HÖLDER CONDITIONS FOR REALIZATIONS OF GAUSSIAN PROCESSES

BY

Z. CIESIELSKI

1. **Introduction.** In this note we will consider real valued Gaussian processes $\{x(t)\}$, $0 \leqq t \leqq 1$ satisfying the following conditions: $x(0)=0$, $E\{x(t)\}=0$ and $E\{x(t)x(s)\}=\rho(t, s)$ for $t, s \in \langle 0, 1\rangle$; $\rho(t, s)$ is the covariance function and hence is semi-definite, symmetric and $\rho(0, t)=0$ for $t \in \langle 0, 1\rangle$ [4, p. 72]. The result of this note is the following:

THEOREM 2. *Let the covariance function $\rho(t, s)$ satisfy the uniform Hölder condition*

$$(1.1) \qquad |\rho(t_1, s) - \rho(t_2, s)| \leqq C|t_1 - t_2|^\alpha$$

for all $t_1, t_2, s \in \langle 0, 1\rangle$, where α, $0<\alpha \leqq 1$, and $C>0$ are absolute constants. Then there exists a Gaussian process $\{x(t)\}$ with covariance $\rho(t, s)$ and an absolute constant $C_1>0$ such that

$$(1.2) \qquad \lim_{\delta \to 0^+} \sup_{|t_1-t_2| \leqq \delta} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^{\alpha/2} \left| \log |t_1 - t_2| \right|^{1/2}} < C_1 \frac{C^{1/2}}{\alpha^2}$$

holds with probability one.

Moreover, we discuss various cases of this theorem, and in §4 we give an application.

It will be convenient to introduce now the problem considered in §4. Let

$$(1.3) \qquad x_n(t) = \frac{1}{n^{1/2}} \sum_{i=0}^{n-1} \{f_t(2^i x) - t\}$$

for $n=1, 2, \cdots$, where $f_t(x)$ is equal to 1 for $0 \leqq x < t$ and to 0 for $t < x < 1$ $(0 \leqq t \leqq 1)$, and for fixed t, $f_t(x)$ is periodic as a function of x with period 1. Now, for fixed n we consider $\{x_n(t)\}$ as a stochastic process with time interval $\langle 0, 1\rangle$ and multivariate distributions defined as follows:

$$(1.4) \qquad P\{x_n(t_1) < \omega_1, \cdots, x_n(t_s) < \omega_s\} = m\{x: x_n(t_i) < \omega_i, i = 1, \cdots, s\},$$

where m denotes the Lebesgue measure in $\langle 0, 1\rangle$. It has been proved by N. J. Fine [5] that for given $t_i \in (0, 1)$, ω_i where $i=1, \cdots, s$, and s is any positive integer (for $s=1$, see [7]), that the probability (1.4) approaches

$$(1.5) \qquad P\{x(t_1) < \omega_1, \cdots, x(t_s) < \omega_s\}$$

Presented to the Society, January 23, 1961; received by the editors January 12, 1961.

as $n\to\infty$, where $\{x(t)\}$ is a Gaussian process associated with the covariance

$$\rho(t, s) = \lim_{n\to\infty} \int_0^1 x_n(t)x_n(s)dx. \tag{1.6}$$

In Fine's paper the explicit formula for the function (1.6) is given. M. Kac conjectured that, as $n\to\infty$, the realizations of the Gaussian process which has (1.6) as its covariance function satisfy, with probability one, a Hölder condition in $\langle 0, 1\rangle$. This conjecture is proved in §4 and the remarkable thing is that the Hölder condition for this process is quite similar to that for the Wiener process.

The author wishes to thank Professor Mark Kac at whose suggestion this work was undertaken.

