

FUNCTION-THEORETIC CHARACTERIZATION OF EINSTEIN SPACES AND HARMONIC SPACES⁽¹⁾

BY
AVNER FRIEDMAN

Introduction. Consider the following general problem: Let P be a function-theoretic property which holds at the points of a (local or global) Euclidean space E_n and which can be stated, formally, for points of a Riemannian space R_n . Find a statement Q which is an intrinsic (geometric) property of R_n such that P implies Q and Q implies P . Conversely, given Q , one may look for P .

In this paper we solve particular problems of the above type. We take for Q the statements that R_n (which is always assumed to have a positive definite metric) is an Einstein space, an harmonic space (to be defined in §3) with a particular fundamental solution, and a space with constant curvature. P then stands for various statements about the mean value of solutions of certain equations. Denoting by $M(u, x^0, R)$ the mean value of u on the geodesic sphere with center x^0 and radius R we obtain the following results:

An Einstein space is characterized (in §1) by

$$(0.1) \quad M(u, x^0, R) = u(x^0)(1 + O(R^3)), \quad \frac{\partial}{\partial R} M(u, x^0, R) = u(x^0)O(R^2)$$

for every $u \neq 0$, $\Delta u = 0$ in a neighborhood of any of its points x^0 , where Δ is the Laplace-Beltrami operator. Its scalar curvature ρ is determined (in §2) by (2.28) which holds for any $u \neq 0$, $\Delta u \neq 0$, $\Delta^2 u = 0$ in a neighborhood of any of its points x^0 .

The characterization of an harmonic space by

$$(0.2) \quad M(u, x^0, R) = u(x^0)$$

for every solution u of $\Delta u = 0$ in a neighborhood of any of its points x^0 was already proved by Willmore [13]. The fundamental solution $\phi(r)$ of an harmonic space is determined (in §3) by a function $A(r)$ via the equation (3.4), whereas $A(R)$ is characterized by the mean value formula

$$(0.3) \quad M(u, x^0, R) = u(x^0) + A(R)\Delta u(x^0)$$

which holds for every solution u of $\Delta^2 u = 0$ in a neighborhood of any point x^0 .

A space of constant curvature K is characterized (in §4) by (0.3) with $A(R)$ defined by

$$(0.4) \quad A'(R) = \int_0^R \left[\frac{\sin(RK^{1/2})}{\sin(rK^{1/2})} \right]^{n-1} dr, \quad A(0) = 0.$$

Presented to the Society, January 24, 1961; received by the editors December 29, 1960 and, in revised form, February 17, 1961.

⁽¹⁾ Prepared under Contract Nonr 710(16) (NR 044 004) between the Office of Naval Research and the University of Minnesota.

In §5 we discuss the characterization of Einstein metrics by means of mean value theorems with regard to solutions of $\Delta u + \lambda u = 0$ (λ constant) and also with regard to the set of eigenfunctions of Δ . Analogous results are derived in §6 for harmonic metrics.

General assumptions and notations. All the spaces are assumed to have a positive definite metric. The metric tensors are assumed to be sufficiently smooth. The dimension n is taken to be ≥ 3 ; all the results however can easily be extended to the case $n = 2$. We denote by R_n a Riemannian space, by A_n an Einstein space, by H_n an harmonic space and by K_n a space of constant curvature. All the considerations of this paper are local.

1. **Characterization of Einstein spaces.** Let (g_{ij}) be the metric tensor of R_n . The Laplace-Beltrami operator is defined by

$$\Delta u = \frac{1}{g^{1/2}} \frac{\partial}{\partial x^i} \left(g^{1/2} g^{ij} \frac{\partial u}{\partial x^j} \right) = g^{ij} u_{,ij} \quad (g = \det(g_{ij}))$$

where "comma" denotes covariant differentiation. Let x^0 be a fixed point in R_n and let r denote the geodesic distance from x^0 to a variable point x in R_n . Since our considerations are local, we shall assume, for simplicity, throughout this paper, that r exists for all x^0, x in R_n . We say that R_n is *Einsteinian* at x^0 if

$$R_{ij} = k g_{ij} \quad \text{at } x = x^0,$$

where R_{ij} is the Ricci tensor and k is a constant.

LEMMA 1. R_n is Einsteinian at x^0 if and only if

$$(1.1) \quad \Delta r = \frac{n-1}{r} + \beta r + F$$

where β is a constant and $F = O(r^2)$ as $r \rightarrow 0$.

REMARK. The lemma and the proof are valid also when the metric of R_n is indefinite.

Proof(²). Let Θ^i be the unit vector at x^0 tangent to the geodesic which connects x^0 to x . Then

$$(1.2) \quad x^i = \Theta^i r$$

are normal coordinates of x about x^0 as the origin. Let (\bar{g}_{ij}) be the metric tensor in the x^i coordinates. Denoting by $(g_{ij})_0$ the value of (g_{ij}) at x^0 ($(g_{ij})_0 = (\bar{g}_{ij})_0$) we have [11, p. 96]

$$(1.3) \quad r^2 = (g_{ij})_0 x^i x^j = \bar{g}_{ij} x^i x^j$$

and

(²) The author thanks the referee for suggesting the present proof which is somewhat simpler than the author's original proof.

$$(1.4) \quad rr_{,i} = (g_{ij})_0 x^j = \bar{g}_{ij} x^j \quad \left(r_{,i} = \frac{\partial r}{\partial x^i} \right).$$

Hence,

$$(1.5) \quad rr^i = x^i \quad (r^i = \bar{g}^{ij} r_{,j}).$$

Differentiating (1.5) covariantly we get

$$(rr^i)_{,j} = \delta_j^i + \Gamma_{kj}^i x^k$$

where Γ_{kj}^i are Christoffel's symbols with respect to (\bar{g}_{ij}) .

