A SURFACE IS TAME IF ITS COMPLEMENT IS 1-ULC

BY
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1. Introduction. In this paper we show that a 2-manifold $M$ in Euclidean 3-space $E^3$ is tame if $E^3 - M$ is uniformly locally simply connected.

A closed subset $X$ of a triangulated manifold $Y$ is tame if there is a homeomorphism of $Y$ onto itself taking $X$ onto a polyhedron (geometric complex) in $Y$. If there is no such homeomorphism, $X$ is called wild. Examples of 2-manifolds wildly embedded in $E^3$ are found in [1; 8; 6].

An $n$-manifold is a separable metric space each of whose points lies in a neighborhood homeomorphic to Euclidean $n$-space. An $n$-manifold-with-boundary is a separable metric space each of whose points lies in a neighborhood whose closure is a topological $n$-cell. If $M$ is an $n$-manifold-with-boundary, we use $\text{Int } M$ to denote the set of points of $M$ with neighborhoods homeomorphic to Euclidean $n$-space and $\text{Bd } D$ to denote $M - \text{Int } M$. For example, if $D$ is a disk, $\text{Bd } D$ is a simple closed curve which is the rim of the disk. If we have a manifold embedded in a larger space and treat the manifold as a subset rather than a space, we insist that it be closed. If $S$ is a 2-sphere embedded in $E^3$, we use $\text{Int } S$ and $\text{Ext } S$ to denote the bounded and unbounded components of $E^3 - S$. The double meaning of the symbol $\text{Int}$ should not lead to confusion.

A subset $X$ of a manifold-with-boundary is locally tame at a point $p$ of $X$ if there is a neighborhood $N$ of $p$ and a homeomorphism of $N$ (the closure of $N$) onto a cell that takes $X \cdot \overline{N}$ onto a polyhedron. If the manifold-with-boundary is triangulated, we say that $X$ is locally polyhedral at $p$ if there is a neighborhood $N$ of $p$ such that $X \cdot \overline{N}$ is a polyhedron.

Suppose $D$ is a disk. We say that a map of $\text{Bd } D$ into a set $Y$ can be shrunk to a constant in $Y$ if the map can be extended to take $D$ into $Y$. If each map of $\text{Bd } D$ into $Y$ can be shrunk to a constant in $Y$, we say that $Y$ is simply connected. Also, $Y$ is locally simply connected at a point $p$ of $\overline{Y}$ if for each neighborhood $U$ of $p$ there is a neighborhood $V$ of $p$ such that each map of $\text{Bd } D$ into $V \cdot Y$ can be shrunk to a point in $U \cdot Y$. A metric space $Y$ is uniformly locally simply connected (or 1-ULC) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of $\text{Bd } D$ into a $\delta$ subset of $Y$ can be shrunk to a point on an $\varepsilon$ subset of $Y$. If $\overline{Y}$ is a compact subset of a metric space, it can be shown that $Y$ is 1-ULC if it is locally simply connected at each point of $\overline{Y}$.

Presented to the Society, January 28, 1958 under the title A surface $S$ is tame in $E^3$ if $E^3 - S$ is locally simply connected at each point of $S$; received by the editors March 1, 1961.

(*) Work on this paper was supported by contract NSF-G 11665.
The surfaces studied in solid geometry are tame. It is useful to have criteria for determining which surfaces are tame. We mention three such criteria.

1. A closed subset of a triangulated 3-manifold-with-boundary is tame if and only if it is locally tame at each of its points.

2. A 2-sphere in $E^3$ is tame if and only if it can be homeomorphically approximated from both sides.

3. A 2-manifold $M$ in a triangulated 3-manifold $M_3$ is tame if and only if $M_3 - M$ is locally simply connected at each point of $M$.

Criterion 1 is proved in [2] and [10]. Criterion 2 is proved in [5] and restated below as Theorem 0. Criterion 3 is Theorem 7 of the present paper.

The distance function is denoted by $\rho$. We shall make use of the following approximation theorem proved in [3].

**Approximation Theorem.** For each 2-manifold $M$ in a triangulated 3-manifold-with-boundary and each non-negative continuous function $f$ defined on $M$, there is a 2-manifold $M'$ and a homeomorphism $h$ of $M$ onto $M'$ such that

$$\rho(x, h(x)) \leq f(x) \quad (x \in M)$$

and $M'$ is locally polyhedral at $h(x)$ if $f(x) > 0$.

Another theorem that we shall use is the Side Approximation Theorem for 2-Spheres. It is proved by the same methods as the Approximation Theorem and its proof will be given in another paper [7].

