A CHARACTERIZATION OF THE $n$-SPHERE

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1. Introduction. We give a set-theoretic characterization of the topological $n$-sphere. In so doing, we use some notions introduced by R. L. Wilder in his summer lectures of 1948 at the University of Michigan. Also an essential use is made of modified forms of some results [1] of R. H. Bing.

2. Characterization theorem. We first define terms cells, complexes and their simple subdivisions. They are originally due to R. L. Wilder and we use slightly modified forms here.

A complex is a finite collection of elements which we call cells, which in turn are point sets of an underlying topological Hausdorff space. The cells of a complex must satisfy the following conditions.

(1) Each cell is a nonempty point set. In particular, a 0-cell (cf. (3) below) consists of a single point.

(2) No two cells have a point in common.

(3) With each cell is associated a unique non-negative integer $r$ which we call dimension of the cell. We may call the cell an $r$-cell if we desire to specify its dimension.

(4) If $\sigma^r$ is an $r$-cell, $r > 0$, there is a nonempty finite collection of $(r-1)$-cells $\sigma'_1^{r-1}, \sigma'_2^{r-1}, \ldots, \sigma'_k^{r-1}$ such that

$$\text{Cl } \sigma^r = \sigma^r + \text{Cl } \sigma'_1^{r-1} + \text{Cl } \sigma'_2^{r-1} + \ldots + \text{Cl } \sigma'_k^{r-1}.$$ 

The cells $\sigma'_i^{r-1}$ are called the boundary cells of $\sigma^r$. The sum of the closures of the boundary cells of $\sigma^r$ will be denoted by $\text{Fr } \sigma^r$.

(5) A 1-cell has exactly two boundary 0-cells.

If the sum of the cells of a complex $K$ is a space $X$, we say that $K$ is a covering complex of $X$.

A simple subdivision of an $r$-cell $\sigma^r$, $r > 0$, is the complex consisting of the faces of $\sigma^r$ (i.e., the cells on Cl $\sigma^r$) except $\sigma^r$ together with cells $\sigma'_1, \sigma'_2$ and $\sigma'^{-1}$ such that (1) $\sigma^r = \sigma'_1 + \sigma'_2 + \sigma'^{-1}$, (2) Cl $\sigma'^{-1} = \text{Cl } \sigma'_1 \cdot \text{Cl } \sigma'_2$ and (3) Fr $\sigma^r \cdot \text{Fr } \sigma'_i, i = 1, 2$, are homeomorphic to Cl $\sigma'^{-1}$. As an example, one may consider subdividing a disk by an arc that meets the boundary of the disk at its endpoints.

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(*) This paper is essentially a part of the author's dissertation written under Professor R. L. Wilder and submitted to the University of Michigan in 1958. The desirability of publishing a complete proof of the characterization theorem (see §2) has recently been called to the author's attention.
By a simple subdivision of a complex we mean a complex obtained from the original complex by replacing one of its cells by a simple subdivision thereof.

The procedure of obtaining a simple subdivision will also be referred to as simple subdivision.

By a sequence of complexes $K_1, K_2, \ldots$ we mean that each complex $K_i$ is a simple subdivision of $K_{i-1}$. We say that $\tau' \Subset K_j$ is a derived cell of $\sigma' \Subset K_i$ if $\tau' \subseteq \sigma'$. In this case, we say that $\tau'$ is directly derivable from $\sigma'$ if further $Fr \sigma' \cdot Fr \tau'$, $Cl(Fr \sigma' - Fr \tau')$ and $Cl(Fr \tau' - Fr \sigma')$ are all homeomorphic to the closure of one and the same boundary cell of $\sigma'$. As an example, one might consider several steps of subdividing a disk nicely.

**Theorem.** (Characterization of the $n$-sphere). In order that a compact Hausdorff space $Q^n$ be a topological $n$-sphere, it is necessary and sufficient that $Q^n$ have an $\omega$-sequence of covering complexes $K_0, K_1, K_2, \ldots$ satisfying the following conditions:

1. $K_0$ consists of exactly two $r$-cells $\sigma_1$ and $\sigma_2$ for each dimension $r \leq n$ and each of the cells $\sigma_1$ and $\sigma_2$ is, if $r < n$, a boundary cell of each of the cells $\sigma_1^{r+1}$ and $\sigma_2^{r+1}$.

2. For each open covering $U$ of $Q^n$, there exists an integer $i$ such that each cell of $K_i$ is of diameter less than $U$ (i.e., contained in an element of $U$).

