

ON THE RESTRICTED CESÀRO SUMMABILITY OF DOUBLE FOURIER SERIES

BY
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1. We suppose throughout this paper that $\phi(u, v) \in L(0, 0; \pi, \pi)$ and is periodic, with period 2π , and that

$$(1.1) \quad \phi(u, v) \sim \sum_{m, n=0}^{\infty} a_{mn} \cos mu \cos nv.$$

We denote $\sum_{m, n=0}^{\infty} a_{mn}$ by $S[\phi]$ and write

$$(1.2) \quad \sigma_{mn}^{\alpha, \beta} = (A_m^\alpha A_n^\beta)^{-1} \sum_{r=0}^m \sum_{s=0}^n A_{m-r}^\alpha A_{n-s}^\beta a_{rs} = (A_m^\alpha A_n^\beta)^{-1} S_{mn}^{\alpha, \beta},$$

where

$$A_m^\alpha = \binom{m + \alpha}{m}.$$

We also write $u^a v^b \phi_{a,b}(u, v)$ for the fractional integral of order (a, b) ($a \geq 0, b \geq 0$) of $\phi(u, v)$, so that, in particular, $\phi_{0,0}(u, v) = \phi(u, v)$,

$$(1.3) \quad \phi_{a,b}(u, v) = ab u^{-a} v^{-b} \int_0^u \int_0^v (u-x)^{a-1} (v-y)^{b-1} \phi(x, y) dx dy$$

($a > 0, b > 0$),

and $\phi_{0,b}(u, v), \phi_{a,0}(u, v)$ are interpreted in the natural way (cf. [4, p. 413]).

The problem of the convergence, in some sense, of the means $\sigma_{mn}^{\alpha, \beta}$, and its connexion with the behaviour of the functional means $\phi_{a,b}(u, v)$, has been considered by a number of writers. Gergen and Littauer [4, Theorems IV and V] have treated the problem of the boundedness, and convergence in the Pringsheim sense, of $\sigma_{mn}^{\alpha, \beta}$. They also considered the corresponding problem when the restriction of boundedness on $\sigma_{mn}^{\alpha, \beta}$ is removed and proved the following theorem.

THEOREM A. *If $a-2 > \alpha \geq 0, b-2 > \beta \geq 0$, if $\phi_{a,b}(u, v)$ is bounded in $(0, 0; \delta, \delta)$ for some positive δ , and if $\sigma_{mn}^{\alpha, \beta} \rightarrow s$ as $(m, n) \rightarrow (\infty, \infty)$, then $\phi_{a,b}(u, v) \rightarrow s$ as $(u, v) \rightarrow (+0, +0)$.*

The question of whether a "converse" of this theorem is true; i.e., whether for suitably related $a, b, \alpha, \beta, \phi_{a,b}(u, v) \rightarrow s$ together with boundedness of $\sigma_{mn}^{\alpha, \beta}$

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for large m and n imply that $\sigma_{mn}^{\alpha,\beta} \rightarrow s$ as $(m, n) \rightarrow (\infty, \infty)$, was left unanswered. Later Gergen [3, Theorem IV] showed that it is not possible to obtain such a theorem and proved instead [3, Theorem V] the following result which contains a "mixed" boundedness condition.

THEOREM B. *If $0 \leq a < \xi$, $0 \leq \alpha < \xi - 1$; $0 \leq b < \eta$, $0 \leq \beta < \eta - 1$; if $\sigma_{mn}^{\alpha,\eta}$, $\sigma_{mn}^{\xi,\beta}$ are bounded for large m and n and if $\phi_{a,b}(u, v) \rightarrow s$ as $(u, v) \rightarrow (+0, +0)$, then $\sigma_{mn}^{\alpha,\beta} \rightarrow s$ as $(m, n) \rightarrow (\infty, \infty)$.*

These results may be regarded as extensions to double series of well-known theorems of Paley [8] and Bosanquet [1].

A problem of a different character arises if we consider the convergence of $\sigma_{mn}^{\alpha,\beta}$ in a restricted sense instead of in the Pringsheim sense. A double sequence $\{b_{mn}\}$ is said to converge restrictedly to s , in symbols $b_{mn} \rightarrow s(R)$ as $(m, n) \rightarrow (\infty, \infty)$, if, for every $\lambda \geq 1$, $b_{mn} \rightarrow s$ as $(m, n) \rightarrow (\infty, \infty)$ in such a way that $\lambda^{-1} \leq mn^{-1} \leq \lambda$. If $\sigma_{mn}^{\alpha,\beta} \rightarrow s(R)$ as $(m, n) \rightarrow (\infty, \infty)$ we shall say that $S[\phi]$ is summable $(C; \alpha, \beta)(R)$ to s . The concept of restricted summability was introduced by Moore [7] who proved the following theorem.