**Side Approximation Theorem for 2-Spheres.** Each 2-sphere $S$ in $E^3$ can be polyhedrally approximated almost from either side—that is for each $\epsilon > 0$ and each component $U$ of $E^3 - S$ there is a homeomorphism $h$ of $S$ onto a polyhedral 2-sphere such that

- $h$ moves no point more than $\epsilon$ and
- $h(S)$ contains a finite collection of mutually exclusive disks each of diameter less than $\epsilon$ such that $h(S)$ minus the sum of the disks lies in $U$.

If $A$ and $B$ are two sets and there is a homeomorphism of $A$ onto $B$ that moves no point by more than $\epsilon$, we write

$$H(A, B) \leq \epsilon.$$

The following theorem is proved in [5].

**Theorem 0.** A 2-sphere $S$ in $E^3$ is tame if it can be homeomorphically approximated from both sides—that is, for each $\epsilon > 0$ and each component $U$ of $E^3 - S$, there is a 2-sphere $S'$ in $U$ such that

$$H(S, S') < \epsilon.$$

To show that a 2-sphere is tame, a first step might be to show that the hypothesis of Theorem 0 is met. As a step toward proving that a 2-sphere in $E^3$ is tame if its complement is 1-ULC, we prove the following.
**Theorem 1.** If $S$ is a 2-sphere in $E^3$ such that $\text{Int } S$ is 1-ULC, then for each $\epsilon > 0$, $S$ can be homeomorphically approximated from $\text{Int } S$—that is, for each $\epsilon > 0$ there is a 2-sphere $S'$ in $\text{Int } S$ such that

$$H(S, S') < \epsilon.$$ 

Our plan for proving Theorem 1 is to get a special cellular decomposition $T$ of $S$, get a homeomorphism of $S$ onto itself that pulls the boundaries of the cells of the decomposition $T$ into $\text{Int } S$, and finally pull the cells themselves into $\text{Int } S$.

It is hoped that the serious reader will understand the why of the attack as well as the details. Hence, we give our over-all plan of attack first and reserve epsilonics to last so that the reader can see why these particular $\epsilon$'s are used. When we need a close approximation, we let it be an $\epsilon$, approximation and decide later how small the $\epsilon$ would need to be to make the details work.

2. **Proof of Theorem 1.** a. Special cellular decomposition $T$ of $S$. We need a cellular decomposition $T$ of $S$ with the following properties.

The mesh of $T$ is less than $\epsilon_1$ ($\epsilon_1$ is a small number whose size is to be described later).

The collection of 2-cells of $T$ is the sum of three subcollections $A_1, A_2, A_3$ such that no two elements of $A_i$ ($i = 1, 2, 3$) intersect each other.

That for each $\epsilon_1 > 0$ there is such a cellular decomposition $T$ of $S$ follows from a consideration of a triangulation $T'$ of $S$ of mesh less than $\epsilon_1/2$. The vertices of $T'$ are swelled into 2-cells and become the elements of $A_1$. The parts of the 1-simplexes of $T'$ not in elements of $A_1$ are expanded into the elements of $A_2$. The closures of the parts of the 2-simplexes of $T'$ not in elements of $A_1$ or $A_2$ are the elements of $A_3$.

b. Pulling elements of $T$ partially into $S$. Suppose $T$ is a fixed special cellular decomposition of $S$ such as mentioned in the preceding section. The 1-skeleton of $T$ is the sum of the boundaries of the 2-cells in $T$ and is denoted by $K_1$. We select a small number $\epsilon_2$ whose size is described later. Then there is a polyhedral 2-sphere $S_1$ and a homeomorphism $h_1$ of $S$ onto $S_1$ such that $h_1$ moves no point more than $\epsilon_2$,

$h_1(K_1) \subset \text{Int } S$, and

$S_1$ contains a finite collection of mutually exclusive $\epsilon_2$ disks such that $S_1$ minus the sum of the interiors of these disks lies in $\text{Int } S$.

That for each $\epsilon_2$ there are such an $S_1$ and an $h_1$ follows from the Side Approximation Theorem of 2-Spheres. We let $\epsilon_2/2$ be the $\epsilon$ in the statement of that theorem and $S_1$ be the $S'$ guaranteed by the conclusion of that theorem. The homeomorphism $h_1$ is the homeomorphism $h$ guaranteed by that theorem.
followed by a homeomorphism of $S_1$ onto itself that moves no point by more
than $\varepsilon_2/2$ but pulls the image of $K_1$ off of the disks.

c. The next approximations to elements of $T$. For each 2-cell $D$ of $T$,
$h_1(D)$ is a first approximation to $D$. We note that $h_1(D)$ is homeomorphically
close to $D$ and $h_1(\text{Int } D)$ contains a finite collection $E_1, E_2, \ldots, E_n$ of mutually
exclusive $\varepsilon_2$ disks such that $h_1(D) - \sum \text{Int } E_i \subset \text{Int } S$. We suppose that
$h_1(\text{Bd } D)$ is a polygon.