2. **Preliminary lemmas and notations.** The proofs of our theorems are based on special properties of Haar and Schauder functions [8, pp. 44, 50]. The complete orthonormal Haar's set $\{\chi_n\}$ is defined as follows:

$$\chi_1(t) \equiv 1,\ \chi_{2^n+k}(t) = \begin{cases} 2^{n/2} \text{ in } \left\langle \dfrac{2k-2}{2^{n+1}}, \dfrac{2k-1}{2^{n+1}} \right), \\ -2^{n/2} \text{ in } \left(\dfrac{2k-1}{2^{n+1}}, \dfrac{2k}{2^{n+1}} \right\rangle, \\ 0 \text{ elsewhere in } \langle 0, 1\rangle, \end{cases} \tag{2.1}$$

for $n=0, 1, \cdots; k=1, \cdots, 2^n$. Schauder's system $\{\phi_n\}$ is defined by the formulas

$$\phi_0(t) \equiv 1, \qquad \phi_n(t) = \int_0^t \chi_n(\tau)d\tau \qquad \text{for } n = 1, 2, \cdots; t \in \langle 0, 1\rangle. \tag{2.2}$$

Let $C_{\langle 0,1\rangle}$ denote the space of all real continuous functions on $\langle 0, 1\rangle$ vanishing at zero. Schauder's result can be formulated in the form of

LEMMA 1 [1]. *For every $x \in C_{\langle 0,1\rangle}$ we have*

$$x(t) = \sum_{n=1}^{\infty} \int_0^1 \chi_n(\tau)dx(\tau)\phi_n(t). \tag{2.3}$$

Moreover, this series converges uniformly in $\langle 0, 1\rangle$ and the representation (2.3) *is unique.*

Now, let H_α $(0<\alpha\leqq 1)$ denote the set of all functions $x\in C_{\langle 0,1\rangle}$ satisfying the Hölder condition

$$\|x\|_\alpha = \sup_{t_1,t_2\in\langle 0,1\rangle} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\alpha} < \infty,$$

and let $H_{\alpha,0}$ be a subset of H_α such functions x that

$$\lim_{\delta\to 0^+} \sup_{|t_1-t_2|\leqq\delta} \frac{|x(t_1)-x(t_2)|}{|t_1-t_2|^\alpha} = 0.$$

Let us put

$$(2.4)\qquad \begin{aligned} \chi_1^{(\alpha)} &= \chi_1, & \chi_{2^n+k}^{(\alpha)} &= \frac{2^{(n+1)\alpha}}{2(2^{n/2})}\chi_{2^n+k}, \\ \phi_1^{(\alpha)} &= \phi_1, & \phi_{2^n+k}^{(\alpha)} &= \frac{2(2^{n/2})}{2^{(n+1)\alpha}}\phi_{2^n+k}, \end{aligned}$$

for $n=0, 1, \cdots; k=1, \cdots, 2^n$. One can check easily that $\|\phi_n^{(\alpha)}\|_\alpha=1$ for $n=1, 2, \cdots$.

From Lemma 1 we now deduce easily

LEMMA 2 [1]. *For every $x\in H_\alpha$ $(\alpha>0)$*

$$\sum_{n=1}^{\infty}\left|\int_0^1 \chi_n^{(\beta)}(\tau)dx(\tau)\phi_n^{(\beta)}(t)\right|$$

converges uniformly in $\langle 0, 1\rangle$, for any $\beta>0$.

Let us remark that $\int_0^1\chi_n^{(\beta)}(\tau)dx(\tau)\phi_n^{(\beta)}(t)$ does not depend on β.