THEOREM C. *If $\phi(u, v) \rightarrow s$ as $(u, v) \rightarrow (+0, +0)$ then $S[\phi]$ is summable $(C; \alpha, \beta)(R)$ to s whenever $\alpha \geq 1$, $\beta \geq 1$.*

The present paper consists of an elaboration of the observation that the conclusion of Theorem C holds if we replace the hypothesis by restricted continuity and local boundedness, of $\phi(u, v)$ at $(0, 0)$. More precisely: we shall say that $\phi(u, v) \rightarrow s(C; a, b)(R)$ as $(u, v) \rightarrow (+0, +0)$ if, for any $\lambda \geq 1$, $\phi_{a,b}(u, v) \rightarrow s$ as $(u, v) \rightarrow (+0, +0)$ in such a way that $\lambda^{-1} \leq uv^{-1} \leq \lambda$, and we prove the following two theorems.

THEOREM 1. *If $\alpha \geq 1$, $\beta \geq 1$; $\alpha > a \geq 0$, $\beta > b \geq 0$; if $\phi(u, v) \rightarrow s(C; a, b)(R)$, and if $\phi_{a,b}(u, v)$ is bounded in $(0, 0; \delta, \delta)$ for some positive δ , then $S[\phi]$ is summable $(C; \alpha, \beta)(R)$ to s .*

THEOREM 2. *If $a - 2 > \alpha \geq 0$, $b - 2 > \beta \geq 0$; if $S[\phi]$ is summable $(C; \alpha, \beta)(R)$ to s , and, if, for some N , $\sigma_{mn}^{\alpha,\beta}$ is bounded for $m > N$, $n > N$, then $\phi(u, v) \rightarrow s(C; a, b)(R)$ as $(u, v) \rightarrow (+0, +0)$.*

Theorem 1 (which contains Theorem C), and Theorem 2 may also be regarded as extensions of the results of Paley and Bosanquet. Before going on to prove these theorems we mention two noteworthy facts which suggest that, although summability $(C; \alpha, \beta)(R)$ is not a regular method, its application to double Fourier series has some advantages over the method used in Theorems A and B. Firstly, Herriot ([5], cf. Lemma 1 below) has shown that the summability $(C; \alpha, \beta)(R)$ ($\alpha \geq 1$, $\beta \geq 1$) of $S[\phi]$ depends only on the behaviour of $\phi(u, v)$ near $(0, 0)$. Secondly, Zygmund [10, p. 309] has shown that the series in (1.1) is summable $(C; \alpha, \beta)(R)$ ($\alpha \geq 1$, $\beta \geq 1$) to $\phi(u, v)$ for

almost every $(u, v) \in (0, 0; \pi, \pi)$. Both these results reflect well-known (C) -summability properties of single Fourier series and neither holds for summability $(C; \alpha, \beta)$ interpreted as involving the existence of $\lim_{(m,n) \rightarrow (\infty, \infty)} \sigma_{mn}^{\alpha, \beta}$ in the Pringsheim sense, whether or not a boundedness condition is imposed (cf. [10, p. 304; 9]).

2. We first give some further notation, collect some known results and establish three lemmas.

If $\{b_{mn}\}$ is a given double sequence, if $\lambda \geq 1$, and if

$$\alpha_{mn}^{(\lambda)} = \begin{cases} d_{mn} & \text{for } \lambda^{-1} \leq mn^{-1} \leq \lambda \\ 0 & \text{otherwise} \end{cases}$$

then we define

$$\begin{aligned} \lambda - \sup_{(m,n)} b_{mn} &= \sup_{m \geq 0, n \geq 0} d_{mn}^{(\lambda)}, \\ \lambda - \lim_{(m,n)} b_{mn} &= \lim_{(m,n) \rightarrow (\infty, \infty)} d_{mn}^{(\lambda)}, \\ \lambda - \limsup_{(m,n)} b_{mn} &= \limsup_{(m,n) \rightarrow (\infty, \infty)} d_{mn}^{(\lambda)}. \end{aligned}$$

In a similar way, if $f(u, v)$ is a given function, defined for $u > 0, v > 0$, then for $\lambda \geq 1$ we define

$$\lambda - \lim_{(u,v)} f(u, v) = \lim_{(u,v) \rightarrow (+0, +0)} g_{\lambda}(u, v)$$

where $g_{\lambda}(u, v) = f(u, v)$ if $\lambda^{-1} \leq uv^{-1} \leq \lambda$ and is zero otherwise. We define $\lambda - \sup_{(u,v)} f(u, v)$ and $\lambda - \limsup_{(u,v)} f(u, v)$ in a similar way.