Contracting i, j :

$$(1.6) \quad r\Delta r + r_{,i} r^i = n + \Gamma_{ki}^i x^k.$$

Using $r_{,i} r^i = 1$ (which follows from (1.4), (1.3)), we get

$$(1.7) \quad r\Delta r = n - 1 + \Gamma_{ki}^i x^k.$$

But

$$\Gamma_{ki}^i = (\Gamma_{ki}^i)_0 + \left(\frac{\partial \Gamma_{ki}^i}{\partial x^j} \right)_0 x^j + O(r^2)$$

and

$$\frac{\partial \Gamma_{ki}^i}{\partial x^j} = \frac{1}{3} R_{kj} \text{ at } x^0$$

(which follows using the identity $\partial \Gamma_{hm}^i / \partial x^k + \partial \Gamma_{mk}^i / \partial x^h + \partial \Gamma_{kh}^i / \partial x^m = 0$ at x^0 ; see [4, p. 52]). Hence we obtain from (1.7)

$$(1.8) \quad \Delta r = \frac{n-1}{r} + \frac{1}{3} (R_{ij})_0 \Theta^i \Theta^j r + F, \quad F = O(r^2).$$

Using (1.8), the proof of the lemma follows easily.

Indeed, if R_n is Einsteinian at x^0 then

$$(1.9) \quad (R_{ij})_0 = k(g_{ij})_0 \quad (k = \text{const}).$$

Since, by (1.2), (1.3),

$$(1.10) \quad (g_{ij})_0 \Theta^i \Theta^j = 1,$$

(1.1) follows with $\beta = k/3$.

Conversely, if (1.1) holds, then by (1.8)

$$\frac{1}{3} (R_{ij})_0 \Theta^i \Theta^j = \beta$$

for all Θ^k satisfying (1.10). Hence (1.9) follows with $k=3\beta$. This completes the proof of the lemma.

Let u^a ($a=1, \dots, n-1$) be local coordinates on the unit sphere. Then r, u^a form polar geodesic coordinates about x^0 , and we have

$$(1.11) \quad ds^2 = dr^2 + \gamma_{ab}(r, u^c) du^a du^b,$$

$$(1.12) \quad \Delta U = \frac{1}{\gamma^{1/2}} \frac{\partial}{\partial r} \left(\gamma^{1/2} \frac{\partial U}{\partial r} \right) + \frac{1}{\gamma^{1/2}} \frac{\partial}{\partial u^a} \left(\gamma^{1/2} \gamma^{ab} \frac{\partial U}{\partial u^b} \right),$$

where $\gamma = \det(\gamma_{ab})$. Hence,

$$(1.13) \quad \Delta r = \frac{1}{\gamma^{1/2}} \frac{\partial \gamma^{1/2}}{\partial r} = \frac{\partial}{\partial r} (\log \gamma^{1/2}).$$

The last two formulas will be needed later on.

Let S_R be the geodesic sphere $r=R$ and denote its interior by D_R . The mean value of a function u is defined by

$$(1.14) \quad M(u, x^0, R) = \int_{S_R} u d\sigma / \int_{S_R} d\sigma$$

where $d\sigma = \gamma^{1/2} du^1 \dots du^{n-1}$ is the surface element of area. For brevity we also set

$$(1.15) \quad M(R) = M(u, x^0, R), \quad S(R) = \int_{S_R} d\sigma.$$

THEOREM 1. *A necessary and sufficient condition that R_n be Einsteinian at a point x^0 is that for every solution u of $\Delta u = 0$ in some neighborhood D_{R_u} of x^0 such that $u \neq 0$ in D_{R_u} ,*

$$(1.16) \quad M(u, x^0, R) = u(x^0)(1 + O(R^3)), \quad \frac{\partial}{\partial R} M(u, x^0, R) = u(x^0)O(R^2)$$

for $R \leq R_u$, where $O(R^k)$ ($k=2, 3$) is defined for $R \leq R_u$ and $|O(R^k)|/R^k \leq A$ where A is a constant independent of u, R, R_u .

Proof. Differentiating $M(R) \int_{S_R} d\sigma = \int_{S_R} u d\sigma$ and using (1.13) and $\int_{D_R} \Delta u dV = \int_{S_R} (\partial u / \partial \nu) d\sigma$, where ν is the outwardly directed normal to S_R , we obtain

$$(1.17) \quad S(R)M'(R) + M(R) \int_{S_R} \Delta R d\sigma = \int_{S_R} u \Delta R d\sigma + \int_{D_R} \Delta u dV.$$

This identity holds for each point $x^0 \in R_n$ and for any function u (it was used by Willmore [13]).

Suppose now that R_n is Einsteinian at x^0 . Then, we can substitute Δr from (1.1) into (1.17). Using also the fact that $\Delta u = 0$, (1.17) becomes

$$(1.18) \quad S(R)M'(R) + M(R) \int_{S_R} F d\sigma = \int_{S_R} u F d\sigma.$$

Let $u \neq 0$ in a neighborhood D_{R_u} of x^0 . It is enough to consider the case $u > 0$, since in the case $u < 0$ we first obtain the mean value formula for $-u$ and then change the sign of both sides. Using $u > 0$ and $F = O(R^2)$ on S_R , we derive from (1.18),

$$\frac{M'(R)}{M(R)} = O(R^2), \quad \text{or} \quad \log \frac{M(R)}{M(0)} = O(R^3).$$

Since $M(0) = u(x^0)$ and $\exp\{O(R^3)\} = 1 + O(R^3)$, we get

$$M(R) = u(x^0)(1 + O(R^3)).$$

Substituting this into (1.18) we find (using: $u > 0$) that $M'(R) = u(x^0)O(R^2)$, and the proof of (1.16) is completed.

Suppose conversely that (1.16) is satisfied. Then, (1.17) yields, if $\Delta u = 0$,

$$(1.19) \quad u(x^0)S(R)O(R^2) + \frac{\delta(R)}{S(R)} \int_{S_R} u d\sigma = \int_{S_R} u \Delta R d\sigma$$

where

$$(1.20) \quad \delta(R) = \int_{S_R} \Delta R d\sigma.$$

Expressing $u(x^0)$ (by (1.16)) in terms of $M(R)$, we get

$$(1.21) \quad \int_{S_R} u \left[\Delta R - \frac{\delta(R)}{S(R)} + H(R, u^a) \right] d\sigma = 0$$

and $|H(R, u^a)|/R^2 \leq A_0$, where A_0 is a constant independent of u, R .

By Lemma 1, all we have to prove is that (1.1) holds (where β is a constant). If (1.1) does not hold then, using (1.8) and $\delta(r)/S(r) = (n-1)/r + k_0 r + O(r^2)$, $k_0 = \text{const.}$ (which follows (1.8), (1.20)), we conclude that there exists a cone K with positive opening and vertex x^0 such that

$$(1.22) \quad \Delta r - \frac{\delta(r)}{S(r)} > \epsilon_0 r \quad (\text{or } \leq -\epsilon_0 r) \text{ in } K,$$

where $\epsilon_0 > 0$. It will be enough to consider the case $> \epsilon_0 r$.

