ON DIFFERENTIABLY SIMPLE ALGEBRAS(1)

BY

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1. **Introduction.** It is known (see Albert [1]) that every simple commutative power-associative algebra of degree $t>2$ over an algebraically closed field $\mathfrak{F}$ of characteristic $p>5$ is a Jordan algebra. Moreover, in the partially stable case, a characterization of the simple algebras of degree two is given by Albert in [3]. In his theory Albert expresses the structure of simple partially stable algebras in terms of certain commutative associative algebras $\mathfrak{B}$ over $\mathfrak{F}$. These commutative associative algebras have unity elements, and each algebra $\mathfrak{B}$ is differentiably simple relative to some set of derivations of $\mathfrak{B}$ over $\mathfrak{F}$. In this paper we shall determine the structure of the algebras $\mathfrak{B}$ and derive a property of simple partially stable algebras which follows from Albert's characterization.

Let $\mathfrak{B}$ be a commutative associative algebra with unity element e over $\mathfrak{F}$. We shall now define a commutative power-associative algebra $\mathfrak{T}$ over $\mathfrak{F}$ which is the essential subalgebra of a partially stable commutative power-associative algebra $\mathfrak{S}$ as defined by Albert in [3]. Let $m\geqq 2$ and let $y_i\mathfrak{B}$ denote a homomorphic image of the vector space $\mathfrak{B}$ for $i=0,\cdots,m$. Then $\mathfrak{T}$ will be the vector space direct sum

$$\mathfrak{T}=\mathfrak{B}+\mathfrak{L} \tag{1}$$

where $\mathfrak{L}$ is the sum, not necessarily direct, of the component spaces $y_0\mathfrak{B},\cdots$, $y_m\mathfrak{B}$. Select elements b_{ij} in $\mathfrak{B}$ and derivations D_{ij} of $\mathfrak{B}$ over $\mathfrak{F}$ such that

$$b_{ij}=b_{ji},\qquad b_{00}=e,\qquad b_{0j}=0\qquad (j\neq 0), \tag{2}$$

$$D_{ij}=-D_{ji} \tag{3}$$

for $i,j=0,\cdots,m$ where then $D_{ii}=0$ for $i=0,\cdots,m$. We now define products in $\mathfrak{T}$ by assuming that $\mathfrak{B}$ is a subalgebra of $\mathfrak{T}$, that

$$(y_ia)b=y_i(ab)=b(y_ia)\qquad (i=0,\cdots,m) \tag{4}$$

for all elements a and b of $\mathfrak{B}$, and finally that

$$(y_ia)(y_jb)=b_{ij}ab+(aD_{ij})b-a(bD_{ij}) \tag{5}$$

for all a and b of $\mathfrak{B}$ and $i,j=0,\cdots,m$. The result will be a commutative power-associative algebra of degree two over $\mathfrak{F}$.

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We shall also require that the b_{ij} and the D_{ij} be chosen so that:

(A) The algebra $\mathfrak{B}$ is $\{D_{ij}\}$-simple.

(B) If g is in $\mathfrak{L}$ and $gu = 0$ for all u in $\mathfrak{L}$, then $g = 0$.

It is one of the principal results of Albert in [3] that these conditions are equivalent to the simplicity of the partially stable algebra $\mathfrak{S}$ mentioned above. It is known [2] that condition (A) implies that

$$\mathfrak{B} = e\mathfrak{F} + \mathfrak{N} \tag{6}$$

where $\mathfrak{N}$ is the radical of $\mathfrak{B}$ and $x^p=0$ for every element x in $\mathfrak{N}$. We shall completely determine the structure of $\mathfrak{B}$, and we state our main result as

THEOREM 1. *Let $\mathfrak{B}$ be a commutative associative algebra with unity e over an algebraically closed field $\mathfrak{F}$, and let $\mathfrak{B}$ be differentiably simple relative to a set of derivations of $\mathfrak{B}$ over $\mathfrak{F}$. Then $\mathfrak{B}=\mathfrak{F}[e, x_1, \cdots, x_n]$ is an algebra with generators $x_1, \cdots, x_n$ over $\mathfrak{F}$ which are independent except for the relations $x_1^p= \cdots =x_n^p=0$ where $p>0$ is the characteristic of $\mathfrak{F}$.*

In all examples of the algebras $\mathfrak{T}$ given to date the space $\mathfrak{L}$ has been a direct sum of the components $y_0\mathfrak{B}, \cdots, y_m\mathfrak{B}$. As our final result we shall construct a class of examples of the algebras $\mathfrak{T}$ in which $\mathfrak{L}=(y_0\mathfrak{B}, \cdots, y_m\mathfrak{B})$ with $m=2$ and $\mathfrak{L}$ is not a direct sum and cannot be represented as a direct sum in this manner.

