PARTITION FUNCTIONS WHOSE LOGARITHMS ARE SLOWLY OSCILLATING(1)

BY

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I. Introduction

"Let Λ be a denumerable set of distinct positive numbers without finite limit point. Then the elements of the set can be arranged in a sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, where λ_k tends to infinity with k. Let $0 = \nu_0 < \nu_1 < \nu_2 < \cdots$ be the elements of the additive semi-group generated by Λ. Consider the weighted partition function $p(\nu_m)$ defined by the generating function

$$f(s) = \prod_{k=1}^{\infty} (1 - e^{-s\lambda_k})^{-\psi(k)} = \sum_{m=0}^{\infty} p(\nu_m) e^{-s\nu_m},$$

where $\psi(k)$ is a function on the positive integers into the non-negative reals such that the sum and product are both convergent for all positive s. (Actually convergence of either the sum or the product implies convergence of the other.) Here

$$p(\nu_i) = \sum_{m_1\lambda_1 + m_2\lambda_2 + \cdots = \nu_i} \prod_{k=1}^{\infty} \binom{\psi(k) + m_k - 1}{m_k}$$

where the summation is taken over non-negative integers m_k and is finite. (The kth factor in the product is 1 for all k for which $\lambda_k > \nu_i$, since then m_k must be zero.) When $\psi(k) = 1$ for all positive integers k, $p(\nu_i)$ is just the number of unrestricted partitions of ν_i into parts taken from the set Λ." [12, p. 346]

1.1. **Basic relation between $n(u)$ and $P(u)$.** Letting $n(u) = \sum_{\lambda_k \le u} \psi(k)$, we have

$$\log f(s) = \sum_{k=1}^{\infty} \psi(k) \cdot \log\, (1 - e^{-s\lambda_k})^{-1} = \sum_{k=1}^{\infty} \psi(k) \cdot s \int_{\lambda_k}^{\infty} \frac{e^{-su}}{1 - e^{-su}}\, du$$

$$= s \int_{\lambda_1}^{\infty} \frac{e^{-su}}{1 - e^{-su}} \left\{ \sum_{\lambda_k \le u} \psi(k) \right\} du = s \int_{\lambda_1}^{\infty} \frac{e^{-su}}{1 - e^{-su}}\, n(u) du.$$

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Now, letting $P(u)=\sum_{\nu_i\leq u} p(\nu_i)$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} p(\nu_m)e^{-s\nu_m} &= \sum_{m=0}^{\infty} p(\nu_m)\cdot s\int_{\nu_m}^{\infty} e^{-su}du\\ &= s\int_0^{\infty} e^{-su}\left\{\sum_{\nu_m\leq u} p(\nu_m)\right\}du\\ &= s\int_0^{\infty} e^{-su}P(u)du.\end{aligned}$$

Thus we have the following basic relation between the functions $n(u)$ and $P(u)$

$$\exp\left\{s\int_0^{\infty}\frac{e^{-su}}{1-e^{-su}}n(u)du\right\} = s\int_0^{\infty}P(u)e^{-su}du. \tag{1}$$

1.2. **History and statement of the problem.** E. E. Kohlbecker [12] has proved that "an asymptotic relation of the form $n(u)\sim u^{\alpha}L(u)$ as $u\rightarrow+\infty$, where α is a positive constant and L is a slowly oscillating function in the sense of Karamata [10], is equivalent to an asymptotic relation of the form $\log P(u)\sim u^{\alpha/(\alpha+1)}L_1(u)$ as $u\rightarrow+\infty$, where L_1 is a slowly oscillating function related to L by a certain implicit formula which can be solved for L_1 in the cases usually encountered." For an illuminating discussion of some of Kohlbecker's results see de Bruijn [4]. Special cases of Kohlbecker's results had been obtained earlier by Hardy and Ramanujan [9], Knopp [11], Erdös [7], and Brigham [2].

The object of this paper is to consider the case where $\alpha=0$. Here we show that an asymptotic relation of the form $n(u)\sim L(u)$ as $u\rightarrow+\infty$, where $L(u)$ is a slowly oscillating function, implies an asymptotic relation of the form $\log P(u)\sim[M^*(u)]^{-1}$ as $u\rightarrow+\infty$. Here $M^*(x)$ is a slowly oscillating function related to $M(x)=\int_{x_0}^{x}(L(u)/u)du$, and hence to $L(x)$ ultimately, by means of an implicit formula, which can be solved for M^* in the cases usually met with; more specifically, M^* is the conjugate slowly oscillating function to M in the sense of N. G. de Bruijn [4]. We have been able to make the deduction in the reverse direction only under further hypotheses on $L(u)$.

The definitions of $n(u)$ and $P(u)$ in §1.1 show that these functions are obviously nondecreasing and also that, if $n(u)\sim L(u)$ as $u\rightarrow+\infty$ for a slowly oscillating function $L(u)$, then at the very least $\liminf_{u\to+\infty} L(u)>0$ and so $\lim_{x\to+\infty}\int_{x_0}^{x}(L(u)/u)du=+\infty$. The deduction of the asymptotic formula for $\log P(u)$ from that for $n(u)$ uses, in addition to the basic relation (1), only the fact that $\int_{x_0}^{x}(L(u)/u)du\rightarrow+\infty$ with x and the monotonicity of $P(u)$. In arguing in the opposite direction we naturally need, besides the basic relation (1), the monotonicity of $n(u)$ and the fact that $\int_{x_0}^{x}(L(u)/u)du\rightarrow+\infty$ with x, but in addition our argument requires the further condition that there exist a differentiable function $J(x)$ such that $J(x) \sim L(x)$ as $x \to +\infty$ and $x(J'(x)/J(x))$ is nonincreasing. This latter condition is satisfied automatically by almost all familiar slowly oscillating functions.

More generally, we could forget the original interpretation in terms of partitions and start merely with (1), where $n(u)$ and $P(u)$ are functions on the non-negative real numbers such that $\int_0^R (n(u)/u)du$ and $\int_0^R P(u)du$ exist in the Lebesgue sense for every positive R. Then, if $L(u)$ is a slowly oscillating function, $M(x) = \int_{x_0}^x (L(u)/u)du \to +\infty$ with x, and M^* is as before, we obtain the following:

(i) If $n(u) \sim L(u)$ as $u \to +\infty$ and $P(u)$ is nondecreasing, then $\log P(u) \sim [M^*(u)]^{-1}$ as $u \to +\infty$.

(ii) If $\log P(u) \sim [M^*(u)]^{-1}$ as $u \to +\infty$ and if $n(u)$ is nondecreasing, then $n(u) \sim L(u)$ as $u \to +\infty$, provided $L(x) \sim J(x)$ as $x \to +\infty$, where $J(x)$ is differentiable and $x(J'(x)/J(x))$ is nonincreasing.

Most of our results will be stated and proved for functions $n(u)$ and $P(u)$ satisfying the foregoing conditions and we shall specialize to the case of partitions only at the end of the paper. For this case the conditions of integrability are immediate.