The second approximation $h_2(D)$ to $D$ may not be quite as close homeo-
morphically to $D$ as is $h_1(D)$ but it will still be close. However, it will have the
advantage that the components of $S \cdot h_2(D)$ will have diameters much smaller
than $\varepsilon_2$ (which is an upper bound on the diameters of $E_1, E_2, \ldots, E_n$) and
simple closed curves in $\text{Int } S$ near the components of $S \cdot h_2(D)$ can be shrunk
to points in $\text{Int } S$ without hitting $h_1(K_i)$.

Let $\varepsilon_3$ be a very small positive number selected in a fashion to be described
later and $S'$ be a polyhedral 2-sphere which is homeomorphically within $\varepsilon_3$ of $S$ and which contains a finite collection of mutually exclusive $\varepsilon_3$ disks such
that $S'$ minus the sum of the interiors of these disks is contained in $\text{Int } S$. We select $\varepsilon_3$ so that $S' \cdot h_1(D) \subset \sum \text{Int } E_i$ and suppose that $S' \cdot h_2(D)$ is the
sum of a finite collection of mutually exclusive simple closed curves $J_1, J_2, \ldots, J_r$.

The $\varepsilon_2$ and $\varepsilon_3$ were selected so that each $J_i$ bounds a disk $F_i$ on $S'$ of small
diameter. We suppose that these disks $F_i$ are ordered by size with the small
ones first so that no $F_i$ contains an $F_{i+j}$.

The disk in $h_1(D)$ bounded by $J_i$ is first replaced by $F_{i+1}$ and then pushed
slightly to one side of $S'$ so as to reduce the number of components with the
intersection with $S'$. The process is continued by replacing disks in the
adjusted $h_1(D)$ by $F_{i+1}$'s and then pushing slightly so as to get a polyhedral
disk $h_2(D)$ which is close to $D$ homeomorphically and which lies on $\text{Int } S'$. We select $h_2$ so that it agrees with $h_1$ in a neighborhood of $\text{Bd } D$ and such that
the components of $S \cdot h_2(D)$ are not much bigger than those of $S' \cdot h_1(D)$. By
selecting $\varepsilon_3$ very small, we can insure that the components of $h_2(D) \cdot S$
are very small—in fact of diameter less than some preselected positive number $\varepsilon_4$.

Although we could have chosen the sum of the $h_2(D)$'s to be a 2-sphere,
we did not insist on this since at the third approximation of $D$, there seems
to be no easy way to prevent the approximating disks from intersecting at
interior points of each.

d. Third approximation to $D$. Since each component of $S \cdot h_2(D)$ is of
diameter less than $\varepsilon_4$, $h_2(\text{Int } D)$ contains a collection of mutually exclusive
disks $E'_1, E'_2, \ldots, E'_m$ which cover $S \cdot h_2(D)$ such that each $\text{Bd } E'_{i}$ lies close
to $S$ and is of diameter less than $\varepsilon_4$. Each $\text{Bd } E'_{i}$ lies in $\text{Int } S$ since $\text{Bd } D$ does.

Each $\text{Bd } E'_{i}$ is of such small diameter that it can be shrunk to a point on
a small subset of $\text{Int } S$ where $\varepsilon_4$ and $\varepsilon_3$ have been selected so that this subset
will not intersect $h_1(K_1) = h_2(K_1)$. Hence there is a map $g$ of $h_2(D)$ into $\text{Int } S$
such that \( gh_3(D) \) lies close to \( h_3(D) \), and \( g \) is the identity in a neighborhood of \( h_2(\text{Bd } D) \), \( gh_3(D) \) intersects \( h_2(K_i) \) only in \( h_2(\text{Bd } D) \), and \( gh_3(D) \) has no singular points near \( gh_3(\text{Bd } D) \). It follows from Dehn's lemma as proved by Papakyriokopoulos [11] that for each neighborhood \( U \) of the set of singular points of \( gh_3(D) \), there is a homeomorphism \( h_3 \) of \( D \) onto a polyhedral disk \( h_3(D) + U \) that agrees with \( h_2 \) in a neighborhood of \( \text{Bd } D \). The third approximation to \( D \) is \( h_3(D) \). The advantage it has over \( h_2(D) \) is that it lies in \( \text{Int } S \). The only thing that makes \( h_3(D) \) and \( D \) homeomorphically close is that each is of small diameter and their boundaries are homeomorphically close.

e. The fourth approximation to \( D \). Our task is now to untangle the \( h_3(D) \)'s so that their sum forms a 2-sphere in \( \text{Int } S \). We recall that the collection of 2-cells of \( T \) is the sum of three subcollections \( A_1, A_2, A_3 \) so that no two elements of any \( A_i \) have a point in common. We will have enlarged the \( h_1(D) \)'s so little as we changed them to \( h_2(D) \)'s and then to \( h_3(D) \)'s that two \( h_3(D) \)'s will not intersect if the corresponding \( D \)'s do not intersect.