2. **The algebra** $\mathfrak{B}$. Let $\mathfrak{B}$ be a commutative associative algebra with unity element $\bar{e}$ over $\mathfrak{F}$, and let $\mathfrak{B}$ be $\overline{\mathfrak{D}}$-simple for some set $\overline{\mathfrak{D}}$ of derivations of $\mathfrak{B}$ over $\mathfrak{F}$. Then by (6) we may write

$$\mathfrak{B} = \bar{e}\mathfrak{F} + \overline{\mathfrak{N}} \tag{7}$$

where $x^p=0$ for each x in $\overline{\mathfrak{N}}$. The algebra $\mathfrak{B}$, being finite dimensional, is finitely generated. Let $\{\bar{e}, \bar{x}_1, \cdots, \bar{x}_n\}$ be a set of generators of $\mathfrak{B}$ which is minimal in the sense that no set containing $\bar{e}$ and having fewer elements generates $\mathfrak{B}$. Also let

$$\mathfrak{A} = \mathfrak{F}[e, x_1, \cdots, x_n]$$

be the commutative associative algebra generated over $\mathfrak{F}$ by generators $e, x_1, \cdots, x_n$ which are independent except for the relations $e^2=e$, $ex_i=x_i$, and $x_i^p=0$ which hold for $i=1, \cdots, n$. It is clear that the mappings $e\rightarrow\bar{e}$, $x_i\rightarrow\bar{x}_i$ $(i=1, \cdots, n)$ define a homomorphism ϕ of $\mathfrak{A}$ onto $\mathfrak{B}$. We let $\mathfrak{M}$ be the kernel of ϕ. We see that Theorem 1 will be proved if we can show that $\mathfrak{M}=0$.

We now note some properties of $\mathfrak{A}$. We may write

$$\mathfrak{A} = e\mathfrak{F} + \mathfrak{N} \tag{8}$$

where $\mathfrak{N}=\mathfrak{F}[x_1, \cdots, x_n]$ is the radical of $\mathfrak{A}$ and consists of all polynomials in the x_i with constant term zero. We observe that every element of $\mathfrak{A}$ which is not in $\mathfrak{N}$ has an inverse. For if $a=\alpha+u$ with α in $\mathfrak{F}$, u in $\mathfrak{N}$, $\alpha\neq 0$, then $a^{-1}=(\alpha^p)^{-1}(\alpha+u)^{p-1}$. Also it is known [4] that the derivation algebra of $\mathfrak{A}$ consists of all linear transformations $D=D(a_1, \cdots, a_n)$ of $\mathfrak{A}$ defined by

$$aD = (\partial a/\partial x_1)a_1 + \cdots + (\partial a/\partial x_n)a_n \tag{9}$$

where $a_1, \cdots, a_n$ are in $\mathfrak{A}$ and $\partial a/\partial x_i$ denotes the ordinary partial derivative of the polynomial a with respect to x_i ($i=1, \cdots, n$). Thus $x_iD=a_i$ and the derivations of $\mathfrak{A}$ are completely determined by the images of the x_i and these images may be arbitrarily chosen.

THEOREM 2. *Let D be a derivation of $\mathfrak{A}$. Then the transformation $\overline{D}$ defined by*

$$\varphi(u)\overline{D} = \phi(uD) \tag{10}$$

is a derivation of $\mathfrak{B}$ if and only if $\mathfrak{M}D\subseteq\mathfrak{M}$. Moreover, every derivation of $\mathfrak{B}$ is induced in this manner by a derivation of $\mathfrak{A}$.

Proof. Every $\bar{u}$ in $\mathfrak{B}$ is the image under ϕ of some u in $\mathfrak{A}$, whence $\overline{D}$ is defined on all of $\mathfrak{B}$. Now assume $\mathfrak{M}D\subseteq\mathfrak{M}$. Suppose $\bar{u}=\phi(u)=\phi(v)$ for elements u and v in $\mathfrak{A}$. Then $u=v+a$ where a is in $\mathfrak{M}$, $uD=vD+aD$, and $\phi(uD)=\phi(vD)+\phi(aD)$. But aD is in $\mathfrak{M}$, so $\phi(aD)=0$, $\phi(uD)=\phi(vD)$. Thus $\overline{D}$ is well-defined. Conversely, if $\overline{D}$ is well-defined, then $\phi(u)=\phi(v)$ implies $\phi(uD)=\phi(vD)$. Thus, if a is any element of $\mathfrak{M}$ we have

$$\phi(uD) = \phi((u + a)D) = \phi(uD) + \phi(aD)$$

from which it follows that $\phi(aD)=0$ and aD is in $\mathfrak{M}$. We conclude that $\overline{D}$ is well-defined if and only if $\mathfrak{M}D\subseteq\mathfrak{M}$. We will now show that $\overline{D}$ is a derivation of $\mathfrak{B}$.

Let $\bar{u}$, $\bar{v}$ be elements of $\mathfrak{B}$ and let α, β be in $\mathfrak{F}$. Then $\bar{u}=\phi(u)$, $\bar{v}=\phi(v)$ for some u and v in $\mathfrak{A}$, and

$$\begin{aligned}(\alpha\bar{u} + \beta\bar{v})\overline{D} &= [\phi(\alpha u + \beta v)]\overline{D} = \phi((\alpha u + \beta v)D)\\ &= \alpha\phi(uD) + \beta\phi(vD) = \alpha(\bar{u}\overline{D}) + \beta(\bar{v}\overline{D}).\end{aligned}$$

Hence $\overline{D}$ is linear. We also have

$$\begin{aligned}(\bar{u}\bar{v})\overline{D} &= [\phi(uv)]\overline{D} = \phi((uv)D)\\ &= \phi((uD)v + u(vD)) = (\bar{u}\overline{D})\bar{v} + \bar{u}(\bar{v}\overline{D}),\end{aligned}$$

so $\overline{D}$ is a derivation.