To prove (i) we go from the assumption on $n(u)$ to a resulting property of the generating function $f(s)$ by an Abelian argument (Theorem 1) and then from this property of $f(s)$ to the assertion about $P(u)$ by a Tauberian argument (Theorem 3). To prove (ii) we go from the assumption on $P(u)$ to a resulting property of $f(s)$ by an Abelian argument (Theorem 4) and then from this property of $f(s)$ to the assertion about $n(u)$ by a Tauberian argument (Theorem 2).

II. Slowly Oscillating Functions

Here we assemble some definitions, results, etc. from [4], which we require for the ensuing sections.

2.1. **Definition and properties of slowly oscillating functions.** A positive-valued function $L(x)$ defined for $x \geqq a$, where a is some positive number(²), is called *slowly oscillating* if it is measurable and if for every $c > 0$ we have

$$\frac{L(cx)}{L(x)} \to 1 \quad \text{as} \quad x \to +\infty.$$

This notion, with restriction to continuous functions, is due to Karamata [10]. For the general case of measurable functions see [6; 13]. The most important properties of slowly oscillating functions are as follows.

(i) If $0 < b < d < +\infty$, then the relation $(L(cx)/L(x)) \to 1$ as $x \to +\infty$ holds uniformly with respect to c for $b \leqq c \leqq d$ [6; 10; 13].

(²) Throughout we assume a to be some positive number.

(ii) There exists a continuous function $\delta(x)$ such that $\delta(x)$ has limit zero and

$$L(x)\cdot\exp\left\{-\int_a^x \delta(u)\cdot u^{-1}du\right\}$$

has a finite positive limit as $x\to+\infty$. If $L(x)$ is in addition *nondecreasing* then there exists a continuous *non-negative* function $\delta(x)$ with these properties.

In fact the assertions of (ii) follow from (i) upon taking

$$\delta(x) = 6\{\log L(ae^{n+1}) - \log L(ae^n)\}\left\{\log\frac{x}{a} - n\right\}\left\{n+1-\log\frac{x}{a}\right\}$$

for $ae^n \leqq x \leqq ae^{n+1}$, where $n=0, 1, 2, \cdots$ (cf. [4, §4]).

Moreover, if a positive-valued measurable function $L(x)$ on $[a, \infty)$ has the property that there exists a continuous function $\delta(x)$ such that $\delta(x)$ has limit zero and

$$L(x)\cdot\exp\left\{-\int_a^x \delta(u)\cdot u^{-1}du\right\}$$

has a finite positive limit as $x\to+\infty$, then $L(x)$ is automatically slowly oscillating.

2.2. **Two lemmas concerning slowly oscillating functions.**

LEMMA 1. *If $M(x)=\int_a^x L(t)\cdot t^{-1}dt$, where $L(x)$ is a slowly oscillating function defined for $x\geqq a$, then*

(i) $$\frac{M(cx)}{M(x)}\to 1 \quad as\ x\to+\infty, \qquad \textit{for any fixed } c>0,$$

(ii) $$\frac{M(x)}{L(x)}\to+\infty \quad as\ x\to+\infty.$$

Proof. Assertion (i) is trivial if $M(x)$ has a finite limit as $x\to+\infty$. If $M(x)\to+\infty$ as $x\to+\infty$, we have by l'Hospital's rule

$$\lim_{x\to+\infty}\frac{M(cx)}{M(x)} = \lim_{x\to+\infty}\frac{\int_{a/c}^x L(ct)\cdot t^{-1}dt}{\int_a^x L(t)\cdot t^{-1}dt} = \lim_{x\to+\infty}\frac{L(cx)}{x}\bigg/\frac{L(x)}{x} = 1.$$

Thus (i) is proved.

Given any positive number A, however large, we can find a positive number k such that $(1/2)\log k>A$. Now by (i) of §2.1 (with $b=1/k$ and $d=1$), we have for sufficiently large x,

$$M(x) = \int_a^x L(t)\cdot t^{-1}dt > \int_{x/k}^x L(t)\cdot t^{-1}dt > \frac{1}{2} L(x)\cdot \int_{x/k}^x t^{-1}dt$$
$$= \frac{1}{2}\log k\cdot L(x) > A\cdot L(x).$$

Therefore $M(x)/L(x) > A$ for sufficiently large x. Since A is arbitrary,

$$\frac{M(x)}{L(x)} \to +\infty \text{ as } x \to +\infty$$

so that (ii) is proved.

LEMMA 2. *If $L(x)$ is a slowly oscillating function defined for $x \geqq a$, ϵ is an arbitrary positive number, u is sufficiently large, and $x \geqq u$, then*

$$\frac{xL(x)}{uL(u)} > 1 - \epsilon.$$

Proof. By property (ii) of slowly oscillating functions there is a continuous function $\delta(x)$ such that $\delta(x)$ has limit zero and a positive number K such that

$$\lim_{x\to+\infty} L(x)\cdot\exp\left\{-\int_a^x \delta(u)u^{-1}du\right\} = K.$$

Thus for u sufficiently large and $x \geqq u$, we have $\delta(x) > -1$ and

$$K(1 - (\epsilon/2)) < L(x)\cdot\exp\left\{-\int_a^x \delta(t)t^{-1}dt\right\} < K(1 + (\epsilon/2)).$$

Then for $x \geqq u$, we have

$$\frac{xL(x)}{uL(u)} > \frac{x(1 - (\epsilon/2))\exp\left\{\int_a^x \delta(t)t^{-1}dt\right\}}{u(1 + (\epsilon/2))\exp\left\{\int_a^u \delta(t)t^{-1}dt\right\}}$$
$$> (1 - \epsilon)\exp\left\{\int_u^x (1 + \delta(t))t^{-1}dt\right\} > 1 - \epsilon.$$

2.3. **Pairs of conjugate slowly oscillating functions.** The following result and nomenclature are due to N. G. de Bruijn [4].

If $M(x)$ is slowly oscillating, then there exists a slowly oscillating function M^* such that

$$\begin{aligned} &M^*(xM(x))\cdot M(x) \to 1 \text{ as } x \to +\infty, \\ &M(xM^*(x))\cdot M^*(x) \to 1 \text{ as } x \to +\infty. \end{aligned} \tag{2}$$

Moreover, M^* is asymptotically uniquely determined. Further, the relation $M \to M^*$ is involutory in the sense that M and $(M^*)^*$ are asymptotically equivalent. *The function M^* will be called the conjugate of M.*

The proof of the preceding result can be found in de Bruijn's paper [4]. We shall need the following more specific result. The proof that follows is essentially due to de Bruijn [4].

