A CHARACTERIZATION OF THE n-SPHERE(1)

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1. **Introduction.** We give a set-theoretic characterization of the topological n-sphere. In so doing, we use some notions introduced by R. L. Wilder in his summer lectures of 1948 at the University of Michigan. Also an essential use is made of modified forms of some results [1] of R. H. Bing.

2. **Characterization theorem.** We first define terms cells, complexes and their simple subdivisions. They are originally due to R. L. Wilder and we use slightly modified forms here.

A *complex* is a finite collection of elements which we call cells, which in turn are point sets of an underlying topological Hausdorff space. The *cells* of a complex must satisfy the following conditions.

(1) Each cell is a nonempty point set. In particular, a 0-cell (cf. (3) below) consists of a single point.

(2) No two cells have a point in common.

(3) With each cell is associated a unique non-negative integer r which we call *dimension* of the cell. We may call the cell an r-cell if we desire to specify its dimension.

(4) If σ^r is an r-cell, $r>0$, there is a nonempty finite collection of $(r-1)$-cells $\sigma_1^{r-1}, \sigma_2^{r-1}, \cdots, \sigma_k^{r-1}$ such that

$$\mathrm{Cl}\,\sigma^r = \sigma^r + \mathrm{Cl}\,\sigma_1^{r-1} + \mathrm{Cl}\,\sigma_2^{r-1} + \cdots + \mathrm{Cl}\,\sigma_k^{r-1}.$$

The cells σ_i^{r-1} are called the *boundary cells* of σ^r. The sum of the closures of the boundary cells of σ^r will be denoted by $\mathrm{Fr}\,\sigma^r$.

(5) A 1-cell has exactly two boundary 0-cells.

If the sum of the cells of a complex K is a space X, we say that K is a *covering complex* of X.

A *simple subdivision* of an r-cell σ^r, $r>0$, is the complex consisting of the faces of σ^r (i.e., the cells on $\mathrm{Cl}\,\sigma^r$) except σ^r together with cells σ_1^r, σ_2^r and σ^{r-1} such that (1) $\sigma^r=\sigma_1^r+\sigma_2^r+\sigma^{r-1}$, (2) $\mathrm{Cl}\,\sigma^{r-1}=\mathrm{Cl}\,\sigma_1^r\cdot\mathrm{Cl}\,\sigma_2^r$ and (3) $\mathrm{Fr}\,\sigma^r\cdot\mathrm{Fr}\,\sigma_i^r$, $i=1, 2$, are homeomorphic to $\mathrm{Cl}\,\sigma^{r-1}$. As an example, one may consider subdividing a disk by an arc that meets the boundary of the disk at its endpoints.

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(1) This paper is essentially a part of the author's dissertation written under Professor R. L. Wilder and submitted to the University of Michigan in 1958. The desirability of publishing a complete proof of the characterization theorem (see §2) has recently been called to the author's attention.

By a *simple subdivision of a complex* we mean a complex obtained from the original complex by replacing one of its cells by a simple subdivision thereof.

The procedure of obtaining a simple subdivision will also be referred to as simple subdivision.

By a sequence of complexes $K_1, K_2, \cdots$ we mean that each complex K_i is a simple subdivision of K_{i-1}. We say that $\tau^r \in K_j$ is a *derived cell* of $\sigma^r \in K_i$ if $\tau^r \subsetneq \sigma^r$. In this case, we say that τ^r is *directly derivable* from σ^r if further $\operatorname{Fr} \sigma^r \cdot \operatorname{Fr} \tau^r$, $\operatorname{Cl}(\operatorname{Fr} \sigma^r - \operatorname{Fr} \tau^r)$ and $\operatorname{Cl}(\operatorname{Fr} \tau^r - \operatorname{Fr} \sigma^r)$ are all homeomorphic to the closure of one and the same boundary cell of σ^r. As an example, one might consider several steps of subdividing a disk nicely.

