LIKELIHOOD RATIOS FOR DIFFUSION PROCESSES WITH SHIFTED MEAN VALUES

BY

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Let the stochastic process $x(t)$ be the solution of the diffusion equation

$$x(t) = \int_0^t a(\tau, x(\tau))d\tau + \int_0^t \sigma(\tau)dy(\tau), \qquad 0 \leqq t \leqq 1,$$

$$x(0) = 0,$$

where $y(t)$ is Brownian motion and a, a_2 and σ are continuous real-valued functions satisfying

$$0 < \epsilon \leqq \sigma(t) \leqq \frac{1}{\epsilon} \tag{1}$$

and

$$a(t, x) = \int_{-\infty}^{\infty} e^{i\mu x} A(\mu, t)d\mu,$$

$$\frac{\partial}{\partial x} a(t, x) = a_2(t, x) = \int_{-\infty}^{\infty} i\mu e^{i\mu x} A(\mu, t)d\mu, \tag{2}$$

$$\int_{-\infty}^{\infty} (1 + |\mu|) |A(\mu, t)| du \leqq K < \infty.$$

The existence and uniqueness of solutions to such equations is proved in [1, p. 277 ff]. Note that conditions (1) and (2) imply H1, H2, and H3 which are assumed there. Let F be the set of functions on the space of continuous paths from 0 to 1 of the form $f(x) = \hat{f}(x(t_1), \cdots, x(t_n))$ where $\hat{f}$ is a bounded function on R^n with bounded second derivatives and (t_j) are points in $[0, 1]$. For each real-valued function m on $[0, 1]$ satisfying

$$m(t) = \int_0^t m'(s)ds, \qquad \int_0^1 (m'(s))^2 ds < \infty, \tag{3}$$

we define a group T_α of transformations on F by

$$T_\alpha f(x) = \hat{f}(x(t_1) + \alpha m(t_1), \cdots, x(t_n) + \alpha m(t_n))$$

and a set P_α of probability measures by closing the functionals $\int f dP_\alpha = E(T_\alpha f)$

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on F. (We will write P for P_0.) It is not true in general that the P_α are mutually absolutely continuous nor even that the T_α can be extended to measurable transformations [3]. We shall prove that this is true under assumptions (1), (2), and (3) and find a formula for $\log(dP_\alpha/dP)$.

We define an operator D on F by

$$(Df)(x) = \frac{\partial}{\partial\alpha}(T_\alpha f)(x)\big|_{\alpha=0},$$

and a function ϕ_0 in $L_2(P)$ by

$$\phi_0(x) = \int_0^1 \frac{m'(t) - m(t)a_2(t, x(t))}{\sigma(t)}\, dy(t).$$

The existence of the stochastic integral ϕ_0 follows from conditions (1), (2) and (3) [1, p. 426 ff].

LEMMA 1. *For any* $f \in F$

$$\int \phi_0 T_\alpha f dP = \frac{\partial}{\partial\alpha}\int T_\alpha f dP.$$

Proof. Suppose $0 < t_1 < t_2 < \cdots < t_n < 1$ and define

$$f(\lambda, t) = \exp\left\{ i\sum_{j=1}^{n} \lambda_j x(t_j) + i\lambda x(t)\right\}$$

and

$$g(\lambda, t) = \int \phi_0 f(\lambda, t) dP - i\left(\sum_{j=1}^{n}\lambda_j m(t_j) + \lambda m(t)\right)\int f(\lambda, t) dP$$

for all λ and $t_n \leq t \leq 1$. The sample functions of the process $x(t)$ are almost all continuous and this plus the dominated convergence theorem implies that g is continuous in the pair (λ, t). Writing d/dt for the right hand derivative we have

$$\begin{aligned}\frac{d}{dt} g(\lambda, t) = \lim_{\delta\to 0^+}\Bigg[&\int \phi_0 f(\lambda, t)\frac{(e^{i\lambda\delta x} - 1)}{\delta} dP \\ &- i\left(\sum_{j=1}^{n}\lambda_j m(t_j) + \lambda m(t)\right)\int f(\lambda, t)\frac{(e^{i\lambda\delta x} - 1)}{\delta} dP \\ &- i\lambda\frac{(m(t+\delta) - m(t))}{\delta}\int f(\lambda, t) dP \\ &- i\lambda(m(t+\delta) - m(t))\int f(\lambda, t)\frac{(e^{i\lambda\delta x} - 1)}{\delta} dP\Bigg].\end{aligned}$$