Now let $\tilde{D}$ be any derivation of $\mathfrak{B}$. We shall show that $\tilde{D}$ is the induced derivation $\overline{D}$ of some derivation D of $\mathfrak{A}$. Any element $\bar{u}$ of $\mathfrak{B}$ may be written as a polynomial in the generators $\bar{x}_1, \cdots, \bar{x}_n$. And, as in $\mathfrak{A}$, $\tilde{D}$ is completely determined by its action on the $\bar{x}_i$ according to the formula

$$\bar{u}\tilde{D} = (\partial\bar{u}/\partial\bar{x}_1)(\bar{x}_1\tilde{D}) + \cdots + (\partial\bar{u}/\partial\bar{x}_n)(\bar{x}_n\tilde{D}). \tag{11}$$

Choose elements y_i in $\mathfrak{A}$ so that $\phi(y_i) = \bar{x}_i\tilde{D}$ for $i=1, \cdots, n$. We can define a derivation D of $\mathfrak{A}$ by specifying that $x_iD = y_i$ $(i=1, \cdots, n)$. Now let $\bar{D}$ be induced by D according to formula (10). Then $\bar{x}_i\bar{D} = \bar{x}_i\tilde{D}$ for $i=1, \cdots, n$. Thus if $\bar{D}$ is a derivation we shall have $\bar{D} = \tilde{D}$. Therefore it remains only to show that $\mathfrak{M}D \subseteq \mathfrak{M}$.

It is readily seen that if $f = f(x_1, \cdots, x_n)$ is any polynomial over $\mathfrak{F}$ in $x_1, \cdots, x_n$, then $\bar{f} = \phi(f) = f(\bar{x}_1, \cdots, \bar{x}_n)$ is the same polynomial with x_i replaced by $\bar{x}_i$ for $i=1, \cdots, n$. Thus we may write $\partial f/\partial x_i = g_i(x_1, \cdots, x_n)$ and

$$\phi(\partial f/\partial x_i) = \phi(g_i) = g_i(\bar{x}_i, \cdots, \bar{x}_n) = \partial\bar{f}/\partial\bar{x}_i \tag{12}$$

for $i=1, \cdots, n$. Now let u be any element of $\mathfrak{M}$. Then $\bar{u} = \phi(u) = 0$, and by (9), (11), and (12) we have

$$\begin{aligned}\phi(uD) &= \phi(\partial u/\partial x_1)\bar{y}_1 + \cdots + \phi(\partial u/\partial x_n)\bar{y}_n \\ &= (\partial\bar{u}/\partial\bar{x}_1)\bar{y}_1 + \cdots + (\partial\bar{u}/\partial\bar{x}_n)\bar{y}_n = \bar{u}\tilde{D} = 0.\end{aligned}$$

Therefore uD is in $\mathfrak{M}$ and the theorem is proved.

We noted earlier that every element of $\mathfrak{A}$ which is not in $\mathfrak{N}$ has an inverse. From this it follows that every proper ideal of $\mathfrak{A}$ is contained in $\mathfrak{N}$. Thus $\mathfrak{M} \subseteq \mathfrak{N}$. Recalling that $\mathfrak{B}$ is $\bar{\mathfrak{D}}$-simple for a set $\bar{\mathfrak{D}}$ of derivations of $\mathfrak{B}$ we now state

THEOREM 3. *Let $\mathfrak{D}$ be the set of all derivations D of $\mathfrak{A}$ over $\mathfrak{F}$ such that the induced derivations $\bar{D}$ are in $\bar{\mathfrak{D}}$. Then $\mathfrak{M}$ is a maximal $\mathfrak{D}$-ideal of $\mathfrak{A}$, and an element u of $\mathfrak{A}$ is in $\mathfrak{M}$ if and only if u is in $\mathfrak{N}$ and the elements $uD_1 \cdots D_k$ are in $\mathfrak{N}$ for all values of k and all derivations D_i in $\mathfrak{D}$.*

Proof. By Theorem 2, if D induces $\bar{D}$ then $\mathfrak{M}$ is a D-ideal. Thus M is a D-ideal for every D in $\mathfrak{D}$ and hence is a $\mathfrak{D}$-ideal. Let $\mathfrak{R} \neq \mathfrak{A}$ be a $\mathfrak{D}$-ideal properly containing $\mathfrak{M}$ and let $\bar{\mathfrak{R}} = \phi(\mathfrak{R})$. It is easily verified that $\bar{\mathfrak{R}}$ is a nontrivial $\bar{\mathfrak{D}}$-ideal in $\mathfrak{B}$ contradicting $\bar{\mathfrak{D}}$-simplicity. Hence $\mathfrak{M}$ is a maximal $\mathfrak{D}$-ideal. It is also easily seen that the sum of two $\mathfrak{D}$-ideals is a $\mathfrak{D}$-ideal from which it follows that $\mathfrak{M}$ is maximal in the strong sense that it contains all other $\mathfrak{D}$-ideals.