LEMMA 3 [3]. *Let $x\in C_{\langle 0,1\rangle}$, $0<\alpha<1$ and let*

$$(2.5)\qquad x(t) = \sum_{n=1}^{\infty}\xi_n\phi_n^{(\alpha)}(t).$$

Then

$$x\in H_\alpha \text{ if and only if } \xi_n = O(1),$$

and

$$x\in H_{\alpha,0} \text{ if and only if } \xi_n = o(1).$$

Moreover, (2.5) *converges in the norm $\|\ \|_\alpha$ for every $x\in H_{\alpha,0}$.*

We will need also some more general results from [2]. Let $\omega(t)$ be a positive function defined on $\langle 0, 1\rangle$. We shall say that $\omega(t)$ satisfies the condition (i), (ii) or (iii) if respectively:

(i) *$\omega(t)$ is nondecreasing in $\langle 0, 1\rangle$ and $\omega(2t)\leqq K\omega(t)$ for $0\leqq 2t\leqq 1$, where K is a positive constant;*

(ii) *the inequality*

$$\int_0^\delta \frac{\omega(t)}{t}dt \leqq L\omega(\delta)$$

holds for $0<\delta\leqq 1$ *and for some positive constant* L;

(iii) *the inequality*

$$\delta\int_\delta^1 \frac{\omega(t)}{t^2}\,dt \leqq M\omega(t)$$

holds for $0<\delta\leqq 1$, *where* M *is some positive constant.*

We also recall that the modulus of continuity $\omega_x(\delta)$ of $x\in C_{\langle 1,0\rangle}$ is defined by the formula

$$\omega_x(\delta) = \sup_{|t_1-t_2|\leqq\delta} |x(t_1)-x(t_2)|.$$

LEMMA 4 [2]. *Let* $\omega(t)$ *satisfy the conditions* (i) *and* (ii) *and suppose that for some* $x\in C_{\langle 0,1\rangle}$

$$|x(1)| \leqq \omega(1), \qquad \left|\int_0^1 \chi_{2^n+k}(\tau)dx(\tau)\right| \leqq 2^{n/2}\omega\left(\frac{1}{2^{n+1}}\right)$$

for $n=0, 1, \cdots; k=1, \cdots, 2^n$. *Then*

$$\omega_x(\delta) \leqq 6^2K(1+L/\log 2)\delta\left[2\omega(1)+\int_\delta^1\frac{\omega(t)}{t^2}dt\right]$$

for $0<\delta\leqq 1$.

Let us now consider the integral equation

$$f(t) = \lambda\int_0^1 \rho(t,s)f(s)ds \tag{2.6}$$

where $\rho(t,s)$ is a covariance function. We denote by $\{\lambda_n\}$ and $\{f_n\}$ the eigenvalues and eigenfunctions of (2.6), respectively. We shall use the following classical result of Mercer.

LEMMA 5 [6, p. 91]. *If* $\rho(t,s)$ *is continuous then*

$$\rho(t,s) = \sum_{n=1}^\infty \frac{f_n(t)f_n(s)}{\lambda_n},$$

where the convergence is uniform and absolute.

Since we will need the Wiener process in our considerations we give now a simple construction of this process. It is well known that the Wiener process is a Gaussian process with covariance $\rho(t,s)=\min(t,s)$. Let $\{\zeta_n\}$, $n=1,2,\cdots$, be a sequence of independent Gaussian random variables such that $E\{\zeta_n\}=0$, $E\{\zeta_n^2\}=1$ for $n=1,2,\cdots$. Then

$$y(t) = \sum_{n=1}^\infty \zeta_n\phi_n(t)$$

is a Wiener process. Indeed, by the Borel-Cantelli lemma we have

$$P_n\{\zeta_n = O((\log n)^{1/2}), \text{ as } n \to \infty\} = 1;$$

hence almost all $y(t)$ are continuous on $\langle 0, 1\rangle$, and since $\{\chi_n\}$ is complete we get from Parseval's identity

$$E\{y(t)y(s)\} = \sum_{n=1}^{\infty} \phi_n(t)\phi_n(s) = \min(t, s).$$

DEFINITION. *Let $g(t)$ be a square integrable function on $\langle 0, 1\rangle$. If $\{y(t)\}$ is a Wiener process then the random variable $\int_0^1 g(\tau)dy(\tau)$ is defined by*

$$\int_0^1 g(\tau)dy(\tau) = \sum_{n=1}^{\infty} \int_0^1 \chi_n(\tau)dy(\tau) \int_0^1 g_n(t)\chi_n(t)dt, \tag{2.7}$$

where the series converges for almost all y.