The first term is

$$\lim_{\delta\to 0^+}\frac{1}{\delta}\int \phi_0 f(\lambda, t)\left[i\lambda\int_t^{t+\delta} a(\tau, x(\tau))d\tau + i\lambda\int_t^{t+\delta}\sigma(\tau)dy(\tau)\right.$$
$$\left.-\frac{\lambda^2}{2}\left(\int_t^{t+\delta}\sigma(\tau)dy(\tau)\right)^2\right]dP$$

$$= i\lambda\int \phi_0 f(\lambda, t)a(t, x(t))dP$$
$$+\lim_{\delta\to 0^+}\frac{i\lambda}{\delta}\int_t^{t+\delta}d\tau\int f(\lambda, t)[m'(\tau) - m(\tau)a_2(\tau, x(\tau))]dP$$
$$-\lim_{\delta\to 0^+}\frac{\lambda^2}{2\delta}\int_t^{t+\delta}\sigma^2(\tau)d\tau\int f(\lambda, t)\left[\int_0^t \frac{m'(\tau) - m(\tau)a_2(\tau, x(\tau))}{\sigma(\tau)}dy(\tau)\right]dP$$

using dominated convergence to get the first subterm and properties of the stochastic integral to get the other two. Further

$$\lim_{\delta\to 0^+}\frac{i\lambda}{\delta}\int_t^{t+\delta}d\tau\int f(\lambda, t)[m'(\tau) - m(\tau)a_2(\tau, x(\tau))]dP$$
$$= \lim_{\delta\to 0^+}\frac{i\lambda}{\delta}(m(t+\delta) - m(t))\int f(\lambda, t)dP - i\lambda m(t)\int f(\lambda, t)a_2(t, x(t))dP$$

and

$$-\lim_{\delta\to 0^+}\frac{\lambda^2}{2\delta}\int_t^{t+\delta}\sigma^2(\tau)d\tau\int f(\lambda, t)\left[\int_0^t \frac{m'(\tau) - m(\tau)a_2(\tau, x(\tau))}{\sigma(\tau)}dy(\tau)\right]dP$$
$$= \frac{-\lambda^2\sigma^2(t)}{2}\int f(\lambda, t)\phi_0 dP.$$

A similar calculation shows that

$$\lim_{\delta\to 0^+}\int f(\lambda, t)\frac{(e^{i\lambda\delta x} - 1)}{\delta}dP = i\lambda\int f(\lambda, t)a(t,x(t))dP - \frac{\lambda^2\sigma^2(t)}{2}\int f(\lambda,t)dP.$$

Incorporating these in the original formula gives

$$\frac{d}{dt}g(\lambda, t) = \frac{-\lambda^2\sigma^2(t)}{2}g(\lambda, t) + i\lambda\int \phi_0 f(\lambda, t)a(t, x(t))dP$$
$$- i\lambda m(t)\int f(\lambda, t)a_2(t, x(t))dP$$
$$- i\lambda\left(i\sum_{j=1}^n \lambda_j m(t_j) + i\lambda m(t)\right)\int f(\lambda, t)a(t, x(t))dP.$$

Using the Fourier transforms of a and a_2 and interchanging $d\mu$ and dP integrations gives

$$\frac{d}{dt} g(\lambda, t) = i\lambda \int_{-\infty}^{\infty} g(\lambda + \mu, t) A(\mu, t) d\mu - \frac{\lambda^2\sigma^2(t)}{2} g(\lambda, t).$$