LEMMA 3. *If $M(x)$ is a nondecreasing slowly oscillating function, then there exists a nonincreasing slowly oscillating function $M^*(x)$ such that* (2) *holds.*

Proof. By (ii) of §2.1 we can find a continuous non-negative function $\eta(x)$ defined for $x \geqq a$ such that $\eta(x)$ has limit zero and

$$M(x) \cdot \exp\left\{-\int_a^x \eta(t) \cdot t^{-1} dt\right\}$$

has a finite positive limit C as $x \to +\infty$. Put

$$f(u) = u + \int_a^{e^u} \eta(t) \cdot t^{-1} dt \qquad \text{for } u \geqq \log a.$$

Then $f'(u) = 1 + \eta(e^u) \geqq 1$ for all u and $f'(u) \to 1$ as $u \to +\infty$. It follows from the inverse function theorem that there is a unique function $g(v)$ defined for $v \geqq \log a$ such that

$$g(f(u)) = u \qquad \text{for } u \geqq \log a,$$

$$f(g(v)) = v \qquad \text{for } v \geqq \log a,$$

g has a continuous derivative g',

$$0 < g'(v) \leqq 1 \qquad \text{for } v \geqq \log a,$$

and

$$g'(v) \to 1 \text{ as } v \to +\infty.$$

Put $g'(v) = 1 + \eta^*(e^v)$ for $v \geqq \log a$. Then $\eta^*(x)$ is a nonpositive valued, continuous function defined for $x \geqq a$ such that $\eta^*(x) \to 0$ as $x \to +\infty$. Then it is easily verified that if we put

$$M^*(x) = \frac{1}{C} \exp\left\{\int_a^x \eta^*(t) \cdot t^{-1} dt\right\},$$

then M^* is a slowly oscillating function with the desired properties.

Finally we make some remarks on conjugate slowly oscillating functions, which will give an insight into the nature of M^* when M is a slowly oscillating function of a familiar type, e.g., see Corollaries I* and I** in §IV. According to de Bruijn, these remarks are essentially due to A. Békéssy [1, Hilfssatz 2]. Let $M(x)$ be slowly oscillating. Put $M_1(x) = M(x)$,

$$M_2(x) = M_1\left(\frac{x}{M_1(x)}\right), \quad M_3(x) = M_1\left(\frac{x}{M_2(x)}\right), \quad \cdots, \quad M_{n+1}(x) = M_1\left(\frac{x}{M_n(x)}\right).$$

It is easy to show that each M_n is slowly oscillating. If, for some n, we have $M_{n+1}(x) \sim M_n(x)$ as $x \to +\infty$, then $M^*(x) \sim \{M_n(x)\}^{-1}$ as $x \to +\infty$. For, $M_{n+1}(x) \sim M_n(x)$ as $x \to +\infty$ implies that

$$M\left(\frac{x}{M_n(x)}\right) \cdot \frac{1}{M_n(x)} = M_{n+1}(x) \cdot \{M_n(x)\}^{-1} \to 1 \text{ as } x \to +\infty,$$

and now the fact that $M^*(x) \sim \{M_n(x)\}^{-1}$ as $x \to +\infty$ follows from the asymptotic uniqueness of $M^*(x)$, by de Bruijn's result quoted above.

III. Some Auxiliary Theorems

In the introduction we stated that the deduction of the asymptotic formula for $\log P(u)$ from that for $n(u)$ and vice versa are achieved not by a direct method but by the introduction of the generating function $f(s)$ as a stepping stone. Here we establish the connection between $n(u)$ and $\log f(s)$ first and then proceed to get the connection between $\log f(s)$ and $\log P(u)$.

3.1. **The connection between $n(u)$ and $\log f(s)$.**

Theorem I. *If $\int_0^R (n(u)/u)du$ exists in the Lebesgue sense for every $R > 0$, and*

$$f(s) = \exp\left\{s \int_0^\infty \frac{e^{-su}}{1 - e^{-su}} n(u) du\right\}$$

for all positive s, then the relation

$$n(u) \sim L(u) \quad as \quad u \to +\infty$$

implies the relation

$$\log f(s) \sim \int_a^{1/s} \frac{L(u)}{u} du \quad as \quad s \to 0,$$

provided that $L(x)$ is a slowly oscillating function defined for $x \geqq a$ such that

$$\int_a^x \frac{L(u)}{u} du \to +\infty \text{ as } x \to +\infty.$$

Remark. We prove the theorem in two stages in order to emphasize the hypotheses used in each section of the argument and to keep the analogy with the Tauberian counterpart of the theorem closer.

Theorem IA. *If $\int_0^R (n(u)/u)du$ exists in the Lebesgue sense for every $R > 0$, then the relation*

$$n(u) \sim L(u) \quad as \quad u \to +\infty$$

implies the relation

$$\int_0^x \frac{n(u)}{u}\,du \sim \int_a^x \frac{L(u)}{u}\,du \text{ as } x \to +\infty,$$

provided that $L(u)$ is a slowly oscillating function defined for $x \geqq a$ such that $\int_a^x (L(u)/u)du \to +\infty$ as $x \to \infty$.

Proof. Immediate by l'Hospital's rule. Next, we proceed to the second stage.

THEOREM IB. *If $\int_0^R (n(u)/u)du$ exists in the Lebesgue sense for every $R > 0$, and*

$$f(s) = \exp\left\{ s \int_0^\infty \frac{e^{-su}}{1 - e^{-su}}\, n(u)du \right\}$$

for all positive s, then the relation

$$\int_0^x \frac{n(u)}{u}\,du \sim M(x) \text{ as } x \to +\infty$$

implies the relation

$$\log f(s) \sim M(1/s) \text{ as } s \to +0,$$

where $M(x)$ is a slowly oscillating function defined for $x \geqq a$.

Proof. For brevity we write

$$I(u) = \int_0^u n(t) \cdot t^{-1} dt \quad \text{and} \quad E(t) = \frac{-t}{e^t - 1}\cdot$$

Then we have

$$\begin{aligned} \log f(s) &= -\int_0^\infty E(su)\,\frac{n(u)}{u}\,du \\ &= s\int_0^\infty E'(su)I(u)du = \int_0^\infty E'(x)I(xs^{-1})dx \end{aligned}$$

since $E(0+) = -1$, $I(0) = 0$, and $E(t)$ is exponentially small for large t.

According to [5, Lemma 1], we have

$$\left| \frac{I(xs^{-1})}{I(s^{-1})} \right| < C(\epsilon)(x^\epsilon + x^{-\epsilon})$$

and by the Arzela-Lebesgue theorem, we find that

$$\int_0^\infty E'(x)\,\frac{I(xs^{-1})}{I(s^{-1})}\,dx \to \int_0^\infty E'(x)dx = 1,$$

whence

$$\log f(s) \sim I(s^{-1}) \text{ as } s \to +0.$$

Further

$$I(x) \sim M(x) \text{ as } x \to +\infty.$$

Hence the theorem.

Proof of Theorem I. If in Theorem IB we take $M(x) = \int_a^x (L(u)/u)du$, where $L(u)$ is as in Theorem I, then Theorem I is immediate from Theorems IA and IB.

The Tauberian counterpart of Theorem I is the following.

THEOREM II. *If $\int_0^R (n(u)/u)du$ exists in the Lebesgue sense for every positive R,*

$$f(s) = \exp\left\{ s \int_0^\infty \frac{e^{-su}}{1 - e^{-su}} n(u)du \right\}$$

for all positive s, and $n(u)$ is nondecreasing, then the relation

$$\log f(s) \sim \int_a^{1/s} \frac{L(u)}{u} du \text{ as } s \to 0$$

implies the relation

$$n(u) \sim L(u) \text{ as } u \to +\infty,$$

provided $L(x)$ is a slowly oscillating function defined for $x \geqq a$ such that

$$L(x) \sim K \exp\left\{ \int_a^x \frac{\delta(u)}{u} du \right\} \text{ as } x \to +\infty,$$

where δ is a nonincreasing function and K is a positive real number.

