ORTHONORMAL SETS WITH NON-NEGATIVE DIRICHLET KERNELS. II

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1. Introduction. Recently the author has shown [2] that an orthonormal set of functions whose associated Dirichlet kernels are non-negative must be a system of step functions similar in structure to the classical Haar functions. The present paper discusses systems in which infinitely many of the kernels, but not necessarily all, are non-negative. It is shown that such systems also must be composed of step functions of a special type. As an application, a characterization of the Walsh functions is given in §4.

2. Definitions and preliminaries. It will be assumed throughout that \( \mu \) is a totally finite measure on a space \( S \) normalized so that \( \mu(S) = 1 \). All sets mentioned will be subsets of \( S \). All functions will be real-valued, bounded, and \( \mu \)-measurable. For the sake of brevity “almost everywhere” qualifications will be omitted. For example, two functions which differ only on a set of measure zero will be tacitly identified and “essential supremum” will be replaced by “supremum.”

A partition of \( S \) is a finite collection \( P \) of disjoint subsets of \( S \) whose union is \( S \). Given partitions \( P_1 \) and \( P_2 \), \( P_1 \supset P_2 \) if each element of \( P_2 \) is contained in an element of \( P_1 \) and if \( P_1 \) and \( P_2 \) are not identical. A function which is constant on each element of a partition \( P \) will be called a step function \( (P) \).

Lemma 1. Let \( f(t) \) be a function defined on a set \( T \) of measure \( \mu_T \). If
(a) \(-M_2 \leq f(t) \leq M_1, \quad t \in T, \quad -M_2 < M_1,\)
(b) \( \int_T f(t) d\mu(t) = I, \)
then
(c) \( \int_T f^2(t) d\mu(t) \leq M_1M_2\mu_T + (M_1 - M_2)I. \)

Equality holds in (c) if and only if
(d) \( f(t) = \begin{cases} M_1, & t \in T_1, \\ -M_2, & t \in T_2, \end{cases} \)

where \( \{T_1, T_2\} \) is a partition of \( T \) such that

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The proof of the special case $I = 0$ is easy and is given in [2]. We shall omit it here. If $I \neq 0$, the lemma is obtained immediately by applying the special case to $g(t) = f(t) - I \mu T^{-1}$.

**Lemma 2.** Let $f(t)$ be defined on a set $T$ of measure $\mu_T$. Suppose

(a) $-M_2 \leq f(t) \leq M_1$, $t \in T$, $-M_2 < M_1$,

(b) $\int_T f(t) d\mu(t) = 0$,

(c) $\int_T f^2(t) d\mu(t) \geq (M_1 - a/2)M_2 \mu_T$, $0 < a < M_1$.

Then, if $\sigma_a = \{ t \mid f(t) \geq M_1 - a \}$,

(d) $\mu(\sigma_a) \geq \frac{1}{2} \frac{M_2 \mu_T}{M_1 + M_2}$.

**Proof.** On $\sigma_a$, $-(a - M_1) \leq f(t) \leq M_1$, and on the complement $\sigma_a^c$, $-M_2 \leq f(t) < M_1 - a$. Let $I$ and $-I$ denote the integrals of $f(t)$ over $\sigma_a$ and $\sigma_a^c$ and let $\mu_a = \mu(\sigma_a)$. Applying Lemma 1 twice,

$$\int_{\sigma_a} f^2(t) d\mu(t) \leq M_1(a - M_1) \mu_a + (2M_1 - a)I,$$

$$\int_{\sigma_a^c} f^2(t) d\mu(t) \leq (M_1 - a)M_2(\mu_T - \mu_a) + (M_1 - a - M_2)(-I).$$

Adding, and using assumption (c),

$$\left( M_1 - \frac{a}{2} \right) M_2 \mu_T \leq \int_T f^2(t) d\mu(t) \leq (M_1 - a)M_2 \mu_T + (M_1 + M_2)(a - M_1) \mu_a + (M_1 + M_2)I.$$