Since $|g(\lambda, t)| \leqq B + C|\lambda|$

$$\frac{d}{dt}(|g(\lambda, t)|^2) = 2 \operatorname{Re} i\lambda\overline{g(\lambda, t)} \int_{-\infty}^{\infty} g(\lambda + \mu, t) A(\mu, t) d\mu - \frac{\lambda^2\sigma^2(t)}{2} |g(\lambda, t)|^2$$

$$\leqq (B_1|\lambda| + C_1|\lambda|^2)|g(\lambda, t)| - \frac{\lambda^2\epsilon^2}{2} |g(\lambda, t)|^2.$$

Suppose now $n=0$ so that $|g(\lambda, 0)| = 0$. If, for fixed λ, $|g(\lambda, t)|$ takes its maximum at $t(\lambda) > 0$ and not before, then every interval to the left of $t(\lambda)$ contains a point where the right hand derivative of $|g(\lambda, t)|^2$ is non-negative; in particular this must be so for the t where $|g(\lambda, t)|$ is a minimum for this interval. At each of these points $|g(\lambda, t)| \leqq B_2 + C_2|\lambda|^{-1}$ from the above inequality, and since $\max|g(\lambda, t)|$ is a limit of these this inequality holds at all points of the interval. This implies that $g(\lambda, t)$ is bounded so that $h(t) = \sup_\lambda |g(\lambda, t)|$ is a bounded measurable function and

$$\frac{d}{dt}(|g(\lambda, t)|^2) \leqq B_3|\lambda g(\lambda, t)| h(t) - \frac{\epsilon^2}{2} |\lambda g(\lambda, t)|^2.$$

Now by a similar argument $|\lambda g(\lambda, t)| \leqq B_4 h(t) \leqq B_5$ so that $d|g(\lambda, t)|^2/dt \leqq B_4 h^2(t)$. It also follows from this that h is continuous since $|h(t) - h(t_0)| \leqq 2B_5/|\lambda_0| + |\sup_{|\lambda| \leq |\lambda_0|} |g(\lambda, t)| - \sup_{|\lambda| \leq |\lambda_0|} |g(\lambda, t_0)||$ and this can be made arbitrarily small by choosing first $|\lambda_0|$ large enough and then t close enough to t_0 since the function $\sup_{|\lambda| \leq |\lambda_0|} |g(\lambda, t)|$ is continuous. Now it is easily shown that $|g(\lambda, t)|^2 \leqq C_4 \int_0^t h^2(s) ds$ so that $h^2(t) \leqq C_4 \int_0^t h^2(s) ds$ and hence that $h = g = 0$. If we have proved that $g(\lambda, t) = 0$ for $n \leqq N$, then $g(\lambda, t_N) = 0$ and the argument above can be used on the interval $[t_N, 1]$ yielding an inductive proof that $g(\lambda, t) = 0$.

The above argument proves the lemma for $f(x) = \hat{f}(x(t_1), \cdots, x(t_n))$ when $\hat{f} = \exp\{i \sum_j \lambda_j x_j\}$ and it follows from this, for any a and b, that $\int_a^b d\alpha \int \phi_0 T_\alpha f dP = \int (T_b f - T_a f) dP$ for such f. An application of Fubini's theorem extends this relation to all $\hat{f}$ whose Fourier transforms are bounded and compactly supported, and hence to the algebra generated by these functions and the function 1 for which the lemma holds trivially. By the Stone-Weierstrass theorem such functions are uniformly dense in the algebra of functions continuous on R^n and at ∞, so the integrated relation holds for all such functions. Since any $\hat{f}$ in F can be approximated by a uniformly bounded set (f_n) of continuous functions of compact support converging at every (finite) point to $\hat{f}$ this relation holds for all of F. The proof is now completed by differentiating this relation with respect to b which can be done since $T_a f$ is L_2 continuous for f in F.

