A SOLUTION OF CHANDRASEKHAR'S INTEGRAL EQUATION

BY

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1. **Introduction.** Many problems of radiative transfer have been reduced by S. Chandrasekhar, [2, Chapters 4 and 5; 3; 4], to the problem of solving the integral equation

$$\frac{1}{H(t)} = 1 - t\int_0^1 \frac{\psi(u)H(u)}{u+t}\,du \tag{1}$$

where $\psi(u)$ is known and $H(t)$ is to be found. If t is complex or does not satisfy $-1<t<1$ then Chandrasekhar has proved that

$$\frac{1}{H(t)H(-t)} = 1 - \int_0^1 \frac{2t^2\psi(u)}{t^2-u^2}\,du = T(t) \tag{2}$$

and from (2) it can then be proved that

$$\log\{H(t)\} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \log\{T(w)\}\,\frac{tdw}{w^2-t^2} \tag{3a}$$

(M. M. Crum [5], Miss I. Busbridge [1]).

In this paper I shall discuss (1) by methods which differ from those used by the three authors cited above. I use (2) to transform (1) into a singular integral equation and this can be solved, in turn, by methods fully described in the standard work on this subject by Muskhelishvili, [6]. Solutions of singular integral equations differ in appearance very considerably from (3a).

Singular integral equations occur in many branches of mathematical physics, e.g. elasticity, aerodynamics, etc., and the methods used to solve them are now classical. These solutions lend themselves to numerical computation just as readily as (3a) and it is frequently possible to draw important theoretical and practical conclusions from them.

In physical applications of (1) certain restrictions are necessary, such as

$$\int_0^1 \psi(u)du \leqq \frac{1}{2} \tag{3b}$$

This paper has been submitted to and accepted for publication by the Proceedings of the American Mathematical Society. It has been transferred to these Transactions, with the consent of the author, for technical reasons. Presented to the Society, April 8, 1960; received by the editors July 9, 1960 and, in revised form, August 17, 1960 and September 19, 1960.

but from the point of view of pure mathematics restrictions of this type are not essential.

2. **The reduction of (1) to a singular integral equation.** Equation (2) has been proved for the case when t is either complex or, if real, does not satisfy the inequalities $-1<t<1$. The object of this restriction is to avoid the singularity in the integrand of (2) when $-1<t<1$. In the investigation here it is necessary to prove that (2) is true even when $0<t<1$ and in this case we must interpret the right hand integral of (2) to be the principal value of a Cauchy integral. By this is meant that if $0<t<1$ then

$$\int_0^1 \frac{\phi(u)}{u-t}du = \lim_{\epsilon\to+0}\left\{\int_0^{t-\epsilon}+\int_{t+\epsilon}^1\right\}\frac{\phi(u)}{u-t}du. \tag{4}$$

Conditions for the existence of this limit will be found in [6, Chapter 2]. (4) can also be extended to the case when the integration is taken along an arc, or round a contour, and t is a point on this arc, or contour, [6, Chapter 2]. In future all singular integrals are to be interpreted in accordance with (4).

I now repeat Crum's proof of (2) followed by arguments which show that the proof is valid even when $-1<t<1$. We have from (1)

$$\begin{aligned}&\left\{1-\frac{1}{H(t)}\right\}\left\{1-\frac{1}{H(-t)}\right\}\\ &=\int_0^1\int_0^1 \psi(u)H(u)\psi(v)H(v)\,\frac{t^2}{(t+u)(t-v)}\,dudv\end{aligned} \tag{5}$$

$$=\int_0^1\int_0^1 \psi(u)H(u)\psi(v)H(v)\,\frac{t}{(u+v)}\left\{\frac{u}{t+u}+\frac{v}{t-v}\right\}dudv \tag{6}$$

$$=\int_0^1\frac{\psi(u)H(u)}{t+u}t\left\{1-\frac{1}{H(u)}\right\}du+\int_0^1\frac{\psi(v)H(v)}{t-v}t\left\{1-\frac{1}{H(v)}\right\}dv, \tag{7}$$

on using (1) again,

$$=1-\frac{1}{H(t)}-\int_0^1\frac{t\psi(u)}{t+u}du+1-\frac{1}{H(-t)}-\int_0^1\frac{t\psi(v)}{t-v}dv \tag{8}$$

on rearranging and using (1). (2) now follows obviously from (8).