Now let u be any element of $\mathfrak{M}$. Then u is in $\mathfrak{N}$ and $uD_1 \cdots D_k$ is in $\mathfrak{N}$ for all k and all D_i in $\mathfrak{D}$. Conversely, suppose u is in $\mathfrak{N}$ and $uD_1 \cdots D_k$ is in $\mathfrak{N}$ for every product $D_1 \cdots D_k$ of derivations in $\mathfrak{D}$. The ideal generated by u and all $uD_1 \cdots D_k$ is a $\mathfrak{D}$-ideal and hence is contained in $\mathfrak{M}$. Thus u is in $\mathfrak{M}$, and the proof is complete.

To Theorem 3 we have the following immediate

COROLLARY. *If u is an element of $\mathfrak{N}$, then $\bar{u}=\phi(u)$ is a nonzero element of $\mathfrak{B}$ if and only if there is some product $D_1 \cdots D_k$ of derivations D_i in $\mathfrak{D}$ such that $uD_1 \cdots D_k$ is nonsingular.*

THEOREM 4. *Let u be an element of $\mathfrak{N}$ whose terms of degree one are not all zero. Then u is not in $\mathfrak{M}$.*

Proof. We assume without loss of generality that u is in $\mathfrak{M}$ and its term of degree one in x_1 is not zero. Then we may write $u=ax_1+v$ where a is nonsingular and v is in $\mathfrak{F}[x_2, \cdots, x_n]$. Thus

$$x_1 = a^{-1}u - a^{-1}v = u_0 + v_0 \tag{13}$$

where $u_0=a^{-1}u$ is in $\mathfrak{M}$ and $v_0=-a^{-1}v$ is in the ideal $\mathfrak{B}$ generated by $x_2, \cdots, x_n$. We observe that every element f of $\mathfrak{B}$ is a polynomial with terms of the form $x_1^r y$ where y is a monomial in $\mathfrak{F}[x_2, \cdots, x_n]$ of degree $t \geqq 1$. If f is not in $\mathfrak{F}[x_2, \cdots, x_n]$ we associate with f the number $N(f)$ which is the minimum of the degrees of y for all terms $x_1^r y$ of f with $r \neq 0$. Note that $1 \leqq N(f) \leqq (n-1)(p-1)$. Now assume it impossible to write $x_1=u_1+v_1$ where u_1 is in $\mathfrak{M}$ and v_1 is in $\mathfrak{F}[x_2, \cdots, x_n]$. Then we may write $x_1=u_2+v_2$ where u_2 is in $\mathfrak{M}$, v_2 is in $\mathfrak{B}$, and $N(v_2)$ is maximal. Let $x_1^r y$ be any term of v_2 with $r\neq 0$ and y of degree $N(v_2)$. By (13) we have for each such $x_1^r y$

$$x_1^r y = x_1^{r-1}(u_0 + v_0)y = x_1^{r-1}u_0 y + x_1^{r-1}v_0 y \tag{14}$$

where $x_1^{r-1}u_0y$ is in $\mathfrak{M}$ and $x_1^{r-1}v_0y$ is a polynomial each term of which has the form $x_1^s z$ with z in $\mathfrak{F}[x_2, \cdots, x_n]$ and the degree of z greater than $N(v_2)$. Hence by means of substitutions as in (14) we may obtain $x_1=u_3+v_3$ with u_3 in $\mathfrak{M}$, v_3 in $\mathfrak{B}$, and $N(v_3)>N(v_2)$. Thus $x_1=u_1+v_1$ with u_1 in $\mathfrak{M}$ and v_1 in $\mathfrak{F}[x_2, \cdots, x_n]$. But from this it follows that $\bar{x}_1 = \phi(x_1) = \phi(v_1)$ is in $\mathfrak{F}[\bar{x}_2, \cdots, \bar{x}_n]$ contradicting the hypothesis that $\{\bar{x}_1, \cdots, \bar{x}_n\}$ is a minimal set of generators of $\mathfrak{B}$. This proves our theorem.

Before we can prove our next theorem we must develop some notation and prove two lemmas on the combinatorial properties of derivations. Let $S=\{n_1, \cdots, n_s\}$ be an ordered set of positive integers, the ordering being the natural one. Let $\pi_1, \cdots, \pi_r$ be ordered subsets of S such that $\pi_1 \cup \cdots \cup \pi_r = S$ and $\pi_i \cap \pi_j = 0$ (the empty set) if $i \neq j$. We shall call the ordered r-tuple $\pi=(\pi_1, \cdots, \pi_r)$ an *r-partition* of S. We now have

LEMMA 1. *Let $a_1, \cdots, a_r$ be elements of $\mathfrak{A}$ and let $D_1, \cdots, D_s$ be derivations in $\mathfrak{D}$. Let*

$$T(\pi_i) = D_{i_1} \cdots D_{i_t} \qquad (i = 1, \cdots, r)$$

if $\pi_i = \{i_1, \cdots, i_t\}$ is a nonempty ordered subset of the ordered set $S=\{1, \cdots, s\}$; and if $\pi_i=0$, then $T(\pi_i)$ is to be the identity transformation I of $\mathfrak{A}$. We now assert that

$$(15)\qquad (a_1 \cdots a_r)D_1 \cdots D_s = \sum_{\pi} [a_1T(\pi_1)] \cdots [a_rT(\pi_r)]$$

where π ranges over all r-partitions of S.