Lemma 1 also holds for the conditional expectation of ϕ_0 on the field generated by the $x(t)$ for t in $[0, 1]$ which we shall call ϕ. For every f in F and $\alpha \geqq 0$ the operator $V_f(\alpha)$ defined by

$$V_f(\alpha)g = \exp\left\{\frac{1}{2}\int_0^\alpha T_{-\beta}fd\beta\right\} T_{-\alpha}g$$

takes F into itself and satisfies $V_f(\alpha)V_f(\beta) = V_f(\alpha+\beta)$ and $V_f(0) = I$. It is easily shown using Lemma 1 that

$$\frac{\partial}{\partial\alpha}\int V_f(\alpha)gdP = \int\left(\frac{f}{2} - \phi\right)V_f(\alpha)gdP.$$

LEMMA 2. *For any N there exists a sequence (f_n) from F converging to $\min(\phi, N) = \phi_N$ in $L_2(P)$ and satisfying $\sup f_n \leqq N$. For any such sequence (f_n), any g in F, and any $\alpha \geqq 0$ the sequence $V_{f_n}(\alpha)g$ has a unique limit $D_N(\alpha)T_{-\alpha}g$ in $L_2(P)$. This convergence for fixed g is uniform in α on every finite interval.*

Proof. Since F is dense in $L_2(P)$ there is a sequence (g_n), $g_n(x) = \hat{g}_n(x(t_1), \cdots, x(t_j))$ from F converging to ϕ_N. Then $\min(g_n, N)$ also converges to ϕ_N and the desired sequence can be gotten by convoluting $\min(\hat{g}_n, N)$ with a sequence (e_n) of functions having bounded second derivatives and supports contained in sufficiently small neighborhoods of 0. It will be sufficient to prove the remainder of the lemma for $g = 1$. Setting $V_f(\alpha)1 = D_f(\alpha)$, we have

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\int &\left| D_{f_n}(\alpha) - D_{f_m}(\alpha)\right|^2 dP \\
&= \frac{\partial}{\partial\alpha}\int [D_{2f_n}(\alpha) - 2D_{f_n+f_m}(\alpha) + D_{2f_m}(\alpha)]dP \\
&= \int [(f_n - \phi)D_{2f_n}(\alpha) - (f_n + f_m + 2\phi)D_{f_n+f_m}(\alpha) + (f_m \quad \phi)D_{2f_m}(\alpha)]dP \\
&= \int [(f_n - \phi_N) + (\phi_N - \phi)][D_{f_n}(\alpha) - D_{f_m}(\alpha)]^2 \\
&\quad + \int (f_m - f_n)(D_{2f_m}(\alpha) - D_{f_n+f_m}(\alpha))dP \\
&\leqq [\|f_n - \phi_N\| + \|f_m - f_n\|]4e^{N\alpha}
\end{aligned}$$

which implies Lemma 2.

LEMMA 3. *$\|D_N(\alpha)T_{-\alpha}g\| \leqq \|g\|$ for any g in F. If $V_N(\alpha)$ is the extension of* *this operator to $L_2(P)$ then $V_N(\alpha)$ is a strongly continuous semigroup satisfying $\|V_N(\alpha)\| \leq 1$.*

Proof.

$$\frac{\partial}{\partial\alpha}\int |V_{f_n}(\alpha)g|^2 dP = \int (f_n - \phi)(V_{2f_n}(\alpha)(g^2))dP$$
$$\leq \|f_n - \phi_N\| e^{N\alpha} \sup |g(x)|$$

and this implies the first assertion. It is easily shown that $V_{f_n}(\alpha)g$ is strongly continuous for any g in F and this plus uniform convergence in α implies the strong continuity of $V_N(\alpha)g$ for g in F, but since F is dense in $L_2(P)$ and $\|V_N(\alpha)\| \leq 1$ this is sufficient to prove the strong continuity of $V_N(\alpha)$. Finally for g in F

$$\begin{aligned}\|V_N(\alpha)V_N(\beta)g - V_N(\alpha+\beta)g\| &\leq \|V_N(\alpha)(V_N(\beta)g - V_{f_n}(\beta)g)\| \\ &\quad + \|(V_N(\alpha) - V_{f_n}(\alpha))V_{f_n}(\beta)g\| + \|V_{f_n}(\alpha+\beta)g - V_N(\alpha+\beta)g\| \\ &\leq \|V_N(\beta)g - V_{f_n}(\beta)g\| + e^{N\beta}\sup|g(x)|\,\|V_N(\alpha)1 - V_{f_n}(\alpha)1\| \\ &\quad + \|V_{f_n}(\alpha+\beta)g - V_N(\alpha+\beta)g\|,\end{aligned}$$

which can be made arbitrarily small and this trivially implies the semigroup equality for all elements of $L_2(P)$.