Subtracting $(M_1 - a)M_2 \mu_T$,

$$\frac{a}{2} M_2 \mu_T \leq (M_1 + M_2)[a \mu_a + (I - M_1 \mu_a)].$$

Since $0 \leq I \leq M_2 \mu_a$, and $M_1 + M_2 > 0$ (because of (a) and (b)),

$$\frac{a}{2} M_2 \mu_T \leq (M_1 + M_2)a \mu_a.$$
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or,

\[ \frac{1}{2} \frac{M_{2uT}}{M_1 + M_2} \leq \mu_a. \]

3. Main theorem. With any orthonormal set \( \{f_j(s)\}_{j=0}^\infty \) in \( L^2(S, \mu) \) are associated the Dirichlet kernels

\[ D_n(s, t) = \sum_{j=0}^{n-1} f_j(s)f_j(t), \quad n \geq 1. \]

The following theorem gives the structure of an orthonormal set when infinitely many of these kernels are non-negative.

Theorem 1. Let \( \mathcal{F} = \{f_j(s)\}_{j=0}^\infty \) be an orthonormal set in \( L^2(S, \mu) \) with \( f_0(s) = 1 \). Suppose there exists a sequence of integers \( 1 = n_0 < n_1 < n_2 < \cdots \) such that \( D_{n_r}(s, t) \geq 0, \ r \geq 0 \). Then \( \mathcal{F} \) is a system of step functions of the following type. There exists a sequence of partitions \( P_r = \{S_{r,i}\}_{i=1}^{n_r} \), such that if \( 0 \leq j \leq n_r \), then \( f_j(s) \) is a step function \( (P_r), \ r \geq 0 \). Furthermore,

\[ D_{n_r}(s, t) = \begin{cases} p_{r,i}, & (s, t) \in S_{r,i}, 1 \leq i \leq n_r, \\ 0, & \text{otherwise}, \end{cases} \]

where

\[ p_{r,i} = \frac{1}{\mu(S_{r,i})}, \quad 1 \leq i \leq n_r. \]

Proof. It will be shown by induction that for each \( k \geq 0 \), there exists a sequence of partitions \( P_0 > P_1 > P_2 > \cdots > P_k \) of \( S \) such that, if \( 0 \leq r \leq k \),

(i) \( f_j(s) \) is a step function \( (P_r) \) when \( 0 \leq j < n_r \),

(ii) \( P_r \) has exactly \( n_r \) elements, \( \{S_{r,i}\}_{i=1}^{n_r} \),

(iii) (2) and (3) hold.

For \( k = 0 \), the situation is trivial. Since \( f_0(s) = 1, \ D_{n_0}(s, t) = 1 \). Taking \( P_0 \) to be the identity partition \( \{S\} = \{S_{0,1}\} \) all assertions are obviously true. Assuming the proposition is true for \( k \), we must construct a partition \( P_{k+1} < P_k \) and show (i), (ii), (iii) can be extended to \( P_{k+1} \).

For simplicity, let \( D_{n_k}(s, t) = \Delta_k(s, t) \). Define

\[ F_k(s, t) = \Delta_{k+1}(s, t) - \Delta_k(s, t) = \sum_{j=n_k}^{n_{k+1}-1} f_j(s)f_j(t). \]

We shall show that \( F_k(s, t) = 0 \) wherever \( \Delta_k(s, t) = 0 \), namely outside the set

\[ \bigcup_{i=1}^{n_k} S_{k,i}. \]

Let \( t_0 \) be arbitrary but fixed, say \( t_0 \in S_{k,j} \).
where the asterisk denotes complement. This follows from (4) since \( \Delta_{k+1}(s, t) \geq 0 \) and \( \Delta_k(s, t_0) \equiv 0, s \in S_{k,j}^*, \) by the induction hypothesis (2). Also by (2), \( \tilde{p}_{k,j} \Delta_k(s, t_0) \) is the characteristic function of \( S_{k,j}. \) Hence

\[
(6) \quad \int_{S_{k,j}^*} f_k(s, t_0) d\mu(s) = \int_S - \int_{S_{k,j}} f_k(s, t_0) d\mu(s) = \int_S f_k(s, t_0) d\mu(s) - \frac{1}{\tilde{p}_{k,j}} \int_S \Delta_k(s, t_0) f_k(s, t_0) d\mu(s).
\]

