ORTHONORMAL SETS WITH NON-NEGATIVE DIRICHLET KERNELS. II

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J. J. PRICE(1)

1. Introduction. Recently the author has shown [2] that an orthonormal set of functions whose associated Dirichlet kernels are non-negative must be a system of step functions similar in structure to the classical Haar functions. The present paper discusses systems in which infinitely many of the kernels, but not necessarily all, are non-negative. It is shown that such systems also must be composed of step functions of a special type. As an application, a characterization of the Walsh functions is given in §4.

2. Definitions and preliminaries. It will be assumed throughout that \( \mu \) is a totally finite measure on a space \( S \) normalized so that \( \mu(S) = 1 \). All sets mentioned will be subsets of \( S \). All functions will be real-valued, bounded, and \( \mu \)-measurable. For the sake of brevity “almost everywhere” qualifications will be omitted. For example, two functions which differ only on a set of measure zero will be tacitly identified and “essential supremum” will be replaced by “supremum.”

A partition of \( S \) is a finite collection \( P \) of disjoint subsets of \( S \) whose union is \( S \). Given partitions \( P_1 \) and \( P_2 \), \( P_1 > P_2 \) if each element of \( P_2 \) is contained in an element of \( P_1 \) and if \( P_1 \) and \( P_2 \) are not identical. A function which is constant on each element of a partition \( P \) will be called a step function \((P)\).

**Lemma 1.** Let \( f(t) \) be a function defined on a set \( T \) of measure \( \mu_T \). If

\[
(a) \quad -M_2 \leq f(t) \leq M_1, \quad t \in T, \quad -M_2 < M_1,
\]

\[
(b) \quad \int_T f(t) d\mu(t) = I,
\]

then

\[
(c) \quad \int_T f^2(t) d\mu(t) \leq M_1 M_2 \mu_T + (M_1 - M_2)I.
\]

Equality holds in (c) if and only if

\[
(d) \quad f(t) = \begin{cases} M_1, & t \in T_1, \\ -M_2, & t \in T_2, \end{cases}
\]

where \( \{T_1, T_2\} \) is a partition of \( T \) such that

Presented to the Society September 3, 1959; received by the editors August 4, 1960.

(1) This work was supported by the National Science Foundation under research grant NSF G-8818.

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\[ \mu(T_1) = \frac{M_2 \mu_T + I}{M_1 + M_2}, \quad \mu(T_2) = \frac{M_2 \mu_T - I}{M_1 + M_2}. \]

The proof of the special case \( I = 0 \) is easy and is given in [2]. We shall omit it here. If \( I \neq 0 \), the lemma is obtained immediately by applying the special case to \( g(t) = f(t) - I \mu_T^{-1} \).

**Lemma 2.** Let \( f(t) \) be defined on a set \( T \) of measure \( \mu_T \). Suppose

(a) \(-M_2 \leq f(t) \leq M_1, \quad t \in T, -M_2 < M_1, \)

(b) \( \int_T f(t) d\mu(t) = 0, \)

(c) \( \int_T f^2(t) d\mu(t) \geq (M_1 - a/2)M_2 \mu_T, \quad 0 < a < M_1. \)

Then, if \( \sigma_a = \{ t \mid f(t) \geq M_1 - a \}, \)

(d) \( \mu(\sigma_a) \geq \frac{1}{2} \frac{M_2 \mu_T}{M_1 + M_2}. \)

**Proof.** On \( \sigma_a \), \( -M_2 \leq f(t) \leq M_1 \), and on the complement \( \sigma_a^c \), \(-M_2 \leq f(t) < M_1 - a \). Let \( I \) and \(-I\) denote the integrals of \( f(t) \) over \( \sigma_a \) and \( \sigma_a^c \) and let \( \mu_a = \mu(\sigma_a) \). Applying Lemma 1 twice,

\[ \int_{\sigma_a} f^2(t) d\mu(t) \leq M_1 (a - M_1) \mu_a + (2M_1 - a)I, \]

\[ \int_{\sigma_a^c} f^2(t) d\mu(t) \leq (M_1 - a)M_2 (\mu_T - \mu_a) + (M_1 - a - M_2) (-I). \]