THEOREM. (CHARACTERIZATION OF THE n-SPHERE). *In order that a compact Hausdorff space Q^n be a topological n-sphere, it is necessary and sufficient that Q^n have an ω-sequence of covering complexes $K_0, K_1, K_2, \cdots$ satisfying the following conditions*:

1. *K_0 consists of exactly two r-cells σ_1^r and σ_2^r for each dimension $r \leqq n$ and each of the cells σ_1^r and σ_2^r is, if $r < n$, a boundary cell of each of the cells σ_1^{r+1} and σ_2^{r+1}.*

2. *For each open covering $\mathfrak{U}$ of Q^n, there exists an integer i such that each cell of K_i is of diameter less than $\mathfrak{U}$ (i.e., contained in an element of $\mathfrak{U}$).*

3. *If τ^r is a derived cell of σ^r such that* $\operatorname{Fr} \sigma^r \cdot \operatorname{Fr} \tau^r \neq \emptyset$, *then τ^r is directly derivable from σ^r.*

The reader might compare the above characterization with Bing's characterization of 3-space by partitionings [1]. In fact, we make use of analogues of Theorems 4 and 5 of [1]. We do not know whether Condition 3 can be deleted from the theorem.

3. **Some lemmas.** We denote by S^n the unit sphere in E^{n+1}. Explicitly,

$$S^n = \{(x_1, x_2, \cdots, x_{n+1}) \in E^{n+1} \mid \sum x_i^2 = 1\}.$$

By S^{n-1} and S_+^n we denote the subsets of S^n defined by $x_{n+1} = 0$ and $x_{n+1} \geqq 0$, respectively. A homeomorphic image of S_+^n will be called a euclidean n-cell.

An *n-manifold with boundary* is a separable metric space such that each point has a neighborhood whose closure is a euclidean n-cell. If X is an n-manifold with boundary, $\operatorname{Bd} X$ will denote the set of those points which fail to have a neighborhood homeomorphic to E^n. Finally, the set $X - \operatorname{Bd} X$ will be denoted by $\operatorname{Int} X$.

A topological $(n-1)$-sphere S' in a topological n-sphere S is called *tame* if there is a homeomorphism of S into S^n sending S' into S^{n-1}. A euclidean n-cell D in a topological n-sphere S is called *tame* if there is a homeomorphism of S into S^n sending D onto S_+^n. Clearly, D is tame if and only if $\operatorname{Bd} D$ is tame.

The following two lemmas are generalizations to general dimension n of Theorems 4 and 5 of [1].

LEMMA 1. *Suppose C is the unit sphere in E^n and $G_1, G_2, \cdots$ is a sequence of partitionings of $S=$ (C plus its interior) satisfying the following conditions:*

(i) *each element of G_i is an open euclidean n-cell disjoint from C;*

(ii) *the closure of each element g of G_i is a euclidean n-cell with $g=\operatorname{Int}(\operatorname{Cl} g)$;*

(iii) *if g is an element of G_i whose closure intersects C, then given any neighborhood U, with respect to the boundary of g, of the intersection F, there exists a euclidean $(n-1)$-cell D which is tame in the boundary of g such that $U \supset D \supset F$;*

(iv) *each G_{i+1} is a refinement of G_i; and*

(v) *for any positive number ϵ, there is an integer i such that the closure of no element of G_i contains two points of the sum of the boundaries of the elements of G_1 which are farther apart than ϵ.*

Let K be a closed set and R be a proper closed subset of C such that if $g \in G_1$, then $\operatorname{Bd}(\operatorname{Cl} g)\cdot C$ does not meet both R and K. Then there is an integer q and a homeomorphism T of S onto itself satisfying the following conditions:

(vi) *each point of the boundary of each element of G_1 is invariant under T; and*

(vii) *if $g \in G_q$, $T(\operatorname{Cl} g)$ does not meet both R and K.*

LEMMA 2. *Suppose the sequence $G_1, G_2, \cdots$ in Lemma 1 satisfies the conditions* (i)–(iv) *as well as the following:*

(viii) *for each positive integer j and each positive number ϵ, there is a positive integer $p(j, \epsilon)$ such that the closure of no element of G_p contains two points of the sum of the boundaries of the elements of G_j which are farther apart than ϵ.*

Then for each positive number δ, there is an integer q and a homeomorphism T of S onto itself such that T leaves invariant each point of $F(G_1)$ and T carries each element of G_q into a set of diameter less than δ, where $F(G_1)$ denotes the sum of the boundaries of the elements of G_1.

In the strength of our definition of tameness, proofs of these two lemmas can be given in parallel to the proofs of Bing's original theorems. Therefore, we omit proofs. But the following lemma will suggest how the parallelism can be achieved.