Both integrals on the right side of (6) vanish because \( f_k(s, t_0) \) is orthogonal to \( f_0(s) \equiv 1 \) and to \( \Delta_k(s, t_0). \) Therefore,

\[
(7) \quad \int_{S_{k,j}^*} f_k(s, t_0) d\mu(s) = 0. \]

It follows from (5) and (7) that \( f_k(s, t_0) \equiv 0 \) on \( S_{k,j}^* \). Since \( t_0 \) was arbitrary, \( f_k(s, t) \equiv 0 \) outside the sets \( S_{k,j}^*, 1 \leq i \leq n_k. \)

We now begin the construction of the partition \( P_{k+1}. \) This will be done by partitioning separately each of the sets \( S_{k,i}, 1 \leq i \leq n_k. \)

The quantity

\[
F_k(t, t) = \sum_{j=n_k}^{n_k+1-1} f_j(t)
\]

does not vanish identically on \( S. \) It is no loss of generality to assume \( \sup_{S_{k,1}} F_k(t, t) = m_1 > 0. \)

Let us carry out the partitioning of \( S_{k,1} \) in detail. Henceforth, all points mentioned, unless otherwise stated, will be understood to belong to \( S_{k,1}. \)

The first step is to show that \( F_k(t, t) = m_1 \) on a set \( T_1 \) of positive measure. Choose a sequence \( \{t_n\}_{n=1}^\infty \) such that \( F_k(t_n, t_n) \geq m_1(1 - 1/2n). \) Then

\[
(8a) \quad -m_1 \leq F_k(s, t_n) \leq m_1,
\]

\[
(8b) \quad \int_{S_{k,1}} F_k(s, t_n) d\mu(s) = 0,
\]

\[
(8c) \quad \int_{S_{k,1}} F_k^2(s, t_n) d\mu(s) \geq m_1 \left( 1 - \frac{1}{2n} \right).
\]

The upper bound in (8a) is a consequence of the Schwarz inequality,

\[
F_k(s, t) = \sum_{j=n_k}^{n_k+1-1} f_j(s) f_j(t) \leq (F_k(s, s) F_k(t, t))^{1/2}.
\]
The lower bound in (8a) follows from (4) and the facts that \( \Delta_k(s, t_n) \leq p_{k,1} \) by the induction hypothesis and that \( \Delta_{k+1}(s, t_n) \geq 0 \). (8b) is obtained from (7) since \( F_k(s, t_n) \) is orthogonal (over \( S \)) to \( f_0(s) = 1 \). Finally, using orthonormality, one obtains the general identity

\[
(10) \quad \int_{S_{k,i}} F_k(s, t_a)F_k(s, t_b)d\mu(s) = F_k(t_a, t_b).
\]

Putting \( t_a = t_b = t_n \) yields (8c).

Now (8a, b, c) are precisely the conditions needed to apply Lemma 2 to \( F_k(s, t_n) \) with \( T = S_{k,i}, \mu_T = \mu_{k,1}, M_1 = m_1, M_2 = p_{k,1} \) and \( a = m_1/n \). We obtain

\[
\text{no} = \frac{1}{n} - \frac{1}{m_1 + p_{k,1}} = c_i
\]

where \( \sigma_n = \{ s \mid F_k(s, t_n) \geq m_1(1-1/n) \} \). Now let \( \tau_n = \{ s \mid F_k(s, s) \geq m_1(1-1/n)^2 \} \). Then \( \sigma_n \subseteq \tau_n \). For, by (9), if \( s \in \sigma_n \)

\[
\mu(\sigma_n) \geq \mu(\tau_n) \geq c_i > 0, \quad n \geq 1.
\]

We may now obtain some precise information about \( F_k(s, t) \) as follows. Using (8a, b), we may apply Lemma 1 to \( F_k(s, t) \) with \( t \) fixed and obtain

\[
\int_{S_{k,i}} F_k(s, t)d\mu(s) = m_1.
\]