Adding, and using assumption (c),

\[ \left( M_1 - \frac{a}{2} \right) M_2 \mu_T \leq \int_T f^2(t) d\mu(t) \]

\[ \quad \leq (M_1 - a)M_2 \mu_T + (M_1 + M_2)(a - M_1) \mu_a + (M_1 + M_2)I. \]

Subtracting \( (M_1 - a)M_2 \mu_T \),

\[ \frac{a}{2} M_2 \mu_T \leq (M_1 + M_2) \mu_a + (I - M_1 \mu_a) \].

Since \( 0 \leq I \leq M_1 \mu_a \), and \( M_1 + M_2 > 0 \) (because of (a) and (b)),

\[ \frac{a}{2} M_2 \mu_T \leq (M_1 + M_2) \mu_a \]
3. Main theorem. With any orthonormal set \( \{f_j(s)\}_{j=0}^\infty \) in \( L^2(S, \mu) \) are associated the Dirichlet kernels

\[
D_n(s, t) = \sum_{j=0}^{n-1} f_j(s)f_j(t), \quad n \geq 1.
\]

The following theorem gives the structure of an orthonormal set when infinitely many of these kernels are non-negative.

**Theorem 1.** Let \( \Phi = \{f_j(s)\}_{j=0}^\infty \) be an orthonormal set in \( L^2(S, \mu) \) with \( f_0(s) \equiv 1 \). Suppose there exists a sequence of integers \( 1 = n_0 < n_1 < n_2 < \cdots \) such that \( D_{n_r}(s, t) \geq 0, r \geq 0 \). Then \( \Phi \) is a system of step functions of the following type. There exists a sequence \( P_0 > P_1 > P_2 > \cdots \) of partitions of \( S, P_r = \{S_{r,i}\}_{i=1}^{n_r} \), such that if \( 0 \leq j < n_r \), then \( f_j(s) \) is a step function \( (P_r) \), \( r \geq 0 \). Furthermore,

\[
D_{n_r}(s, t) = \begin{cases} 
\phi_{r,i}(s, t) \\ 0,
\end{cases}
\quad (s, t) \in S_{r,i}^2, \quad 1 \leq i \leq n_r,
\]

otherwise,

where

\[
\phi_{r,i} = \frac{1}{\mu(S_{r,i})}, \quad 1 \leq i \leq n_r.
\]

**Proof.** It will be shown by induction that for each \( k \geq 0 \), there exists a sequence of partitions \( P_0 > P_1 > P_2 > \cdots > P_k \) of \( S \) such that, if \( 0 \leq r \leq k \),

(i) \( f_j(s) \) is a step function \( (P_r) \) when \( 0 \leq j < n_r \),

(ii) \( P_r \) has exactly \( n_r \) elements, \( \{S_{r,i}\}_{i=1}^{n_r} \),

(iii) (2) and (3) hold.

For \( k = 0 \), the situation is trivial. Since \( f_0(s) \equiv 1 \), \( D_{n_0}(s, t) \equiv 1 \). Taking \( P_0 \) to be the identity partition \( \{S\} = \{S_{0,1}\} \) all assertions are obviously true. Assuming the proposition is true for \( k \), we must construct a partition \( P_{k+1} < P_k \) and show (i), (ii), (iii) can be extended to \( P_{k+1} \).