Proof. We induce on s. If $s = 1$, each partition has the form $\pi=(0, \cdots, 0, 1, 0, \cdots, 0)$ and formula (15) becomes

$$(a_1 \cdots a_r)D_1 = \sum_{i=1}^{r} a_1 \cdots a_{i-1}(a_iD_1)a_{i+1} \cdots a_r$$

which is correct. Now assume (15) correct for s derivations. Then

$$\begin{aligned}(a_1 \cdots a_r)D_1 \cdots D_{s+1} &= \sum_{\pi} \{[a_1T(\pi_1)] \cdots [a_rT(\pi_r)]\} D_{s+1} \\ &= \sum_{\pi}\sum_{i} [a_1T(\pi_1)] \cdots [a_{i-1}T(\pi_{i-1})] \\ &\quad \cdot [a_iT(\pi_i \cup \{s+1\})][a_{i+1}T(\pi_{i+1})] \cdots [a_rT(\pi_r)].\end{aligned}$$

But if $\pi=(\pi_1, \cdots, \pi_r)$ is a general r-partition of $\{1, \cdots, s\}$, then $\theta=(\pi_1, \cdots, \pi_{i-1}, \pi_i\cup\{s+1\}, \pi_{i+1}, \cdots, \pi_r)$ is a general r-partition of $\{1, \cdots, s+1\}$. Hence

$$(a_1 \cdots a_r)D_1 \cdots D_{s+1} = \sum_{\theta} [a_1T(\theta_1)] \cdots [a_rT(\theta_r)]$$

where $\theta=(\theta_1, \cdots, \theta_r)$ ranges over all r-partitions of $\{1, \cdots, s+1\}$. This is formula (15) for $s+1$ derivations, and the lemma is proved.

We also have

LEMMA 2. *Let $S_1=\{i_1, \cdots, i_q\}$ and S_2 be ordered subsets of the set $S=\{1, \cdots, s\}$ such that $S_1\cap S_2=0$ and $S_1\cup S_2=S$, and let R be the set of all r-partitions π of S with $\pi_t=S_2$ for some fixed t. If $a_1, \cdots, a_r$ are in $\mathfrak{A}$ and $D_1, \cdots, D_s$ in $\mathfrak{D}$, then*

$$(16)\qquad \begin{aligned}&\sum_{\pi \text{ in } R} [a_1T(\pi_1)] \cdots [a_rT(\pi_r)] \\ &\quad = [(a_1 \cdots a_{t-1}a_{t+1} \cdots a_r)D_{i_1} \cdots D_{i_q}][a_tT(\pi_t)].\end{aligned}$$

Proof. If $\pi=(\pi_1, \cdots, \pi_r)$ with $\pi_t=S_2$, then

$$\theta = (\pi_1, \cdots, \pi_{t-1}, \pi_{t+1}, \cdots, \pi_r)$$

is an $(r-1)$-partition of S_1. Moreover, the correspondence $\pi\leftrightarrow\theta$ is a 1-1 correspondence of R with the set of all $(r-1)$-partitions of S_1. Our result now follows from Lemma 1.

We are now able to prove

THEOREM 5. *The ideal $\mathfrak{M}$ contains no monomial.*

Proof. Theorem 4 asserts that $\mathfrak{M}$ contains no monomial of total degree one in $x_1, \cdots, x_n$. Assume that $\mathfrak{M}$ contains no monomial of degree $r-1$ but that $u=x_1^{r_1}\cdots x_n^{r_n}$ has degree $r=r_1+\cdots+r_n$ and is in $\mathfrak{M}$. Then for each i for which $r_i\neq 0$ we may write $u=a_ix_i$ where a_i is not in $\mathfrak{M}$. Thus, by the corollary to Theorem 3, there is a product G_i of t_i derivations in $\mathfrak{D}$ such that a_iG_i is nonsingular. We let i_0 be a value of i for which $t=t_i$ is minimal. There is clearly no loss of generality if we assume $i_0=1$ and $G=G_1=D_1\cdots D_t$ so that a_1G is nonsingular. We now apply G to the element u, and by Lemma 1 we obtain

$$\begin{aligned} uG &= (x_1^{r_1}\cdots x_n^{r_n})D_1\cdots D_t \\ &= \sum_\pi [x_1T(\pi_{11})]\cdots[x_1T(\pi_{1r_1})]\cdots[x_nT(\pi_{n1})]\cdots[x_nT(\pi_{nr_n})]. \end{aligned} \tag{17}$$