For each element \( D \) of \( A_1 \), the fourth approximation \( h(D) \) to \( D \) is \( h_3(D) \).

We may suppose that for each element \( D \) of \( A_1 \), the intersection of \( h_3(\text{Int } D) = h(\text{Int } D) \) and the sum of the images under \( h_3 \) of the elements of \( A_2 \) is the sum of a finite collection of mutually exclusive simple closed curves \( J_1, J_2, \ldots, J_s \) such that \( J_i \) bounds a disk \( G_i \) in \( \text{Int } h(D) \) and the \( J_i \)'s are ordered according to the size of the disks \( G_i \) they bound in \( h(D) \), with the small ones coming first.

If \( J_i \) lies in an \( h_3(D') \) for an element \( D' \) of \( A_2 \), we replace the disk in \( h_3(D') \) by \( G_i \) and shove this replaced disk slightly to one side of \( h_3(D) \). This process is continued until for each element \( D' \) of \( A_2 \), there is an \( h \) defined on \( D' \) so that \( h(D) \) and the \( h(D') \)'s fit together only along their boundaries as they should.

Finally we turn to the elements of \( A_3 \). We suppose that for each element \( D \) of \( A_1 + A_2 \), \( h(\text{Int } D) \) intersects the sum of the images under \( h_3 \) of the elements of \( A_3 \) in the sum of a collection of mutually exclusive simple closed curves. These simple closed curves are eliminated one by one, starting at the inside in a manner already described. For each element \( D' \) of \( A_3 \), the resulting adjustment of \( h_3(D') \) is called \( h(D') \).

The sum of the \( h(D) \)'s is a 2-sphere in \( \text{Int } S \). We note that \( h = h_1 = h_2 = h_3 \) on \( K_1 \). We will cause \( h \) to be near the identity by picking the \( \epsilon_i \)'s so that the \( D \)'s and the \( h(D) \)'s are small and \( h \) is near the identity on \( K_1 \).

f. Epsilontics. In this section we explain the sizes for \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \) and the reasons for these selections. We recall that

\( \epsilon_1 \) limits the mesh of \( T \),

\( \epsilon_2 \) limits \( h_1 \) to be near the identity and limits the sizes of the \( E_i \)'s,

\( \epsilon_3 \) limits \( S' \) to be near \( S \) and limits the sizes of the disks in \( S' \) such that \( S' \) minus these disks lies in \( \text{Int } S \), and
\( \varepsilon_4 \) limits the sizes of the components of \( h_3(D) \cdot S \).

We choose

\[ \varepsilon_1 < \varepsilon/8, \]

where \( \varepsilon \) is the number mentioned in the statement of Theorem 1. This is the only restriction we place on \( \varepsilon_1 \).

Let \( \delta_1 \) be a positive number so small that the distance between two elements of \( T \) without a common point is more than \( \delta_1 \).

Our goal is to choose \( \varepsilon_2, \varepsilon_3, \varepsilon_4 \) so that for each 2-cell \( D \) of \( T \), the following conditions are satisfied.

- \( h_1(D) \) lies in a \( \delta_1/6 \) neighborhood of \( D \).
- \( h_2(D) \) lies in a \( \delta_1/6 \) neighborhood of \( h_1(D) \).
- \( h_3(D) \) lies in a \( \delta_1/6 \) neighborhood of \( h_2(D) \).

This will insure that if \( D_1, D_2 \) are two 2-cells of \( T \) without a common point, then \( h_3(D_1) \cdot h_3(D_2) = 0 \).

Since each point of \( h_3(D) \) lies within \( 3\delta_1/6 \) of a point of \( D \) and \( \delta_1 < \varepsilon_1 \),

\[ \text{diameter } h_3(D) < \varepsilon_1 + \delta_1 < 2\varepsilon_1. \]

Since \( h = h_3 \) on elements of \( A_1 \),

\[ \text{diameter } h(D) < 2\varepsilon_1 \text{ if } D \text{ is an element of } A_1. \]

In changing from \( h_3 \) to \( h \) on elements \( D \) of \( A_2 \), the adjustment was made so that \( h(D) \) lies within \( 2\varepsilon_1 \) of \( h_3(D) \). Hence,

\[ \text{diameter } h(D) < 6\varepsilon_1 \text{ if } D \text{ is an element of } A_2. \]

For each element \( D \) of \( A_3 \),

\[ h(D) \text{ lies within } 6\varepsilon_1 \text{ of } h_3(D). \]

Since each point of \( h(D) \) lies within \( 6\varepsilon_1 \) of a point of \( h_3(D) \), this point in turn lies within \( 3\delta_1/6 \) of a point of \( D \), and diameter \( D < \varepsilon_1 \), we find that \( h \) moves no point as much as \( 8\varepsilon_1 \). This is the reason we selected \( \varepsilon_1 < \varepsilon/8 \).