The change in the order of integration from (6) to (7) causes no difficulty when t is complex or when t is real but lies outside the interval $(-1, 1)$. But if $-1<t<1$ then one of the integrals involved in the change is a singular integral, although the other is not singular. The singular integral, however, is to be interpreted as a Cauchy principal value, according to (4), and in this case the change of order of integration from (6) to (7) is still correct, as is proved in [6, p. 59]. Thus (2) still holds even when $-1<t<1$ (which includes the case of physical importance when $0<t<1$).

We now replace t by $-t$ in (1) and substitute for $H(-t)$ from (2). The result is that (1) is equivalent to

$$T(t)H(t) - t\int_0^1 \frac{\psi(u)H(u)}{u-t}\,du = 1, \tag{9}$$

where, if $0<t<1$, the integral is interpreted to be a Cauchy principal value. The singular integral equation (9) can be solved by the methods described in [6] and we now proceed to give a brief account of this solution.

3. **The generalized Cauchy integral.** Consider the function $G(z)$ defined by the equation

$$G(z) = \frac{1}{2\pi i}\int_C \frac{g(u)}{u-z}\,du, \tag{10}$$

where C denotes the arc of integration and $g(z)$ is assumed to satisfy a Hölder condition in a region which encloses C [6, p. 11]. If the point z is not on C then $G(z)$ is evidently an analytic function of z which tends to zero as $z\to\infty$. If z is on C then the integral for $G(z)$ is taken to be a Cauchy principal value, as in (4) suitably generalized for application to an arc. Evidently the line C must be a line of discontinuity for the function $G(z)$, i.e., as z crosses C the function $G(z)$ must pass through a discontinuity.

The nature of this discontinuity is important. Let A and B be the end points of the arc C, let the direction from A to B be taken as the positive direction and let the tangent at the point t on C have the same direction as C, i.e., near t they have the same direction. Then the immediate neighbourhood of t can be partitioned into points to the left of C and points to the right of C. We shall denote the limit of $G(z)$ as z tends to t (on C) from the left by $G_L(t)$ and as z tends to t from the right by $G_R(t)$. These limits are related to the principal value of $G(t)$ by the Plemelj formulae, [6, p. 42], as follows:

$$\begin{aligned} G_L(t) &= \frac{1}{2}\,g(t) + \frac{1}{2\pi i}\int_C \frac{g(u)}{u-t}\,du, \\ G_R(t) &= -\frac{1}{2}\,g(t) + \frac{1}{2\pi i}\int_C \frac{g(u)}{u-t}\,du. \end{aligned} \tag{11}$$

Here the point t is on the curve C and the integral is taken to be a Cauchy principal value.

Equation (9) can be solved for $H(t)$ by means of (11). First we multiply (9) by $\pi i\psi(t)$ and then write

$$g(u) = \pi i\psi(u)H(u). \tag{12}$$

It follows from (11) that (9) is equivalent to

$$T(t)\{G_L(t) - G_R(t)\} - \pi i t\psi(t)\{G_L(t) + G_R(t)\} = \pi i\psi(t) \tag{13}$$

or

$$G_L(t) = \frac{T(t) + \pi i t\psi(t)}{T(t) - \pi i t\psi(t)} G_R(t) + \frac{\pi i \psi(t)}{T(t) - \pi i t\psi(t)}. \tag{14}$$

Our problen now is to solve (14) for $G(z)$, given by (10), using properties of $\psi(t)$ and $T(t)$ only, where C, the arc of integration, is the straight line real segment from 0 to 1. $G_L(t)$ denotes the limit of $G(z)$ as z tends to some point t on this segment from above and $G_R(t)$ the limit as z tends to such a point from below.