On the other hand, if \( t \in T_1 \), we have from (10)

\[
\int_{S_{k,i}} F^2(s, t)d\mu(s) = F_k(t, t) = m_1.
\]

Thus, \( F_k(s, t) \) is an extremal function in the sense of Lemma 1 when \( t \in T_1 \). Therefore, for each \( t \in T_1 \), there exists a set \( V(t) \) such that

\[
(11) \quad F_k(s, t) = \begin{cases} m_1, & s \in V(t), \\ -p_{k,1}, & \text{otherwise,} \end{cases}
\]

where

\[
(12) \quad \mu(V(t)) = \frac{1}{m_1 + p_{k,1}}.
\]

We are going to show that there are only a finite number of distinct sets \( V(t) \), that these form a partition \( P(T_1) \) of \( T_1 \), and that the functions \( f_j(s) \), \( 0 \leq j < n_{k+1} \), are step functions \( (P(T_1)) \).

Let \( t \in T_1 \) and \( s \in V(t) \). From (9) and (11),
(13) \[ m_1 = F_k(s, t) \leq (F_k(s, s)F_k(t, t))^{1/2} \leq m_1. \]

Hence, \( F_k(s, s) = m_1 \) which shows that \( V(t) \subseteq T_1 \). Since \( t \in V(t) \), \( T_1 = \bigcup_{t \in T_1} V(t) \).

Also from (13),

\[
\sum_{j=m_k}^{m_{k+1}-1} f_j(s)f_j(t) = \left( \sum_{j=m_k}^{m_{k+1}-1} f_j(s) \sum_{j=m_k}^{m_{k+1}-1} f_j(t) \right)^{1/2}.
\]

Consequently, for \( t \) fixed, there exists a proportionality factor \( \lambda(s) \) such that \( f_j(s) = \lambda(s)f_j(t) \) for all \( s \in V(t) \), \( n_k \leq j < n_{k+1} \). Therefore, \( F_k(s, t) = \lambda(s)F_k(t, t) \).

But \( F_k(s, t) = F_k(t, t) = m_1 \), \( s \in V(t) \), by (11). It follows that \( \lambda(s) = 1 \) on \( V(t) \).

Therefore, the functions \( f_j(s) \), \( n_k \leq j < n_{k+1} \), are constant on \( V(t) \). (By the induction hypothesis, this is also true for \( 0 \leq j < n_k \) since \( f_j(s) \) is then constant on the superset \( S_{k,1} \).)

It is now clear that \( F_k(s, \tau) = F_k(t, t) = m_1 \), \( s, \tau \in V(t) \). Hence, if \( t' \in V(t) \), \( F_k(s, t') = m_1 \) for \( s \in V(t) \). Since \( V(t') \) is that set of measure \( (m_1 + p_{k,1})^{-1} \) on which \( F_k(s, t') = m_1 \), we have that \( V(t') = V(t) \). It follows that if \( t_1, t_2 \in T_1 \), then \( V(t_1) \) and \( V(t_2) \) are either disjoint or identical. In other words, the sets \( V(t) \) form a partition \( P(T_1) \) of \( T_1 \). Since they all have the same positive measure and \( \mu(T_1) \) is finite, \( P(T_1) \) is a finite partition. Let us call the elements of this partition \( S_{k+1,i}, 1 \leq i \leq q_1 \). Define \( p_{k+1,i} = \mu(S_{k+1,i})^{-1} = m_1 + p_{k,1} \).

To summarize, we have established the following facts. There exists a set of positive measure \( T_1 \) on which \( F_k(t, t) = m_1 \). There is a partition \( P(T_1) = \{ S_{k+1,i} \}_{i=1}^{q_1} \) of \( T_1 \) into sets of equal measure such that \( f_j(s) \) is a step function \( (P(T_1)) \) for \( 0 \leq j < n_{k+1} \). Furthermore,

(14) \[ F_k(s, t) = \begin{cases} m_1 - p_{k+1,i} & (s, t) \in S_{k+1,i}, 1 \leq i \leq q_1, \\ -p_{k,1}, & \text{elsewhere on } S_{k,1} \times T_1 \cup T_1 \times S_{k,1}. \end{cases} \]