We shall require the functions

$$K_{\alpha}(m, u) = \frac{1}{2} + (A_m^{\alpha})^{-1} \sum_{r=1}^m A_{m-r}^{\alpha} \cos ru$$

which are known [2, p. 64] to satisfy

$$(2.1) \quad |K_{\alpha}^{(r)}(m, u)| \leq \begin{cases} Am^{r+1} \\ Am^{r+1} \max[(mu)^{-r-2}, (mu)^{-\alpha-1}], \end{cases}$$

for $\alpha > 0, r = 0, 1, \dots$, and $0 \leq u \leq \pi$; where A is independent of m and u . It follows easily from (2.1) that for $r = 0, 1, \dots$ and $m \geq 0$

$$(2.2) \quad \int_0^{\pi} u^r |K_{\alpha}^{(r)}(m, u)| du < A \quad (\alpha > r)$$

where A is independent of m .

We also require Young's functions $\gamma_p(t)$ defined by

$$\gamma_p(t) = p \int_0^1 (1-u)^{p-1} \cos ut \, du \quad (p > 0).$$

It is known [2, p. 64] that, for $p > 0, k = 0, 1, \dots; n = 0, 1, \dots$ and $t \geq 0$

$$(2.3) \quad |\Delta^k \gamma_p(nt)| \leq \begin{cases} At^k \\ At^k \max[(nt)^{-k-2}, (nt)^{-p}], \end{cases}$$

where A is independent of t and n , and Δ is the usual difference operator. It easily follows from (2.3) that for $u > 0, \rho > -1$,

$$(2.4) \quad \sum_{m=1}^{\infty} m^\rho |\Delta^{\alpha+1} \gamma_a(mu)| < Au^{\alpha-\rho} \quad (\alpha > \rho - 2, a > \rho + 1),$$

where A is independent of u .

We require the following three lemmas.

LEMMA 1. *The summability $(C; \alpha, \beta)(R)$ of $S[\phi]$ depends only on the behaviour of $\phi(u, v)$ in an arbitrary neighbourhood $(0, 0; \delta, \delta)$ ($0 < \delta < \pi$) of the origin.*

This is due to Herriot [5].

LEMMA 2. *If $\alpha > a \geq 0, \beta > b \geq 0$ and if $0 < \delta \leq \pi$ then for each $\lambda \geq 1$,*

$$(2.5) \quad \limsup_{\mu \rightarrow \infty} \{ \lambda - \limsup_{(m,n)} \int_0^\delta u^a |K_\alpha^{(a)}(m, u)| \, du \int_0^{u\mu^{-1}} v^b |K_\beta^{(b)}(n, v)| \, dv \} = 0.$$

Proof. Denoting the integral in (2.5) by J , and choosing (as we clearly may) $\mu \geq \lambda$, we have

$$J = \int_0^{m^{-1}} du \int_0^{u\mu^{-1}} dv + \int_{m^{-1}}^{\mu n^{-1}} du \int_0^{u\mu^{-1}} dv + \int_{\mu n^{-1}}^\delta du \int_0^{n^{-1}} dv \\ + \int_{\mu n^{-1}}^\delta du \int_{n^{-1}}^{u\mu^{-1}} dv = \sum_{r=1}^4 J_r$$

(say).

By (2.1),

$$J_1 \leq Am^{a+1}n^{b+1} \int_0^{m^{-1}} u^a du \int_0^{u\mu^{-1}} v^b dv = \frac{Am^{a+1}n^{b+1}}{m^{a+b+2}\mu^{b+1}} \leq A \left(\frac{\lambda}{\mu}\right)^{b+1},$$

whenever $\lambda^{-1} \leq mn^{-1} \leq \lambda$, so that

$$(2.6) \quad \limsup_{\mu \rightarrow \infty} (\lambda - \limsup_{(m,n)} J_1) = 0.$$

Next, using (2.1) again,

$$\begin{aligned}
 J_2 &= An^{b+1} \int_{m^{-1}}^{\mu n^{-1}} (m^{-1}u^{-2} + m^{\alpha-\alpha}u^{\alpha-\alpha-1})du \int_0^{u\mu^{-1}} v^b dv \\
 &= \frac{An^{b+1}}{\mu^{b+1}} \left\{ m^{-1} \int_{m^{-1}}^{\mu n^{-1}} u^{b-1} du + m^{\alpha-\alpha} \int_{m^{-1}}^{\mu n^{-1}} u^{\alpha+b-\alpha} du \right\} \\
 &= J_{21} + J_{22}
 \end{aligned}$$