Let \( \varepsilon_2 \) be a positive number so small that each subset of \( S \) of diameter \( 3\varepsilon_2 \) lies in a disk on \( S \) of diameter less than \( \delta_1/10 \). We note that

\[ \varepsilon_2 < \delta_1/30 < \delta_1/6. \]

Since \( \varepsilon_2 < \delta_1/6 \), for each 2-cell \( D \) of \( T \), \( h_1(D) \) lies in a \( \delta_1/6 \) neighborhood of \( D \).

The more stringent condition that \( \varepsilon_2 < \delta_1/30 \) is used later to help insure that \( h_3(D) \) lies in a \( \delta_1/6 \) neighborhood of \( h_1(D) \).

Let \( \delta_2 \) be a positive number so small that for each 2-cell \( D \) of \( T \), \( \delta_2 \) is less than the distance between \( S \) and \( h_1(D) - \sum E_i \). We note that

\[ \delta_2 < \varepsilon_2. \]
Although we shall place more stringent conditions on $e_3$, we first consider the sizes of the components of $S' \cdot h_1(D)$ if we merely suppose 

$$e_3 < \delta_2.$$

Let $X$ be a component of $S' \cdot h_1(D)$ and $h'$ be a homeomorphism of $S'$ onto $S'$ that moves no point more than $e_3$. Then $X$ lies in an $E_i$ (which is of diameter less than $e_3$), $h'^{-1}(X)$ is of diameter less than $e_2 + 2e_3 < 3e_3$, $h'^{-1}(X)$ lies in a disk of diameter less than $\delta_i/10$, and the image of this disk under $h'$ is a disk in $S'$ of diameter less than $\delta_i/10 + 2e_3 < 5\delta_i/10$. Hence, the restrictions we have placed on $e_2$, $e_3$ are enough to insure that we can select $h_2$ so that $h_2(D)$ lies in a $\delta_i/6$ neighborhood of $h_1(D)$.

Let $\epsilon_4$ be a positive number so small that each simple closed curve in $E^3 - S$ of diameter less than $\epsilon_4$ can be shrunk to a point on a subset of $E^3 - S$ of diameter less than the minimum of $\delta_i/6$ and $\delta_i/2$. Note that $\epsilon_4$ does not depend on $e_3$. The $\text{Bd } E_i$'s are selected to lie within $\epsilon_4 < \delta_i/2$ of $S$. Also, the $\text{Bd } E_i$'s have diameters less than $\epsilon_4$ so that $gh_2(D)$ intersects $h_2(K_i)$ only in $h_2(\text{Bd } D)$. Furthermore, $h_3(D)$ lies in a $\delta_i/6$ neighborhood of $h_3(D)$.

The final restriction we place on $e_3$ is to insure that each component of $S' \cdot h_2(D)$ is of diameter less than $\epsilon_4$. Let $\delta_i$ be a number so small that each $\delta_i$ subset of $S$ lies in a disk in $E^3$ of diameter less than $\epsilon_4$. We suppose 

$$e_3 < \delta_2.$$

Each component $X$ of $S \cdot h_2(D)$ lies on $\text{Int } S'$ and is separated from the big component of $S - S'$ by a disk in $S'$ of diameter less than $e_4 < \delta_i$. Hence, diameter $X < \epsilon_4$.

3. **Conditions under which a 2-sphere is tame.** Tame 2-spheres in $E^3$ have uniformly locally simply connected complements. A converse is as follows.

**Theorem 2.** A 2-sphere $S$ in $E^3$ is tame if each component of $E^3 - S$ is 1-ULC.

**Proof.** It follows from Theorem 1 that for each positive number $\epsilon$ there is a 2-sphere $S'$ in $\text{Int } S$ such that 

$$H(S, S') < \epsilon.$$

Similarly, it follows that there is a 2-sphere $S''$ in $\text{Ext } S$ such that 

$$H(S, S'') < \epsilon.$$

That $S$ is tame then follows from Theorem 0.

**Corollary.** A 2-sphere $S$ in $E^3$ is tame if $E^3 - S$ is locally simply connected at each point of $S$.

4. **Retaining simple connectivity.** We shall be applying the Approximation Theorem to a 2-manifold to make part of the 2-manifold locally poly-
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hedral. We are interested in seeing what this does to the local simple connectivity of the complement. First we examine the effect of throwing away part of the 2-manifold.

**Theorem 3.** Suppose $M$ is a 2-manifold in $E^3$, $D$ is a disk in $M$, and $p$ is a point of $D$ at which $E^3 - M$ is locally simply connected. Then $E^3 - D$ is locally simply connected at $p$.