The definition is introduced in [9].

3. **Main result.** The first theorem which we shall prove will give us a representation of the realizations of Gaussian processes with the covariance functions satisfying (1.1).

We use the following notation:

$$\int_0^1 \int_0^1 \chi_n(\sigma)\chi_m(\tau)\rho(d\sigma, d\tau) = \int_0^1 \chi_n(\sigma)\, d\int_0^1 \chi_m(\tau)d\rho(\sigma, \tau). \tag{3.1}$$

THEOREM 1. *Let the covariance $\rho(t, s)$ satisfy the condition* (1.1) *for some fixed $\alpha \in (0, 1\rangle$ and all $t_1, t_2, s \in \langle 0, 1\rangle$, and let $\beta \in (0, \alpha/2)$ be fixed. Then there exists a sequence $\{\xi_n\}$ of Gaussian random variables such that the process*

$$x(t) = \sum_{n=1}^{\infty} \xi_n \phi_n^{(\beta)}(t) \tag{3.2}$$

is Gaussian and has $\rho(t, s)$ as its covariance function. Moreover, for almost all $x(t)$ (3.2) *converges in the norm $\| \ \|_\beta$, and almost all $x(t)$ are from H_β. The random variables $\{\xi_n\}$ satisfy the following conditions:*

$$E\{\xi_n\} = 0, \qquad E\{\xi_n\xi_m\} = \int_0^1 \int_0^1 \chi_n(\tau)\chi_m(\sigma)\rho(d\tau, d\sigma) \tag{3.3}$$

for $n, m = 1, 2, \cdots$.

Proof. Since $\rho(t, s)$ satisfies Hölder condition (1.1) and $\rho(0, s) = 0$ we have by Lemmas 1 and 2

$$\rho(t, s) = \sum_{n=1}^{\infty} \int_0^1 \chi_n^{(\beta)}(\tau)d\rho(\tau, s)\phi_n^{(\beta)}(t), \tag{3.4}$$

where the series converges absolutely and uniformly. Since

$$\int_0^1 \chi_1(\tau)d\rho(\tau, s) = \rho(1, s),$$

$$(3.5)\quad \int_0^1 \chi_{2^n+k}^{(\beta)}(\tau)d\rho(\tau, s) = 2^{(n+1)\beta}\left\{\rho\left(\frac{2k-1}{2^{n+1}}, s\right) - \frac{\rho((k-1)/2^n, s) + \rho(k/2^n, s)}{2}\right\},$$

for $n=0, 1, \cdots; k=1, \cdots, 2^n$, we see that $\int_0^1 \chi_m^{(\beta)}(\tau)d\rho(\tau, s)$ satisfies a Hölder condition. Again, using Lemmas 1 and 2 we get absolute and uniform convergence of

$$(3.6)\quad \int_0^1 \chi_n^{(\beta)}(\tau)d\rho(\tau, s) = \sum_{m=1}^{\infty} \int_0^1 \int_0^1 \chi_n^{(\beta)}(\tau)\chi_m^{(\beta)}(\sigma)\rho(d\tau, d\sigma)\phi_m^{(\beta)}(s)$$

for $n=1, 2, \cdots$. From (3.4) and (3. 6) we obtain

$$(3.7)\quad \rho(t, s) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \int_0^1 \int_0^1 \chi_n^{(\beta)}(\tau)\chi_m^{(\beta)}(\sigma)\rho(d\tau, d\sigma)\phi_n^{(\beta)}(t)\phi_m^{(\beta)}(s).$$

It should be remembered that Lemma 1 insures the uniqueness of the expansion (3.7).