Now \( \Delta_{k+1}(s, t) = \Delta_k(s, t) + F_k(s, t) \). We obtain from (14) and the induction hypothesis (2) that

(15) \[ \Delta_{k+1}(s, t) = \begin{cases} p_{k+1,i} \mu(S_{k+1,i})^{-1}, & (s, t) \in S_{k+1,i}, 1 \leq i \leq q_1, \\ 0, & \text{elsewhere on } S_{k,1} \times T_1 \cup T_1 \times S_{k,1}. \end{cases} \]

where

(16) \[ p_{k+1,i} = \frac{1}{\mu(S_{k+1,i})}. \]

(15) and (16) represent a partial extension of (2) and (3) to the case \( r = k+1 \). If \( T_1 = S_{k,1} \) we have the desired partition of \( S_{k,1} \).

Suppose then, that \( T_1 \) is a proper subset of \( S_{k,1} \). Let \( m_2 = \sup F_k(t, t), t \in T_1^2 \). \( m_2 \) must be positive. For otherwise \( f_j(t) = 0 \) on \( T_1^2 \), \( n_k \leq j < n_{k+1} \). Hence \( F_k(s, t) = 0 \) outside of \( T_1^2 \) which contradicts (14). (Note that \( m_2 \leq m_1 \).)
Using the above arguments, we can easily establish the following. There exists a set \( T_1 \subset T_2^* \), \( \mu(T_1) \geq c_1 = (1/2)(m_2 + p_{k,1})^{-1} \), such that \( F_k(t, t) = m_2 \) on \( T_1 \). There is a partition \( P(T_2) = \{ S_{k+1, i} \}_{i=1}^{n_{k+1}} \) of \( T_2 \) into sets of equal measure such that the functions \( f_j(s) \), \( 0 \leq j < n_{k+1} \) are step functions (\( P(T_2) \)) and the analogues of (15) and (16) hold.

Continuing in this way, we obtain sets \( T_1, T_2, \ldots \) such that \( F_k(t, t) = m_2 > 0 \) on \( T_1 \), and \( \mu(T_i) \geq c_i = 2^{-i}(m_2 + p_{k,1})^{-1} \). The process terminates after a finite number of steps since \( \mu(T_i) \geq 2^{-i}(m_2 + p_{k,1})^{-1} \geq 2^{-i}(m_1 + p_{k,1})^{-1} = c_i \) while \( \mu(S_{k,i}) \) is finite. The sets \( T_i \) form a finite partition of \( S_{k,i} \). Each of these is partitioned in the same way as \( T_1 \). The result is a partition of \( S_{k,i} \) possessing all the required properties.

(We now drop the convention that all points named belong to \( S_{k,i} \).) Each of the sets \( S_{k,i} \), \( 2 \leq i \leq n_k \), can be partitioned in the same way provided \( \sup S_{k,i} F_k(t, t) > 0 \). If \( \sup S_{k,i} F_k(t, t) = 0 \), then \( f_j(s) = 0 \) on \( S_{k,i} \), \( n_k \leq j < n_{k+1} \). In this case \( \Delta_{k+1}(s, t) = \Delta_k(s, t) \) on \( S_{k,i} \), and (2) and (3) trivially carry over if we take the identity partition \( \{ S_{k,i} \} \).

Combining these partitions we obtain a partition \( P_{k+1} = \{ S_{k+1, i} \}_{i=1}^{n_{k+1}} \) of \( S \) such that \( P_{k+1} < P_k \), \( f_j(s) \) is a step function \( (P_{k+1}) \) if \( 0 \leq j < n_{k+1} \), and

\[
\Delta_{k+1}(s, t) = \begin{cases} 
\hat{p}_{k+1,i}, & (s, t) \in S_{k+1,i}, 1 \leq i \leq N, \\
0, & \text{otherwise,}
\end{cases}
\]

where

\[
\hat{p}_{k+1,i} = \frac{1}{\mu(S_{k+1,i})}.
\]

To complete the induction, it remains only to show that \( N = n_{k+1} \). By orthonormality,

\[
\int_S \left( \int_S \Delta_{k+1}(s, t) d\mu(s) \right) d\mu(t) = n_{k+1}.
\]

On the other hand this integral is easily computed as a double integral from (17) and (18). Its value is

\[
\sum_{i=1}^{n_{k+1}} \hat{p}_{k+1,i} \mu(S_{k+1,i})^2 = \sum_{i=1}^{n_{k+1}} 1 = N.
\]

Hence, \( N = n_{k+1} \).