**Proof.** We only consider the case where $p \in \text{Bd } D$. Suppose $U$ is a given neighborhood of $p$. Let $V'$ be a neighborhood of $p$ such that each map of a circle into $V' \cdot (E^3 - M)$ can be shrunk to a point in $U \cdot (E^3 - M)$ and $V$ be a neighborhood of $p$ such that each pair of points of $V' \cdot (M - D)$ lie in an arc in $V' \cdot (M - D)$. We show that if $E$ is a plane circular disk and $f$ is a map of $\text{Bd } E$ into $V' \cdot (E^3 - D)$, then there is a map of $E$ into $U \cdot (E^3 - D)$ that agrees with $f$ on $\text{Bd } E$.

We want to simplify $f$ so that $M \cdot f(\text{Bd } E)$ does not have infinitely many components. Suppose $aa'$ is an arc on $\text{Bd } E$ and $bb'$ is an arc on $V' \cdot (M - D)$ such that $f(a) = b, f(a') = b'$, and $bb' + f(aa')$ lies in a convex subset of $V' \cdot (M - D)$. Then there is a homotopy $F_t$ on $aa'$ such that $F_0 = f, F_1$ is constant on $a$ and $a'$, $F_t(aa') = bb'$, and each $F_t(aa')$ lies in $V'.D$. Hence, we suppose with no loss of generality that $f^{-1}(M \cdot f(\text{Bd } E))$ is the sum of a finite number of arcs $a_1a_2, a_2a_3, \ldots, a_{2n-1}a_{2n}$ on $\text{Bd } E$ ordered so that there is no $a_i$ between any $a_i$ and $a_{i+1}$.

Let $q$ be the center of $E$ and $f(q)$ be any point of $V' \cdot (M - D)$. Extend $f$ to map the radius $qa_i$ of $E$ onto an arc in $V' \cdot (M - D)$ from $f(q)$ to $f(a_i)$. Since each component of $M$ divides $E^3$ into two pieces and any arc in $M$ can be approximated by arcs on either side of $M$, the map $f$ on the boundary of the sector $a_1a_{i+1}$ of $E$ can be extended to map an annulus ring in the sector one of whose boundary components is the boundary of the sector into $V' \cdot (E^3 - D)$ so that the image of the other boundary component of the annulus misses $M$. The map $f$ can be further extended to take the rest of the sector into $U \cdot (E^3 - M)$.

**Theorem 4.** Suppose $M$ is a 2-manifold in $E^3$, $D$ is a disk in $M$, $p$ is a point of $M$ at which $E^3 - M$ is locally simply connected, and $h$ is a homeomorphism of $M$ into $E^3$ such that $h$ is the identity on $D$ and $h(M)$ is locally polyhedral at each point of $h(M) - D$. Then $E^3 - h(M)$ is locally simply connected at $p$.

**Proof.** The only case we consider is the one in which $p$ is a point of $\text{Bd } D$. Suppose $U$ is a given neighborhood of $p$. Let $V'$ be a neighborhood of $p$ such that each closed set in $V' \cdot (h(M) - D)$ lies in a disk in $U \cdot (h(M) - D)$. It follows from Theorem 3 that there is a neighborhood $V$ of $p$ such that each map of a circle into $V' \cdot (E^3 - D)$ can be shrunk to a point in $V' \cdot (E^3 - D)$. We show
that if $E$ is a disk and $f$ is a map of $\text{Bd } E$ into $V \cdot (E^3 - h(M))$, then $f$ can be extended to map $E$ into $U \cdot (E^3 - h(M))$.

Let $f'$ be a map of $E$ into $V' \cdot (E^3 - D)$ that is an extension of $f$ on $\text{Bd } E$. Let $E'$ be the component of $E - f'^{-1}(h(M) \cdot f(E))$ containing $\text{Bd } E$. With no loss of generality we suppose that $E'$ is $E$ minus a finite collection of mutually exclusive subdisks $E_1, E_2, \ldots, E_n$ of $E$.

Since $f'(\text{Bd } E)$ lies on the interior of a polyhedral disk in $U \cdot (h(M) - D)$, it is possible to adjust $f'$ on disks in $E$ slightly larger than the $E_i$'s so as to take the larger disks slightly to one side of $h(M)$. The adjusted $f'$ is $f$ and takes $E$ into $U \cdot (E^3 - h(M))$.

**Question.** Would Theorem 4 be true if we supposed that $D$ were merely a closed subset of $M$ with only nondegenerate components rather than actually a disk in $M$?