We give the proof in two stages.

THEOREM IIA. *If $\int_0^R n(u)u^{-1}du$ exists in the Lebesgue sense for every positive R,*

$$f(s) = \exp\left\{ s \int_0^\infty \frac{e^{-su}}{1 - e^{-su}} n(u)du \right\}$$

for all positive s, and $n(u)$ is nondecreasing, then the relation

$$\log f(s) \sim M\left(\frac{1}{s}\right) \text{ as } s \to 0,$$

implies the relation

$$\int_0^x \frac{n(u)}{u} du \sim M(x) \text{ as } x \to +\infty,$$

provided that $M(x)$ *is a nondecreasing slowly oscillating function defined for* $x \geqq a$.

The assumption that $M(x)$ is nondecreasing is not really needed, but involves no loss of generality, since $M(1/s) \sim \log f(s)$ and $f(s)$ is monotonic, in view of the monotonicity of $n(u)$.

Proof. Define $N(u) = \sum_{m=1}^{\infty} (1/m) n(u/m)$. This exists, since

$$\sum_{m=2}^{\infty} \frac{1}{m} n\left(\frac{u}{m}\right) \leqq \int_1^{\infty} \frac{1}{x} n\left(\frac{u}{x}\right) dx = \int_0^u \frac{n(t)}{t} dt,$$

the non-negativity of $n(u)$ being implied by the hypothesis. Clearly $N(u)$ is a non-negative, nondecreasing function of u. Now

$$\begin{aligned} \log f(s) &= s \int_0^{\infty} \frac{e^{-su}}{1 - e^{-su}} n(u) du \\ &= \sum_{m=1}^{\infty} s \int_0^{\infty} e^{-mus} n(u) du \\ &= \sum_{m=1}^{\infty} s \int_0^{\infty} e^{-su} \frac{1}{m} n\left(\frac{u}{m}\right) du \\ &= s \int_0^{\infty} e^{-su} N(u) du. \end{aligned}$$

To this last expression we apply a Tauberian theorem of Karamata [8, Theorem 108] and obtain

$$N(u) \sim M(u) \text{ as } u \to +\infty. \tag{3}$$

Next we put $I(u) = \int_0^u n(t) \cdot t^{-1} dt$. Then, since $n(u)$ is nondecreasing,

$$I(u) = \sum_{m=1}^{\infty} \int_{u/(m+1)}^{u/m} \frac{n(t)}{t} dt \leqq \sum_{m=1}^{\infty} n\left(\frac{u}{m}\right) \log \frac{m+1}{m} \leqq \sum_{m=1}^{\infty} \frac{1}{m} n\left(\frac{u}{m}\right) = N(u).$$

On the other hand, since $n(u)$ is nondecreasing,

$$\begin{aligned} I(eu) &= \int_0^{eu} \frac{n(t)}{t} dt = \int_u^{eu} \frac{n(t)}{t} dt + \sum_{m=2}^{\infty} \int_{u/m}^{u/(m-1)} \frac{n(t)}{t} dt \\ &\geqq n(u) \int_u^{eu} \frac{dt}{t} + \sum_{m=2}^{\infty} n\left(\frac{u}{m}\right) \log \frac{m}{m-1} \\ &\geqq n(u) + \sum_{m=2}^{\infty} \frac{1}{m} n\left(\frac{u}{m}\right) = N(u). \end{aligned}$$

Combining the two inequalities, we get

$$N(u) \geqq I(u) \geqq N\left(\frac{u}{e}\right).$$

Since $M(u)$ is slowly oscillating, we have by (3)

$$N(u) \sim M(u) \sim M\left(\frac{u}{e}\right) \sim N\left(\frac{u}{e}\right) \text{ as } u \to +\infty$$

and so

$$I(u) \sim M(u) \text{ as } u \to +\infty.$$

Next we turn to the second stage.

THEOREM IIB. *If $\int_0^R n(u)u^{-1}du$ exists in the Lebesgue sense for every positive R, and $n(u)$ is nondecreasing, then the relation*

$$\int_0^x \frac{n(u)}{u} du \sim \int_a^x \frac{L(u)}{u} du \quad as \quad x \to +\infty,$$

implies the relation

$$n(u) \sim L(u) \text{ as } u \to +\infty,$$

provided $L(x)$ is a slowly oscillating function defined for $x \geqq a$ and subject to the further condition that

$$L(x) \sim K \exp\left\{\int_a^x \delta(u)u^{-1}du\right\},$$

where $\delta(x)$ is a nonincreasing function and K is a positive real number.

Proof. Put

$$\int_1^x J(t)\cdot t^{-1}dt = y, \qquad J(x) = \psi(y), \qquad n(x) = f(y),$$

whence $d\psi/dy = \delta(x)$. Then the theorem becomes:

If $\psi'(y)$ decreases to 0, if f increases, then

$$\int_0^y \frac{f(s)}{\psi(s)} ds \sim y \quad implies \quad f(y) \sim \psi(y) \quad as \quad y \to \infty.$$

The proof can be imitated from the well-known case where $\psi(s) = 1$ identically.

The condition that $\psi'(y)$ decreases to 0 can be replaced by a weaker one, viz. that $y\psi'(y)/\psi(y)$ is bounded. This means that in the original version of Theorem IIB the condition that $\delta(x)$ is nonincreasing can be replaced by

$$\delta(x) \int_1^x J(t) \cdot t^{-1} dt = O(J(x)).$$

Theorem II follows from Theorems IIA and IIB.

3.2. **The connection between** $\log f(s)$ **and** $\log P(u)$.

THEOREM III. *Suppose that* (i) $\int_0^R P(u)du$ *exists in the Lebesgue sense for every positive* R, (ii) $f(s) = s\int_0^\infty P(u)e^{-su}du$ *for all positive* s, (iii) $(M(x), M^*(x))$ *form a pair of conjugate slowly oscillating functions, each defined for* $x \geqq a$, *say*, (iv) $M(x)$ *is nondecreasing, and* (v) $P(x)$ *is nondecreasing. Then the relation*

$$\log f(s) \sim M\left(\frac{1}{s}\right) \text{ as } s \to 0,$$

implies the relation

$$\log P(u) \sim [M^*(u)]^{-1} \text{ as } u \to +\infty.$$

The assumption (iv) is not really needed, but involves no loss of generality, since $M(1/s) \sim \log f(s)$ and $f(s)$ is monotonic, in view of the monotonicity of $P(x)$.

Proof. The theorem is immediate if $M(x)$ has a finite limit as $x \to +\infty$. Thus we shall assume throughout the rest of the proof that $M(x) \to +\infty$ as $x \to +\infty$.