For simplicity, let \( D_{n_k}(s, t) = \Delta_k(s, t) \). Define

\[
F_k(s, t) = \Delta_{k+1}(s, t) - \Delta_k(s, t) = \sum_{j=n_k}^{n_{k+1}-1} f_j(s)f_j(t).
\]

We shall show that \( F_k(s, t) = 0 \) wherever \( \Delta_k(s, t) = 0 \), namely outside the set

\[
\bigcup_{i=1}^{n_k} S_{k,i}.
\]

Let \( t_0 \) be arbitrary but fixed, say \( t_0 \in S_{k,i} \).
where the asterisk denotes complement. This follows from (4) since $\Delta_{k+1}(s, t) \geq 0$ and $\Delta_k(s, t_0) = 0$, $s \in S_{k,j}^*$, by the induction hypothesis (2). Also by (2), $\delta_{k,j}\Delta_k(s, t_0)$ is the characteristic function of $S_{k,j}$. Hence

$$\int_{S_{k,j}^*} F_k(s, t_0) d\mu(s) = \int_S - \int_{S_{k,j}}$$

Both integrals on the right side of (6) vanish because $F_k(s, t_0)$ is orthogonal to $f_0(s) = 1$ and to $\Delta_k(s, t_0)$. Therefore,

$$\int_{S_{k,j}} F_k(s, t_0) d\mu(s) = 0.$$

It follows from (5) and (7) that $F_k(s, t_0) = 0$ on $S_{k,j}^*$. Since $t_0$ was arbitrary, $F_k(s, t) = 0$ outside the sets $S_{k,j}^*$, $1 \leq i \leq n_k$.

We now begin the construction of the partition $P_{k+1}$. This will be done by partitioning separately each of the sets $S_{k,i}$, $1 \leq i \leq n_k$.

The quantity

$$F_k(t, t) = \sum_{j=n_k}^{n_{k+1}} f_j(t)$$

does not vanish identically on $S$. It is no loss of generality to assume $\sup_{S_{k,1}} F_k(t, t) = m_1 > 0$.

Let us carry out the partitioning of $S_{k,1}$ in detail. Henceforth, all points mentioned, unless otherwise stated, will be understood to belong to $S_{k,1}$.

The first step is to show that $F_k(t, t) = m_1$ on a set $T_1$ of positive measure. Choose a sequence $\{t_n\}_{n=1}^\infty$ such that $F_k(t_n, t_n) \geq m_1(1 - 1/2n)$. Then

$$\int_{S_{k,1}} F_k(s, t_n) d\mu(s) = 0,$$

$$\int_{S_{k,1}} F_k^2(s, t_n) d\mu(s) \geq m_1 \left(1 - \frac{1}{2n}\right).$$

The upper bound in (8a) is a consequence of the Schwarz inequality,

$$F_k(t, t) = \sum_{j=n_k}^{n_{k+1}} f_j(t) \leq (F_k(s, s) F_k(t, t))^{1/2}.$$
The lower bound in (8a) follows from (4) and the facts that $\Delta_k(s, t_n) \leq p_{k,1}$ by the induction hypothesis and that $\Delta_{k+1}(s, t_n) \geq 0$. (8b) is obtained from (7) since $F_k(s, t_n)$ is orthogonal (over $S$) to $f_0(s) \equiv 1$. Finally, using orthonormality, one obtains the general identity

\[(10) \int_{S_{k,1}} F_k(s, t_n) F_k(s, t_\beta) d\mu(s) = F_k(t_n, t_\beta).\]

Putting $t_n = t_\beta = t_n$ yields (8c).

Now (8a, b, c) are precisely the conditions needed to apply Lemma 2 to $F_k(s, t_n)$ with $T = S_{k,1}$, $\mu_T = p_{k,1}^{1/n}$, $M_1 = m_1$, $M_2 = p_{k,1}$ and $a = m_1/n$. We obtain

\[\mu(\sigma_n) \geq \frac{1}{2} \frac{1}{m_1 + p_{k,1}} = c_1\]

where $\sigma_n = \{ s \mid F_k(s, t_n) \geq m_1(1 - 1/n) \}$. Now let $\tau_n = \{ s \mid F_k(s, s) \geq m_1(1 - 1/n)^2 \}$. Then $\sigma_n \subseteq \tau_n$. For, by (9), if $s \in \sigma_n$

\[F_k(s, s) m_1 \geq F_k(s, s) F_k(t_n, t_n) \geq F_k(s, s) \geq m_1^2(1 - 1/n)^2.\]

Consequently $\mu(\tau_n) \geq \mu(\sigma_n) \geq c_1 > 0$, $n \geq 1$. If $T_1 = \{ s \mid F_k(s, s) = m_1 \}$ then $T_1 = \cap_{n=1}^{\infty} \tau_n$. But $\tau_1 \supset \tau_2 \supset \tau_3 \supset \cdots$. Therefore, $\mu(T_1) \geq c_1 > 0$.