5. Enlarging a disk to a 2-sphere. Not each disk in $E^3$ lies on a 2-sphere. An example of such a disk is obtained by taking a horizontal disk $D$ in $E^3$; removing two circular holes from Int $D$; adding tubes from the holes, one tube going up and the other down and around; and finally adding hooked disks as shown in [1]. The disk does not lie on a 2-sphere since its boundary cannot be shrunk to a point in the complement of the disk. Although it does not lie on a 2-sphere, it does lie on a torus as was pointed out to me by David Gillman. If instead of removing a pair of holes from the horizontal disk and replacing the holes with hooked wild disks, one had removed an infinite collection of pairs of holes converging to a boundary point of $D$ and replaced each pair of holes with wild disks hooked over the boundary of $D$, there would have resulted a wild disk in $E^3$ that does not lie on any 2-manifold in $E^3$.

The following result shows that each disk contains many small disks each of which lies on a 2-sphere.

**Theorem 5.** Suppose $M$ is a 2-manifold in $E^3$, $p$ is a point of $M$, and $U$ is a neighborhood of $p$. Then there is a disk $D$ in $M \cdot U$ and a 2-sphere $S$ in $U$ such that $p \in \text{Int } D \subset S$ and $S$ is locally polyhedral at each point of $S - D$.

**Proof.** Let $E$ be a disk in $M$ such that $p \in \text{Int } E$ and $C$ be a cube in $U$ whose interior contains $p$ and whose exterior contains $\text{Bd } E$. Let $D$ be a disk in $M \cdot \text{Int } C$ such that $p \in \text{Int } D$. It follows from the Approximation Theorem that there is a homeomorphism $h$ of $E$ into $E^3$ such that $h$ is the identity on $D$, $h$ takes $\text{Bd } E$ into Ext $C$, and $h(E)$ is locally polyhedral at each point of $h(E) - D$. We suppose with no loss of generality that $\text{Bd } C \cdot h(E)$ is the sum of a finite collection of mutually exclusive polygons.

Let $E'$ be the component of $h(E) - \text{Bd } C$ containing $D$. It is topologically equivalent to a 2-sphere minus the sum of a finite collection of mutually exclusive disks. By adding polygonal disks in $U \cdot (C + \text{Ext } C)$ to $E'$, one obtains the required 2-sphere $S$.

6. Locally tame subsets of 2-manifolds. A 2-sphere $S$ in $E^3$ may not be
locally tame at a point \( p \) even if \( E^3 - S \) is locally simply connected at \( p \) as shown by the following example.

**Example.** Consider a spherical 2-sphere \( S' \) and a sequence of mutually exclusive spherical disks \( E_1, E_2, \ldots \) in \( S' \) converging to a point \( p \) of \( S' \). Fox and Artin have described [8] a wild arc which is locally polyhedral except at one end point. For each \( E_i \), let \( A_i \) be such an arc reaching out from the center of \( E_i \) such that the arc is of diameter less than the radius of \( E_i \) and such that \( A_i \) intersects \( S \) only at the polyhedral end of \( A_i \). By replacing each \( E_i \) in \( S \) by a disk obtained by swelling up \( A_i \) as done in [7] one can obtain a 2-sphere \( S \) such that \( E^3 - S \) is locally simply connected at \( p \) even though \( S \) is not locally tame at \( p \).

**Theorem 6.** Suppose \( M_2 \) is a 2-manifold embedded in a 3-manifold \( M_3 \) and \( U \) is an open subset of \( M_2 \) such that \( M_3 - M_2 \) is locally simply connected at each point of \( U \). Then \( M_2 \) is locally tame at each point of \( U \).

**Proof.** Since local tameness is only a local property, we suppose that \( M_4 \) is \( E^3 \) and \( U \) is all of \( M_2 \). If this were not the case already we would take a homeomorphism \( h \) of a neighborhood of \( p \) in \( M_3 \) into \( E^3 \) such that this neighborhood did not intersect \( M_3 - U \).

It follows from Theorem 5 that there is a disk \( D \) in \( M_2 \) and a 2-sphere \( S \) such that \( p \in \text{Int} \ D \subseteq S \) and \( S \) is locally polyhedral at each point of \( S - D \). It follows from Theorem 4 that \( E^3 - S \) is locally simply connected at each point of \( S \). Since \( S \) is compact, \( E^3 - S \) is \( 1 \)-ULC. Theorem 2 implies that \( S \) is tame.

Since \( S \) is tame, \( M_2 \) is locally tame at \( p \).

Since a closed set in a 3-manifold is tame if it is locally tame [2; 10] we have the following result.

**Theorem 7.** A 2-manifold \( M_2 \) in a triangulated 3-manifold \( M_3 \) is tame if and only if \( M_3 - M_2 \) is locally simply connected at each point of \( M_2 \).

**Corollary.** A 2-manifold \( M_2 \) in a 3-manifold \( M_3 \) is tame if \( M_3 - M_2 \) is \( 1 \)-ULC.

7. **Tame 2-manifolds-with-boundaries.** In this section we extend our results about 2-manifolds to 2-manifolds-with-boundaries.