Equation (14) is called by Muskhelishvili, [6, §37], the nonhomogeneous Hilbert problem and its solution, for the unknown function $G(z)$, can be obtained by two simple applications of (11).

4. The solution of the homogeneous Hilbert problem associated with (14). The homogeneous problem associated with (14) is that of finding a function $X(z)$ for which

$$X_L(t) = \frac{T(t) + \pi i t\psi(t)}{T(t) - \pi i t\psi(t)} X_R(t). \tag{15}$$

In order to obtain a simple solution of (15) we now assume that for $0 \leqq t \leqq 1$:

(i) $\psi(t)$ is real, one-valued and satisfies a Hölder condition and

(ii) $T(t) \neq 0$.

Physically (i) is always true and (ii) is certainly true in the conservative case, when the light is perfectly scattered, and it is also true in many nonconservative cases.

It is known, [6, p. 13], that if $f(t)$ and $g(t)$ satisfy Hölder conditions then $f(t) \pm g(t)$, $f(t)g(t)$ and $f(t)/g(t)$ $(g(t) \neq 0)$ also satisfy Hölder conditions. Hence, from (i) and (2) we immediately deduce that $T(t)$ is real, one-valued and satisfies a Hölder condition for $0 \leqq t \leqq 1$.

We may now solve (15) by taking logarithms. Since, from (i) and the reality of $T(t)$, the modulus of the coefficient of $X_R(t)$ is unity we have

$$\log\{X_L(t)\} - \log\{X_R(t)\} = 2i\theta(t), \tag{16}$$

where

$$\tan\{\theta(t)\} = \frac{\pi t\psi(t)}{T(t)} \qquad (0 \leqq t \leqq 1). \tag{17}$$

On choosing $\theta(0) = 0$ we have, from (ii) and (17), $-\pi/2 < \theta(t) < \pi/2$ $(0 \leqq t \leqq 1)$ and so $\theta(t)$ is one-valued. With this choice of $\theta(0)$ we now prove that the solution of (15) is

$$\log\{X(z)\} = \frac{1}{2\pi i}\int_0^1 \frac{2i\theta(u)}{u-z}\,du. \tag{18}$$

To prove this we first note that since $\psi(t)$ and $T(t)$ both satisfy Hölder conditions it follows easily from (17) that $\theta(t)$ must also satisfy a Hölder condition when $0 \leqq t \leqq 1$. Hence the integral in (18) exists as a Cauchy principal value when $0<z<1$. Consequently the Plemelj formulae (11) hold for $\log\{X(z)\}$.

Again $X(z)$ (using the Cauchy principal value when $0<z<1$) is a unique function of z and so $\log\{X_L(t)\} = \{\log X(t)\}_L$ (together with a similar result for the R limit). The Plemelj formulae (11) then become

$$\begin{aligned} \log\{X_L(t)\} &= i\theta(t) + \frac{1}{\pi}\int_0^1 \frac{\theta(u)}{u-t}\,du, \\ \log\{X_R(t)\} &= -\,i\theta(t) + \frac{1}{\pi}\int_0^1 \frac{\theta(u)}{u-t}\,du, \end{aligned} \tag{19}$$

where $0<t<1$ and the integrals are Cauchy principal value integrals. It is immediately verified that the expressions in (19) satisfy (16) and it therefore follows that (18) is the solution of (15).

5. **The solution of** (14). On using (15) we can rewrite (14) in the form

$$\frac{G_L(t)}{X_L(t)} - \frac{G_R(t)}{X_R(t)} = \frac{\pi i\psi(t)}{X_L(t)\{T(t) - \pi it\psi(t)\}}\,. \tag{20}$$

On raising (19) to exponential powers it is clear that $X_L(t)$ is one-valued and cannot vanish for $0 \leqq t \leqq 1$. Again $\psi(t)$ and $T(t)$ are one-valued in this interval and $T(t) \neq 0$, (ii) §4. Since $\psi(t)$ and $T(t)$ are real it follows that $T(t) - \pi\, it\, \psi(t) \neq 0$ and since $\psi(t)$ satisfies a Hölder condition it must be continuous and bounded away from zero. Consequently the right-hand side of (20) is a one-valued and bounded function of t when $0 \leqq t \leqq 1$.