Since adding a positive constant to $P(u)$ merely adds a constant to $f(s)$, we may assume $P(u) \geqq 0$ for all positive u. Then for $u > 0$, $s > 0$, we have, since P is nondecreasing,

$$P(u)e^{-su} = s\int_u^\infty P(u)e^{-sx}dx$$
$$\leqq s\int_u^\infty P(x)e^{-sx}dx.$$

Further, since $P(x) \geqq 0$ for $x > 0$, we have

$$P(u)e^{-su} \leqq s\int_0^\infty P(x)e^{-sx}dx = f(s).$$

Then, for a given positive $\epsilon(<1/25$, say), by hypothesis on $f(s)$, we have for all positive u and sufficiently small s,

$$P(u) < \exp\left\{(1+\epsilon)M\left(\frac{1}{s}\right) + su\right\}.$$

For *fixed* ϵ, we may take $1/s = uM^*(u)/\epsilon$, provided u is sufficiently large. Then, for sufficiently large u,

$$P(u) < \exp\left\{(1+\epsilon)M\left(\frac{uM^*(u)}{\epsilon}\right) + \epsilon[M^*(u)]^{-1}\right\}$$
$$< \exp\{(1+\epsilon)^2 M(uM^*(u)) + \epsilon[M^*(u)]^{-1}\},$$

since, for sufficiently large u

$$M\left(\frac{uM^*(u)}{\epsilon}\right) < (1+\epsilon)M(uM^*(u)).$$

Again, since

$$M(uM^*(u))\cdot M^*(u) \to 1 \text{ as } u \to +\infty,$$

we have, for sufficiently large u

$$M(uM^*(u)) < (1+\epsilon)[M^*(u)]^{-1}.$$

Then the above result reduces to

$$P(u) < \exp\{(1+\epsilon)^3[M^*(u)]^{-1} + \epsilon[M^*(u)]^{-1}\}$$

for sufficiently large u. Since $\epsilon < 1/25$, we get

$$P(u) < \exp\{(1+5\epsilon)[M^*(u)]^{-1}\}, \tag{4}$$

for sufficiently large u.

Now we proceed to get a lower estimate for $P(u)$ for large u. For this we consider,

$$f\left(\frac{1}{uM^*(u)}\right) = \frac{1}{uM^*(u)}\left(\int_0^u + \int_u^{2u} + \int_{2u}^\infty\right) P(x)\exp\left\{-\frac{x}{uM^*(u)}\right\}dx$$
$$= J_1(u) + J_2(u) + J_3(u).$$

First of all, using (4), we get, for sufficiently large u,

$$J_3(u) < \frac{1}{uM^*(u)}\int_{2u}^\infty \exp\left\{(1+5\epsilon)[M^*(x)]^{-1} - \frac{x}{uM^*(u)}\right\}dx$$
$$= \frac{1}{uM^*(u)}\int_{2u}^\infty \exp\left\{-\frac{x}{uM^*(u)}\left[1 - \frac{(1+5\epsilon)uM^*(u)}{xM^*(x)}\right]\right\}dx.$$

Now, by Lemma 2, we have, for $x \geqq 2u$, and u sufficiently large,

$$xM^*(x) \geqq (1-\epsilon)2uM^*(2u) \geqq (1-\epsilon)^2\, 2uM^*(u)$$

and so

$$\frac{uM^*(u)}{xM^*(x)} \leqq \frac{1}{2(1-\epsilon)^2} < \frac{5}{8}.$$

Therefore, for $x \geqq 2u$ and u sufficiently large,

$$1 - \frac{(1 + 5\epsilon)uM^*(u)}{xM^*(x)} \geqq 1 - \frac{(1 + 5\epsilon)5}{8} > \frac{1}{4} \cdot$$

Hence, for sufficiently large u,

$$J_3(u) < \frac{1}{uM^*(u)} \int_{2u}^{\infty} \exp\left\{ -\frac{x}{4uM^*(u)} \right\} dx \leqq 4.$$

Next, for sufficiently large u, we have

$$\begin{aligned} J_2(u) &\leqq \frac{1}{uM^*(u)} \int_{u}^{2u} \exp\left\{ (1 + 5\epsilon)[M^*(x)]^{-1} - \frac{x}{uM^*(u)} \right\} dx \\ &\leqq \frac{1}{uM^*(u)} \exp\left\{ (1 + 5\epsilon)(1 + \epsilon)[M^*(u)]^{-1} \right\} \cdot \int_{u}^{\infty} \exp\left(-\frac{x}{uM^*(u)} \right) dx, \end{aligned}$$

since $[M^*(x)]^{-1} < (1+\epsilon)[M^*(u)]^{-1}$ for $u \leqq x \leqq 2u$, provided u is sufficiently large. Hence, for sufficiently large u,

$$\begin{aligned} J_2(u) &\leqq \exp\{ (1 + 7\epsilon)[M^*(u)]^{-1} - [M^*(u)]^{-1} \} \\ &\leqq \exp\{ 7\epsilon[M^*(u)]^{-1} \}. \end{aligned}$$

Thus, for sufficiently large u we have

$$\begin{aligned} J_2(u) + J_3(u) &\leqq \exp\{ 7\epsilon[M^*(u)]^{-1} \} + 4 \\ &< \exp\{ 8\epsilon[M^*(u)]^{-1} \}. \end{aligned} \tag{5}$$

Now, since P is non-negative and nondecreasing,

$$\begin{aligned} P(u) &= s \int_0^{\infty} P(u) e^{-xs} dx \\ &\geqq s \int_0^{u} P(x) e^{-xs} dx \end{aligned}$$

for any positive s. In particular,

$$P(u) \geqq J_1(u) = f\left(\frac{1}{uM^*(u)} \right) - J_2(u) - J_3(u).$$

By hypothesis on $f(s)$, we have, for sufficiently large u,

$$f\left(\frac{1}{uM^*(u)} \right) > \exp\{ (1 - \epsilon) M(uM^*(u)) \}.$$

Since

$$M(uM^*(u))\cdot M^*(u)\to 1 \text{ as } u\to+\infty,$$

we have

$$f\left(\frac{1}{uM^*(u)}\right) > \exp\{(1-2\epsilon)[M^*(u)]^{-1}\} \tag{6}$$

for sufficiently large u. Using (5) and (6), we get

$$\begin{aligned}P(u) &\geqq f\left(\frac{1}{uM^*(u)}\right) - \{J_2(u)+J_3(u)\}\\ &> \exp\{(1-2\epsilon)[M^*(u)]^{-1}\} - \exp\{8\epsilon[M^*(u)]^{-1}\}\end{aligned}$$

for sufficiently large u. Hence, for sufficiently large u,

$$P(u) > \exp\{(1-3\epsilon)[M^*(u)]^{-1}\}. \tag{7}$$

Combining (4) and (7), we get

$$\log P(u) \sim [M^*(u)]^{-1} \text{ as } u\to+\infty,$$

which proves Theorem III.

The following is the Abelian counterpart of Theorem III.