Now let us find the representation of $\rho(t, s)$ in terms of eigenvalues and eigenfunctions of the integral equation (2.6). By Lemma 5 we get

$$(3.8)\quad \rho(t, s) = \sum_{i=1}^{\infty} \frac{f_i(t)f_i(s)}{\lambda_i}.$$

It is obvious that the eigenfunctions $f_i(t)$ satisfy a Hölder condition. Once more by Lemmas 1 and 2 we have for each integer i

$$(3.9)\quad f_i(t) = \sum_{n=1}^{\infty} a_n^{(i)}\phi_n^{(\beta)}(t),$$

where

$$a_n^{(i)} = \int_0^1 \chi_n^{(\beta)}(\tau)df_i(\tau) \qquad \text{for } n = 1, 2, \cdots$$

and where the series converges absolutely and uniformly. Now, we substitute (3.9) in (3.8), change the order of summation and get the second representation

$$(3.10)\quad \rho(t, s) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i} a_n^{(i)} a_m^{(i)}\right)\phi_n^{(\beta)}(t)\phi_m^{(\beta)}(s).$$

Since both representations (3.7) and (3.10) are unique we have

$$\int_0^1 \int_0^1 \chi_n^{(\beta)}(\tau)\chi_m^{(\beta)}(\sigma)\rho(d\tau, d\sigma) = \sum_{i=1}^{\infty} \frac{a_n^{(i)} a_m^{(i)}}{\lambda_i} \tag{3.11}$$

for $n, m = 1, 2, \cdots$.

Let us put for $n = 1, 2, \cdots$

$$g_n(t) = \sum_{i=1}^{\infty} \frac{1}{(\lambda_i)^{1/2}} a_n^{(i)} \chi_i(t). \tag{3.12}$$

We shall show that $g_n(t)$ are square integrable over $\langle 0, 1\rangle$. From (3.8) it follows that

$$\sum_{i=1}^{\infty} \frac{f_i^2(t)}{\lambda_i} < \infty \text{ in } \langle 0, 1\rangle. \tag{3.13}$$

The coefficients $a_n^{(i)}$, $i = 1, 2, \cdots$, defined in (3.9) for a given n represent the second differences of functions $f_i(t)$, respectively, at three fixed points which depend only on n. Thus, it is easy to see that (3.13) implies

$$\sum_{i=1}^{\infty} \left(\frac{1}{(\lambda_i)^{1/2}} a_n^{(i)}\right)^2 < \infty.$$

Hence by the Riesz-Fisher theorem $g_n(t)$ are square integrable.

We also note that by (3.11), (3.12) and by Parseval's identity we have

$$(g_n, g_m) = \int_0^1 \int_0^1 \chi_n^{(\beta)}(\tau)\chi_m^{(\beta)}(\sigma)\rho(d\tau, d\sigma) \tag{3.14}$$

for $n, m = 1, 2, \cdots$.

Let $\{y(t)\}$, $0 \leqq t \leqq 1$, be a Wiener process. Using this process and the functions (3.12) we define a new process

$$x(t) = \sum_{n=1}^{\infty} \int_0^1 g_n(\tau) dy(\tau) \phi_n^{(\beta)}(t), \tag{3.15}$$

where the random variables

$$\xi_n = \int_0^1 g_n(\tau) dy(\tau), \qquad n = 1, 2, \cdots \tag{3.16}$$

are defined by (2.7) if we replace $g(t)$ by $g_n(t)$. One can easily get (3.3) from (3.14) and (3.16).

Since $E(\xi_n^2) = (g_n, g_n) = \|g_n\|^2$ we have from (1.1), (3.14) and (2.4),

$$\|g_n\| \leqq 2C^{1/2} n^{\beta - \alpha/2} \text{ for } n \geqq 2. \tag{3.17}$$

Since

$$P\{|\xi_n| > \omega\|g_n\|\} \simeq \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{\omega e^{\omega^2/2}} \tag{3.18}$$

for large ω, we see that for $\omega = (2\gamma \log n)^{1/2}$, $\gamma > 1$,

$$\sum_{n=2}^{\infty} P\{|\xi_n| > (2\gamma \log n)^{1/2}\|g_n\|\} < \infty.$$

By the Borel-Cantelli lemma we get

$$P\left(\limsup_{n\to\infty} \frac{|\xi_n|}{(2\log n)^{1/2}\|g_n\|} \leqq \gamma^{1/2}\right) = 1. \tag{3.19}$$

Now, since $\beta < \alpha/2$ we have by (3.17) and (3.19) that

$$P\{\xi_n = o(1)\} = 1.$$

Applying Lemma 3 we complete the proof of Theorem 1.