Again $X_L(t)$ is expressed in terms of a Cauchy principal value and this, in turn, depends upon logarithms in addition to nonsingular integrals [6, p. 27]. Hence $X_L(t)$ satisfies a Hölder condition and, since $\psi(t)$ and $T(t)$ each satisfy such a condition, the right-hand side of (20) must also satisfy a Hölder condition.

On using arguments similar to those of the previous section, §4, the solution of (20) is found to be

$$\frac{G(z)}{X(z)} = \frac{1}{2\pi i}\int_0^1 \frac{\pi i\psi(u)du}{X_L(u)\{T(u) - \pi iu\psi(u)\}(u-z)}\,. \tag{21}$$

From (21) formulae can be obtained for $G_L(z)/X_L(z)$ and $G_R(z)/X_R(t)$ $(0<t<1)$ by using (11) (analogous to but more complex than (19)).

6. **The solution of** (9). From (12) we have

$$H(t) = g(t)/\pi i\psi(t) \qquad (0 < t < 1)$$
$$\tag{22} = \{G_L(t) - G_R(t)\}/\pi i\psi(t)$$

on using (11). Now $G_L(t)/X_L(t)$ and $G_R(t)/X_R(t)$ can be computed from (21) by means of (11) and again $X_L(t)$ and $X_R(t)$ have already been computed in (19). Hence we can compute $G_L(t)$ and $G_R(t)$ in turn. The final formula for $H(t)$ then becomes

$$\tag{23} \begin{aligned} H(t) = {} & \frac{\{X_L(t) + X_R(t)\}}{2X_L(t)\{T(t) - \pi it\psi(t)\}} \\ & + \frac{\{X_L(t) - X_R(t)\}}{2\pi i\psi(t)} \int_0^1 \frac{\psi(u)du}{X_L(u)\{T(u) - \pi iu\psi(u)\}(u - t)}, \end{aligned}$$

where $0<t<1$, the integral is a Cauchy principal value and $X_L(t)$ and $X_R(t)$ are given by (19). If $T(t) \neq 0$ when $0 \leqq t \leqq 1$ then this is the only solution.

In this solution we encounter integrals, $I(t)$, of the type

$$\tag{24} I(t) = \int_0^1 \frac{g(u)}{u - t} du \qquad (0 < t < 1)$$

and these have meanings only when defined as in (4). We may write (24) in the form

$$\tag{25} I(t) = \int_0^1 \frac{g(u) - g(t)}{u - t} du + \int_0^1 \frac{g(t)}{u - t} du$$

where, if $g(u)$ satisfies a Hölder condition when u is in the interval $[0, 1]$, the first integral is no longer singular. The second integral, however, is still a Cauchy integral and must be evaluated from definition (4) as follows:

$$\tag{26} \int_0^1 \frac{g(t)}{u - t} du = g(t) \lim_{\epsilon \to +0} \left\{ \int_0^{t-\epsilon} \frac{1}{u - t} du + \int_{t+\epsilon}^1 \frac{1}{u - t} du \right\}$$

$$\tag{27} = g(t) \log\left\{\frac{1 - t}{t}\right\}.$$

Hence

$$\tag{28} I(t) = \int_0^1 \frac{g(u) - g(t)}{u - t} du + g(t) \log\left\{\frac{1 - t}{t}\right\}$$

where the integral in (28) is a nonsingular integral. Consequently the Cauchy principal values encountered in the solution (23) present no greater difficulties than ordinary integrals do when applied to problems either of a theoretical or a practical nature.

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