(say), where, if $\lambda^{-1} \leq mn^{-1} \leq \lambda$,

$$J_{21} \leq \begin{cases} A \left(\frac{\lambda}{\mu} \right), & b \neq 0, \\ A \frac{\lambda}{\mu} \log \lambda \mu, & b = 0, \end{cases}$$

and

$$J_{22} \leq \begin{cases} A \left(\frac{\lambda}{\mu} \right)^{\alpha-\alpha} + A \left(\frac{\lambda}{\mu} \right)^{b+1}, & a + b - \alpha \neq -1, \\ A \left(\frac{\lambda}{\mu} \right)^{\alpha-\alpha} \log \lambda \mu, & a + b - \alpha = -1, \end{cases}$$

so that in any case,

$$(2.7) \quad \limsup_{\mu \rightarrow \infty} (\lambda - \limsup_{(m,n)} J_2) = 0.$$

Next, by (2.1),

$$\begin{aligned}
 J_3 &\leq An^{b+1} \int_{\mu n^{-1}}^{\delta} (m^{-1}u^{-2} + m^{\alpha-\alpha}u^{\alpha-\alpha-1})du \int_0^{n^-} v^b dv \\
 &\leq A \frac{\lambda}{\mu} + A \left(\frac{\lambda}{\mu} \right)^{\alpha-\alpha},
 \end{aligned}$$

if $\lambda^{-1} \leq mn^{-1} \leq \lambda$ so that

$$(2.8) \quad \limsup_{\mu \rightarrow \infty} \lambda - \limsup_{(m,n)} J_3 = 0.$$

Finally by (2.1),

$$\begin{aligned}
 J_4 &\leq A \int_{\mu n^{-1}}^{\infty} (m^{-1}u^{-2} + m^{\alpha-\alpha}u^{\alpha-\alpha-1})du \int_{n^{-1}}^{\infty} (n^{-1}v^2 + n^{b-\beta}v^{b-\beta-1})dv \\
 &\leq A \frac{\lambda}{\mu} + A \left(\frac{\lambda}{\mu} \right)^{\alpha-\alpha},
 \end{aligned}$$

if $\lambda^{-1} \leq mn^{-1} \leq \lambda$, so that

$$(2.9) \quad \limsup_{\mu \rightarrow \infty} (\lambda - \limsup_{(m,n)} J_4) = 0.$$

Combining (2.6)–(2.9) we obtain the required result.

LEMMA 3. *If $a > \alpha + 1 \geq 1$, $b > \beta + 1 \geq 1$, then for every $\lambda \geq 1$,*

$$(2.10) \quad \limsup_{\mu \rightarrow \infty} \left\{ (\lambda - \limsup_{(u,v)} \sum_{m=1}^{\infty} m^{\alpha} |\Delta^{\alpha+1} \gamma_{\alpha}(mu)| \sum_{n < m\mu-1} n^{\beta} |\Delta^{\beta+1} \gamma_{\beta}(nv)|) \right\} = 0.$$

Since the proof of this lemma is quite similar to the proof of Lemma 2 we shall just outline the procedure.

The series in (2.10) is, by (2.3), absolutely convergent, for fixed positive u and v and we denote its sum by $S(u, v)$. We then choose $\lambda (\geq 1)$ freely and then regard it fixed and take (as we clearly may) $\mu \geq \lambda$. We then write

$$\begin{aligned} S(u, v) &= \sum_{m=1}^{[u^{-1}]} \sum_{n=1}^{[m\mu^{-1}]} + \sum_{m=[u^{-1}]+1}^{[v^{-1}]} \sum_{n=1}^{[m\mu^{-1}]} + \sum_{m=[\mu v^{-1}]+1}^{\infty} \sum_{n=1}^{[v^{-1}]} + \sum_{m=[\mu v^{-1}]+1}^{\infty} \sum_{n=[v^{-1}]+1}^{[m\mu^{-1}]} \\ &= \sum_{r=1}^4 S_r \end{aligned}$$

(say). Using (2.3) (as we used (2.1) to estimate the J_r in the proof of Lemma 2) we can show that each of the S_r satisfy a relation

$$S_r \leq f_r \left(\frac{\lambda}{\mu} \right)$$

for every positive u and v satisfying $\lambda^{-1} \leq uv^{-1} \leq \lambda$, and that $f_r(\lambda/\mu) \rightarrow 0$ as $\mu \rightarrow \infty$. This is sufficient to establish (2.10).