3. If $\tau'$ is a derived cell of $\sigma'$ such that $Fr \sigma' \cdot Fr \tau' \neq \emptyset$, then $\tau'$ is directly derivable from $\sigma'$.

The reader might compare the above characterization with Bing’s characterization of 3-space by partitionings [1]. In fact, we make use of analogues of Theorems 4 and 5 of [1]. We do not know whether Condition 3 can be deleted from the theorem.

3. **Some lemmas.** We denote by $S^n$ the unit sphere in $E^{n+1}$. Explicitly,

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in E^{n+1} | \sum x_i = 1\}.$$ 

By $S^{n-1}$ and $S^*_+$ we denote the subsets of $S^n$ defined by $x_{n+1} = 0$ and $x_{n+1} \geq 0$, respectively. A homeomorphic image of $S^*_+$ will be called a euclidean $n$-cell.

An $n$-manifold with boundary is a separable metric space such that each point has a neighborhood whose closure is a euclidean $n$-cell. If $X$ is an $n$-manifold with boundary, $Bd X$ will denote the set of those points which fail to have a neighborhood homeomorphic to $E^n$. Finally, the set $X - Bd X$ will be denoted by Int $X$.

A topological $(n-1)$-sphere $S'$ in a topological $n$-sphere $S$ is called tame if there is a homeomorphism of $S$ into $S^n$ sending $S'$ into $S^{n-1}$. A euclidean $n$-cell $D$ in a topological $n$-sphere $S$ is called tame if there is a homeomorphism of $S$ into $S^n$ sending $D$ onto $S^*_+$. Clearly, $D$ is tame if and only if $Bd D$ is tame.

The following two lemmas are generalizations to general dimension $n$ of Theorems 4 and 5 of [1].
Lemma 1. Suppose $C$ is the unit sphere in $E^n$ and $G_1, G_2, \cdots$ is a sequence of partitionings of $S=(C \text{ plus its interior})$ satisfying the following conditions:

(i) each element of $G_i$ is an open euclidean $n$-cell disjoint from $C$;
(ii) the closure of each element $g$ of $G_i$ is a euclidean $n$-cell with $g = \text{Int}(\text{Cl} g)$;
(iii) if $g$ is an element of $G_i$ whose closure intersects $C$, then given any neighborhood $U$, with respect to the boundary of $g$, of the intersection $F$, there exists a euclidean $(n-1)$-cell $D$ which is tame in the boundary of $g$ such that $U \supset D \supset F$;
(iv) each $G_{i+1}$ is a refinement of $G_i$; and
(v) for any positive number $e$, there is an integer $i$ such that the closure of no element of $G_i$ contains two points of the sum of the boundaries of the elements of $G_i$ which are farther apart than $e$.

Let $K$ be a closed set and $R$ be a proper closed subset of $C$ such that if $g \in G_i$, then $\text{Bd}(\text{Cl} g) \cap C$ does not meet both $R$ and $K$. Then there is an integer $q$ and a homeomorphism $T$ of $S$ onto itself satisfying the following conditions:

(vi) each point of the boundary of each element of $G_i$ is invariant under $T$; and
(vii) if $g \in G_q$, $T(\text{Cl} g)$ does not meet both $R$ and $K$.

Lemma 2. Suppose the sequence $G_1, G_2, \cdots$ in Lemma 1 satisfies the conditions (i)-(iv) as well as the following:

(viii) for each positive integer $j$ and each positive number $e$, there is a positive integer $p(j, e)$ such that the closure of no element of $G_p$ contains two points of the sum of the boundaries of the elements of $G_j$ which are farther apart than $e$.

Then for each positive number $\delta$, there is an integer $q$ and a homeomorphism $T$ of $S$ onto itself such that $T$ leaves invariant each point of $F(G_i)$ and $T$ carries each element of $G_q$ into a set of diameter less than $\delta$, where $F(G_i)$ denotes the sum of the boundaries of the elements of $G_i$.

In the strength of our definition of tameness, proofs of these two lemmas can be given in parallel to the proofs of Bing's original theorems. Therefore, we omit proofs. But the following lemma will suggest how the parallelism can be achieved.