THEOREM IV. *Suppose that* (i) $\int_0^R P(u)du$ *exists in the Lebesgue sense for every positive* R, (ii) $f(s)=s\int_0^\infty P(u)e^{-su}\,du$ *for all positive* s, (iii) $(M(x), M^*(x))$ *form a pair of conjugate slowly oscillating functions, each defined for* $x\geqq a$, *say,* (iv) $M(x)$ *is nondecreasing. Then the relation*

$$\log P(u) \sim [M^*(u)]^{-1} \text{ as } u\to+\infty,$$

implies the relation

$$\log f(s) \sim M\left(\frac{1}{s}\right) \text{ as } s\to 0.$$

Proof. As in the proof of Theorem III we may assume that $M(x)\to+\infty$ as $x\to+\infty$. By Lemma 3 we may assume $M^*(x)$ to be nonincreasing. We may also assume $P(x)\geqq 0$ for $x>0$(8). Let ϵ be a given positive number ($<1/25$, say). Then, by hypothesis, we have

$$P(x) \geqq \exp\{(1-\epsilon)[M^*(x)]^{-1}\}$$

for sufficiently large x. Hence, if u is sufficiently large,

(8) It easily follows from the assumptions of the theorem that $P(u)\to\infty$ as $u\to\infty$, whence $f(s)\to\infty$ as $s\to 0$. Since adding a positive constant to $P(u)$ merely adds a constant to $f(s)$, we may assume $P(u)\geq 0$ for all positive u.

$$f(s) \geqq s \int_u^{u/\epsilon} P(x) e^{-sx} dx$$

$$\geqq s \int_u^{u/\epsilon} \exp\{(1-\epsilon)[M^*(x)]^{-1} - sx\} dx$$

$$\geqq \exp\{(1-\epsilon)[M^*(u)]^{-1}\} \cdot s \int_u^{u/\epsilon} \exp(-xs) dx,$$

since $M^*(u) \geqq M^*(x)$ for $x \geqq u$. Hence, for sufficiently large u,

$$f(s) \geqq \exp\{(1-\epsilon)[M^*(u)]^{-1} - us\}\{1 - \exp[-(1/\epsilon - 1)su]\}.$$

Now, for a *fixed* ϵ, we may take

$$u = \epsilon \frac{1}{s} M\left(\frac{1}{s}\right),$$

provided s is sufficiently small. Since $M(x) \to +\infty$ with x and $us = \epsilon M(1/s)$, we note that $us \to +\infty$ as $s \to 0$. Hence, for sufficiently small s,

$$f(s) \geqq \exp\left((1-2\epsilon)\left[M^*\left\{\epsilon \frac{1}{s} M\left(\frac{1}{s}\right)\right\}\right]^{-1} - \epsilon M\left(\frac{1}{s}\right)\right)$$

$$\geqq \exp\left((1-2\epsilon)(1-\epsilon)\left[M^*\left\{\frac{1}{s} M\left(\frac{1}{s}\right)\right\}\right]^{-1} - \epsilon M\left(\frac{1}{s}\right)\right),$$

since $(M^*(x)/M^*(\epsilon x)) > 1-\epsilon$ for sufficiently large x, by the definition of a slowly oscillating function. From the fact that

$$M^*\left\{\frac{1}{s} M\left(\frac{1}{s}\right)\right\} \cdot M\left(\frac{1}{s}\right) \to 1 \text{ as } s \to 0$$

we get, for sufficiently small s,

$$\left[M^*\left\{\frac{1}{s} M\left(\frac{1}{s}\right)\right\}\right]^{-1} > (1-\epsilon) M\left(\frac{1}{s}\right).$$

Hence

$$f(s) \geqq \exp\left\{(1-2\epsilon)(1-\epsilon)^2 M\left(\frac{1}{s}\right) - \epsilon M\left(\frac{1}{s}\right)\right\}$$

for sufficiently small s. Since $\epsilon < 1/25$, we get, for sufficiently small s,

$$f(s) \geqq \exp\left\{(1-5\epsilon) M\left(\frac{1}{s}\right)\right\}. \tag{8}$$

Next we proceed to obtain the estimate from above. Here again, we split $f(s)$ into three parts,

$$f(s) = s\left(\int_0^{(1/s)M(1/s)} + \int_{(1/s)M(1/s)}^{(2/s)M(1/s)} + \int_{(2/s)M(1/s)}^{\infty}\right)P(x)e^{-xs}dx$$

$$= J_1(s) + J_2(s) + J_3(s).$$

By hypothesis on $P(u)$, for sufficiently small s, we have

$$J_2(s) = s\int_{(1/s)M(1/s)}^{(2/s)M(1/s)} P(x)e^{-xs}dx$$

$$\leqq s\int_{(1/s)M(1/s)}^{(2/s)M(1/s)} \exp\{(1+\epsilon)[M^*(x)]^{-1} - xs\}dx$$

$$\leqq s\exp\left((1+\epsilon)^2\left[M^*\left\{\frac{1}{s}M\left(\frac{1}{s}\right)\right\}\right]^{-1}\right)\int_{(1/s)M(1/s)}^{\infty}\exp(-xs)dx,$$

since $[M^*(x)]^{-1} \leqq (1+\epsilon)[M^*\{(1/s)M(1/s)\}]^{-1}$ for $(1/s)M(1/s) \leqq x \leqq (2/s)M(1/s)$, if s is sufficiently small. Hence for sufficiently small s,

$$J_2(s) \leqq \exp\left((1+3\epsilon)\left[M^*\left\{\frac{1}{s}M\left(\frac{1}{s}\right)\right\}\right]^{-1} - M\left(\frac{1}{s}\right)\right)$$

$$\leqq \exp\left\{(1+5\epsilon)M\left(\frac{1}{s}\right) - M\left(\frac{1}{s}\right)\right\},$$

since $[M^*\{(1/s)M((1/s))\}]^{-1} < (1+\epsilon)M((1/s))$ for sufficiently small s. Hence, for sufficiently small s,

$$J_2(s) < \exp\left\{(5\epsilon)M\left(\frac{1}{s}\right)\right\}.$$

Next, by hypothesis on $P(u)$, for sufficiently small s, we have

$$J_3(s) \leqq s\int_{(2/s)M(1/s)}^{\infty}\exp\{(1+\epsilon)[M^*(x)]^{-1} - xs\}dx$$

$$= s\int_{(2/s)M(1/s)}^{\infty}\exp\left[-xs\left\{1 - \frac{1+\epsilon}{xsM^*(x)}\right\}\right]dx.$$

Now, by Lemma 2, for $x \geqq (2/s)M(1/s)$ and s sufficiently small, we have

$$\begin{aligned} xM^*(x) &> (1-\epsilon)\left(\frac{2}{s}\right)M\left(\frac{1}{s}\right)M^*\left\{\frac{2}{s}M\left(\frac{1}{s}\right)\right\} \\ &> (1-\epsilon)^2\left(\frac{2}{s}\right)M\left(\frac{1}{s}\right)M^*\left\{\frac{1}{s}M\left(\frac{1}{s}\right)\right\} \\ &> \frac{2}{s}(1-\epsilon)^3 \end{aligned}$$

by the definition of M^*. Hence, in the preceding integral

$$xsM^*(x) > 2(1-\epsilon)^3 > \frac{8}{5},$$

for sufficiently small s, since $\epsilon < 1/25$; and so for $x \geqq (2/s)M(1/s)$

$$1 - \frac{1+\epsilon}{xsM^*(x)} \geqq 1 - \frac{5(1+\epsilon)}{8} > \frac{1}{4},$$

for sufficiently small s. Hence, for sufficiently small s,

$$\begin{aligned} J_3(s) &\leqq s\int_{(2/s)M(1/s)}^{\infty} \exp\left(-\frac{xs}{4}\right)dx \\ &< 4. \end{aligned}$$