We observe that the constant term of uG is zero since u is in $\mathfrak{M}$ and uG is in $\mathfrak{M}$. Let us now compute the linear term in x_1 of uG. Consider first all summands in (17) with $\pi_{1j}=0$ for some fixed index j. By Lemma 2 the sum of these summands is $(a_1G)x_1$ which has a term αx_1 where $\alpha\neq 0$ is the constant term of the nonsingular element a_1G. Letting $j=1,\cdots,r_1$ we find that the total coefficient of x_1 from this source is $r_1\alpha\neq 0$. Note that any summand in (17) in which $\pi_{ij}=0$ with $i\neq 1$ has x_i as a factor and therefore does not have a linear term in x_1. Thus there remains only the consideration of those summands of (17) in which all π_{ij} are nonempty. For such a summand to have a linear term in x_1 it must be that some $x_iT(\pi_{ij})$ has a linear term in x_1 and all other $x_hT(\pi_{hk})$ are nonsingular. But again it follows from Lemma 2 that the sum of all summands in (17) having $x_iT(\pi_{ij})$ as a factor is $w=(a_iH)[x_iT(\pi_{ij})]$ where H is a product of fewer than t derivations. Hence a_iH is singular and w has no linear term. We conclude that uG has a linear term $r_1\alpha x_1$ contrary to Theorem 4.

We are now essentially through. Albert has shown [2] that for any nonzero element u of $\mathfrak{N}$ there exists an element v of $\mathfrak{A}$ such that $uv=x_1^{p-1}\cdots x_n^{p-1}$. Thus if $\mathfrak{M}\neq 0$ then $\mathfrak{M}$ contains a monomial, contrary to Theorem 5. Therefore $\mathfrak{M}=0$ from which Theorem 1 follows.

3. **Some consequences of condition** (B). Let $\mathfrak{T}$ be the commutative power-associative algebra described in §1. By (1) we see that

$$\mathfrak{T} = \mathfrak{B} + \mathfrak{L} = \mathfrak{B} + (y_0\mathfrak{B}, \cdots, y_m\mathfrak{B}),$$

and, having determined the structure of $\mathfrak{B}$, we are now in a position to investigate that of $\mathfrak{T}$.

Let u be any element of $\mathfrak{L}$. Then $u=\sum_{j=0}^n y_jb_j$ where b_j is in $\mathfrak{B}$. From condition (B) we see that $u=0$ if and only if $(y_ia)u=\sum_{j=0}^n (y_ia)(y_jb_j)=0$ for all a in $\mathfrak{B}$ and $i=0,1,\cdots,m$. From this it follows that $u=0$ if and only if the relations

$$\sum_{j=0}^m (b_{ij}b_j - b_jD_{ij}) = 0 \qquad (i=0,\cdots,m), \tag{18}$$

$$\sum_{j=0}^{m} (aD_{ij})b_j = 0 \qquad (i = 0, \cdots, m) \tag{19}$$

hold for every a in $\mathfrak{B}$.

It should be noted that the requirement that the algebra $\mathfrak{T}$ satisfy condition (B) is never inconsistent with the definition of multiplication in $\mathfrak{T}$. The effect of condition (B) is to completely determine the algebra $\mathfrak{T}$ by determining the kernels of the vector space homomorphisms $\mathfrak{B}\rightarrow y_i\mathfrak{B}$ for $i=0, \cdots, m$ and the nature of the sum $\mathfrak{L}$. To demonstrate this we let

$$\mathfrak{T}^* = \mathfrak{B} + \mathfrak{L}^* = \mathfrak{B} + z_0\mathfrak{B} + \cdots + z_m\mathfrak{B}$$

where each vector space $z_i\mathfrak{B}$ is an isomorphic copy of $\mathfrak{B}$. Let products in $\mathfrak{T}^*$ be defined in terms of the same elements b_{ij} and derivations D_{ij} of $\mathfrak{B}$ which determined products in $\mathfrak{T}$. Since $\mathfrak{T}^*$ is a direct sum we see that multiplication is well-defined. Now let $\mathfrak{U}$ be the set of all elements u in $\mathfrak{L}^*$ such that $uw=0$ for all w in $\mathfrak{L}^*$. The set $\mathfrak{U}$ is an ideal of $\mathfrak{T}^*$. The algebra $\mathfrak{T}$ is equivalent to $\mathfrak{T}^*-\mathfrak{U}$ and hence exists and is uniquely determined by condition (B) and the choice of the elements b_{ij} and derivations D_{ij} of $\mathfrak{B}$.

4. **A special case with** $m=2$. In this section we shall construct a class of examples of the algebras $\mathfrak{T}$ in which $\mathfrak{L}=(y_0\mathfrak{B}, \cdots, y_m\mathfrak{B})$ with $m=2$ and $\mathfrak{L}$ is *not a direct sum*. We let $\mathfrak{B}=\mathfrak{F}[e, x, y]$, $\mathfrak{T}=\mathfrak{B}+(y_0\mathfrak{B}, y_1\mathfrak{B}, y_2\mathfrak{B})$ and let

$$xD_{01} = e, \qquad yD_{01} = x^{p-1}, \tag{20}$$

$$xD_{02} = x^2y, \qquad yD_{02} = xy, \tag{21}$$

$$xD_{12} = -x, \qquad yD_{12} = xy^2, \tag{22}$$

$$b_{11} = 0, \qquad b_{12} = e, \qquad b_{22} = -x^2. \tag{23}$$

The algebra $\mathfrak{B}$ is D_{01}-simple [2] and hence is $\{D_{ij}\}$-simple. Thus $\mathfrak{T}$ satisfies condition (A). We complete the definition of $\mathfrak{T}$ by imposing condition (B). As a routine consequence of formulas (18) and (19) we now have Lemma 3 which we state without proof.