On F define $\hat{A}_N f = (1/2)\phi_N f - Df$ and $\hat{A} f = (1/2)\phi f - Df$, and let A_N and A be their respective closures.

LEMMA 4. *A_N is the generator of $V_N(\alpha)$ and for any $\lambda > 0$, f in F and finite a and b*

$$(\lambda - A_N)\int_a^b e^{-\alpha\lambda}V_N(\alpha)f\,d\alpha = e^{-a\lambda}V_N(a)f - e^{-b\lambda}V_N(b)f.$$

Proof. $\int_a^b e^{-\alpha\lambda}V_{f_n}(\alpha)g\,d\alpha$ is in F and converges to $\int_a^b e^{-\alpha\lambda}V_N(\alpha)g\,d\alpha$ for all g in F. We have

$$\begin{aligned}(\lambda - A_N)&\int_a^b e^{-\alpha\lambda}V_{f_n}(\alpha)g\,d\alpha \\ &= \frac{1}{2}(f_n - \phi_N)\int_a^b e^{-\alpha\lambda}V_{f_n}(\alpha)g\,d\alpha + e^{-a\lambda}V_{f_n}(a)g - e^{-b\lambda}V_{f_n}(b)g \\ &\rightarrow e^{-a\lambda}V_N(a)g - e^{-b\lambda}V_N(b)g\end{aligned}$$

since

$$\left\|(f_n - \phi_N)\int_a^b e^{-\alpha\lambda}V_{f_n}(\alpha)g\,d\alpha\right\| \leq \|f_n - \phi_N\| \int_a^b e^{-\alpha\lambda}e^{\alpha N}\sup|g(x)|\,d\alpha \rightarrow 0.$$

Let $\bar{A}$ be the generator of $V_N(\alpha)$ and suppose that $\lambda > N$, then as $a \rightarrow 0$ and

$b \to \infty$ in the above we get $(\lambda - A_N)(\lambda - \bar{A})^{-1}g = g$ for all g in F and hence for all g and this implies that A_N contains $\bar{A}$. Finally, for g in F

$$\begin{aligned} V_N(\alpha)g &= g + \lim_{n\to\infty} \int_0^\alpha V_{f_n}(\beta)[(1/2)f_n g - Dg]d\beta \\ &= g + \lim_{n\to\infty} \int_0^\alpha V_{f_n}(\beta)[(1/2)\phi_N g - Dg]d\beta \\ &= g + \int_0^\alpha V_N(\beta) A_N g d\beta \end{aligned}$$

so $\bar{A}g = \lim_{\epsilon\to 0} ((V_N(\epsilon)g - g)/\epsilon) = A_N g$, which shows that $\bar{A}$ contains A_N and completes the proof.

LEMMA 5. *$V_N(\alpha)$ converges strongly to a strongly continuous semigroup $V(\alpha)$ with generator A.*

Proof. It is easily seen that $V_N(\alpha)1$ is a nondecreasing set of functions and since $\int V_N(\alpha)1dP \leqq \| V_N(\alpha)1\| \leqq 1$ it converges almost everywhere. This implies that $V_N(\alpha)f$ converges for all f in F and this plus the uniform boundedness of $\| V_N(\alpha)\|$ implies that all $V_N(\alpha)f$ converge. If $\lim_{N\to\infty} V_N(\alpha)f = V(\alpha)f$, then $\| V(\alpha)f\| \leqq \|f\|$ and $\| V(\alpha+\beta)f - V(\alpha)V(\beta)f\| = \| V(\alpha+\beta)f - V_N(\alpha+\beta)f\| + \| V_N(\alpha)(V_N(\beta) - V(\beta))f\| + \|(V_N(\alpha) - V(\alpha))V(\beta)f\|$ can be made arbitrarily small. For f in F