We may now obtain some precise information about $F_k(s, t)$ as follows. Using (8a, b), we may apply Lemma 1 to $F_k(s, t)$ with $t$ fixed and obtain

\[\int_{S_{k,1}} F_k(s, t) d\mu(s) = m_1.\]

On the other hand, if $t \in T_1$, we have from (10)

\[\int_{S_{k,1}} F^2(s, t) d\mu(s) = F_k(t, t) = m_1.\]

Thus, $F_k(s, t)$ is an extremal function in the sense of Lemma 1 when $t \in T_1$. Therefore, for each $t \in T_1$, there exists a set $V(t)$ such that

\[(11) F_k(s, t) = \begin{cases} m_1, & s \in V(t), \\ -p_{k,1}, & \text{otherwise}, \end{cases}\]

where

\[\mu(V(t)) = \frac{1}{m_1 + p_{k,1}}. \tag{12}\]

We are going to show that there are only a finite number of distinct sets $V(t)$, that these form a partition $P(T_1)$ of $T_1$, and that the functions $f_j(s)$, $0 \leq j < n_{k+1}$, are step functions ($P(T_1)$).

Let $t \in T_1$ and $s \in V(t)$. From (9) and (11),
\begin{align*}
F_k(s, t) &= \left( \sum_{j=n_k}^{n_{k+1}-1} f_j(s) f_j(t) \right)^{1/2}.
\end{align*}

Consequently, for \( t \) fixed, there exists a proportionality factor \( \lambda(s) \) such that 
\( f_j(s) = \lambda(s) f_j(t) \) for all \( s \in V(t) \), \( n_k \leq j < n_{k+1} \). Therefore, \( F_k(s, t) = \lambda(s) F_k(t, t) \).

But \( F_k(s, t) = F_k(t, t) = m_1 \) for \( s \in V(t) \). Since \( V(t') \) is the set of measure \( m_1 + p_{k,i} \) on which \( F_k(s, t') = m_1 \), we have that \( V(t') = V(t) \). It follows that if \( t_1, t_2 \in T_1 \), then \( V(t_1) \) and \( V(t_2) \) are either disjoint or identical. In other words, the sets \( V(t) \) form a partition \( P(T_1) \) of \( T_1 \). Since they all have the same positive measure and \( \mu(T_1) \) is finite, \( P(T_1) \) is a finite partition. Let us call the elements of this partition \( S_{k+1,i}, 1 \leq i \leq q_1 \). Define 
\( p_{k+1,i} = \mu(S_{k+1,i})^{-1} = m_1 + p_{k,i} \).

To summarize, we have established the following facts. There exists a set of positive measure \( T_1 \) on which \( F_k(t, t) = m_1 \). There is a partition \( P(T_1) = \{ S_{k+1,i}\}_{i=1}^{q_1} \) of \( T_1 \) into sets of equal measure such that \( f_j(s) \) is a step function \( (P(T_1)) \) for \( 0 \leq j < n_{k+1} \). Furthermore,

\begin{align*}
F_k(s, t) &= \begin{cases} 
  m_1 = p_{k+1,i} - p_{k,i}, & (s, t) \in S_{k+1,i}, 1 \leq i \leq q_1, \\
  -p_{k,i}, & \text{elsewhere on } S_{k,1} \times T_1 \cup T_1 \times S_{k,1}.
\end{cases}
\end{align*}

