1)

BY

PETER L. DUREN

1. **Introduction.** In a fundamental paper [1], A. Beurling has characterized the invariant subspaces of the shift operator $S: (x_0, x_1, x_2, \cdots) \to (x_1, x_2, x_3, \cdots)$ in the Hilbert space l_2 of all complex square-summable sequences. Transforming to an equivalent problem in the analytic function space H_2, Beurling made use of a factorization theorem(2) to derive results implying that the lattice of invariant subspaces of S is isomorphic to the lattice of "inner functions."

It is the aim of the present paper to extend Beurling's result to the more general "tridiagonal" operator $T: (x_0, x_1, \cdots) \to (y_0, y_1, \cdots)$ for which $y_0 = \alpha x_0 + \beta x_1$ and $y_n = \gamma x_{n-1} + \alpha x_n + \beta x_{n+1}$, $n = 1, 2, \cdots$. Here α, β, γ are complex numbers such that $\beta \neq 0$ and $|\beta| \neq |\gamma|$. The term *tridiagonal* is suggested by the form of the infinite matrix generating T. Since the value of α does not affect the invariant subspaces of T, it will be assumed $\alpha = 0$. The adjoint of such an operator T is given by $T^* x = y$, where $y_0 = \bar{\gamma} x_1$ and $y_n = \bar{\beta} x_{n-1} + \bar{\gamma} x_{n+1}$, $n = 1, 2, \cdots$.

2. **Eigenvalues of** T. The vector equation $Tx = \lambda x$ is equivalent to the recurrence relation

$$(1) \qquad -\lambda x_0 + \beta x_1 = 0; \quad \gamma x_{n-1} - \lambda x_n + \beta x_{n+1} = 0, \qquad n = 1, 2, \cdots.$$

Let $x_n = p_n(\lambda)$, $n \geqq 0$, denote the solution to (1) normalized by $x_0 = 1$. Note that $p_n(\lambda)$ is a polynomial of degree n. The number λ is an eigenvalue of T if and only if $p(\lambda) = (p_0(\lambda), p_1(\lambda), \cdots) \in l_2$.

Let z_1, z_2 denote the roots of $f_T(z; \lambda) = \gamma - \lambda z + \beta z^2$, the characteristic polynomial of (1). Then $p(\lambda) \in l_2$ if and *only* if $|z_k| = |z_k(\lambda)| < 1$, $k = 1, 2$. Hence it is important to study the functions $z_k(\lambda)$, which are the inverses of $\lambda = \lambda(z) = \beta z + \gamma/z$.

To this end, set $\gamma/\beta = re^{i\theta}$ ($0 \leqq \theta < 2\pi$; $\theta = 0$ if $\gamma = 0$), $z = \rho e^{i\psi}$, and $w = u + iv = (1/\beta) e^{-i\theta/2} \lambda$. Then

This paper has been submitted to and accepted for publication by the Proceedings of the American Mathematical Society. It has been transferred to these Transactions, with the consent of the author, for technical reasons. Presented to the Society, September 2, 1960; received by the editors July 16, 1960 and, in revised form, October 24, 1960.

(1) This paper contains part of a doctoral dissertation written at the Massachusetts Institute of Technology under the guidance of Professor G.-C. Rota. The author also wishes to acknowledge two very helpful conversations with Professor J. L. Walsh. The research was sponsored in part by a National Science Foundation fellowship and in part by the Office of Naval Research, Contract Nonr-1841(38).

(2) A more detailed proof of Beurling's factorization theorem has been given by Rudin [6].

$$u = (\rho + r/\rho)\cos(\psi - \theta/2); \qquad v = (\rho - r/\rho)\sin(\psi - \theta/2).$$

Thus the two circles $|z| = \rho$, $|z| = r/\rho$ correspond to the ellipse $u^2(\rho + r/\rho)^{-2} + v^2(\rho - r/\rho)^{-2} = 1$ in the w-plane. When $\rho = r^{1/2}$, the two circles coincide, and the ellipse degenerates into the line segment $-2r^{1/2} \leqq u \leqq 2r^{1/2}$, $v = 0$ joining the foci. Since $\lambda = \beta e^{i\theta/2} w$ is a magnification followed by a rotation, the two circles correspond also to an ellipse $\mathcal{E}_\rho = \mathcal{E}_{r/\rho}$ in the λ-plane. Each $\mathcal{E}_\rho$ has as foci the two points $\lambda = \pm 2(\beta\gamma)^{1/2}$ for which $f_T(z;\ \lambda)$ has a multiple root. The ellipse $\mathcal{E}_1$ will be denoted(3) $\mathcal{E}_T$, and will be called the *spectral ellipse* of T. Int $\mathcal{E}_T$ will denote the open set bounded by $\mathcal{E}_T$.