Set $K_R = K \cap S_R$, $K(R) = \text{area of } K_R$ and let g_R be a non-negative smooth function defined on S_R such that $g_R = 1$ on K_R and $g_R = 0$ outside a sufficiently small neighborhood of K_R (on S_R) in which the first inequality of (1.22) still holds. Let u_R be the solution of $\Delta u_R = 0$ in D_R which assumes the values of g_R on S_R . Then u_R is positive in D_R , and an application of (1.21) gives

$$(1.23) \quad K(R)\epsilon_0 R - S(R)A_0 R^2 < 0$$

which is a contradiction if R is sufficiently small.

Using $\Delta r = (n-1)/r + O(r)$ in the proof of the first part of Theorem 1, we get:

THEOREM 2. *At any point x^0 of R_n ,*

$$(1.24) \quad M(u, x^0, R) = u(x^0)(1 + O(R^2)), \quad \frac{\partial}{\partial R} M(u, x^0, R) = u(x^0)O(R)$$

for any solution u of $\Delta u = 0$ in D_{R_u} ($u \neq 0$ in D_{R_u}). $O(R^k)$ is defined for $R \leq R_u$ and $|O(R^k)|/R^k \leq A$ ($k = 1, 2$) where A is independent of u, R, R_u .

REMARK 1. From the proof of the first part of Theorem 1 it follows that for every R_n , if $u \geq 0$, $\Delta u = 0$, then

$$(1.25) \quad M'(R) + M(R)O(R) = 0.$$

Suppose now that $u(x^0) = 0$. Then $M(0) = 0$ and we can write (1.25) as a Volterra type integral equation

$$M'(R) + O(R) \int_0^R M'(t) dt = 0.$$

We conclude that $M'(R) \equiv 0$ and, hence, $M(R) \equiv 0$. Since $u \geq 0$, it follows that $u \equiv 0$. We have thus proved: *In every R_n , if $u(x^0) = 0$, $u \geq 0$, $\Delta u = 0$, then $u \equiv 0$.* This is the well known maximum principle for solutions of the Laplace-Beltrami equation.

REMARK 2. From the proof of Theorem 1 it follows (see (1.23)) that for the "sufficiency" part of the theorem it is enough to assume that (1.16) holds with $O(R^2)$ and $O(R^3)$ replaced by $o(R)$ and $o(R^2)$, respectively.

REMARK 3. The conditions (1.16) are consequences of the single condition

$$(1.26) \quad LM(R) \equiv M''(R) + \frac{n-1}{R} M'(R) = M(R)O(R).$$

Indeed, integrating (1.26) we get $M'(R) = (\int_0^R M(t) dt)O(R)$. Hence, $M(R) = u(x^0) + (\int_0^R M(t) dt)O(R^2)$. This leads to $M(R) = u(x^0)(1 + O(R^3))$. Finally, $M'(R) = (\int_0^R M(t) dt)O(R) = u(x^0)O(R^2)$.

Note that L is the radial part of the Laplace operator in n variables.

2. **Determination of the scalar curvature.** In this section we assume that R_n is Einsteinian at a point x^0 and determine the scalar curvature at this point. We shall need a well known integral formula (see [3, p. 112]) which holds in any R_n : Let

$$\gamma(x, x^0) = \gamma_n r^{2-n} + O(r^{2-n}), \quad \frac{\partial}{\partial x^i} \gamma(x, x^0) = (2 - n)\gamma_n r^{-n} x^i + O(r^{2-n})$$

as $r \rightarrow 0$, where $O(r^0)$ is always understood to mean $O(\log(1/r))$, and where

$$\gamma_n = \frac{1}{(n - 2)\omega_n}, \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and let $\Delta\gamma(x, x^0)$ be integrable. γ is assumed to be smooth for $x \neq x^0$, x^0 being a fixed point in a domain D with smooth boundary B . Then, for any smooth function u in $D + B$,

$$\begin{aligned} u(x^0) &= \int_D [u(x)\Delta\gamma(x, x^0) - \gamma(x, x^0)\Delta u(x)] dV_x \\ (2.1) \quad &+ \int_B \left[\gamma(y, x^0) \frac{\partial u(y)}{\partial \nu} - u(y) \frac{\partial \gamma(y, x^0)}{\partial \nu} \right] d\sigma_y \end{aligned}$$

where $\partial/\partial \nu$ is the derivative in the direction of the outward normal to B at y .

Now let $\phi(r)$ be a solution of the equation

$$(2.2) \quad \phi'' + \left(\frac{n - 1}{r} + \beta r \right) \phi' = 0$$

where β appears in §1. ϕ is determined up to an additive constant if we require that $\phi(r)r^{n-2} \rightarrow$ prescribed number, as $r \rightarrow 0$. For the sake of definiteness we always take this number to be 1. Then

$$(2.3) \quad \phi'(r) = (2 - n)r^{1-n} \exp \left\{ -\frac{1}{2} \beta r^2 \right\}$$

and

$$(2.4) \quad \phi(r) = r^{2-n} + O(r^{4-n}), \quad \phi'(r) = (2 - n)r^{1-n} + O(r^{2-n}).$$

By (1.12), (1.13), $\Delta\psi(r) = \psi''(r) + \psi'(r)\Delta r$ for any function $\psi(r)$. Using (2.2), (1.1) we find that

$$(2.5) \quad \Delta\phi = \phi'' + \phi'\Delta r = \phi'F = O(r^{3-n}).$$

Applying (2.1) with

$$(2.6) \quad \gamma(x, x^0) = \Phi(r) - \Phi(R) \quad \text{where} \quad \Phi(r) = \gamma_n \phi(r)$$

and with D being D_R , we obtain

$$(2.7) \quad u(x^0) = \int_{DR} \Phi' F u dV - \int_{DR} [\Phi(r) - \Phi(R)] \Delta u dV - \int_{SR} \left[u \frac{d\Phi}{dr} \right]_{r=R} d\sigma,$$

since $\partial\Phi/\partial\nu = d\Phi/dr$ on $r=R$. Taking $u \equiv 1$ in (2.7) we get

$$(2.8) \quad 1 + O(R^3) = - \left[\frac{d\Phi}{dr} \right]_{r=R} \int_{SR} d\sigma.$$