Thus, for sufficiently small s,

$$\begin{aligned} J_2(s) + J_3(s) &< \exp\left\{5\epsilon M\left(\frac{1}{s}\right)\right\} + 4 \\ &< \exp\left\{6\epsilon M\left(\frac{1}{s}\right)\right\}. \end{aligned} \tag{9}$$

Now

$$\begin{aligned} J_1(s) &= s\int_0^{(1/s)M(1/s)} P(x)e^{-xs}dx \\ &= s\int_0^{t_0} P(x)e^{-xs}dx + s\int_{t_0}^{(1/s)M(1/s)} P(x)e^{-xs}dx, \end{aligned}$$

where t_0 is chosen such that

$$P(u) < \exp\{(1+\epsilon)[M^*(u)]^{-1}\} \qquad \text{for } u \geqq t_0.$$

Then, if s is small enough so that $(1/s)M(1/s) > t_0$, we have

$$J_1(s) \leqq s\int_0^{t_0} P(x)e^{-xs}dx + s\int_{t_0}^{(1/s)M(1/s)} \exp\{(1+\epsilon)[M^*(x)]^{-1} - xs\}dx$$
$$\leqq K + \exp\left((1+\epsilon)\left[M^*\left\{\frac{1}{s}M\left(\frac{1}{s}\right)\right\}\right]^{-1}\right)\cdot s\int_0^\infty e^{-xs}dx,$$

where K is the maximum value of $P(x)$ on the interval $[0, t_0]$, since $M^*\{(1/s)M(1/s)\} \leqq M^*(x)$ for $t_0 \leqq x \leqq (1/s)M(1/s)$, by the monotonicity of M^*. Hence for sufficiently small s,

$$J_1(s) \leqq K + \exp\left((1+\epsilon)\left[M^*\left\{\left(\frac{1}{s}\right)M\left(\frac{1}{s}\right)\right\}\right]^{-1}\right)$$
$$\leqq \exp\left((1+2\epsilon)\left[M^*\left\{\left(\frac{1}{s}\right)M\left(\frac{1}{s}\right)\right\}\right]^{-1}\right).$$

Since

$$M^*\left\{\frac{1}{s}M\left(\frac{1}{s}\right)\right\}\cdot M\left(\frac{1}{s}\right) \to 1 \text{ as } s \to 0,$$
$$\left[M^*\left\{\frac{1}{s}M\left(\frac{1}{s}\right)\right\}\right]^{-1} \leqq (1+\epsilon)M\left(\frac{1}{s}\right)$$

for sufficiently small s. Hence

$$J_1(s) \leqq \exp\left\{(1+4\epsilon)M\left(\frac{1}{s}\right)\right\} \tag{10}$$

for sufficiently small s. Combining (9) and (10), we have

$$f(s) = J_1(s) + J_2(s) + J_3(s)$$
$$\leqq \exp\left\{(1+4\epsilon)M\left(\frac{1}{s}\right)\right\} + \exp\left\{6\epsilon M\left(\frac{1}{s}\right)\right\}$$

for sufficiently small s. Hence, for sufficiently small s,

$$f(s) \leqq \exp\left\{(1+5\epsilon)M\left(\frac{1}{s}\right)\right\}. \tag{11}$$

From (8) and (11) it follows that

$$\log f(s) \sim M\left(\frac{1}{s}\right) \text{ as } s \to 0,$$

which is the conclusion of Theorem IV.

IV. The Main Theorem and Its Consequences

4.1. The Main Theorem itself.

Main Theorem. *Suppose that*

(i) *$n(u)$ and $P(u)$ are functions on the non-negative reals and that $\int_0^R n(u) \cdot u^{-1}du$ and $\int_0^R P(u)du$ exist in the Lebesgue sense for every positive R,*

(ii) $\exp\{s\int_0^\infty e^{-su}/(1-e^{-su})\, n(u)du\} = s\int_0^\infty P(u)e^{-su}du$ *for all positive s,*

(iii) *$M(x)$ is defined by $M(x)=\int_a^x L(u)\cdot u^{-1}du$, where $L(u)$ is a slowly oscillating function defined for $u \geqq a$,*

(iv) *$(M(x), M^*(x))$ form a pair of conjugate slowly oscillating functions, and*

(v) *$M(x) \to +\infty$ as $x \to +\infty$.*

(A) *If $n(u) \sim L(u)$ as $u \to +\infty$ and $P(u)$ is nondecreasing, then* $\log P(u) \sim [M^*(u)]^{-1}$ *as $u \to +\infty$.*

(B) *Suppose L has the property that*

$$L(u) \sim C \exp\left\{\int_a^u \delta(t) \cdot t^{-1}dt\right\},$$

where $\delta(u)$ is nonincreasing, i.e., suppose $L(u) \sim J(u)$ as $u \to +\infty$, where $J(u)$ is differentiable and $u(J'(u)/J(u))$ is nonincreasing. Then if

$$\log P(u) \sim [M^*(u)]^{-1} \text{ as } u \to +\infty,$$

and $n(u)$ is nondecreasing, it follows that

$$n(u) \sim L(u) \text{ as } u \to +\infty.$$

Proof. Lemma 1 guarantees that $M(x)$ (cf. (iii)) is itself slowly oscillating. The existence and asymptotic uniqueness of $M^*(x)$ for a given M is guaranteed by de Bruijn's result mentioned in §2.3. Further the monotonicity of $M(x)$ follows from the fact that $L(x)$ is positive for $x \geqq a$.

Now, (A) follows from Theorems I and III and (B) follows from Theorems IV and II.

Remark. The supplementary condition in (B) is needed only in going from $I(u)=\int_0^u n(t)t^{-1}dt$ to $n(u)$, that is, in proving Theorem IIB. If we consider merely the relation between $I(u)$ and $P(u)$, we have the following more symmetrical result.

Weaker form of the main theorem. *Suppose that hypotheses* (i), (ii), *and* (iv) *of the Main Theorem hold and that $M(x)$ is nondecreasing.*

(A′) *If $\int_0^u n(t)t^{-1}dt \sim M(u)$ as $u \to +\infty$ and $P(u)$ is nondecreasing, then* $\log P(u) \sim [M^*(u)]^{-1}$ *as $u \to +\infty$.*

(B′) *If* $\log P(u) \sim [M^*(u)]^{-1}$ *as $u \to +\infty$ and $n(u)$ is nondecreasing, then $\int_0^u n(t)t^{-1}dt \sim M(u)$ as $u \to +\infty$.*

Proof. (A′) follows from Theorems IB and III, while (B′) follows from Theorems IV and IIA.

4.2. Consequences of the main theorem in partition problems.

In the case of partitions the functions $n(u)$ and $P(u)$ are automatically nondecreasing and trivially satisfy the required conditions of integrability. Thus we have the following.