**Theorem 8.** Suppose \( M_2 \) is a 2-manifold-with-boundary embedded in a 3-manifold \( M_3 \) and \( U \) is an open subset of \( M_2 \) such that \( M_3 - M_2 \) is locally simply connected at each point of \( U \). Then \( M_2 \) is locally tame at each point of \( U \).

**Proof.** Since we only look at \( M_2 \) locally, we suppose that \( M_4 \) is \( E^3 \) and \( U \) is \( M_2 \). Since Theorem 6 takes care of points of \( \text{Int} \ M_2 \), we only show that \( M_2 \) is locally tame at a point \( p \) of \( \text{Bd} \ M_2 \).

Let \( D \) be a disk in \( M_2 \) such that \( \text{Bd} \ D \cdot \text{Bd} \ M_2 \) is an arc containing \( p \) as a non end point. An argument like that used in the proof of Theorem 3 shows
that $E^3 - D$ is locally simply connected at each point of $D$. We finish the proof of Theorem 8 by showing that $D$ is tame.

Since Theorem 6 shows that $D$ is locally tame at each point of Int $D$, it follows from [4; 9] that there is a homeomorphism $h$ of $E^3$ onto itself such that $h(D)$ is locally polyhedral at each point of $h($Int $D)$. Hence, we suppose with no loss of generality that $D$ is locally polyhedral at each point of Int $D$.

By pushing $D$ to one side at points of Int $D$, one obtains a disk $D'$ such that $\text{Bd} \ D = \text{Bd} \ D'$, $D'$ is locally polyhedral at points of Int $D'$, and $D + D'$ bounds a topological cube $C$. Since $E^3 - \text{Bd} \ C$ is locally simply connected at each point of $\text{Bd} \ C$, it follows from either Theorem 2 or 6 that $\text{Bd} \ C$ is tame. Hence $D$ is tame and $M_2$ is locally tame at $p$.

8. Extensions of Theorem 0. As pointed out in [5], we can use Theorem 6 to extend Theorem 0 as follows.

**Theorem 9.** A 2-manifold $M_2$ in a 3-manifold $M_3$ is locally tame at a point $p$ of $M_2$ if there is a disk $D$ with $p \in \text{Int} \ D \subset M_2$ such that for each positive number $\epsilon$, there are disks $D', D''$ on opposite sides of $M_2$ such that

$$H(D, D') < \epsilon, \quad H(D, D'') < \epsilon.$$ 

**Proof.** Theorem 9 follows from Theorem 6 when we show that $M_3 - M_2$ is locally simply connected at each point of Int $D$.

Suppose $E$ is a disk, $q$ is a point of Int $D$, and $f$ is a map of $\text{Bd} \ E$ into a small subset of $M_3 - M_2$ near $q$. Suppose $f$ is extended to map $E$ into a small subset of $M_3$. Let $D', D''$ be disks on opposite sides of $M_2$ which are homeomorphically close to $D$ and whose sum separates $f(\text{Bd} \ E)$ from $M_3 - f(E)$ in $f(E)$. If $E'$ is the component of $E - f^{-1}(D' + D'')$ containing $\text{Bd} \ E$, $f$ on $E'$ can be extended to map $E$ into a small subset of $f(E') + D' + D''$. Hence, $M_3 - M_2$ is locally simply connected at $q$.

Since locally tame closed subsets are tame in triangulated 3-manifolds [2; 10], we have the following as a corollary of Theorem 9.

**Corollary.** A 2-manifold $M_2$ in a triangulated 3-manifold $M_3$ is tame if and only if for each positive number $\epsilon$ the appropriate one of the following conditions is satisfied:

**Case 1.** If $M_2$ is two sided in some neighborhood $N$ of $M_2$ there are 2-manifolds $M', M''$ on opposite sides of $M_2$ in $N$ such that

$$H(M_2, M') < \epsilon, \quad H(M_2, M'') < \epsilon.$$ 

**Case 2.** If $M_2$ is one sided in each neighborhood of $M_2$ there is a connected double covering $M'$ of $M_2$ with projection map $\pi$ and a homeomorphism $h'$ of $M'$ into $M_3 - M_2$ such that

$$\rho(h(x), \pi(x)) < \epsilon, \quad x \in M'.$$
Neither Theorem 9 nor its corollary can be extended to 2-manifolds-with-boundaries. Besides having to speak with care about the two sides of a 2-manifold-with-boundary in a 3-manifold, one would have to contend with the example of Stallings [12] in which he describes an uncountable family of mutually exclusive wild disks in $E^3$. It would follow from an application of Theorem 9 that most of these disks are locally tame except on their boundaries.

References

6. ———, *A wild sphere each of whose arcs is tame*, Duke Math. J.

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