Substituting $d\Phi/dr$ from (2.8) into (2.7) we obtain

$$(2.9) \quad \begin{aligned} M(u, x^0, R)(1 + O(R^3)) \\ = u(x^0) - \int_{DR} \Phi' F u dV + \int_{DR} [\Phi(r) - \Phi(R)] \Delta u dV. \end{aligned}$$

In order to evaluate the third term on the right side of (2.9), we introduce a function v which satisfies:

$$(2.10) \quad v'' + \left(\frac{n-1}{r} + \beta r \right) v' = \Phi(r) - \Phi(R),$$

$$(2.11) \quad v(R) = 0, \quad v'(R) = 0.$$

We can write (2.10) in the form (see (2.2))

$$(2.12) \quad v''(r) - \frac{\phi''(r)}{\phi'(r)} v'(r) = \gamma_n [\phi(r) - \phi(R)],$$

or

$$(2.13) \quad \left(\frac{v'(r)}{\phi'(r)} \right)' = \gamma_n \frac{\phi(r)}{\phi'(r)} - \gamma_n \frac{\phi(R)}{\phi'(r)}.$$

Integrating and making use of (2.11) we get

$$(2.14) \quad -v'(r) = \gamma_n \phi'(r) \int_r^R \frac{\phi(\lambda)}{\phi'(\lambda)} d\lambda - \gamma_n \phi(R) \phi'(r) \int_r^R \frac{d\lambda}{\phi'(\lambda)},$$

$$(2.15) \quad \begin{aligned} v(t) = \gamma_n \int_t^R \phi'(r) \left(\int_r^R \frac{\phi(\lambda)}{\phi'(\lambda)} d\lambda \right) dr \\ - \gamma_n \int_t^R \phi(R) \phi'(r) \left(\int_r^R \frac{d\lambda}{\phi'(\lambda)} \right) dr. \end{aligned}$$

Using *only* the fact that ϕ satisfies (2.4), one easily derives from (2.14) and (2.15):

$$(2.16) \quad v(r) = \gamma_n A(R) r^{2-n} + O(r^{3-n}), \quad v'(r) = (2-n)\gamma_n A(R) r^{1-n} + O(r^{2-n})$$

where

$$(2.17) \quad A(R) = - \int_0^R \frac{\phi(\lambda)}{\phi'(\lambda)} d\lambda + \phi(R) \int_0^R \frac{d\lambda}{\phi'(\lambda)},$$

or, equivalently,

$$(2.18) \quad A'(R) = \phi'(R) \int_0^R \frac{d\lambda}{\phi'(\lambda)}, \quad A(0) = 0.$$

Later we shall use the estimate

$$(2.16') \quad v'(r) = A(R)O(r^{1-n})$$

which follows from (2.16), noting that (by (2.18), (2.4)) $A(R) = \text{const. } R^2 + O(R^3)$.

Substituting ϕ from (2.3) into (2.18), we obtain

$$(2.19) \quad A(R) = \frac{R^2}{2n} - \frac{\beta B^4}{4n(n+2)} + O(R^3).$$

We now use (2.1) with u replaced by Δu , γ being v and $D = D_R$. Using (2.11), (2.16) and

$$(2.20) \quad \Delta v(r) = v''(r) + v'(r)\Delta r = [\Phi(r) - \Phi(R)] + v'(r)F,$$

we obtain

$$(2.21) \quad A(R)\Delta u(x^0) = \int_{D_R} [\Phi(r) - \Phi(R)]\Delta u dV + \int_{D_R} v'F\Delta u dV - \int_{D_R} v\Delta^2 u dV.$$

If $\Delta^2 u = 0$ then, by comparing (2.21) with (2.9), we get

$$(2.22) \quad \begin{aligned} M(u, x^0, R)(1 + O(R^3)) \\ = u(x^0) - \int_{D_R} \phi'FudV + A(R)\Delta u(x^0) + \int_{D_R} v'F\Delta u dV. \end{aligned}$$

We now assume that $u \neq 0$, $\Delta u \neq 0$ in some D_{R_u} and take $R \leq R_u$. We then have:

$$(2.23) \quad \begin{aligned} - \int_{D_R} \phi'FudV &= \int_0^R O(r^{3-n}) \left(\int_{S_r} u d\sigma \right) dr = \int_0^R O(r^{3-n})S(r)M(r)dr \\ &= O(R^2) \int_0^R M(r)dr, \\ \int_{D_R} v'F\Delta u dV &= \int_{D_R} A(r)O(r^{1-n})O(r^2)\Delta u dV \\ &= A(R) \int_0^R O(r^{3-n}) \left(\int_{S_r} \Delta u d\sigma \right) dr \end{aligned}$$

(using (2.16')), and since $\Delta(\Delta u) = 0$, $\Delta u \neq 0$, we can apply Theorem 1 and get

$$(2.24) \quad \int_{D_R} v' F \Delta u dV = A(R) O(R^3) \Delta u(x^0).$$

Substituting (2.23), (2.24) into (2.22), (2.22) simplifies to

$$(2.25) \quad M(R) + O(R^2) \int_0^R M(r) dr = u(x^0)(1 + O(R^3)) + A(R) \Delta u(x^0)(1 + O(R^3)).$$

Integrating (2.25) over R we find that

$$\int_0^R M(r) dr = u(x^0) O(R) + A(R) \Delta u(x^0) O(R).$$

Hence, (2.25) reduces to

$$(2.26) \quad M(R) = u(x^0)(1 + O(R^3)) + A(R) \Delta u(x^0)(1 + O(R^3)),$$

and substituting $A(R)$ from (2.19), we get

$$(2.27) \quad M(u, x^0, R) = u(x^0)(1 + O(R^3)) + u(x^0) \left\{ \frac{R^2}{2n} - \frac{\beta R^4}{4n(n+2)} + O(R^5) \right\}.$$

Note that the proof of (2.27) remains true also in case $u(x^0) = 0$, provided either $u \geq 0$ or $u \leq 0$.