It is worth noting the following facts, all of which follow directly from the proof of Theorem 1 but are not given in the statement of the theorem. \( P_{k+1} \) is obtained from \( P_k \) by partitioning each set \( S_{k,i} \) into two or more subsets unless \( F_k(t, t) = 0 \) on \( S_{k,i} \). In particular, if \( F_k(t, t) > 0 \) for all \( t \) and \( n_{k+1} = 2n_k \), then each element of \( P_k \) splits into exactly two parts. If \( n_{k+1} < 2n_k \), then
$F_k(t, t) \equiv 0$ on a set of positive measure. Finally we observe that if $F_k(t, t)$ is constant on $S_{k,i}$, then $S_{k,i}$ is partitioned into sets of equal measure.

4. **An application to Walsh functions.** In this section the unit interval \( \{ x \mid 0 \leq x < 1 \} \) will be denoted by \( I \), the dyadic interval \( \{ x \mid r \cdot 2^{-k} \leq x < (r + 1)2^{-k} \} \) by \( I(r, k) \), and the dyadic partition \( \{ I(r, k) \}_{r=0}^{2^k-1} \) of \( I \) by \( J_k \).

The Walsh functions(2) are step functions related to the sequence of partitions \( \mathcal{J}_0 = \mathcal{J}_1 = \mathcal{J}_2 = \cdots \) in the sense of Theorem 1. This suggests a characterization of the Walsh system by its Dirichlet kernels.

**Theorem 2.** Let \( \mathcal{F} = \{ f_n(x) \}_{n=0}^{\infty} \) be an orthonormal set on \( I \) with the following properties.

(a) \( f_0(x) = 1 \).

(b) \( D_2^n(x, y) \geq 0, k \geq 0 \).

(c) For each \( n \geq 0 \), there is a partition \( Q_n = \{ Q_n, j \}_{j=0}^{n} \) of \( I \) into \( n+1 \) sub-intervals on which \( f_n(x) \) is alternately non-negative and non-positive.

\( f_n(x) \) is non-negative on the sub-interval containing \( 0 \).

(d) For each \( n \), \( \sup_{Q_n} f_n(x) \) is independent of \( j \).

Then \( \mathcal{F} \) is the set of Walsh functions.

**Proof.** By Theorem 1, assumptions (a) and (b) imply that \( \mathcal{F} \) is a system of step functions relative to a sequence of partitions \( P_0 > P_1 > P_2 > \cdots \) of \( I \), \( P_k \) having \( 2^k \) elements. It follows from (d) and the fact that \( f_n(x) \) is normalized that \( |f_n(x)| = 1 \). Consequently,

$$ F_k(x, x) = \sum_{n=2^k}^{2^{k+1}-1} f_n(x) = 2^k. $$

By the remarks following Theorem 1, \( P_{k+1} \) arises by splitting each element of \( P_k \) into two subsets of equal measure. These must be intervals because of (c). Therefore, \( \{ P_k \}_{k=0}^{\infty} \) is the sequence of dyadic partitions \( \{ J_k \}_{k=0}^{\infty} \).

To complete the proof of Theorem 2, it suffices to prove the following assertion. If \( \{ f_n(x) \}_{n=0}^{2^k-1} \) is an orthonormal set of step functions \( (J_k) \) satisfying (c) such that \( |f_n(x)| = 1, 0 \leq n \leq 2^k - 1 \), then the given set is the set of Walsh functions \( \{ \psi_n(x) \}_{n=0}^{2^k-1} \) (in some order).

The proof is by induction. When \( k = 0 \), the assumptions imply \( f_0(x) = 1 \) \( \equiv \psi_0(x) \) and the assertion is true. Assuming it true for \( k \), consider a set \( \{ f_n(x) \}_{n=0}^{2^k-1} \) satisfying the given conditions.