Proof of Theorem 2. According to (2.4) we can write (3.2) in the form

$$x(t) = \sum_{n=1}^{\infty} \eta_n \phi_n(t), \tag{3.20}$$

where $\eta_1 = \xi_1$, $\eta_{2^m+v} = \xi_{2^m+v} 2^{m/2}/2^{(m+1)\beta}$ for $m = 0, 1, \cdots; v = 1, \cdots, 2^m$. Combining (3.17) and (3.19) we have

$$P\left\{\limsup_{n\to\infty} \frac{\eta_n}{n^{1/2-\alpha/2}(\log n)^{1/2}} \leqq 4(2C\gamma)^{1/2}\right\} = 1 \tag{3.21}$$

where $\gamma > 1$. We see from (3.21) that for almost every $\{\eta_n\}$ there exists a positive integer N, depending only on this sequence, such that

$$|\eta_n| \leqq 8(Cn)^{1/2} \frac{(\log n)^{1/2}}{n^{\alpha/2}} \tag{3.22}$$

for $n > 2^N$.

Let now $x(t) = S_N(t) + x_N(t)$, where

$$S_N(t) = \sum_{n=1}^{2^N} \eta_n \phi_n(t) \quad \text{and} \quad x_N(t) = \sum_{n=2^N+1}^{\infty} \eta_n \phi_n(t).$$

Since $S_N(t)$ is a polygonal line we have

$$\lim_{\delta\to 0^+} \sup_{|t_1-t_2|\leqq\delta} \frac{|S_N(t_1) - S_N(t_2)|}{|t_1 - t_2|^{\alpha/2} |\log|t_1 - t_2||^{1/2}} = 0,$$

and hence

$$\tag{3.23} \begin{aligned} &\lim_{\delta \to 0^+} \sup_{|t_1 - t_2| \leq \delta} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^{\alpha/2} |\log|t_1 - t_2||^{\alpha/2}} \\ &\qquad \leq \lim_{\delta \to 0^+} \sup_{|t_1 - t_2| \leq \delta} \frac{|x_N(t_1) - x_N(t_2)|}{|t_1 - t_2|^{\alpha/2} |\log|t_1 - t_2||^{1/2}} . \end{aligned}$$

Let us consider now the function $x_N(t)$. From the definition of $x_N(t)$ and from Lemma 1 we have

$$\int_0^1 \chi_n(\tau) dx_n(\tau) = \begin{cases} 0 & \text{for } n \leq 2^N, \\ \eta_n & \text{for } n > 2^N. \end{cases}$$

Also (3.22) implies that

$$\tag{3.24} \int_0^1 \chi_{2^m+\nu}(\tau) dx_N(\tau) \leq C_0 2^{m/2} \frac{\log 2^{m+1}}{2^{(m+1)\alpha/2}}$$

for $m=0, 1, \cdots; \nu=1, \cdots, 2^m$, where C_0 is an absolute constant.

We put now $\omega(t)=C_0 t^{\alpha/2}(1+|\log t|^{1/2})$ for $t \in \langle 0, 1\rangle$. One checks easily that $\omega(t)$ satisfies the conditions (i), (ii) and (iii) with the constants $K=2$, $L=6/\alpha^2$ and $M=2$, respectively. Now using (3.24) we apply Lemma 4 to the function $x_N(t)$ and we get at last for some absolute constant C_1

$$\omega_{x_N}(\delta) \leq C_1 \frac{C^{1/2}}{\alpha^2} \omega(\delta),$$

hence by (3.23) our Theorem 2 is proved.