Specifically, $\mathcal{E}_T$ is the curve $\lambda = \beta e^{i\phi} + \gamma e^{-i\phi}$, $0 \leqq \phi < 2\pi$. Note that $\mathcal{E}_{T^*} = \overline{\mathcal{E}}_T$.

LEMMA 1. *Let* $\lambda \in$ Int $\mathcal{E}_T$. *Then* λ *is a simple eigenvalue of* T *if* $r < 1$, *while* $\bar{\lambda}$ *is a simple eigenvalue of* T^* *if* $r > 1$. *Furthermore, for* $r < 1$ *the convergence of* $\sum |p_n(\lambda)|^2$ *is uniform in each closed subset of* Int $\mathcal{E}_T$.

3. **Generalized H_2 spaces.** The polynomials $p_n(\lambda)$ possess a remarkable orthogonality property, first discovered by Szegö [7].

LEMMA 2. *For each* $\rho > 0$,

$$\frac{1}{2\pi}\int_{\mathcal{E}_\rho} p_n(\lambda)\overline{p_m(\lambda)}\omega(\lambda)\,|d\lambda| = \begin{cases} 0, & n \neq m, \\ \rho^{2(n+1)} + (r/\rho)^{2(n+1)}, & n = m, \end{cases}$$

where(4) $\omega(\lambda) = |\beta|^{-2}|\lambda^2 - 4\beta\gamma|^{1/2}$.

The proof is essentially the same as that of Szegö for the Chebyshev polynomials of the second kind, of which the $p_n(\lambda)$ are a generalization; the mapping $\lambda = \beta z + \gamma/z$ is used to transform the path of integration to the unit circle in the z-plane. The details are worked out in [3]. As an illustration, let $\beta = 1$, $\gamma = 0$, $\rho = 1$. Then $p_n(\lambda) = \lambda^n$, $\mathcal{E}_\rho$ is the circle $|\lambda| = 1$, $\omega(\lambda) \equiv 1$ on $\mathcal{E}_\rho$; and Lemma 2 reduces to the familiar relation

$$\frac{1}{2\pi}\int_0^{2\pi} e^{in\phi}e^{-im\phi}\,d\phi = \delta_{nm}.$$

The tridiagonal operator with $\beta = 1$, $\gamma = 0$ is precisely the shift operator S.

For $r < 1$, let $\mathfrak{F}$ denote the set of all functions $f(\lambda) = \sum_{n=0}^{\infty} a_n p_n(\lambda)$ for which the complex sequence $\{a_n\}$ is square-summable. In view of Lemma 1 and Schwarz' inequality, each such $f(\lambda)$ is analytic in Int $\mathcal{E}_T$. Indeed, $\mathfrak{F}$ is the generalized Hardy space [5] $H_2(\mathcal{E}_T)$ consisting of all functions $f(\lambda)$ analytic in Int $\mathcal{E}_T$ with the property that $|f(\lambda)|^2$ has a harmonic majorant there.

---

(3) $\mathcal{E}_T$ is nondegenerate on account of the assumption $|\beta| \neq |\gamma|$, or $r \neq 1$. Otherwise T would be a scalar multiple of a self-adjoint operator.

(4) Geometrically interpreted, $\omega(\lambda)$ is a constant multiple of the geometric mean of the distances from λ to the foci of $\mathcal{E}_\rho$.

LEMMA 3. *$\mathfrak{F} = H_2(\mathcal{E}_T)$. Moreover, there exist positive constants A and B such that*

$$Au_f(0) \leqq \sum_{n=0}^{\infty} |a_n|^2 \leqq Bu_f(0),$$

where $u_f(\lambda)$ is the least harmonic majorant of $|f(\lambda)|^2 = |\sum a_n p_n(\lambda)|^2$.

