# SIMPLIFYING THE STRUCTURE OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS(1)

BY

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1. **The problem.** Consider the differential equation

$$\Delta_2 A + b^i A_i + cA = 0 \tag{1}$$

where the summation convention is being used and

$$\Delta_2 A = \frac{1}{g^{1/2}} \frac{\partial}{\partial x^i}\left(g^{1/2} g^{ij} \frac{\partial A}{\partial x^j}\right) = g^{ij} A_{,ij} = g^{ij}(A_{.ij} - \Gamma_{ij}^k A_k) \tag{2}$$

is the Laplace-Beltrami operator. Here a comma indicates covariant differentiation, a dot indicates (ordinary) differentiation, $A_i = A_{.i}$, $g = \det(g_{ij})$ and is always assumed to be positive, and $\Gamma_{ij}^k$ are the Christoffel symbols of the second kind with respect to $(g_{ij})$. All the functions in this paper are assumed to be sufficiently smooth. If (1) has a *positive* solution $A_0$ in a domain $D$, then the structure of (1) can be simplified. Indeed, setting $A = A_0\bar{A}$, we reduce (1) to

$$\Delta_2 \bar{A} + \bar{b}^i \bar{A}_i = 0 \tag{3}$$

where the $\bar{b}^i$ depend on $b^j$, $A_0$.

In this paper we consider the equation

$$\Delta_2 A + cA = 0 \qquad (c \not\equiv 0) \tag{4}$$

and wish to obtain an equivalent equation

$$\bar{\Delta}_2 \bar{A} + c_0 \bar{A} = 0 \tag{5}$$

where $c_0$ is a constant, preferably zero. Here, $\bar{\Delta}_2$ is the Laplace-Beltrami operator with respect to a different metric. We shall prove that it is possible to reduce (4) to (5), and that we can take $c_0 = 0$ if $c \leqq 0$. We also get similar results for parabolic equations.

Ishii [6] has considered conformal mappings which transfer solutions of $\Delta_2 A = 0$ into solutions of $\bar{\Delta}_2 \bar{A} = 0$. Ingraham [4] considered the question of eliminating the $b^i$ from equation (1) when $c \equiv 0$, on account of replacing the

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covariant derivatives in $\Delta_2$ by covariant derivatives with respect to some non-Riemannian affine connections. In [5] he considered the parabolic case.

2. **The transformation.** We try to perform conformal mapping

$$\bar{g}_{ij} = e^{2\sigma} g_{ij} \tag{6}$$

and set

$$\bar{A} = e^{h(\sigma)} A \tag{7}$$

where $\sigma$ and $h(\sigma)$ are to be determined. Using the formulas [3, p. 89]

$$\bar{g}^{ij} = e^{-2\sigma} g^{ij},$$
$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \sigma_j + \delta^k_j \sigma_i - g_{ij} g^{km} \sigma_m$$

we find, after some elementary calculation,

$$\begin{aligned} \bar{\Delta}_2 \bar{A} + c_0 \bar{A} = e^{-2\sigma} e^{h(\sigma)} \{ (\Delta_2 A + cA) + (2h'(\sigma) + n - 2) g^{jk} \sigma_j A_k \\ + [h''(\sigma)\Delta_1\sigma + h'(\sigma)\Delta_2\sigma + h'(\sigma)(h'(\sigma) + n - 2)\Delta_1\sigma - c + e^{2\sigma} c_0] A \} \end{aligned} \tag{8}$$

where $\Delta_1\sigma = g^{ij}\sigma_i\sigma_j$. Hence, (4) and (5) are equivalent in any subdomain of $D$ if and only if the coefficients of $A$ and $A_k$ vanish. The vanishing of the coefficients of the $A_k$ is equivalent to

$$h(\sigma) = \frac{2-n}{2}\sigma. \tag{9}$$

Assuming $n \geqq 3$ we find that the coefficient of $A$ vanishes if and only if

$$\Delta_2\sigma + \frac{n-2}{2}\Delta_1\sigma + \frac{2c}{n-2} - \frac{2c_0}{n-2} e^{2\sigma} = 0. \tag{10}$$