3. **Proof of Theorem 1.** We suppose throughout that the conditions $\alpha \geq 1$, $\beta \geq 1$, $\alpha > a \geq 0$, $\beta > b \geq 0$, are satisfied, and (without loss of generality) that $s = 0$. We also suppose that a and b are integers, the proof may be completed in the general case by standard methods.

It is easily shown that

$$(3.1) \quad \sigma_{mn}^{\alpha, \beta} = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \phi(u, v) K_{\alpha}(m, u) K_{\beta}(n, v) dudv.$$

If we split the range of integration on the right of (3.1) into $(0, 0; \delta, \delta)$, $(0, \delta; \delta, \pi)$, $(\delta, 0; \pi, \delta)$, $(\delta, \delta; \pi, \pi)$; $(0 < \delta < \pi)$ then it follows from Lemma 1 that the corresponding integrals over the last three ranges have restricted limit zero as $(m, n) \rightarrow (\infty, \infty)$. Hence it is sufficient to show that

$$(3.2) \quad \left\{ \limsup_{\delta \rightarrow +0} \lambda - \limsup_{(m,n)} \left| \int_0^{\delta} \int_0^{\delta} \phi(u, v) K_{\alpha}(m, u) K_{\beta}(n, v) dudv \right| \right\} = 0,$$

for each $\lambda \geq 1$.

Denoting the integral in (3.2) by J we have, on integrating by parts,

$$\begin{aligned}
 J &= \sum_{r=1}^a \sum_{s=1}^b (-)^{r+s} \delta^{r+s} (r!s!)^{-1} K_\alpha^{(r-1)}(m, \delta) K_\beta^{(s-1)}(n, \delta) \phi_{r,s}(\delta, \delta) \\
 &+ (-)^{a-1} (a!)^{-1} \sum_{s=1}^b (-)^{s-1} \delta^s (s!)^{-1} K_\beta^{(s-1)}(n, \delta) \int_0^\delta \phi_{a,s}(u, \delta) u^a K_\alpha^{(a)}(u) du \\
 &+ (-)^{b-1} (b!)^{-1} \sum_{r=1}^a (-)^{r-1} \delta^r (r!)^{-1} K_\alpha^{(r-1)}(m, \delta) \int_0^\delta \phi_{r,b}(\delta, v) v^b K_\beta^{(b)}(v) dv \\
 &+ (-)^{a+b} (a!b!)^{-1} \int_0^\delta \int_0^\delta \phi_{a,b}(u, v) u^a v^b K_\alpha^{(a)}(m, u) K_\beta^{(b)}(n, v) dudv \\
 &= \sum_{r=1}^4 J_r
 \end{aligned}$$

(say).

It follows immediately from (2.1) that, for any positive δ ,

$$J_1 = \sum_{r=1}^a \sum_{s=1}^b O(m^{-1} + m^{r-1-\alpha}) O(n^{-1} + n^{s-1-\beta}) = o(1),$$

as $(m, n) \rightarrow (\infty, \infty)$ in any manner.

Next, using (2.1) again, for any positive δ ,

$$\begin{aligned}
 J_2 &= \sum_{s=1}^b O(n^{-1} + n^{s-1-\beta}) O \left\{ m^{a+1} \int_0^{m^{-1}} |\phi_{a,s}(u, \delta)| u^a du \right. \\
 &\quad \left. + \int_{m^{-1}}^\delta |\phi_{a,s}(u, \delta)| (m^{-1}u^{-2} + m^{a-\alpha}u^{a-\alpha-1}) du \right\} \\
 &= O(n^{-1} + n^{b-\beta-1}) o(m) = o(1)(R),
 \end{aligned}$$

as $(m, n) \rightarrow (\infty, \infty)$.

Similarly $J_3 = o(1)(R)$ as $(m, n) \rightarrow (\infty, \infty)$, for any positive δ , and hence to establish (3.2) it is sufficient to show that

$$(3.3) \quad \limsup_{\delta \rightarrow +0} \left(\lambda - \limsup_{(m,n)} |J_4| \right) = 0,$$

for arbitrary $\lambda \geq 1$.

We now choose $\lambda (\geq 1)$ freely and then regard it as fixed. Choosing $\mu > \lambda$ we write

$$J_4 = \left\{ \iint_{\substack{0 < u < \delta, 0 < v < \delta \\ \mu^{-1} < uv^{-1} < \mu}} + \int_0^\delta du \int_0^{u\mu^{-1}} dv + \int_0^\delta dv \int_0^{v\mu^{-1}} du \right\} = \sum_{r=1}^2 I_r$$

(say).