Proof. It follows from Lemma 2 that

$$\lim_{\rho\to 1-0} \frac{1}{2\pi}\int_{\mathcal{E}_\rho} |f(\lambda)|^2 \omega(\lambda) |d\lambda| = \sum_{n=0}^{\infty} |a_n|^2 [1 + r^{2(n+1)}],$$

where $f(\lambda) = \sum a_n p_n(\lambda)$. On the other hand, it is known [5] that

$$\lim_{\rho\to 1-0} \frac{1}{2\pi}\int_{\mathcal{E}_\rho} |f(\lambda)|^2 \frac{\partial g_\rho(\lambda, 0)}{\partial n} |d\lambda| = u_f(0),$$

provided $f \in H_2(\mathcal{E}_T)$. Here $g_\rho(\lambda, \mu)$ is Green's function for Int $\mathcal{E}_\rho$ with pole at $\lambda = \mu$, and the derivative is taken along the inner normal. Let $g(\lambda, \mu) = g_1(\lambda, \mu)$ denote Green's function for Int $\mathcal{E}_T$.

To complete the proof, therefore, it suffices to establish the existence of positive constants a and b, independent of ρ, such that $a \leqq |\partial g_\rho(\lambda, 0)/\partial n| \leqq b$, $\lambda \in \mathcal{E}_\rho$, for all sufficiently large $\rho < 1$. Since [5] $\partial g(\lambda, 0)/\partial n > 0$ on $\mathcal{E}_T$, this follows from

$$\lim_{\rho\to 1-0} \left\{ \frac{\partial g_\rho(\lambda, 0)}{\partial n} - \frac{\partial g(\lambda, 0)}{\partial n} \right\} = 0 \tag{2}$$

uniformly in $\lambda \in \mathcal{E}_\rho$.

It remains to prove (2). Application of the inversion principle [4; 2, pp. 87–88] to the algebraic curves $\mathcal{E}_\rho$ shows that the functions $g_\rho(\lambda, 0)$ have, for all $\rho > \rho_0$, harmonic extensions to a fixed domain Int $\mathcal{E}_{\rho_1}$ ($\rho_1 > 1$) with $\lambda = 0$ deleted. Let $\kappa = 1/2(1+\rho_1)$. Because [4, pp. 42 ff.] the sequence of harmonic functions $\{g_\rho(\lambda, 0) - g(\lambda, 0)\} \to 0$ as $\rho \to 1-0$ uniformly in each closed subdomain of Int $\mathcal{E}_T$, it follows from Vitali's theorem that the same is true uniformly on $\mathcal{E}_\kappa$. Thus differentiation of the formula

$$g_\rho(\lambda, 0) - g(\lambda, 0) = \frac{1}{2\pi}\int_{\mathcal{E}_\kappa} \{g_\rho(\nu, 0) - g(\nu, 0)\} \frac{\partial g_\kappa(\nu, \lambda)}{\partial n_\nu} |d\nu|$$

under the integral sign yields (2).

The norm of $f \in H_2(\mathcal{E}_T)$ will be defined as $\|f\| = [u_f(0)]^{1/2}$.

4. Invariant subspaces. Assuming $r < 1$, let τ denote the one-one antilinear(5) transformation of l_2 onto $H_2(\mathcal{E}_T)$ defined by

(5) $\tau(ax+by) = \bar{a}\tau(x) + \bar{b}\tau(y)$ for all $x, y \in l_2$ and complex a, b.

$$(p(\lambda), x) = \sum_{n=0}^{\infty} \bar{x}_n p_n(\lambda) = f(\lambda), \qquad \lambda \in \text{Int } \mathcal{E}_T.$$

Since $Tp(\lambda) = \lambda p(\lambda)$ by construction, it is clear that $\tau(T^*x) = \lambda f(\lambda)$, where $f(\lambda) = \tau(x)$. This observation, together with Lemma 3, shows that the lattice of invariant subspaces of T^* in l_2 is isomorphic to that of the operator $M: f(\lambda) \to \lambda f(\lambda)$ in $H_2(\mathcal{E}_T)$. It suffices, therefore, to consider the invariant subspaces of M.