Setting

$$u = \exp\left\{\frac{n-2}{2}\sigma\right\}, \tag{11}$$

(10) becomes

$$\Delta_2 u + cu = c_0 u^{(n+2)/(n-2)}, \qquad u > 0. \tag{12}$$

We have thus proved:

THEOREM 1. *Let $n \geqq 3$. Equation* (4) *is reducible in a domain $D$ to equation* (5) *by means of the conformal transformation* (6), (7) *if and only if equation* (12) *has a positive solution in $D$.*

Due to the invariance of the operators of (4), (5) with respect to local change of the $x$-coordinates, *Theorem* 1 *holds also when $D$ is a domain on a manifold.*

Quite incidentally, if $c$ and $c_0$ are the scalar curvatures corresponding to the metric tensors $g_{ij}$ and $\bar{g}_{ij}$ related by (6), then (10) is known to hold (see [3, p. 90]). Recently Yamabe [9] proved that on a compact manifold, for any positive definite metric there exists a conformal transformation which yields a new metric with *constant* scalar curvature. His proof does not make use of the specific nature of the function $c$, that is, he proves that for *any* smooth function $c$ on a compact Riemannian manifold with a positive definite metric there exists a constant $c_0$ and a positive smooth function $u$, defined on the whole manifold, such that (12) is satisfied. Hence we have:

THEOREM 2. *Given a compact Riemannian manifold $R_n$, $n \geqq 3$, with a positive definite metric, equation* (4) *can globally be reduced to equation* (5) *by means of* (6), (7), (9), *where $c_0$ is a constant depending on $c$.*

If $c_0 \leqq 0$ then the transformation is uniquely determined, up to a constant multiple $\neq 0$. Indeed, if $u_1$, $c_1$ is another solution, then $w = u_1/u$ satisfies $\bar{\Delta}_2 w + c_0 w = c_1 w^{(n+2)/(n-2)}$. Setting $w(x^0) = \max w(x)$, we then have $\bar{\Delta}_2 w \leqq 0$, $c_0 w \leqq 0$ at $x^0$; hence $c_1 \leqq 0$. It follows that $\bar{\Delta}_2 w + c_0 w \leqq 0$. The minimum principle now yields $w \equiv$ const.

COROLLARY. *Let $D$ be an $n$-dimensional bounded domain, $n \geqq 3$, and let $(g_{ij})$ be a positive matrix. Then* (4) *is reducible to* (5), *in the whole domain $D$, by means of* (6), (7), (9), *where the constant $c_0$ depends on $c$.*

Under some conditions on $c(x)$ we can even take $c_0 = 0$. Thus we have:

THEOREM 3. *Let $D$ be an $n$-dimensional domain, $n \geqq 3$, either bounded or unbounded but with finite boundary, and let $(g_{ij})$ be a positive matrix. If $c(x) \leqq 0$ then* (4) *is reducible, in the whole domain $D$, to* (5) *with $c_0 = 0$, by means of a transformation* (6), (7), (9).

**Proof.** We only have to establish the existence, in $D$, of a positive solution of the equation

$$\Delta_2 u + cu = 0. \tag{13}$$

If $D$ is bounded, we solve (13) for any positive boundary values and thus obtain a positive solution, using the maximum principle. If $D$ is unbounded then the existence of a positive solution follows by recent results of Meyers and Serrin [7].

We note that if the diameter of $D$ is sufficiently small it is not necessary to make any assumption on $c$. We also remark that for a given bounded domain $D$ we can get $c_0 = 0$ if $c(x) \leqq \epsilon$ where $\epsilon$ is sufficiently small, depending on $D$. However, *if $\epsilon$ is not small there are in general no positive solutions of* (13) *in $D$.*

For simplicity, we produce a counter-example for $n = 3$, $\Delta_2$ being the Laplace operator $\Delta$, and $c(x) \equiv k^2 > 0$. Let $x^0$ be a point in $D$ whose distance from the boundary of $D$ is $H$. We claim that there are no positive solutions of $\Delta u+k^2u=0$ in $D$ if $k>\pi/H$.