Since $\phi_{a,b}(u, v)$ is bounded in $(0, 0; \delta, \delta)$ for small δ it follows that there exists K such that

$$\begin{aligned} \limsup_{\mu \rightarrow \infty} \limsup_{\delta \rightarrow +0} \left(\lambda - \lim_{(m,n)} |I_2 + I_3| \right) &\leq K \limsup_{\mu \rightarrow \infty} \left[\lambda - \limsup_{(m,n)} \right. \\ &\cdot \left. \left\{ \int_0^\pi du \int_0^{u\mu^{-1}} dv + \int_0^\pi dv \int_0^{v\mu^{-1}} du \right\} u^a v^b K_\alpha^{(a)}(m, u) K_\beta^{(b)}(n, v) \right] \\ &= 0, \end{aligned}$$

by Lemma 2.

Finally,

$$\begin{aligned} \limsup_{\mu \rightarrow \infty} \limsup_{\delta \rightarrow +0} \left(\lambda - \limsup_{(m,n)} |I_1| \right) &\leq \limsup_{\mu \rightarrow \infty} \limsup_{\delta \rightarrow +0} \left[\lambda - \limsup_{(m,n)} \right. \\ &\cdot \left. \left\{ \mu - \sup_{0 < u \leq \delta; 0 < v \leq \delta} |\phi_{a,b}(u, v)| \int_0^\pi u^a |K_\alpha^{(a)}(m, u)| du \int_0^\pi v^b |K_\beta^{(b)}(n, v)| dv \right\} \right] \\ &= 0, \end{aligned}$$

by (2.2) and the fact that $\mu - \lim_{(u,v)} |\phi_{a,b}(u, v)| = 0$.

Thus

$$\begin{aligned} \lim_{\mu \rightarrow \infty} \limsup_{\delta \rightarrow +0} \left(\lambda - \limsup_{(m,n)} |J_4| \right) \\ \leq \limsup_{\mu \rightarrow \infty} \limsup_{\delta \rightarrow +0} \left[\lambda - \limsup_{(m,n)} \{ |I_1| + |I_2 + I_3| \} \right] \\ = 0, \end{aligned}$$

and this establishes (3.3).

4. Proof of Theorem 2. We suppose throughout that $a - 2 > \alpha \geq 0, b - 2 > \beta \geq 0$ and (without loss of generality) that $s = 0$. We also suppose that α and β are integers, the proof may be completed in the general case by standard methods.

Since $a \geq 1, b \geq 1$ it follows that $(u - x)^{a-1}, (v - y)^{b-1}$ are of bounded variation in $0 \leq x \leq u$ and $0 \leq y \leq v$ respectively and hence from (1.1), by a straightforward extension of a well-known result [6, p. 583], that for $u > 0, v > 0$

$$\begin{aligned} \phi_{a,b}(u, v) &= ab u^{-a} v^{-b} \int_0^u (u - x)^{a-1} dx \int_0^v (v - y)^{b-1} \phi(x, y) dy \\ &= \sum_{m,n=0}^\infty a_{mn} a u^{-a} \int_0^u (u - x)^{a-1} \cos mx dx b v^{-b} \int_0^v (v - y)^{b-1} \cos ny dy \\ &= \sum_{m,n=0}^\infty a_{mn} \gamma_a(mu) \gamma_b(nv). \end{aligned}$$

By repeated application of Abel's lemma we obtain

$$\begin{aligned} \phi_{a,b}(u, v) &= \lim_{N \rightarrow \infty} \left\{ \sum_{m,n=0}^N S_{mn}^{\alpha,\beta} \Delta^{\alpha+1} \gamma_a(mu) \Delta^{\beta+1} \gamma_b(nv) \right. \\ &\quad + \sum_{m=0}^N \sum_{s=0}^{\beta} S_{mN-1}^{\alpha,s} \Delta^{\alpha+1} \gamma_a(mu) \Delta^s \gamma_b(Nv) \\ &\quad + \sum_{n=0}^N \sum_{r=0}^{\alpha} S_{N-1n}^{r,\beta} \Delta^r \gamma_a(Nu) \Delta^{\beta+1} \gamma_b(nv) \\ &\quad \left. + \sum_{r=0}^{\alpha} \sum_{s=0}^{\beta} S_{N-1N-1}^{r,s} \Delta^r \gamma_a(Nu) \Delta^s \gamma_b(Nv) \right\} \\ &= \lim_{N \rightarrow \infty} \sum_{i=1}^4 T_i \end{aligned}$$

(say).