Remarks. (1) Using the same methods one may slightly generalize Theorems 1 and 2. Let us replace the condition (1.1) for the covariance $\rho(t, s)$ by

$$\tag{3.25} |\rho(t_1, s) - \rho(t_2, s)| \leq \omega(|t_1 - t_2|),$$

where the function $(\omega(t)(1+|\log t|))^{1/2}$ satisfies the conditions (i), (ii) and (iii). Then the assertion of Theorem 2 will be the following:

There is a Gaussian process $\{x(t)\}$, $0 \leq t \leq 1$, associated with the covariance $\rho(t, s)$ such that for some constant C_2 (depending on $\omega(t)$)

$$\tag{3.26} \lim_{\delta \to 0^+} \sup_{|t_1 - t_2| \leq \delta} \frac{|x(t_1) - x(t_2)|}{(\omega(|t_1 - t_2|)|\log|t_1 - t_2||)^{1/2}} < C_2$$

holds with probability one.

(2) From the methods of proofs of Theorems 1 and 2 it follows that these theorems will still be valid if we replace the condition (1.1) by one of the following conditions:

(a) The inequality

$$\tag{3.27} \left| \int_0^1 \int_0^1 \chi_m^{(\alpha/2)}(\tau) \chi_n^{(\alpha/2)}(\sigma) \rho(d\tau, d\sigma) \right| \leq C_3 \left(\frac{m}{n}\right)^{\alpha/2}$$

holds for $n \geqq m \geqq 2$, where C_3 is an absolute constant and ρ is continuous.

(b) The inequality

$$|\Delta_\tau^2 \Delta_\sigma^2 \rho(t, s)| \leqq C_4 \min(\tau^\alpha, \sigma^\alpha) \tag{3.28}$$

holds for some absolute constant C_4 in the square $\langle 0, 1\rangle \times \langle 0, 1\rangle$.

The symbols Δ_τ^2, Δ_σ^2 denote the nonsymmetric second differences with increments τ, σ and acting respectively on the variables t and s.

(3) Let $\{F_n\}$ denote the orthonormal system which one gets by applying Schmidt's orthonormalization procedure to Schauder's system $\{\phi_n\}$. These functions are known as Franklin's functions [8, p. 122]. One can prove that the properties of $\{\phi_n\}$ and $\{F_n\}$ are so similar that the results of this section can be obtained using $\{F_n\}$ instead of $\{\phi_n\}$. These properties of $\{F_n\}$ were investigated by the author, and the results were announced at the Conference on Functional Analysis in Warsaw in September 1960. The tedious proofs are not yet published, but will appear in Studia Mathematica.

4. Application. The purpose of this section is to examine some properties of the Gaussian process associated with the covariance given by (1.6).

LEMMA 5. *Let $\rho(t, s)$ be given by* (1.6). *Then*

$$|\rho(t_1, s) - \rho(t_2, s)| \leqq 5(12)^2 |t_1 - t_2| \{1 + (\log |t_1 - t_2|)^2\}$$

for $t_1, t_2, s \in \langle 0, 1\rangle$ and

$$\left| \int_0^1 \chi_n(\tau)\chi_m(\sigma)\rho(d\tau, d\sigma) \right| \leqq 2^{1/2} \left(\frac{n}{m}\right)^{1/2}$$

for $2 \leqq n \leqq m$.

Proof. It is easy to show that

$$\begin{aligned} \rho(t, s) = \min(t, s) - ts + \sum_{i=1}^{\infty} \frac{1}{2^i} \{&\min(s, 2^i t - [2^i t]) - s(2^i t - [2^i t]) \\ &+ \min(t, 2^i s - [2^i s]) - t(2^i s - [2^i s])\} \end{aligned} \tag{4.1}$$

where $[t]$ denotes the greatest integer less than or equal to t.