Indeed, we apply the Pizetti formula [1, p. 259]

$$\frac{1}{4\pi R^2}\int_{S_R} u(x)dS = \sum_{h=0}^{m} \frac{R^{2h}\Delta^h u(x^0)}{(2h+1)!} + \frac{1}{4\pi(2m+1)!}\int_{K_R} \frac{(R-r)^{2m+1}\Delta^{m+1}u(x)}{Rr}dV$$

where $K_R$ is a ball of radius $R$, center $x^0$ and surface $S_R$, and $r=|x-x^0|$. Taking $m\to\infty$ we obtain

$$\frac{1}{4\pi R^2}\int_{S_R} u(x)dS = \sin(kR)u(x^0)/kR.$$

Now, if $u$ is positive in $D$ then it follows that $\sin kR>0$; hence $kH\leqq\pi$ which is a contradiction.

THE CASE $n=2$. From (8) we conclude that a reduction of (4) to (5) is possible if $h(\sigma)=0$ and if

$$c = e^{2\sigma}c_0. \tag{14}$$

Hence, the reduction by (6), (7) is possible if and only if $\operatorname{sgn} c(x)=\text{const.}$ and $c_0$ can be taken to be either any positive or any negative number, depending on $\operatorname{sgn} c(x)$. Note that $(g_{ij})$ can be taken to be any nonsingular matrix.

3. **Parabolic equations.** Consider the parabolic equation

$$a\frac{\partial A}{\partial t} = \Delta_2 A + cA \qquad (a>0) \tag{15}$$

where $x$ varies in an $n$-dimensional domain $D$ ($n\geqq 3$) and $0\leqq t<\infty$. The coefficients are functions of $(x, t)$ and $\Delta_2$ is elliptic. Performing the transformation (6), (7), (9) with $\sigma$ depending also on $t$, we find that (15) is reduced to

$$\bar{a}\frac{\partial \bar{A}}{\partial t} = \bar{\Delta}_2\bar{A} + c_0\bar{A} \qquad (\bar{a}>0) \tag{16}$$

if and only if there exists a positive solution of

$$a\frac{\partial u}{\partial t} = \Delta_2 u + cu - c_0u^{(n+2)/(n-2)} \tag{17}$$

and then $\sigma$ is defined by (11). Also,

$$\bar{a} = e^{-2\sigma}a. \tag{18}$$

From now on we assume that if $D$ is unbounded then the functions $a$, $1/a$, $c$, $g_{ij}$, $g^{ij}$, $\partial a/\partial x^i$, $\partial^2 a/\partial x^i\partial x^j$, $\partial a/\partial t$, $\partial g^{ij}/\partial x^h$, $\partial^2 g^{ij}/\partial x^h\partial x^k$, $\partial g^{ij}/\partial t$ are Hölder continuous and bounded for $x$ in $D$ and $t$ in finite intervals. We then claim that for any $D$, there exists a positive solution of (17) with $c_0=0$. It is enough to establish it when $D$ is the whole $n$-dimensional space $E_n$.

Let $K(x, t; \xi, \tau)$ be the fundamental solution of (17) constructed by Dressel [2]. Then for any positive function $\phi(\xi)$ which tends to a positive limit $B$ as $|\xi|\to\infty$ we have a solution of (17) with $c_0=0$ for $x\in E_n$, $0<t<\infty$, in the form

$$u(x, t) = \int_{E_n} K(x, t; \xi, 0)\phi(\xi)d\xi.$$

This solution is positive on $t=0$, and it tends to $B$ as $|x|\to\infty$, uniformly in $t$ in finite intervals $0\leqq t\leqq T$, $T>0$. Hence by appropriately applying the maximum principle [8] we conclude that $u(x, t)>0$. We have thus proved:

THEOREM 4. *Let $D$ be any domain in $E_n$, $n\geqq 3$ and let the coefficients of the parabolic equation* (15) *satisfy the boundedness assumptions mentioned above. Then equation* (15) *is reducible to equation* (16) *with $c_0=0$, by means of* (6), (7), (9).

Note that no assumption is being made on the signature of $c$.

If $n=2$, we first make a transformation of the form $\hat{A}=e^{\alpha t}A$ and obtain a new coefficient $c$ which is positive. We then can reduce (15) to (16) since equation (14) can then be solved.

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