Let $z = \xi(\lambda)$, $\lambda = \eta(z)$ be a fixed conformal mapping of Int $\mathcal{E}_T$ onto $|z| < 1$ which carries 0 into 0: $\eta(0) = \xi(0) = 0$. Then [5] the transformation $\pi: f(\lambda) \to F(z) = f(\eta(z))$ is an isometric isomorphism of $H_2(\mathcal{E}_T)$ onto the classical space H_2.

For $f, g \in H_2(\mathcal{E}_T)$ let $\mathcal{P}[f, g]$ denote the (closed) subspace spanned by the functions $\lambda^n f(\lambda)$ and $\lambda^n g(\lambda)$, $n = 0, 1, \cdots$. Let $\mathcal{P}[f] = \mathcal{P}[f, 0]$.

Lemma 4. *A subspace of H_2 is invariant under multiplication by z if and only if it is invariant under multiplication by $\eta(z)$.*

Proof. Suppose first that $zU \subseteq U$, and let $F(z) \in U$. Since $\eta(z)$ is analytic in $|z| \leq 1$, there exists a polynomial $Q(z)$ such that $|\eta(z) - Q(z)| < \epsilon$ for $|z| \leq 1$. Thus $\|\eta F - QF\| \leq \epsilon \|F\|$. But $Qf \in U$, by hypothesis. The converse is proved similarly.

Corollary. $\pi(\mathcal{P}[f, g]) = \mathcal{P}[\pi(f), \pi(g)]$.

Lemma 4 and its corollary, together with Beurling's Theorems I, III, and IV, now yield the following extensions of those theorems.

Theorem 1. *Let $f, g \in H_2(\mathcal{E}_T)$ be $\neq 0$, and let $F = \pi(f)$, $G = \pi(g)$. Then $g \in \mathcal{P}[f]$ if and only if the inner factor*(6) *F_0 of F divides the inner factor G_0 of G.*

Theorem 2. *Let f, g, F, G be as in Theorem 1. Then $\mathcal{P}[f, g] = \mathcal{P}[\pi^{-1}(H_0)]$, where H_0 is the greatest common divisor of F_0 and G_0.*

Theorem 3. *Let $U \subseteq H_2(\mathcal{E}_T)$ be a non-null invariant subspace of M. Then there exists a unique inner function $G_0(z)$ such that $\mathcal{P}[\pi^{-1}(G_0)] = U$.*

From Theorems 1, 2, 3 one deduces that the lattice of invariant subspaces of T^* is isomorphic to the lattice of inner functions. Implicit is the assumption $r = |\gamma/\beta| < 1$. When $r > 1$, one derives the same result for $(T^*)^* = T$. But the invariant subspaces of T^* are the orthocomplements of the invariant subspaces of T; indeed, interchanging "unions" and intersections, the lattice of invariant subspaces of T^* is isomorphic to the lattice of invariant subspaces of T. Consequently:

Theorem 4. *The lattice of invariant subspaces of a tridiagonal operator ($r \neq 1$) is isomorphic to the lattice of inner functions.*

(6) The reader is referred to Beurling's paper for definitions of terms.

REFERENCES

1. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. vol. 81 (1949) pp. 239–255.

2. C. Carathéodory, *Conformal representation*, 2nd ed., Cambridge, University Press, 1952.

3. P. L. Duren, *Spectral theory of a class of non-self-adjoint infinite matrix operators*, doctoral thesis, Massachusetts Institute of Technology, 1960.

4. Z. Nehari, *Conformal mapping*, New York, McGraw-Hill, 1952.

5. W. Rudin, *Analytic functions of class H_p*, Trans. Amer. Math. Soc. vol. 78 (1955) pp. 46–66.

6. ———, *The closed ideals in an algebra of analytic functions*, Canad. J. Math. vol. 9 (1957) pp. 426–434.

7. G. Szegö, *A problem concerning orthogonal polynomials*, Trans. Amer. Math. Soc. vol. 37 (1935) pp. 196–206.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASSACHUSETTS
STANFORD UNIVERSITY,
STANFORD, CALIFORNIA