Let us note that from Parseval's identity we get

$$\min(t, s) - ts = \sum_{n=1}^{\infty} \phi_n(t)\phi_n(s). \tag{4.2}$$

Moreover, by definitions (2.1) and (2.2) we have

$$\phi_{2^m+\nu}(2^i t - [2^i t]) = 2^{i/2} \sum_{j=1}^{2^i} \phi_{2^{m+i}+2^m(j-1)+\nu}(t). \tag{4.3}$$

Combining (4.1), (4.2) and (4.3) we obtain

$$
\begin{aligned}
\rho(t, s) = & \sum_{n=0}^{\infty} \sum_{v=1}^{2^n} \phi_{2^n+v}(t)\phi_{2^n+v}(s) + \sum_{i=1}^{\infty} \frac{1}{2^{i/2}} \\
& \cdot \sum_{n=0}^{\infty} \sum_{v=1}^{2^n} \left\{ \phi_{2^n+v}(t) \sum_{j=1}^{2^i} \phi_{2^{n+i}+2^n(j-1)+v}(s) + \phi_{2^n+v}(s) \sum_{j=1}^{2^i} \phi_{2^{n+i}+2^n(j-1)+v}(t) \right\}.
\end{aligned} \tag{4.5}
$$

It is easy to see that from (4.5) we can get the second part of our lemma. The first part of Lemma 5 one can prove using (4.5) and Lemma 4. The by now standard computations will be omitted.

THEOREM 3. *There exists a Gaussian process* $\{x(t)\}$, $0 \leqq t \leqq 1$, *with covariance function* $\rho(t, s)$ *given by* (1.6) *such that for some absolute constant* C_5

$$P\left\{ \lim_{\delta \to 0^+} \sup_{|t_1 - t_2| \leqq \delta} \frac{|x(t_1) - x(t_2)|}{(|t_1 - t_2| \, |\log|t_1 - t_2||)^{1/2}} < C_5 \right\} = 1.$$

REMARK. One can prove also that there exists a positive constant C_6 such that

$$P\left\{ \lim_{\delta \to 0^+} \sup_{|t_1 - t_2| \leqq \delta} \frac{|x(t_1) - x(t_2)|}{(|t_1 - t_2| \, |\log|t_1 - t_2||)^{1/2}} > C_6 \right\} = 1.$$

The proof is based on (4.5) and on some lemmas given by K. L. Chung, P. Erdös and T. Sirao (see J. Math. Soc. Japan vol. 11 (1959) pp. 263–274, Lemmas 2, 3 and 4).

The proof of Theorem 3 follows immediately from Lemma 5, Theorem 2 and Remark 2a given in §2.

REFERENCES

1. Z. Ciesielski, *On Haar functions and on the Schauder basis of the space* $C_{\langle 0,1 \rangle}$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. vol. 7 (1959) pp. 227–332.

2. ———, *Some properties of Schauder basis of the space* $C_{\langle 0,1 \rangle}$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. vol. 8 (1960) pp. 141–144.

3. ———, *On the isomorphisms of the space* H_α *and* m, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. vol. 8 (1960) pp. 217–222.

4. J. L. Doob, *Stochastic processes*, New York, John Wiley and Sons, 1953.

5. N. J. Fine, *On the asymptotic distributions of certain sums*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 243–252.

6. G. Hamel, *Integralgleichungen*, Berlin, Julius Springer, 1937.

7. M. Kac, *On the distribution of values of sums of type* $\sum f(2^k t)$, Ann. of Math. vol. 47 (1946) pp. 33–49.

8. S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen*, Warsaw, 1935.

9. R. E. A. C. Paley, N. Wiener and A. Zygmund, *Notes on random functions*, Math. Z. vol. 37 (1933) pp. 647–668.

CORNELL UNIVERSITY,
ITHACA, NEW YORK