Since $a_{mn} = o(1)$ as $(m, n) \rightarrow (\infty, \infty)$ it follows that $S_{mn}^{r,s} = o(m^{r+1}n^{s+1})$ as $(m, n) \rightarrow (\infty, \infty)$ for $r=0, 1, \dots, s=0, 1, \dots$ and hence, from (2.3), that for fixed positive u and v

$$\begin{aligned} T_4 &= \sum_{r=0}^{\alpha} \sum_{s=0}^{\beta} O(N^{r+s+2}) O(N^{-r-2} + N^{-s}) O(N^{-s-2} + N^{-b}) \\ &= o(1), \end{aligned}$$

as $N \rightarrow \infty$.

Next, since $S_{mN-1}^{\alpha,s} = o(m^{\alpha+1}N^{s+1})$ it follows, using (2.3) and (2.4), that, for fixed positive u and v ,

$$\begin{aligned} T_2 &= O\{(N^{-1} + N^{\beta-b+1}) \sum_{m=1}^N m^{\alpha} |\Delta^{\alpha+1} \gamma_a(mu)|\} \\ &= O(N^{-1})O(1) = o(1) \end{aligned}$$

as $N \rightarrow \infty$.

Similarly $T_3 = o(1)$ as $N \rightarrow \infty$ for fixed positive u and v so that, for $u > 0, v > 0$,

$$\begin{aligned} \phi_{a,b}(u, v) &= \sum_{m,n=0}^{\infty} S_{mn}^{\alpha,\beta} \Delta^{\alpha+1} \gamma_a(mu) \Delta^{\beta+1} \gamma_b(nv) \\ &= \sum_{m=0}^N \sum_{n=0}^N + \sum_{m=N+1}^{\infty} \sum_{n=0}^N + \sum_{m=0}^N \sum_{n=N+1}^{\infty} + \sum_{m=N+1}^{\infty} \sum_{n=N+1}^{\infty} \\ &= \sum_{r=1}^4 U_r \end{aligned}$$

(say).

It follows immediately from (2.3) that for any positive N ,

$$|U_1| = O\left(u^{\alpha+1} v^{\beta+1} \sum_{m=0}^N \sum_{n=0}^N |S_{mn}^{\alpha,\beta}|\right) = o(1),$$

as $(u, v) \rightarrow (+0, +0)$ in any manner.

Next since $\sigma_{mn} \rightarrow 0$ as $m \rightarrow \infty$ for each $n \geq 0$ it follows that $S_{mn}^{\alpha,\beta} = o(m^{\alpha+1})$ for $n = 0, 1, \dots$ and hence, using (2.3), that for each positive N ,

$$\begin{aligned} |U_2| &= O(v^{\beta+1}) \sum_{m=1}^{\infty} o(m^{\alpha+1}) |\Delta^{\alpha+1}\gamma_a(mu)| \\ &= O(v^{\beta+1})o(u^{-1}) \qquad \qquad \qquad \text{(by 2.4)} \\ &= o(1)(R) \end{aligned}$$

as $(u, v) \rightarrow (+0, +0)$.

In a similar way we can show that $U_3 = o(1)(R)$ as $(u, v) \rightarrow (+0, +0)$. Hence in order to establish the theorem it is sufficient to show that for each $\lambda \geq 1$

$$(4.1) \qquad \limsup_{N \rightarrow \infty} (\lambda - \limsup_{(u,v)} |U_4|) = 0.$$

To this end we first choose λ freely and then regard it as fixed. We then choose $\mu > \lambda$ and write

$$\begin{aligned} U_4 &= \left\{ \sum_{\substack{m>N, n>N \\ \mu^{-1} \leq mn^{-1} \leq \mu}} + \sum_{\substack{m>N, n>N \\ mn^{-1} > \mu}} + \sum_{\substack{m>N, n>N \\ mn^{-1} < \mu^{-1}}} \right\} S_{mn}^{\alpha,\beta} \Delta^{\alpha+1} \gamma_a(mu) \Delta^{\beta+1} \gamma_b(nv) \\ &= \sum_{r=1}^3 V_r \end{aligned}$$

(say).

Since $\sigma_{mn}^{\alpha,\beta}$ is bounded for large m and n it follows that there exists K such that

$$\begin{aligned} \limsup_{\mu \rightarrow \infty} \limsup_{N \rightarrow \infty} (\lambda - \limsup_{(u,v)} |V_2 + V_3|) &\leq K \limsup_{\mu \rightarrow \infty} \left[\lambda - \limsup_{(u,v)} \right. \\ &\quad \left. \cdot \left\{ \sum_{m=1}^{\infty} \sum_{n < m\mu^{-1}} + \sum_{n=1}^{\infty} \sum_{m < n\mu^{-1}} \right\} m^{\alpha} n^{\beta} |\Delta^{\alpha+1}\gamma_a(mu)| |\Delta^{\beta+1}\gamma_b(nv)| \right] \\ &= 0, \end{aligned}$$

by Lemma 3.