LEMMA 3. *Let C be the n-cube in E^n defined by $0 \leqq x_i \leqq 1$, $i=1, 2, \cdots, n$. Let D be the subset of C defined by $x_n=1/2$. We finally denote by C' the subset of C defined by $x_n \geqq 1/2$. If F is a closed subset of C such that $F\cdot C'\cdot \operatorname{Bd} C=\emptyset$, then there exists an isotopy f_t, $0\leqq t\leqq 1$, such that f_0 is the identity, each f_t is a homeomorphism leaving* Bd C *pointwise fixed and moving the x_i-coordinates, $i\neq n$, of no point of C and $f_1(F)\cdot C'=\emptyset$.*

Proof. With respect to the x_n-coordinate, we may use such words like "above" and "below." Let F_{ij}, $i=1, 2, \cdots, n$; $j=0, 1$, be the $(n-2)$-cube in C determined by $x_i=j$ and $D_{ij}=D\cdot F_{ij}$. For each $i\neq n$ and j and any positive number $d<1/2$, let G_{ij} be the $(n-2)$-cube on F_{n_1} determined by $x_i=j+(-1)^j d$. We finally denote by H^d_{ij} the convex body which is a subset of

C and is the set of the points lying not lower than the $(n-1)$-plane spanned by the sets G_{ij}^{d} and D_{ij}. Consider the sequence $H_{ij}^{1/k}$, $k=3, 4, \cdots$. It is a decreasing sequence of compact sets whose intersection is $F_{ij}\cdot C'$. Since $F\cdot F_{ij}\cdot C'=\emptyset$, there is an integer k_{ij} such that $H_{ij}^{1/k_{ij}}\cdot F=\emptyset$. Also there exists a positive number $e<1/2$ such that no point of F has the x_n-coordinate as large as $1-e$. We now let H_n be the subset of C determined by $x_n\geqq 1-e$.

Let $C_1=H_n+\sum_{i,j} H_{ij}^{1/k_{ij}}$. The set $D'=C_1\cdot \mathrm{Cl}(C-C_1)$ is a topological $(n-1)$-cube such that Bd $D'=$ Bd D and each vertical segment meeting C meets D at exactly one point. The desired isotopy is obtained by defining an isotopy on each vertical segment using proportion.

The following two lemmas will also be necessary.

Lemma 4. *Let A and B be two n-manifolds with boundary. If $X=A+B$ is an n-manifold with boundary and* dim $A\cdot B=n-1$, *then*

$$A\cdot B\subset \mathrm{Bd}\, A\cdot \mathrm{Bd}\, B.$$

Proof. Suppose there exists a point p in $A\cdot B$ that is not contained, say, in Bd A. Then a euclidean neighborhood U_A of p in A is an open subset of X (invariance of domain). Consequently, $U_A\cdot B$ is a nonempty open subset of B. Hence, dim $U_A\cdot B=n$. On the other hand,

$$\dim U_A\cdot B \leqq \dim A\cdot B = n-1.$$

This contradiction proves the lemma.

Lemma 5. *Let A and B be two n-manifolds with boundary with* dim $A\cdot B=n-1$. *If $X=A+B$ is an n-manifold without boundary, then* $A\cdot B=$ Bd $A=$ Bd B.

Proof. By Lemma 4, $A\cdot B\subset$ Bd $A\cdot$ Bd B. But if a point in Bd A or in Bd B did not lie in $A\cdot B$, then X would fail to have a euclidean open neighborhood at the point. Therefore, Bd $A+$ Bd $B\subset A\cdot B$. Consequently, $A\cdot B=$ Bd A $=$ Bd B.

Lemma 6. *If σ^1 is a 1-cell of any K_i of the characterization theorem, then* Cl σ^1 *is an arc,* Fr σ^1 *being its end-points.*

Lemma 7. *If σ^2 is a 2-cell of any K_i of the characterization theorem, then* Cl σ^2 *is a disk,* Fr σ^2 *being its boundary.*

Proof. The lemma can be proved by a simplified version of what we are going to give for the proof of a similar statement about cells of higher dimensions.

Let Q^n denote, hereafter, a space satisfying the three conditions of the characterization theorem. Then by Lemmas 6 and 7, we have

Lemma 8. *Q^1 and Q^2 are 1-sphere and 2-sphere, respectively.*

4. The proof of the characterization theorem. In this section, K_i will denote a covering complex of Q^n in the statement of the characterization theorem.