Now \( \Delta_{k+1}(s, t) = \Delta_k(s, t) + F_k(s, t) \). We obtain from (14) and the induction hypothesis (2) that

\begin{align*}
\Delta_{k+1}(s, t) &= \begin{cases} 
  p_{k+1,i}, & (s, t) \in S_{k+1,i}, 1 \leq i \leq q_1, \\
  0, & \text{elsewhere on } S_{k,1} \times T_1 \cup T_1 \times S_{k,1},
\end{cases}
\end{align*}

where

\begin{align*}
p_{k+1,i} &= \frac{1}{\mu(S_{k+1,i})}.
\end{align*}

(15) and (16) represent a partial extension of (2) and (3) to the case \( r = k+1 \). If \( T_1 = S_{k,1} \) we have the desired partition of \( S_{k,1} \).

Suppose then, that \( T_1 \) is a proper subset of \( S_{k,1} \). Let \( m_2 = \sup F_k(t, t), \ t \in T_1 \). \( m_2 \) must be positive. For otherwise \( f_j(t) = 0 \) on \( T_1^* \), \( n_k \leq j < n_{k+1} \). Hence \( F_k(s, t) = 0 \) outside of \( T_1^* \) which contradicts (14). (Note that \( m_2 \leq m_1 \).)
Using the above arguments, we can easily establish the following. There exists a set \( T_2 \subset \mathbb{T}^*, \mu(T_2) \geq \alpha_3 = (1/2)(m_2 + \rho_k)^{-1} \), such that \( F_k(t, t) = m_2 \) on \( T_2 \). There is a partition \( P(T_2) = \{ S_{k+1,i} \}_{i=0}^{n_k+1} \) of \( T_2 \) into sets of equal measure such that the functions \( f_j(s) \), \( 0 \leq j < n_{k+1} \) are step functions \( (P(T_2)) \) and the analogues of (15) and (16) hold.

Continuing in this way, we obtain sets \( T_1, T_2, \ldots \) such that \( F_k(t, t) = m_2 > 0 \) on \( T_1 \), and \( \mu(T_n) \geq \alpha_n = 2^{-1}(m_n + \rho_k)^{-1} \). The process terminates after a finite number of steps since \( \mu(T_n) \geq 2^{-1}(m_n + \rho_k)^{-1} \geq 2^{-1}(m_1 + \rho_k)^{-1} = \alpha_1 \) while \( \mu(S_{k,1}) \) is finite. The sets \( T_k \) form a finite partition of \( S_{k,1} \). Each of these is partitioned in the same way as \( T_1 \). The result is a partition of \( S_{k,1} \) possessing all the required properties.

(We now drop the convention that all points named belong to \( S_{k,1} \).) Each of the sets \( S_{k,i}, 2 \leq i \leq n_k \), can be partitioned in the same way provided \( \sup S_{k,i} F_k(t, t) > 0 \). If \( \sup S_{k,i} F_k(t, t) = 0 \), then \( f_j(s) = 0 \) on \( S_{k,i} \), \( n_k \geq j < n_{k+1} \). In this case \( \Delta_{k+1}(s, t) = \Delta_k(s, t) \) on \( S_{k,i}^2 \) and (2) and (3) trivially carry over if we take the identity partition \( \{ S_{k,i} \} \).

Combining these partitions we obtain a partition \( P_{k+1} = \{ S_{k+1,i} \}_{i=1}^{N} \) of \( S \) such that \( P_{k+1} < P_k, f_j(s) \) is a step function \( (P_{k+1}) \) if \( 0 \leq j < n_{k+1} \), and

\[
\Delta_{k+1}(s, t) = \begin{cases} 
   p_{k+1,i} & (s, t) \in S_{k+1,i}, 1 \leq i \leq N, \\
   0, & \text{otherwise},
\end{cases}
\]

where

\[
p_{k+1,i} = \frac{1}{\mu(S_{k+1,i})}.
\]

To complete the induction, it remains only to show that \( N = n_{k+1} \). By orthonormality,

\[
\int_S \left( \int_S \Delta_{k+1}(s, t) d\mu(s) \right) d\mu(t) = n_{k+1}.
\]

On the other hand this integral is easily computed as a double integral from (17) and (18). Its value is

\[
\sum_{i=1}^{n_{k+1}} p_{k+1,i} \mu(S_{k+1,i})^2 = \sum_{i=1}^{n_{k+1}} 1 = N.
\]

Hence, \( N = n_{k+1} \).