LEMMA 3. *In this special case an element* $y_0b_0+y_1b_1+y_2b_2$ *of* $\mathfrak{L}$ *is zero if and only if* $b_0=-b_2x$, $b_1=0$, *and* $b_2=x^{p-1}f(y)+y^{p-1}g(x)$ *where* $f(y)$ *and* $g(x)$ *are polynomials over* $\mathfrak{F}$ *in* y *and* x *respectively.*

It follows from Lemma 3 that $\mathfrak{T}$ is not a direct sum. In fact we may write

$$\mathfrak{T} = \mathfrak{B} + (y_0\mathfrak{B} + y_1\mathfrak{B}, y_2\mathfrak{B}) \tag{24}$$

and we see that $(y_0\mathfrak{B}+y_1\mathfrak{B})\cap y_2\mathfrak{B}$ is spanned by the independent vectors $y_2x^{p-1}y^j$ and $y_2x^iy^{p-1}$ where $i=0, \cdots, p-1$ and $j=0, \cdots, p-2$. Hence $\mathfrak{T}$ has dimension $4p^2-2p+1$. We will show next that $\mathfrak{T}$ not only fails to be a direct sum as presently represented, but, furthermore, *Albert's construction cannot yield a representation of* $\mathfrak{T}$ *as a direct sum.*

Let $\overline{\mathfrak{B}}=\mathfrak{F}[\bar{e}, \bar{z}_1, \cdots, \bar{z}_r]=\bar{e}\mathfrak{F}+\overline{\mathfrak{N}}$ be a polynomial algebra over $\mathfrak{F}$ with unity $\bar{e}$ and generators $\bar{z}_1, \cdots, \bar{z}_r$ such that $\bar{z}_i^p=0$ for $i=1, \cdots, r$ but which are otherwise independent. Suppose there exist $x_0, \cdots, x_m$ such that $\mathfrak{T}=\overline{\mathfrak{B}}+\overline{\mathfrak{L}}$ where $\overline{\mathfrak{L}}=x_0\overline{\mathfrak{B}}+\cdots+x_m\overline{\mathfrak{B}}$ is a direct sum. We will denote by $\bar{a}$, $\bar{b}$, etc. elements of $\overline{\mathfrak{B}}$ and by $\bar{b}_{ij}$ and $\overline{D}_{ij}$ the elements and derivations of $\overline{\mathfrak{B}}$ which define multiplication in this new representation of $\mathfrak{T}$. We observe that $\bar{e}=e$ since $\overline{\mathfrak{B}}$ and $\mathfrak{B}$ have the same unity element as $\mathfrak{T}$. We may write expressions for the x_ke in terms of the original representation of $\mathfrak{T}$. Thus

$$x_ke = a_k + y_0b_k + y_1c_k + y_2d_k \tag{25}$$

where a_k, b_k, c_k, d_k are in $\mathfrak{B}$ and $k=0, \cdots, m$. Since $\mathfrak{T}$ is power-associative and p is an odd prime we have

$$(x_k\bar{b})^p = [(x^k\bar{b})^2]^{(p-1)/2}(x_k\bar{b}) = x_k(\bar{b}_{kk}^{(p-1)/2}\bar{b}^p) \tag{26}$$

for $k=0, \cdots, m$, and similarly

$$(y_kb)^p = y_k(b_{kk}^{(p-1)/2}b^p) \tag{27}$$

for $k=0, 1, 2$. From (23), (25), (26), (27) and Lemma 3 we see that

$$x_0e = (x_0e)^p = a_0^p + y_0b_0^p = \alpha + y_0\beta$$

where α and β are in $\mathfrak{F}$. Since $(x_0e)^2=e$ it follows that $\alpha=0$ and $\beta=\pm 1$. Thus $x_0e=\pm y_0e$ and we can now prove

LEMMA 4. *The algebras $\mathfrak{B}$ and $\overline{\mathfrak{B}}$ coincide as do the spaces $\mathfrak{L}$ and $\overline{\mathfrak{L}}$.*

Proof. Since $x_0e=\pm y_0e$ and $(x_0e)(x_ke)=0$ for $k=1, \cdots, m$, we see by (25) that $a_k=0$ for $k=0, \cdots, m$. Hence $\overline{\mathfrak{L}}\subseteq\mathfrak{L}$. Since $\overline{\mathfrak{B}}$ is $\{\overline{D}_{ij}\}$-simple there is a derivation $\overline{D}_{st}$ of $\overline{\mathfrak{B}}$ such that $\overline{\mathfrak{N}}$ is not a $\overline{D}_{st}$-ideal. From this it follows that each of $\overline{\mathfrak{B}}$, $x_s\overline{\mathfrak{B}}$, and $x_t\overline{\mathfrak{B}}$ has dimension p^r. Thus $3p^r\leqq 4p^2-2p+1$ which implies $r\leqq 2$. If $r<2$ the dimension of $\overline{\mathfrak{L}}$ is seen to be greater than that of $\mathfrak{L}$. Thus $r=2$ and $\overline{\mathfrak{L}}=\mathfrak{L}$.