THEOREM 9. (1) *For each r-cell* σ^r *of any* K_i, $\operatorname{Fr} \sigma^r$ *is a topological* $(r-1)$-*sphere.*

(2) *If* $\sigma^r = \sigma_1^r + \sigma_2^r + \sigma^{r-1}$ *is the simple subdivision of* σ^r, *then* $\operatorname{Fr} \sigma^{r-1}$ *is a tame* $(r-2)$-*sphere in* $\operatorname{Fr} \sigma^r$.

(3) *If* τ^r *is a derived cell of* σ^r *such that* $\operatorname{Fr} \sigma^r \cdot \operatorname{Fr} \tau^r$ *is not empty, then* $\operatorname{Fr} \sigma^r \cdot \operatorname{Fr} \tau^r$ *is a euclidean* $(r-1)$-*cell tame in* $\operatorname{Fr} \sigma^r$.

(4) *For each r-cell* σ^r *of any* K_i, $\operatorname{Cl} \sigma^r$ *is a euclidean r-cell and* $\operatorname{Fr} \sigma^r = \operatorname{Bd}(\operatorname{Cl} \sigma^r)$.

Proof. Since the theorem is certainly true for $r=2$, we apply induction on r. Suppose the theorem has been proved for all cells of dimension less than s.

The proof of Statement 1. Let σ^s be a given s-cell of given K_i. By Condition 2 of the characterization theorem, σ^s will be simply subdivided sooner or later. Let $\sigma^s = \sigma_1^s + \sigma_2^s + \sigma^{s-1}$ be the simple subdivision of σ^s. Then $\operatorname{Fr} \sigma^s = A + B$ with $A \cdot B = \operatorname{Fr} \sigma^{s-1}$, where $A = \operatorname{Fr} \sigma^s \cdot \operatorname{Fr} \sigma_1^s$ and $B = \operatorname{Fr} \sigma^s \cdot \operatorname{Fr} \sigma_2^s$. We first claim that $\operatorname{Fr} \sigma^s$ is an $(s-1)$-manifold with boundary. Note that A and B are, by the definition of simple subdivision, homeomorphic to $\operatorname{Cl} \sigma^{s-1}$, which in turn is, by induction hypothesis, a euclidean $(s-1)$-cell. To justify our claim, it suffices to show that each point of $A \cdot B$ has a neighborhood whose closure is a euclidean $(s-1)$-cell in $A+B$, since each point of $A+B-A\cdot B$ has this property already. We show this by showing that there exists an $(s-1)$-manifold X with boundary containing $A \cdot B$ such that there is an open subset of X containing $A \cdot B$ which is at the same time an open set of $A+B$. Let q be a point on $\operatorname{Fr} \sigma^s - A \cdot B$. There is an integer k such that a derived cell τ^s of σ^s in K_k has the closure containing q and not meeting $A \cdot B$. By Condition (3) of the characterization theorem and induction hypothesis, $\operatorname{Cl}(\operatorname{Fr} \sigma^s - \operatorname{Fr} \tau^s)$ is a euclidean $(s-1)$-cell X. The set $X - \operatorname{Fr} \tau^s$ equals $\operatorname{Fr} \sigma^s - \operatorname{Fr} \tau^s$. This set contains $A \cdot B$ and is open in X and in $\operatorname{Fr} \sigma^s$. We have proved that $A+B$ is an $(s-1)$-manifold with boundary. By Lemma 4, $A \cdot B \subset \operatorname{Bd} A \cdot \operatorname{Bd} B$. Since $A \cdot B = \operatorname{Fr} \sigma^{s-1}$ is an $(s-2)$-sphere by induction hypothesis, it follows that $\operatorname{Bd} A = A \cdot B = \operatorname{Bd} B$. On the other hand, given two euclidean k-cells C and D, any homeomorphism of $\operatorname{Bd} C$ into $\operatorname{Bd} D$ can be extended to a homeomorphism of C onto D. Therefore, $\operatorname{Fr} \sigma^s (= A+B)$ has a homeomorphism onto S^{s-1}.