$$\begin{aligned} V(\alpha)f &= f + \lim_{N\to\infty} \int_0^\alpha V_N(\beta) A_N f d\beta \\ &= f + \lim_{N\to\infty} \int_0^\alpha V_N(\beta) A f d\beta \\ &= f + \int_0^\alpha V(\beta) A f d\beta \end{aligned}$$

which shows that $V(\alpha)f$ is strongly continuous for f in F and hence for all f. This equation also shows, as in the proof of Lemma 4, that the generator of $V(\alpha)$ contains A.

Now if K_N is any sequence which converges to ∞, $\int_0^{K_N} e^{-\alpha\lambda} V_N(\alpha)fd\alpha$ converges to $\int_0^\infty e^{-\alpha\lambda} V(\alpha)fd\alpha$. The proof of the lemma will be complete if we can show that $(\lambda - A)\int_0^\infty e^{-\alpha\lambda} V(\alpha)fd\alpha = f$ for all f and this will be implied if we can show it for f in F.

$$\begin{aligned} &(\lambda - A)\int_0^{K_N} e^{-\alpha\lambda} V_N(\alpha)fd\alpha \\ &\qquad = \frac{1}{2}(\phi_N - \phi)\int_0^{K_N} e^{-\alpha\lambda} V_N(\alpha)fd\alpha - e^{-K_N\lambda} V_N(K_N)f + f. \end{aligned}$$

Since the second term goes to 0 it will be sufficient to show that $\|\phi_N-\phi\|$ $\int_0^{K_N} e^{\alpha N} \sup |f| d\alpha$ goes to 0 for some K_N converging to ∞. If $(\phi_0)_N$ is the original ϕ_0 chopped at N then the conditional expectation of $(\phi_0)_N$ on the field generated by the $x(t)$'s is less than or equal to ϕ_N so $\|\phi-\phi_N\|$ $\leqq \|\phi_0-(\phi_0)_N\|$. We have $\phi_0=\psi+\eta(1)$ where $\psi=\int_0^1(m'(s)/\sigma(s))dy(s)$ and $\eta(t)=-\int_0^t(m(s)a_2(s, x(s))/\sigma(s))dy(s)$. Since $0\leqq\phi_0-(\phi_0)_{2N}\leqq\psi-\psi_N+\eta(1)$ $-\eta_N(1)$ the two pieces can be handled separately. ψ is a Gaussian random variable so $\|\psi-\psi_N\| \leqq e^{-BN^2}$ for some $B>0$. Also setting $F(t)$ $=-m(t)a_2(t, x(t))/\sigma(t)$ and defining $\xi(t)$ to be 1 if $\int_0^t F(s)dy(s)\geqq N$ and 0 otherwise we have

$$\eta(t) - \eta_N(t) = \int_0^t F(s)\xi(s)dy(s).$$

Hence, using $|F(t)| \leqq C$,

$$\begin{aligned} P(\eta(t) \geqq N+1) &\leqq E(|\eta(t) - \eta_N(t)|^2) \\ &= \int_0^t E(F^2(s)\xi(s))ds \leqq C^2\int_0^t P(\eta(s) \geqq N)ds \end{aligned}$$

and this relation plus $P(\eta(t)\geqq 0)\leqq 1$ yields by induction $P(\eta(t)\geqq N)$ $\leqq (At)^N/N!$. This gives

$$\|\eta(1) - \eta(1)_N\|^2 \leqq \sum_{k=N}^{\infty} \frac{A^k}{k!}(k+1-N)^2 \leqq \frac{A^N e^A}{(N-2)!}$$

and by Stirling's formula this goes to 0 like $e^{DN} \log^N$ for some D.