From (1.9) and $k = 3\beta$ it follows that $\rho = 3\beta n$ is the scalar curvature at x^0 . We have thus proved:

THEOREM 3. *Let R_n be Einsteinian at a point x^0 . Then for any function u defined in some neighborhood R_{R_u} of x^0 and satisfying $u \geq 0$ or $u \leq 0$, $\Delta u \neq 0$, $\Delta^2 u = 0$ in D_{R_u} , we have:*

$$(2.28) \quad M(u, x^0, R) = u(x^0)(1 + O(R^3)) + \Delta u(x^0) \left(\frac{R^2}{2n} + \frac{\rho R^4}{12n^2(n+2)} + O(R^5) \right)$$

where $O(R^k)$ ($k = 3, 5$) is defined for $R \leq R_u$ and satisfies $|O(R^k)|/R^k \leq B$, where B is a constant independent of u, R, R_u . The coefficient ρ is the scalar curvature of R_n at x^0 .

REMARK 1. In order to calculate ρ from (2.28) one should use it with a function u which vanishes at x^0 .

REMARK 2. Set

$$(2.29) \quad \begin{aligned} f(\Theta) &= \frac{1}{3} (R_{ij})_0 \Theta^i \Theta^j, & f_0 &= \lim_{R \rightarrow 0} M(f, x^0, R), \\ f_1 &= \text{g.l.b. } f(\Theta), & f_2 &= \text{l.u.b. } f(\Theta). \end{aligned}$$

Then, using (2.1) with $\gamma = r^{2-n} - R^{2-n}$ we easily obtain, if $\Delta u = 0$ and $u > 0$,

$$(2.30) \quad 1 + \frac{R^2}{2\gamma_n} (f_1 - f_0) + O(R^3) \leq \frac{M(u, x^0, R)}{u(x^0)} \leq 1 + \frac{R^2}{2\gamma_n} (f_2 - f_0) + O(R^3)$$

which sharpens Theorem 2. Note that $\rho = 3nf_0$.

3. Characterization of harmonic spaces. A Riemannian space R_n is said to be *harmonic* at a point x^0 if the equation $\Delta u = 0$ has a solution $u = \phi(r)$ in some neighborhood of x^0 . The space is *trivially harmonic* (or *simply harmonic*) at x^0 if $\phi(r) = r^{2-n}$. If R_n is (trivially)harmonic at each of its points, then we say that R_n is (trivially)harmonic. It is easily seen that $\phi(r)$ is a fundamental solution of $\Delta u = 0$. The theory of harmonic spaces was developed by Ruse, Walker, Lichnerowicz, and others. For a general survey see [7] and for related references, see [14]. We mention that H_n is necessarily an A_n , whereas every space with constant curvature K_n is necessarily an H_n . For $n = 2, 3$, $A_n = H_n = K_n$. The last statement is also true in the case of indefinite metric tensors.

Willmore [13] proved that a necessary and sufficient condition for R_n to be harmonic at a point x^0 is that for every solution u of $\Delta u = 0$, in some neighborhood of x^0 ,

$$(3.1) \quad M(u, x^0, R) = u(x^0).$$

We now want to find a characterization for the fundamental solution $\phi(r)$.

THEOREM 4. *If R_n is harmonic at a point x^0 , then for any solution u of $\Delta^2 u = 0$, in some neighborhood D_{R_u} of x^0 ,*

$$(3.2) \quad M(u, x^0, R) = u(x^0) + A(R)\Delta u(x^0) \quad (R \leq R_u)$$

where $A(R)$ is related to $\phi(R)$ by

$$(3.3) \quad A(R) = - \int_0^R \frac{\phi(\lambda)}{\phi'(\lambda)} d\lambda + \phi(R) \int_0^R \frac{d\lambda}{\phi'(\lambda)}$$

or, equivalently, by

$$(3.4) \quad A'(R) = \phi'(R) \int_0^R \frac{d\lambda}{\phi'(\lambda)}, \quad A(0) = 0.$$

If R_n is an H_n then [7] $\phi(r)$ is independent of the initial point x^0 . Therefore, the same is true of $A(R)$.

REMARK. Since we assume that $\phi(r)$ is normalized by $\phi(r) = r^{2-n} + O(r^{3-n})$, it is determined by (3.4) up to an additive constant. Indeed, this follows from the equation

$$(3.5) \quad \frac{\phi''}{\phi'} = \frac{A'' - 1}{A'}$$

which is derived from (3.4).

Proof of Theorem 4. The proof is similar to the proof of Theorem 3 and we therefore only indicate the modifications which one has to make. (2.5) is replaced by $\Delta\phi=0$ and hence in (2.7), (2.9) $F\equiv 0$. Also $O(R^3)\equiv 0$ in (2.8), (2.9). As for v , the only difference is that instead of $(n-1)/r+\beta r$ (in (2.10)) we now write Δr or $-\phi''/\phi'$. Consequently, in (2.20) we have $F\equiv 0$.

REMARK 1. Helgason [6] has recently extended Ásgeirson's mean value theorem [1] to solutions of $\Delta_x u = \Delta_y u$, where Δ_x, Δ_y are the Laplace-Beltrami operators in the x -space and the y -space of n -dimensions. He assumed that the spaces are two-point homogeneous spaces (and, hence, harmonic). It is immediately seen (using [13]) that for Ásgeirson's theorem to be valid it is necessary that the two spaces be harmonic.

REMARK 2. In 1909 Pizetti proved (see [2, p. 261]) that for any smooth function u and a non-negative integer m ,

$$\frac{1}{S(R)} \int_{S_R} u d\sigma = \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^m \left(\frac{R}{2}\right)^{2k} \frac{\Delta^k u(x^0)}{k! \Gamma(k + n/2)} - \int_{D_R} v_m \Delta^{m+1} u dV$$

where

$$v_0 = \gamma_n (r^{2-n} - R^{2-n}),$$

$$v_k(r) = \gamma_n \int_r^R \rho v_{k-1}(\rho) [\rho^{n-2} - r^{n-2}] d\rho,$$

if $n \neq 2$. For $n = 2$ the definition of v_k is slightly different. The proof can easily be extended to R_n harmonic at x^0 , provided

$$(3.6) \quad 0 < A_1 \leq r^{n-1} \phi'(r) \leq A_2,$$

$$(3.7) \quad 0 < A_3 \leq \frac{r^{n-2} R}{R-r} [\phi(r) - \phi(R)] \leq A_4.$$

The formula obtained is

$$(3.8) \quad M(u, x^0, R) = \sum_{k=0}^m A_k(R) \Delta^k u(x^0) - \int_{D_R} v_m \Delta^{m+1} u dV$$

and

$$(3.9) \quad 0 < B_0 B_1^k \leq \frac{A_k(R)}{R^{2k} (k!)^2} \leq B_2 B_3^k,$$

$$(3.10) \quad 0 < v_m(r) \leq B_4 B_5^m \frac{(R-r)^{2m+1}}{R(m!)^2} r^{2-n}$$

where the B_i are constants depending only on the A_i .