It is worth noting the following facts, all of which follow directly from the proof of Theorem 1 but are not given in the statement of the theorem. \( P_{k+1} \) is obtained from \( P_k \) by partitioning each set \( S_{k,i} \) into two or more subsets unless \( F_k(t, t) = 0 \) on \( S_{k,i} \). In particular, if \( F_k(t, t) > 0 \) for all \( t \) and \( n_{k+1} = 2n_k \), then each element of \( P_k \) splits into exactly two parts. If \( n_{k+1} < 2n_k \), then
$F_k(t, t) \equiv 0$ on a set of positive measure. Finally we observe that if $F_k(t, t)$ is constant on $S_{k,i}$, then $S_{k,i}$ is partitioned into sets of equal measure.

4. An application to Walsh functions. In this section the unit interval \( \{x \mid 0 \leq x < 1\} \) will be denoted by $I$, the dyadic interval \( \{x \mid r \cdot 2^{-k} \leq x < (r + 1)2^{-k}\} \) by $I(r, k)$, and the dyadic partition \( \{I(r, k)\}_{r=0}^{2^k-1} \) of $I$ by $J_k$.

The Walsh functions(\textsuperscript{2}) are step functions related to the sequence of partitions $J_0 > J_1 > J_2 > \ldots$ in the sense of Theorem 1. This suggests a characterization of the Walsh system by its Dirichlet kernels.

Theorem 2. Let $\mathcal{F} = \{f_n(x)\}_{n=0}^{\infty}$ be an orthonormal set on $I$ with the following properties.

(a) $f_0(x) \equiv 1$.
(b) $D_k^2(x, y) \geq 0$, $k \geq 0$.
(c) For each $n \geq 0$, there is a partition $Q_n = \{Q_{n,i}\}_{i=0}^{n}$ of $I$ into $n+1$ sub-intervals on which $f_n(x)$ is alternately non-negative and non-positive.
(d) For each $n$, $\sup_{Q_{n,i}} |f_n(x)|$ is independent of $j$.

Then $\mathcal{F}$ is the set of Walsh functions.

Proof. By Theorem 1, assumptions (a) and (b) imply that $\mathcal{F}$ is a system of step functions relative to a sequence of partitions $P_0 > P_1 > P_2 > \ldots$ of $I$, $P_k$ having $2^k$ elements. It follows from (d) and the fact that $f_n(x)$ is normalized that $|f_n(x)| \equiv 1$. Consequently,

$$F_k(x, x) = \sum_{n=2^k}^{2^{k+1}-1} f_n(x) \equiv 2^k.$$  

By the remarks following Theorem 1, $P_{k+1}$ arises by splitting each element of $P_k$ into two subsets of equal measure. These must be intervals because of (c). Therefore, $\{P_k\}_{k=0}^{\infty}$ is the sequence of dyadic partitions $\{J_k\}_{k=0}^{\infty}$.

To complete the proof of Theorem 2, it suffices to prove the following assertion. If $\{f_n(x)\}_{n=0}^{2^k-1}$ is an orthonormal set of step functions $(J_k)$ satisfying (c) such that $|f_n(x)| \equiv 1$, $0 \leq n \leq 2^k-1$, then the given set is the set of Walsh functions $\{\psi_n(x)\}_{n=0}^{2^k-1}$ (in some order).