Let \( 2^k \leq n \leq 2^{k+1} - 1 \). We claim that on two successive intervals of the form \( I(2r, k+1) \) and \( I(2r+1, k+1) \), \( f_n(x) \) takes values \( \epsilon \) and \( -\epsilon \) respectively \( (\epsilon = \pm 1) \). To see this let \( \chi(x) \) be the characteristic function of \( I(r, k) = I(2r, k+1) \cup I(2r+1, k+1) \). Since \( \{ f_i(x) \}_{i=0}^{2^k-1} \) is clearly a basis for the space of step functions \( (J_k) \),

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(2) For particulars on the Walsh functions see [1].
\[ \chi(x) = \sum_{j=0}^{2^k-1} a_j f_j(x) \]

for appropriate coefficients \( a_j \). Suppose \( f_n(x) \) takes the values \( \epsilon \) and \( \epsilon' \) on \( I(2r, k+1) \) and \( I(2r+1, k+1) \) respectively. Then by orthogonality,

\[
0 = \sum_{j=0}^{2^k-1} a_j \int_I f_j(x) f_n(x) \, dx = \int_I \chi(x) f_n(x) \, dx = \int_{I(r,k)} f_n(x) \, dx = \frac{\epsilon + \epsilon'}{2^{k+1}}.
\]

Hence \( \epsilon' = -\epsilon \).

The above property is also possessed by the Rademacher function \( \phi_k(x) \) defined by \( \phi_k(x) \equiv (-1)^r \) on \( I(r, k+1) \). Thus while \( f_n(x) \) is a step function \( (P_{k+1}) \), the product \( \phi_k(x) f_n(x) \) is a step function \( (P_k) \).

Consider the functions \( \{ g_n'(x) = \phi_k(x) f_{2^k+n'}(x) \}_{n'=0}^{2^k-1} \). We claim they satisfy all the conditions of our assertion. Since \( \| \phi_k(x) \| = 1 \), it is clear that \( \| g_n'(x) \| = 1 \) and that the \( g_n'(x) \) form an orthonormal set of step functions \( (P_k) \). It remains to show that (c) holds.

Let \( n = 2^k + n' \) where \( 0 \leq n' \leq 2^k - 1 \). From our assumptions \( f_n(x) \) has \( n+1 \) intervals of constancy, or equivalently, \( n \) discontinuities. The latter occur among the dyadic rationals \( r \cdot 2^{-(k+1)} \), \( 1 \leq r \leq 2^{k+1} - 1 \). Multiplication by \( \phi_k(x) \) removes these discontinuities (since \( \| f_n(x) \| = 1 \)), but introduces new ones at the remaining dyadic rationals \( r \cdot 2^{-(k+1)} \). Therefore, \( g_n'(x) = \phi_k(x) f_n(x) \) has exactly \( 2^{k+1} - 1 - n = 2^k - 1 - n' \) jumps. Hence, the set \( \{ g_n'(x) \}_{n'=0}^{2^k-1} \) can be re-ordered so that (c) is satisfied (set \( h_n(x) = g_{2^k-1-n'}(x) \)). By the induction hypothesis, this set as well as \( \{ f_j(x) \}_{j=0}^{2^k-1} \) is the set \( \{ \psi_j(x) \}_{j=0}^{2^k-1} \).

Since \( \phi_k^{-1}(x) = \phi_k(x), f_n(x) = \phi_k(x) g_n(x) \). Thus, the set \( \{ f_j(x) \}_{j=0}^{2^k-1} \) is obtained when the Walsh functions \( \{ \psi_j(x) \}_{j=0}^{2^k-1} \) are multiplied by \( \phi_k(x) \). But this is precisely the definition of the Walsh functions \( \{ \psi_j(x) \}_{j=0}^{2^k-1} \). Therefore \( \{ f_j(x) \}_{j=0}^{2^k-1} \) is the set of Walsh functions \( \{ \psi_j(x) \}_{j=0}^{2^k-1} \) (in some order) which completes the induction.

**REFERENCES**


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