Formula (3.8) with the estimates (3.9) can be used to derive a Liouville type theorem, namely: if $\Delta^p u = 0$ (p a positive integer) in the whole Euclidean space, where Δ is the Laplace-Beltrami operator, and if u is bounded, then $u \equiv \text{const}$. (We assume that (3.6), (3.7) hold for all $0 < r < R < \infty$ and that Δ is uniformly elliptic.) The proof is obtained by first deducing that $\Delta^{p-1} u = 0$, $\Delta^{p-2} u = 0$, \dots , $\Delta u = 0$ and then applying Nash's estimates [9] which immediately give $u \equiv \text{const}$.

4. Characterization of spaces of constant curvature. We need the following lemma:

LEMMA 2. *If in a space H_n*

$$(4.1) \quad \phi'(r) = (2-n) \left[\frac{K^{1/2}}{\sin(rK^{1/2})} \right]^{n-1}, \quad K \neq 0 \text{ (} K \text{ constant)}$$

then H_n is a space of constant curvature K .

The converse of this lemma is well known [7] and is proved by direct calculation. Lemma 2 for $K=0$ ((4.1) is then understood to mean: $\phi'(r) = (2-n)r^{1-n}$) is due to Thomas and Titt [10].

Proof. Set

$$\chi(r) = -\frac{\phi''(r)}{\phi'(r)}, \quad f(\Omega) = r\chi(r) + 1 \quad \text{where} \quad \Omega = \frac{1}{2} r^2.$$

Lichnerowicz [7] (and later also Willmore [12]) proved that for any H_n

$$(4.2) \quad \frac{5}{2} f''(0) + \frac{(f'(0))^2}{n-1} \leq 0$$

and equality holds if and only if H_n is a K_n ; its curvature K is then found to be

$$(4.3) \quad K = -\frac{3f'(0)}{2(n-1)}.$$

A simple calculation shows that for ϕ' as given by (4.1) we have

$$f'(0) = -\frac{2(n-1)}{3} K, \quad f''(0) = -\frac{8(n-1)}{45} K^2.$$

Hence equality holds in (4.2) and the proof of Lemma 2 is completed.

Combining Theorem 4 and Lemma 2, we get:

THEOREM 5. *A necessary and sufficient condition for an R_n to be a K_n with curvature K is that (3.2) holds at each point x^0 of R_n and for every solution u of $\Delta^2 u = 0$ in a neighborhood of x^0 , where $A(R)$ is given by*

$$(4.4) \quad A'(R) = \int_0^R \left[\frac{\sin(rK^{1/2})}{\sin(RK^{1/2})} \right]^{n-1} dr, \quad A(0) = 0.$$

REMARK. Using (3.5) we have

$$f(\Omega) = r \frac{1 - A''(r)}{A'(r)} + 1, \quad \Omega = \frac{1}{2} r^2.$$

We can use this formula to calculate $f'(0)$, $f''(0)$ in terms of $A^{(4)}(0)$, $A^{(6)}(0)$ (note that $A^{(2m+1)}(0) = 0$ for $m = 0, 1, 2, \dots$). We then can express (4.2) as an inequality involving $A^{(4)}(0)$, $A^{(6)}(0)$. In particular, if $A^{(4)}(0) = A^{(6)}(0) = 0$ then H_n is flat.

5. Characterization of Einstein metrics, using solutions of $\Delta u + \lambda u = 0$. The following theorem can be derived by the method of §1:

R_n is Einsteinian at x^0 if and only if for every solution $u \geq 0$ of $\Delta u + \lambda u = 0$ (λ fixed) in some D_{R_u} ,

$$(5.1) \quad M'(R) + \frac{\lambda}{S(R)} \int_0^R S(r)M(r)dr = O(R^2)M(R)$$

for $R \leq R_u$, where $|O(R^2)| \leq AR^2$, A being independent of u , R_u , R .

Let $\{\lambda_m\}$, $\{\phi_m\}$ be the sets of eigenvalues and orthonormal eigenfunctions of Δ on R_n , where R_n is, from now on, taken to be a compact Riemannian manifold. As is well known, the set $\{\phi_m\}$ is complete in the L^2 sense. Furthermore, from the asymptotic behavior of the λ_m , ϕ_m (see, for instance, [8]) it follows that for every function g on R_n , $g = \sum a_m \phi_m$ (a_m are Fourier's coefficients) and the series is uniformly convergent together with any preassigned number of its term-by-term derivatives, provided g is sufficiently smooth. We wish to characterize Einsteinian metrics by means of properties of the ϕ_m . Since the ϕ_m , however, are not in general non-negative functions, the above cited theorem is not helpful. We shall instead use the following theorem whose proof, which is similar to that of the above cited theorem, is omitted.

THEOREM 6. *A necessary and sufficient condition that R_n be Einsteinian at x^0 is that for every solution of $\Delta u + \lambda u = 0$ in some D_{R_u} , ($R_u < \bar{R}$, for some $\bar{R} > 0$)*

$$(5.2) \quad M'(R) + \frac{\lambda}{S(R)} \int_0^R S(r)M(r)dr = \mu(R)O(R^2), \quad \mu(R) = \text{l.u.b.} \left| \frac{u}{s} \right|$$

for $R \leq R_u$, where $|O(R^2)| \leq AR^2$ and A is independent of u , R_u , R .

REMARK 1. Theorem 6 remains true if $\mu(R)$ is replaced by $\hat{\mu}(R) = \text{l.u.b.}_{D_R} |u|$. In both cases, the "sufficiency" part remains true if $O(R^2)$ is replaced by $o(R)$.

REMARK 2. From the proof of Theorem 6 it follows that A and \bar{R} are independent of λ .