The proof is by induction. When $k = 0$, the assumptions imply $f_0(x) \equiv 1 \equiv \psi_0(x)$ and the assertion is true. Assuming it true for $k$, consider a set $\{f_n(x)\}_{n=0}^{2^{k+1}-1}$ satisfying the given conditions.

Let $2^k \leq n \leq 2^{k+1}-1$. We claim that on two successive intervals of the form $I(2r, k+1)$ and $I(2r+1, k+1)$, $f_n(x)$ takes values $\epsilon$ and $-\epsilon$ respectively ($\epsilon = \pm 1$). To see this let $\chi(x)$ be the characteristic function of $I(r, k) = I(2r, k+1) \cup I(2r+1, k+1)$. Since $\{f_i(x)\}_{i=0}^{2^k-1}$ is clearly a basis for the space of step functions $(J_k)$,

\textsuperscript{2} For particulars on the Walsh functions see [1].
for appropriate coefficients $a_j$. Suppose $f_n(x)$ takes the values $\epsilon$ and $\epsilon'$ on $I(2r, k+1)$ and $I(2r+1, k+1)$ respectively. Then by orthogonality,

$$0 = \sum_{j=0}^{2^k-1} a_j \int_I f_j(x) f_n(x) dx = \int_I \chi(x) f_n(x) dx = \int_{I(r,k)} f_n(x) dx = \frac{\epsilon + \epsilon'}{2^{k+1}}.$$ 

Hence $\epsilon' = -\epsilon$.

The above property is also possessed by the Rademacher function $\phi_k(x)$ defined by $\phi_k(x) \equiv (-1)^r$ on $I(r, k+1)$. Thus while $f_n(x)$ is a step function $(P_{k+1})$, the product $\phi_k(x) f_n(x)$ is a step function $(P_k)$.

Consider the functions \{ $g_{n'}(x) = \phi_k(x) f_{2^k+n'}(x)$ \}. We claim they satisfy all the conditions of our assertion. Since $|\phi_k(x)| = 1$, it is clear that $|g_{n'}(x)| = 1$ and that the $g_{n'}(x)$ form an orthonormal set of step functions $(P_k)$. It remains to show that (c) holds.

Let $n = 2^k + n'$ where $0 \leq n' \leq 2^k - 1$. From our assumptions $f_n(x)$ has $n + 1$ intervals of constancy, or equivalently, $n$ discontinuities. The latter occur among the dyadic rationals $r \cdot 2^{-(k+1)}$, $1 \leq r \leq 2^{k+1} - 1$. Multiplication by $\phi_k(x)$ removes these discontinuities (since $|f_n(x)| = 1$), but introduces new ones at the remaining dyadic rationals $r \cdot 2^{-(k+1)}$. Therefore, $g_{n'}(x) = \phi_k(x) f_n(x)$ has exactly $2^{k+1} - 1 - n = 2^k - 1 - n'$ jumps. Hence, the set \{ $g_{n'}(x)$ \}$_{n'=0}^{2^k-1}$ can be reordered so that (c) is satisfied (set $h_{n'}(x) = g_{2^k-1-n'}(x)$). By the induction hypothesis, this set as well as \{ $f_j(x)$ \}$_{j=0}^{2^k-1}$ is the set \{ $\psi_j(x)$ \}$_{j=0}^{2^k-1}$.

Since $\phi_k^{-1}(x) = \phi_k(x)$, $f_n(x) = \phi_k(x) g_{n'}(x)$. Thus, the set \{ $f_j(x)$ \}$_{j=0}^{2^k-1}$ is obtained when the Walsh functions \{ $\psi_j(x)$ \}$_{j=0}^{2^k-1}$ are multiplied by $\phi_k(x)$. But this is precisely the definition of the Walsh functions \{ $\psi_j(x)$ \}$_{j=0}^{2^k-1}$. Therefore \{ $f_j(x)$ \}$_{j=0}^{2^k-1}$ is the set of Walsh functions \{ $\psi_j(x)$ \}$_{j=0}^{2^k-1}$ (in some order) which completes the induction.

**References**


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