The proof of Statement 2. Let $\sigma^s = \sigma_1^s + \sigma_2^s + \sigma^{s-1}$ be the simple subdivision of σ^s. Since $\operatorname{Fr} \sigma^{s-1} = \operatorname{Bd} A = \operatorname{Bd} B = A \cdot B$ (see the proof of Statement 1), any homeomorphism of $\operatorname{Fr} \sigma^{s-1}$ into S^{s-2} can be extended to that of $A+B$ into S^{s-1}.

The proof of Statement 3. Entirely similar to the proof of Statement 2 after one goes through an argument similar to the proof of Statement 1.

The proof of Statement 4. In this section, if K is a complex, we denote by K' the sum of all cells of K except those cells of the highest dimension.

Let σ^s be an s-cell of any K_i. Then for any integer $j \geqq i$, the cells of K_j on $\mathrm{Cl}\,\sigma^s$ form a complex which we might call the subcomplex of K_j on $\mathrm{Cl}\,\sigma^s$. Let the subcomplex of K_i on $\mathrm{Cl}\,\sigma^s$ be denoted by L_1. The sequence $K_i, K_{i+1}, \cdots,$ induces a sequence $L_1, L_2, \cdots,$ of covering complexes of $\mathrm{Cl}\,\sigma^s$ which satisfies Conditions (2) and (3) of the characterization theorem. Let C be the unit sphere and S the sum of C and its interior in E^s. We will show that $\mathrm{Cl}\,\sigma^s$ is homeomorphic to S. For this purpose, we first construct a sequence $M_1, M_2, \cdots$ of covering complexes of S.

Since $L_1' = \mathrm{Fr}\,\sigma^s$ is a topological $(s-1)$-sphere by Statement 1, there exists a homeomorphism h_1 of $\mathrm{Fr}\,\sigma^s$ into C. Denote by π^s the set $\mathrm{Int}\,S$. If σ^r, $r \neq s$, is an element of L_1, we denote by π^r the set $h_1(\sigma^r)$. In other words, the image of a cell on $\mathrm{Fr}\,\sigma^s$ under h_1 will be expressed by changing the letter σ to the letter π. The π's form a covering complex of S which we denote by M_1. Let f_1 be a 1-1 map of L_1 onto M_1 defined by $f_1(\sigma) = \pi$. The map f_1 has the property that given two cells σ and σ' of L_1, σ is on $\mathrm{Cl}\,\sigma'$ if and only if $f_1(\sigma)$ is on $\mathrm{Cl}(f_1(\sigma'))$.

Suppose we have constructed $M_1, M_2, \cdots, M_{p-1}$ together with $h_1, h_2, \cdots, h_{p-1}$ and $f_1, f_2, \cdots, f_{p-1}$, where each M_{i+1} is a simple subdivision of M_i, h_i is a homeomorphism of L_i' onto M_i' and f_i is a 1-1 map of L_i onto M_i such that

(1) σ is on $\mathrm{Cl}\,\sigma'$ if and only if $f_i(\sigma)$ is on $\mathrm{Cl}(f_i(\sigma'))$ for any σ and σ' of L_i,

(2) $f_i(\sigma) = h_i(\sigma)$ for any cell σ on L_i',

(3) h_{i+1} is an extension of h_i,

(4) for $\sigma \in L_i$ and $\sigma' \in L_j$, $\sigma \subset \sigma'$ if and only if $f_i(\sigma) \subset f_j(\sigma')$ and

(5) for each s-cell $\pi \in M_i$, $\mathrm{Cl}\,\pi$ is a euclidean s-cell with $\pi = \mathrm{Int}(\mathrm{Cl}\,\pi)$.

We now construct M_p from M_{p-1}' as follows. Let $\sigma_0^r = \sigma_1^r + \sigma_2^r + \sigma^{r-1}$ be the simple subdivision of L_{p-1} to L_p.