THEOREM 7. A necessary and sufficient condition that R_n be Einsteinian at x^0 is that $\{\phi_m\}$, $\{\lambda_m\}$ satisfy for all $R < \bar{R}$ (for some $\bar{R} > 0$)

$$(5.3) \quad \frac{\partial}{\partial R} M(\phi_m, x^0, R) + \frac{\lambda_m}{S(R)} \int_0^R S(r) M(\phi_m, x^0, r) dr = \mu_m(R) O(R^2)$$

where $\mu_m(R) = \text{l.u.b.}_{S_R} |\phi_m|$, $|O(R^2)| \leq AR^2$ and A is independent of m, R .

Proof. The "necessary" part is a consequence of Theorem 6 and Remark 2. To prove the "sufficiency" part, we need the relation:

$$(5.4) \quad S(R)M'(R) + M(R) \int_{S_R} \Delta R d\sigma = \int_{S_R} u \Delta R d\sigma - \lambda \int_0^R S(r)M(r)dr$$

valid for any solution of $\Delta u + \lambda u = 0$. Its derivation is similar to that of (1.17). Taking $u = \phi_m$, $\lambda = \lambda_m$ in (5.4) and making use of (5.3) we obtain

$$(5.5) \quad M_m(R) \int_{S_R} \Delta R d\sigma = \int_{S_R} \phi_m \Delta R d\sigma + \mu_m(R) O(R^2)$$

where we set $M_m(R) = M(\phi_m, x^0, R)$. Inserting ΔR from (1.8) into (5.5) and using the notation (2.29), we get

$$(5.6) \quad \left[\frac{1}{S(R)} \int_{S_R} f(\Theta) d\sigma \right] \left[\frac{1}{S(R)} \int_{S_R} \phi_m(R, \Theta) d\sigma \right] \\ = \frac{1}{S(R)} \int_{S_R} \phi_m(R, \Theta) f(\Theta) d\sigma + \mu_m O(R)$$

for $0 < R < R_0$ (for some R_0), where $\mu_m = \text{l.u.b.}_{0 < R < R_0} |\phi_m(R)|$.

The function $f(\Theta)$ is smooth in some domain containing \bar{D}_{R_0} , if R_0 is sufficiently small. Let \hat{f} be a smooth function on R_n which coincides with f on D_{R_0} . Then

$$\hat{f} = \sum c_m \phi_m,$$

and the series is uniformly convergent. Hence, for any $\epsilon > 0$ there exists a k such that

$$(5.7) \quad \left| f - \sum_{m=1}^k c_m \phi_m \right| < \epsilon \text{ in } D_{R_0}.$$

We now multiply both sides of (5.6) by c_m and sum over $m = 1, \dots, k$. We obtain

$$\begin{aligned}
 (5.8) \quad & \left[\frac{1}{S(R)} \int_{S_R} f(\Theta) d\sigma_R \right] \left[\frac{1}{S(R)} \int_{S_R} (f(\Theta) + h_\epsilon(R, \Theta)) d\sigma_R \right] \\
 & = \frac{1}{S(R)} \int_{S_R} f(\Theta)(f(\Theta) + h_\epsilon(R, \Theta)) d\sigma_R + \left(\sum_{m=1}^k |c_m| \mu_m \right) O(R),
 \end{aligned}$$

where, for clarity, we set $d\sigma = d\sigma_R$ on S_R , and where $|h_\epsilon(R, \Theta)| < \epsilon$.

Denoting $d\omega = \lim_{R \rightarrow 0} (d\sigma_R/S(R))$ and letting $R \rightarrow 0$ in (5.8), we obtain the inequality

$$\left(\int f(\Theta) d\omega \right)^2 + \epsilon \left| \int f(\Theta) d\omega \right| \geq \int (f(\Theta))^2 d\omega - \epsilon \int |f(\Theta)| d\omega.$$

Taking $\epsilon \rightarrow 0$, we get

$$(5.9) \quad \left(\int f(\Theta) d\omega \right)^2 \geq \int (f(\Theta))^2 d\omega.$$

Noting that $d\omega > 0$, $\int d\omega = 1$, we can use Schwarz's inequality and conclude that $(\int f d\omega)^2 \leq \int f^2 d\omega$ and equality holds if and only if $f(\Theta) \equiv \text{const}$. In view of (5.9), $f(\Theta)$ is indeed a constant. Hence, by Lemma 1, R_n is Einsteinian at x^0 .

REMARK. Theorem 7 remains true if, in (5.3), $O(R^2)$ is replaced by $o(R)$.

6. Characterization of harmonic metrics, using solutions of $\Delta u + \lambda u = 0$.

Using (5.4) one can show that R_n is harmonic at x^0 if and only if

$$(6.1) \quad M(u, x^0, R) = \delta(R)u(x^0) \quad \text{for } R < R_u,$$

for any solution u of $\Delta u + \lambda u = 0$ in D_{R_u} , where $\delta(R)$ is defined by

$$(6.2) \quad \delta'(R) + \frac{\lambda}{S(R)} \int_0^R S(r)\delta(r)dr = 0, \quad \delta(0) = 1.$$

We shall prove:

THEOREM 8. *A necessary and sufficient condition that R_n be harmonic at x^0 is that $\{\phi_m\}$, $\{\lambda_m\}$ satisfy, for all $R < \bar{R}$ (for some $\bar{R} > 0$)*

$$(6.3) \quad M(\phi_m, x^0, R) = \delta_m(R)\phi_m(x^0)$$

where $\delta_m(R)$ is defined by

$$(6.4) \quad \delta'_m(R) + \frac{\lambda_m}{S(R)} \int_0^R S(r)\delta_m(r)dr = 0, \quad \delta_m(0) = 1.$$

Proof. We only have to prove the "sufficiency" part of the theorem. By [13] it is enough to prove that

$$(6.5) \quad M(u, x^0, R) = u(x^0) \quad \text{for } R < R_u$$

for any solution u of $\Delta u = 0$ in D_{R_u} . Now every local solution can be extended into a smooth function on R_n . Hence we have

$$(6.6) \quad u(x) = \sum_{m=0}^{\infty} a_m \phi_m(x), \quad \Delta u = - \sum_{m=0}^{\infty} \lambda_m a_m \phi_m(x) = 0$$

uniformly in x for $0 \leq r \leq R'_u$ for any $R'_u < R_u$ (r is the geodesic distance from x^0 to x).