THEOREM. *Under assumptions* (1), (2), *and* (3):

(i) *The measures P_α are mutually absolutely continuous.*

(ii) *(T_α) can be extended to a group of measurable linear transformations on all measurable functions preserving bounds and satisfying $T_\alpha(fg)=(T_\alpha f)(T_\alpha g)$.*

(iii) $\log dP_\alpha/dP=\int_0^\alpha T_{-\beta}\phi d\beta$ *for any measurable version of $T_{-\beta}\phi$.*

(iv) *$V(\alpha)f=(dP_\alpha/dP)^{1/2}T_{-\alpha}f$ is a strongly continuous unitary group with generator A.*

Proof. Since A generates a semigroup the range of $iA-iI$ is all of $L_2(P)$. A similar argument using $\hat{T}_\alpha=T_{-\alpha}$ and $\hat{\phi}=-\phi$ will prove that the range of $iA+iI$ is all of $L_2(P)$. These facts plus the symmetry of iA show that iA is self adjoint and this proves (iv). The proof of (i) follows from this for if (f_n) is a sequence from F decreasing to 0 almost everywhere with respect to P_α then $T_\alpha f_n$ decreases to 0 almost everywhere with respect to P and $\int|f_n|^2dP_\beta$ $=\int|T(_\beta f_n)|^2dP=\int|V\beta-\alpha T_\beta f_n|^2dP=\int D(\beta-\alpha)^2|T_\alpha f_n|^2dP\to 0$ by the dominated convergence theorem. The extension of T_α to all functions measurable on the extended field is now straightforward. To prove (iii) assuming $\alpha>0$, choose a uniformly bounded sequence $(f_{n,M})$ from F with $\lim_{n\to\infty} f_{n,M} = \phi_{N,M}$ almost everywhere where

$$\phi_{N,M}(x) = \begin{cases} N \text{ if } \phi(x) > N, \\ \phi(x) \text{ if } -M \leqq \phi(x) \leqq N, \\ -M \text{ if } \phi(x) < -M. \end{cases}$$

Now $\int_0^\alpha T_{-\beta} f_{n,M} d\beta$ converges on n to $\int_0^\alpha T_{-\beta}\phi_{N,M} d\beta$ in $L_1(P)$ so a subsequence converges almost everywhere. n can be chosen as a function of M to make the sequence $f_{n(M),M}$ converge to ϕ_N in $L_2(P)$ and

$$\left\| \exp\left\{\frac{1}{2}\int_0^\alpha T_{-\beta} f_{n(M),M} d\beta\right\} - \exp\left\{\frac{1}{2}\int_0^\alpha T_{-\beta}\phi_{N,M} d\beta\right\}\right\| \to 0 \quad \text{as} \quad M \to \infty.$$

Thus

$$D_N(\alpha) = \lim \exp\left\{\frac{1}{2}\int_0^\alpha T_{-\beta} f_{n(M),M} d\beta\right\} = \exp\left\{\frac{1}{2}\int_0^\alpha T_{-\beta}\phi_N d\beta\right\}$$

and

$$D(\alpha) = \left[\frac{dP_\alpha}{dP}\right]^{1/2} = \lim \exp\left\{\frac{1}{2}\int_0^\alpha T_{-\beta}\phi_N d\beta\right\} = \exp\left\{\frac{1}{2}\int_0^\alpha T_{-\beta}\phi d\beta\right\}.$$

The restriction in the above theorem that σ not depend on $x(t)$ seems essential for without it one is led intuitively to the formula

$$\phi(x) = \int_0^1 \frac{m(t)\sigma_2(t, x(t))y'(t)}{\sigma(t)} dy(t) + \int_0^1 \frac{m'(t) - m(t)a_2(t, x(t))}{\sigma(t)} dy(t)$$

and the first integral is rather strongly "divergent."

The technique used in this paper was suggested by the solution to the corresponding problem where $x(t)$ is a Gaussian stochastic process. There, if $R(s, t) = E(x(s)x(t))$ is the autocorrelation function for $x(t)$ and if $m(s) = \int_0^1 R(s, t)dF(t)$ the infinitesimal generator again has the form $Af = (1/2)\phi f - Df$ where now $\phi(x) = \int_0^1 x(t)dF(t)$ [2]. We hope in future papers to apply this technique to other similar problems.

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