Case 1. $r < s$. Since $L_p' = L_{p-1}'$, we let $h_p = h_{p-1}$ and M_p and f_p be determined by means of h_{p-1} in the obvious way

Case 2. $r = s$. By Statement 2, $h_{p-1}(\mathrm{Fr}\,\sigma^{s-1})$ is a tame $(s-2)$-sphere in $h_{p-1}(\mathrm{Fr}\,\sigma_0^s)$. There exists a euclidean $(s-1)$-cell P in $\mathrm{Cl}\,\pi_0^s$, where $\pi_0^s = f_{p-1}(\sigma_0^s)$, such that P is the intersection of two euclidean s-cells Q and R that make up $\mathrm{Cl}\,\pi_0^s$, $\mathrm{Bd}\,P$ is tame on $\mathrm{Bd}\,Q$ and $\mathrm{Bd}\,R$ and P meets $\mathrm{Bd}(\mathrm{Cl}\,\pi_0^s)$ exactly at $\mathrm{Bd}\,P$. Of Q and R, suppose Q meets $h_{p-1}(\mathrm{Fr}\,\sigma_1^s - \mathrm{Fr}\,\sigma_2^s)$. Then we let $\pi_1^s = \mathrm{Int}\,Q$, $\pi_2^s = \mathrm{Int}\,R$ and finally $\pi^{s-1} = \mathrm{Int}\,P$. Then these π's form a covering complex M_p of S which is a simple subdivision of M_{p-1}. The map f_p is nothing but changing the letter σ to the letter π. Since any homeomorphism of $\mathrm{Fr}\,\sigma^{s-1}$ into $\mathrm{Bd}\,P$ has an extension to a homeomorphism of $\mathrm{Cl}\,\sigma^{s-1}$ onto P, we obtain a homeomorphism h_p of L_p' onto M_p' which is an extension of h_{p-1}.

Thus-constructed $M_1, M_2, \cdots, M_p$ together with $h_1, h_2, \cdots, h_p$ and $f_1, f_2, \cdots, f_p$ satisfy Conditions (1)–(5) in the preceding paragraph. Thus we have inductively constructed M's, h's and f's subject to the four conditions cited.

We note that mesh M_i does not necessarily converge to zero. We observe that, however, given any s-cell σ of any M_i mesh 0_j, $j=i, i+1, \cdots$, converges to zero, where 0_j is the subcomplex of M_j on Fr σ. This is true because $h_i=h_{i+1}= \cdots$ on Fr σ and this homeomorphism is uniformly continuous, Fr σ being compact.

For any M_i, the collection of the s-cells of M_i is a partitioning G_i of S. These $G_1, G_2, \cdots$ satisfy all conditions of Lemmas 1 and 2. Therefore, by Lemma 2, we find an integer m_1 and a homeomorphism T_1 of S onto itself such that T_1 leaves invariant each point of M_1' and T_1 carries each element of M_{m_1} into a set of diameter less than 1/2. Let $N_1=T_1(M_{m_1})$. We now apply Lemma 2 to the sequence $N_1=T_1(M_{m_1})$, $T_1(M_{m_1+1}), \cdots$ to find an integer m_2 and a homeomorphism T_2 of S onto itself such that T_2 leaves each point of N_1' and carries each element of $T_1(M_{m_2})$ into a set of diameter less than 1/4. We let $N_2=T_2T_1(M_{m_2})$. Now we apply Lemma 2 to $T_2T_1(M_{m_2})$, $T_2T_1(M_{m_2+1}), \cdots$, and so forth.

There are integers $m_1, m_2, \cdots$, complexes $N_1, N_2, \cdots$ and homeomorphisms $T_1, T_2, \cdots$ of S onto itself such that $N_p=T^p(M_{m_p})$, where $T^p = T_pT_{p-1} \cdots T_2T_1$, T_p leaves each point of N'_{p-1} and carries each element of $T^{p-1}(M_{m_p})$ into a set of diameter less than $1/2^p$. This shows that each N_{p+1} is obtainable from N_p by a finite number of simple subdivisions.

We now consider a map g_i of L_{m_i} onto N_i defined by $g_i(\sigma)=T^i(f_{m_i}(\sigma))$. We note that a cell σ in L_{m_i} is contained in a cell σ' in L_{m_j} if and only if $g_i(\sigma)$ is contained in $g_j(\sigma')$. Using these g_i a homeomorphism g of Cl σ^s onto S can be readily constructed. Under the map g, Fr σ^s is mapped into C. This proves Statement 4.

The proof of the characterization theorem. The necessity can be proved by a suitable construction and the sufficiency follows from Statement 4 of Theorem 9.

ADDED IN PROOF. It can be shown that all the n-cells of any K_i in the characterization theorem are tame in the n-sphere Q^n.

REFERENCE

1. R. H. Bing, *Characterization of 3-space by partitionings*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 15–27.

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