From (6.6) we obtain

$$(6.7) \quad M(u, x^0, R) = \sum_{m=1}^{\infty} a_m M(\phi_m, x^0, R) = \sum_{m=1}^{\infty} a_m \delta_m(R) \phi_m(x^0),$$

$$(6.8) \quad M(u, x^0, R) = - \sum_{m=1}^{\infty} \lambda_m a_m \delta_m(R) \phi_m(x^0) = 0,$$

and both series converge uniformly for $0 < R < R'_u$. Multiplying (6.8) by $S(R)/S(\rho)$ and integrating over R , we obtain

$$(6.9) \quad \sum_{m=1}^{\infty} a_m \phi_m(x^0) \left[\frac{\lambda_m}{S(\rho)} \int_0^\rho S(R) \delta_m(R) dR \right] = 0$$

and the convergence is uniform in ρ , $0 < \rho \leq R'_u$.

Using (6.4), (6.9) reduces to

$$(6.10) \quad \sum_{m=1}^{\infty} a_m \delta'_m(\rho) \phi_m(x^0) = 0.$$

Finally, integrating the series (6.10), which is uniformly convergent for $0 < \rho < R'_u$, we find that $\sum_{m=1}^{\infty} a_m \delta_m(R) \phi_m(x^0)$ is a function independent of R . Hence, recalling (6.7), the proof of (6.5) follows.

REMARK 1. The fundamental solution $\phi(R)$ can be determined, up to an additive constant, by the function $\delta(R)$ defined in (6.2). Indeed, using (2.1) with γ defined by (2.6) and ϕ being a solution of $\Delta \phi + \lambda \phi = 0$, we get

$$(6.11) \quad u(x^0) = - \lambda \gamma_n \phi(R) \int_0^R S(r) M(r) dr - \lambda \gamma_n \int_{S_R} u \frac{\partial \phi}{\partial \nu} d\sigma \\ - \int_{D_R} \gamma (\Delta + \lambda) u dV.$$

Employing (6.1) and $(\Delta + \lambda)u = 0$, we obtain

$$- \gamma_n \frac{d\phi(R)}{dR} \int_{S_R} u d\sigma = u(x^0) \left[1 + \lambda \gamma_n \phi(R) \int_0^R S(r) \delta(r) dr \right].$$

Comparing this with (6.1) we get an equation which, after using (6.2), simplifies to

$$(6.12) \quad \delta'(R)\phi(R) - \delta(R)\phi'(R) = \frac{1}{\gamma_n S(R)}.$$

If we can express $S(R)$ in terms of $\phi(R)$, then (6.12) can be used to solve $\phi(R)$ in terms of $\delta(R)$. Now setting $u \equiv 1$ in (6.11) gives

$$(6.13) \quad S(R)\phi'(R) + \lambda \int_0^R S(r)\phi(r)dr + \frac{1}{\gamma_n} = 0$$

which can be used to express $S(R)$ in terms of $\phi(R)$.

REMARK 2. We have proved in §§5, 6 that some properties of $\{\phi_m\}$, $\{\lambda_m\}$ determine some properties of the metric of the manifold. A direct construction of the metric from the sequences of eigenvalues and eigenfunctions seems to be hard to accomplish. All we can prove is: *If $\{\phi_m\}$, $\{\lambda_m\}$ are the sequences of eigenfunctions and eigenvalues of a metric tensor (g_{ij}) on a compact Riemannian manifold, then the $\{\phi_m\}$, $\{\lambda_m\}$ determine the metric tensor (g_{ij}) in a unique manner* (i.e., they cannot be the sequences corresponding to another metric tensor).

The proof is based on the following observations:

- (a) The ϕ_m , λ_m determine the set of all local solutions (that is: in neighborhoods of a point) by the set of all sequences $\{a_m\}$ for which $\sum a_m \phi_m$, $\sum \lambda_m a_m \phi_m$ are uniformly (locally) convergent and the second series vanishes.
- (b) The local solutions of an elliptic equation determine the coefficients up to a common factor (this follows from [5, proof of Theorem 2]). To find this factor one uses the relations $\Delta \phi_m + \lambda_m \phi_m = 0$.

REFERENCES

1. L. Åsgeirson, *Über eine Mittelwertseigenschaft von Lösungen homogener linearer partieller Differentialgleichungen 2. Ordnung mit konstanten Koeffizienten*, Math. Ann. vol. 113 (1937) pp. 321-346.
2. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. 2, Berlin, Julius Springer, 1937.
3. G. F. D. Duff, *Partial differential equations*, University of Toronto Press, 1956.
4. L. P. Eisenhart, *Riemannian geometry*, Princeton University Press, 1949.
5. A. Friedman, *Linear partial differential systems with an additional differential equation at one point*, J. Math. Mech. vol. 7 (1958) pp. 173-190.
6. S. Helgason, *Differential operators on homogeneous spaces*, Acta Math. vol. 102 (1960) pp. 239-299.
7. M. A. Lichnerowicz, *Equations de Laplace et espaces harmoniques*, Premier colloque sur les équations aux dérivées partielles, Louvain (1953) pp. 9-23. Paris, Masson, 1954.

8. S. Minakshisundaram, *Eigenfunctions on Riemannian manifolds*, J. Indian Math. Soc. vol. 17 (1953) pp. 159–165.
9. J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. vol. 80 (1958) pp. 931–954.
10. T. Y. Thomas and E. W. Titt, *On the elementary solution of the general linear differential equation of the second order with analytic coefficients*, J. Math. Pures Appl. vol. 18 (1939) pp. 217–248.
11. O. Veblen, *Invariants of quadratic differential forms*, Cambridge University Press, 1927.
12. T. J. Willmore, *Some properties of harmonic Riemannian manifolds*, Trieste, Convegno Internaz. Geometria Differenziale Italia, 1953, pp. 141–147.
13. ———, *Mean value theorems*, J. London Math. Soc. vol. 25 (1950) pp. 54–57.
14. ———, *An introduction to differential geometry*, Oxford University Press, 1959.

UNIVERSITY OF MINNESOTA, INSTITUTE OF TECHNOLOGY,
MINNEAPOLIS, MINNESOTA