We have shown that $\mathfrak{B}+\mathfrak{L}=\overline{\mathfrak{B}}+\mathfrak{L}$. Now let b be any element of $\mathfrak{B}$. Then $b=\bar{b}+u$ for elements $\bar{b}$ in $\overline{\mathfrak{B}}$ and u in $\mathfrak{L}$. Let w be an arbitrary element of $\mathfrak{L}$. We see that $uw=bw-\bar{b}w$ is in $\mathfrak{B}$ and in $\mathfrak{L}$. Thus $uw=0$ for all w in $\mathfrak{L}$. Therefore $u=0$, and the lemma is proved.

We now have $\overline{\mathfrak{B}}=\mathfrak{B}=\mathfrak{F}[e, x, y]$ and

$$\mathfrak{T} = \mathfrak{B} + (y_0\mathfrak{B} + y_1\mathfrak{B}, y_2\mathfrak{B}) = \mathfrak{B} + x_0\mathfrak{B} + \cdots + x_m\mathfrak{B}.$$

We shall show that this leads to a contradiction. Setting $a_k=0$ $(k=0, \cdots, m)$ in (25) we obtain

$$x_ke = y_0b_k + y_1c_k + y_2d_k \qquad (k = 0, \cdots, m). \tag{28}$$

LEMMA 5. *If $\mathfrak{T}=\mathfrak{B}+x_0\mathfrak{B}+\cdots+x_m\mathfrak{B}$ is a direct sum, then $m=2$ and there exist elements v and w in $\mathfrak{B}$ such that $b_2=-d_2x+xyv$ and $c_2=xyw$.*

Proof. The dimension of $x_0\mathfrak{B}$ is always the same as that of $\mathfrak{B}$, and we noted earlier that this must also be true for at least one other $x_k\mathfrak{B}$. Hence we may assume that $x_1\mathfrak{B}$ has dimension p^2. It follows that $x_k\mathfrak{B}$ has dimension less than p^2 for $k=2, \cdots, m$. Hence if $U=x^{p-1}y^{p-1}$, then $x_kU=0$ for $k\geqq 2$, and we see from (28) and Lemma 3 that b_k and c_k are in $\mathfrak{N}$ for $k\geqq 2$. We see also that $x_1U\mp x_0b_1U=y_1c_1U\neq 0$ since $x_0e=\pm y_0e$ and $x_0\mathfrak{B}+x_1\mathfrak{B}$ is a direct sum. Hence c_1 is not in $\mathfrak{N}$.

Now let $u_0, \cdots, u_m$ be elements of $\mathfrak{B}$ such that $y_2e=x_0u_0+\cdots+x_mu_m$. From (28) and Lemma 3 we see that $c_1u_1+\cdots+c_mu_m=0$, and, since c_1 is not in $\mathfrak{N}$ but $c_2, \cdots, c_m$ are in $\mathfrak{N}$, this implies that u_1 is in $\mathfrak{N}$. Now, also by Lemma 3, $d_1u_1+\cdots+d_mu_m-e$ is in $\mathfrak{N}$ and hence d_k is not in $\mathfrak{N}$ for some $k\geqq 2$. Without loss of generality we assume d_2 is not in $\mathfrak{N}$.

Let $\mathfrak{E}$ be the subspace of $\mathfrak{B}$ consisting of all elements u such that $x_2u=0$. Since d_2 is nonsingular, it follows from (28) and Lemma 3 that $u=x^{p-1}F(y)+y^{p-1}G(x)$ for some polynomials $F(y)$ and $G(x)$. Let s be the dimension of $\mathfrak{E}$. Then $x_2\mathfrak{B}$ has dimension p^2-s, and we see that $s\geqq 2p-1$. Thus $x_2u=0$ for all possible choices of the polynomials $F(y)$ and $G(x)$. Thus $x_2\mathfrak{B}$ has dimension p^2-2p+1 and $m=2$.

We have shown that the space $\mathfrak{E}$ consists of all u in $\mathfrak{B}$ of the form $u=x^{p-1}F(y)+y^{p-1}G(x)$. From Lemma 3 we see that $c_2u=0$ and $(b_2+d_2x)u=0$ for all u in $\mathfrak{E}$. It therefore follows that $c_2=xyw$ and $b_2=-d_2x+xyv$ for some v and w in $\mathfrak{B}$. We can now obtain our main result which we state as

THEOREM 6. *The algebra $\mathfrak{T}$ cannot be represented as a direct sum.*

Proof. We compute $\bar{b}_{02}$ and single out those terms which possibly give rise to a linear term in x alone. We recall that $\bar{b}_{02}=0$ by (2). Using (20), $\cdots$, (23) and Lemma 5 we see that

$$\begin{aligned}\bar{b}_{02} &= (x_0e)(x_2e) = \pm (y_0e)(y_0b_2 + y_1c_2 + y_2d_2) \\ &= \pm (b_2 - c_2D_{01} - d_2D_{02}) = \mp d_2x + \Omega\end{aligned}$$

where Ω is a sum of terms each having x^2 or y as a factor. Since d_2 is nonsingular $\bar{b}_{02}\neq 0$. This contradiction proves the theorem.

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