Finally,

$$\begin{aligned} \limsup_{\mu \rightarrow \infty} \limsup_{N \rightarrow \infty} \left(\lambda - \lim_{(u,v)} |V_1| \right) &\leq \limsup_{N \rightarrow \infty} \left[\lambda - \limsup_{(u,v)} \right. \\ &\cdot \left. \left\{ \lambda - \sup_{m > N, n > N} | \sigma_{mn}^{\alpha, \beta} | \sum_{m=1}^{\infty} m^{\alpha} | \Delta^{\alpha+1} \gamma_a(mu) | \sum_{n=1}^{\infty} n^{\beta} | \Delta^{\beta+1} \gamma_b(nv) | \right\} \right] \\ &= 0, \end{aligned}$$

by (2.4) and the fact that $\lambda - \lim_{(m,n)} | \sigma_{mn}^{\alpha, \beta} | = 0$.

Consequently

$$\begin{aligned} \limsup_{\mu \rightarrow \infty} \limsup_{N \rightarrow \infty} (\lambda - \limsup_{(u,v)} |U_4|) \\ \leq \limsup_{\mu \rightarrow \infty} \limsup_{N \rightarrow \infty} \left[\lambda - \limsup_{(u,v)} \{ |V_1| + |V_2 + V_3| \} \right] \\ = 0, \end{aligned}$$

which establishes (4.1).

5. An example. In view of the single Fourier series theorem on which Theorem 1 is based (cf. [1, Theorem 1]) and in view of the facts (i) that the series in (1.1) is summable almost everywhere in $(0, 0; \pi, \pi)$ to $\phi(u, v)$ and (ii)

$$abs^{-a}t^{-b} \int_0^s \int_0^t (s-x)^{a-1}(t-y)^{b-1} \phi(u+x, v+y) dx dy \rightarrow \phi(u, v)(R)$$

as $(s, t) \rightarrow (+0, +0)$ for almost every $(u, v) \in (0, 0; \pi, \pi)$; it is attractive to conjecture that $\phi(u, v) \rightarrow s(C; a, b)(R)$ is alone sufficient for the conclusion of Theorem 1. That this conjecture is false, i.e., that the boundedness condition on $\phi_{a,b}(u, v)$ in Theorem 1 cannot be entirely removed, is shown by the following example.

Let $\phi(u, v)$ be even and periodic with period 2π and such that

$$\phi(r \cos \theta, r \sin \theta) = \begin{cases} r^{-2} e^{-\theta r^{-1}} & 0 < r < \pi, 0 \leq \theta \leq \pi/2, \\ 0 & \text{elsewhere in } (0, 0; \pi, \pi). \end{cases}$$

It is easily verified that $\phi(u, v) \in L(0, 0; \pi, \pi)$ and that $\phi(u, v) \rightarrow 0(R)$ as $(u, v) \rightarrow (+0, +0)$.

Since $\phi(u, v)$ is positive and since $\sin^2 mu/2 \geq (4m^2/\pi^2) \sin^2 u/2$ for $0 < u \leq 1/2m$ it follows that

$$\begin{aligned} \sigma_{mm}^{1,1} &> \frac{1}{\pi^2} \int_0^{1/2m} \int_0^u \phi(u, v) \frac{\sin^2 mu/2}{m \sin^2 u/2} \frac{\sin^2 mv/2}{m \sin^2 v/2} dudv \\ &\geq \frac{16m^2}{\pi^6} \int_0^{1/2m} r^{-1} dr \int_0^{\pi/4} e^{-\theta r^{-1}} d\theta \\ &= \frac{16m^2}{\pi^6} \int_0^{1/2m} (1 - e^{-\pi/4r}) dr > \frac{8m^2}{\pi^6} \int_0^{1/2m} dr \rightarrow \infty \end{aligned}$$

as $m \rightarrow \infty$. Consequently $\sigma_{mm}^{1,1} \rightarrow \infty$ as $m \rightarrow \infty$ and a fortiori $S[\phi]$ is not summable $(C; 1, 1)$ (R) even though $R\text{-}\lim_{(u,v) \rightarrow (+0,+0)} \phi(u, v)$ exists.

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