COROLLARY I. *Suppose $n(u)$ and $P(u)$ are defined from a set Λ of positive real numbers and a non-negative-valued function $\psi(k)$ on the positive integers as mentioned in the introduction. Suppose $L(u)$, $M(u)$, $M^*(u)$ are as in the Main Theorem.*

(i) *If $n(u) \sim L(u)$ as $u \to +\infty$, then $\log P(u) \sim [M^*(u)]^{-1}$ as $u \to +\infty$.*

(ii) *If L has the property that*

$$L(u) \sim C \exp\left\{\int_a^u \delta(t) \cdot t^{-1} dt\right\}$$

where $\delta(u)$ is nonincreasing, i.e., if $L(u) \sim J(u)$ as $u \to +\infty$, where $J(u)$ is differentiable and $u(J'(u)/J(u))$ is nonincreasing, then

$$\log P(u) \sim [M^*(u)]^{-1} \quad \text{as} \quad u \to +\infty$$

implies

$$n(u) \sim L(u) \quad \text{as} \quad u \to +\infty.$$

It should be pointed out that the condition on L required in (ii) is automatically satisfied by many of the familiar slowly oscillating functions, e.g., if

(a) $$L(u) = K(\log u)^{\alpha_1}(\log_2 u)^{\alpha_2} \cdots (\log_k u)^{\alpha_k}$$

with the first nonvanishing α positive, or

(b) $$L(u) = K_1 \exp\{K_2(\log u)^{\beta_1}(\log_2 u)^{\beta_2} \cdots (\log_m u)^{\beta_m}\}$$

with $0 < \beta_1 < 1$. Here $\log_k u$ denotes the kth iterated logarithm.

4.3. We shall consider the two types of functions mentioned above, in some detail.

(a) It is easy to see that for

$$L(u) = K(\log u)^{\alpha_1}(\log_2 u)^{\alpha_2} \cdots (\log_k u)^{\alpha_k}$$

with the first nonvanishing α positive, we have

$$\begin{aligned} M(u) &= \int_a^u \frac{L(t)}{t}\, dt \\ &\sim \frac{K}{\alpha_1 + 1} (\log u)^{\alpha_1+1}(\log_2 u)^{\alpha_2} \cdots (\log_k u)^{\alpha_k} \text{ as } u \to \infty. \end{aligned}$$

(Applying l'Hospital's rule to the fraction formed with $\int_a^u (L(t)/t)dt$ as numerator and the expression on the right as denominator, we get the result.)

It is easy to verify that

$$M\left(\frac{u}{M(u)}\right)\cdot\frac{1}{M(u)} \to 1 \text{ as } u \to +\infty.$$

Hence,

$$M^*(u) \sim \{M(u)\}^{-1} \text{ as } u \to +\infty.$$

This is an example of the case where $n=1$ in Békéssy's remark mentioned in §2.3. In this special case, Corollary I reduces to the following form.

COROLLARY I*. *Suppose $n(u)$ and $P(u)$ are defined from a set of positive real numbers and a non-negative-valued function $\psi(k)$ on the positive integers as mentioned in the introduction. Suppose*

$$L(u) = K(\log u)^{\alpha_1}(\log_2 u)^{\alpha_2} \cdots (\log_k u)^{\alpha_k},$$

where the first nonvanishing α is positive and $\log_k u$ denotes the kth iterated logarithm, and

$$M(u) = \int_a^u \frac{L(t)}{t}\,dt$$

$$\sim \frac{K}{\alpha_1 + 1}(\log u)^{\alpha_1+1}(\log_2 u)^{\alpha_2} \cdots (\log_k u)^{\alpha_k} \text{ as } u \to +\infty.$$

Then

$$n(u) \sim L(u) \text{ as } u \to +\infty,$$

if and only if

$$\log P(u) \sim M(u) \text{ as } u \to +\infty.$$

It is easily seen (from Corollary I) that the result of Corollary I* is still valid in the case where all the α's are zero, i.e., when $L(u)=K$.

As an application of Corollary I* we can take $\Lambda = \{1, r, r^2, \cdots\}$, $r>1$. Here

$$n(u) = \left[\frac{\log u}{\log r} + 1\right].$$

Then we get

$$\log P(u) \sim \frac{(\log u)^2}{2 \log r} \text{ as } u \to +\infty.$$

For more precise results in this particular case see [3; 7; 14; 15].

(b) Let us turn to the second example. If

$$L(u) = K_1 \exp\{K_2(\log u)^{\beta_1}(\log_2 u)^{\beta_2} \cdots (\log_m u)^{\beta_m}\}$$

with $0<\beta_1<1$, then

$$M(u) = \int_a^u \frac{L(t)}{t}\,dt$$

$$\sim \frac{1}{K_2\beta_1} L(u)\cdot\frac{(\log u)^{1-\beta_1}}{(\log_2 u)^{\beta_2}\cdots(\log_m u)^{\beta_m}} \text{ as } u \to +\infty$$

by l'Hospital's rule. Further if $(l-1)/l \leqq \beta_1 < l/(l+1)$, then it is possible to show that

$$\frac{M_{r+1}(u)}{M_r(u)} \to \begin{Bmatrix} \text{a limit} \neq 1 & \text{if } r < l \\ 1 & \text{if } r = l \end{Bmatrix} \text{ as } u \to +\infty,$$

where $M_r(u)$ is as defined in §2.3. Therefore

$$M^*(u) \sim \{M_l(u)\}^{-1} \text{ as } u \to +\infty.$$

This is an example of the case where $n = l$ in Békéssy's remark.

In this case, Corollary I can be rewritten as follows.

COROLLARY I**. *Suppose $n(u)$ and $P(u)$ are defined from a set of positive real numbers and a non-negative-valued function $\psi(k)$ on the positive integers as mentioned in the introduction. Suppose*

$$L(u) = K_1 \exp\{K_2(\log u)^{\beta_1}(\log_2 u)^{\beta_2} \cdots (\log_m u)^{\beta_m}\}$$

where $\beta_1>0$ and $(l-1)/l \leqq \beta_1 < l/(l+1)$, l being a positive integer, and

$$M(u) = \int_a^u \frac{L(t)}{t}\,dt$$

$$\sim \frac{1}{K_2\beta_1}\cdot L(u)\cdot\frac{(\log u)^{1-\beta_1}}{(\log_2 u)^{\beta_2}\cdots(\log_m u)^{\beta_m}} \textit{ as } u \to +\infty,$$

then

$$n(u) \sim L(u) \textit{ as } u \to +\infty$$

if and only if

$$\log P(u) \sim M_l(u) \textit{ as } u \to +\infty,$$

where $M_l(u)$ is as defined in §2.3.

4.4. Following Kohlbecker we remark that if $\lambda_1, \lambda_2, \cdots$ are positive *integers* and if the greatest common divisor of those λ_i for which $\psi(i) \geqq 1$ is unity, then an asymptotic relation for $\log P(n)$ of the sort given in Corollaries I or I* or I** (except for the case $L(u) \sim$constant in Corollary I or I*) is equivalent to the same asymptotic relation for $\log p(n)$ as n goes to infinity through *integral* values. For in this case there is a positive integer c such that

$$P(n) \geqq p(n) \geqq \frac{P(n-c)}{n-c+1}$$

for integral $n \geqq c$, from which the asserted equivalence follows. *We must assume that* $L(u) \to +\infty$ *for this*, since if $L(u)$ is bounded we have $\log P(n) = O(\log n)$ and then a term in $\log n$ is not negligible.

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