Lemma 3. Let $C$ be the $n$-cube in $E^n$ defined by $0 \leq x_i \leq 1$, $i=1, 2, \cdots, n$. Let $D$ be the subset of $C$ defined by $x_n = 1/2$. We finally denote by $C'$ the subset of $C$ defined by $x_n \geq 1/2$. If $F$ is a closed subset of $C$ such that $F \cdot C' \cdot \text{Bd} C = \emptyset$, then there exists an isotopy $f_t$, $0 \leq t \leq 1$, such that $f_0$ is the identity, each $f_t$ is a homeomorphism leaving $\text{Bd} C$ pointwise fixed and moving the $x_i$-coordinates, $i \neq n$, of no point of $C$ and $f_1(F) \cdot C' = \emptyset$.

Proof. With respect to the $x_n$-coordinate, we may use such words like "above" and "below." Let $F_{ij}$, $i=1, 2, \cdots, n$; $j=0, 1$, be the $(n-2)$-cube in $C$ determined by $x_j = j$ and $D_{ij} = D \cdot F_{ij}$. For each $i \neq n$ and $j$ and any positive number $d < 1/2$, let $G_{ij}$ be the $(n-2)$-cube on $F_{n, j}$ determined by $x_i = j + (-1)^j d$ and $x_n = 1/2$. We finally denote by $H_0^j$ the convex body which is a subset of
C and is the set of the points lying not lower than the \((n-1)\)-plane spanned by the sets \(\mathcal{C}_n\) and \(D_{ij}\). Consider the sequence \(H_{ij}^{k}\), \(k=3, 4, \ldots\). It is a decreasing sequence of compact sets whose intersection is \(F_{ij} \cdot C'\). Since \(F \cdot F_{ij} \cdot C' = \emptyset\), there is an integer \(k_{ij}\) such that \(H_{ij}^{k_{ij}} \cdot F = \emptyset\). Also there exists a positive number \(\epsilon < 1/2\) such that no point of \(F\) has the \(x_n\)-coordinate as large as \(1-\epsilon\). We now let \(H_n\) be the subset of \(C\) determined by \(x_n \geq 1-\epsilon\).

Let \(C_{1} = H_n + \sum_{i,j} H_{ij}^{k_{ij}}\). The set \(D' = C_{1} \cdot \text{Cl}(C-C_{1})\) is a topological \((n-1)\)-cube such that \(\text{Bd} \, D' = \text{Bd} \, D\) and each vertical segment meeting \(C\) meets \(D\) at exactly one point. The desired isotopy is obtained by defining an isotopy on each vertical segment using proportion.

The following two lemmas will also be necessary.

**Lemma 4.** Let \(A\) and \(B\) be two \(n\)-manifolds with boundary. If \(X = A \cup B\) is an \(n\)-manifold with boundary and \(\dim A \cdot B = n-1\), then

\[ A \cdot B \subset \text{Bd} \, A \cdot \text{Bd} \, B. \]

**Proof.** Suppose there exists a point \(p\) in \(A \cdot B\) that is not contained, say, in \(\text{Bd} \, A\). Then a euclidean neighborhood \(U_{A}\) of \(p\) in \(A\) is an open subset of \(X\) (invariance of domain). Consequently, \(U_{A} \cdot B\) is a nonempty open subset of \(B\). Hence, \(\dim U_{A} \cdot B = n\). On the other hand,

\[ \dim U_{A} \cdot B \leq \dim A \cdot B = n - 1. \]

This contradiction proves the lemma.

**Lemma 5.** Let \(A\) and \(B\) be two \(n\)-manifolds with boundary with \(\dim A \cdot B = n-1\). If \(X = A \cup B\) is an \(n\)-manifold without boundary, then \(A \cdot B = \text{Bd} \, A = \text{Bd} \, B\).

**Proof.** By Lemma 4, \(A \cdot B \subset \text{Bd} \, A \cdot \text{Bd} \, B\). But if a point in \(\text{Bd} \, A\) or in \(\text{Bd} \, B\) did not lie in \(A \cdot B\), then \(X\) would fail to have a euclidean open neighborhood at the point. Therefore, \(\text{Bd} \, A + \text{Bd} \, B \subset A \cdot B\). Consequently, \(A \cdot B = \text{Bd} \, A = \text{Bd} \, B\).

**Lemma 6.** If \(\sigma^1\) is a 1-cell of any \(K_i\) of the characterization theorem, then \(\text{Cl} \, \sigma^1\) is an arc, \(\text{Fr} \, \sigma^1\) being its end-points.

**Lemma 7.** If \(\sigma^2\) is a 2-cell of any \(K_i\) of the characterization theorem, then \(\text{Cl} \, \sigma^2\) is a disk, \(\text{Fr} \, \sigma^2\) being its boundary.

**Proof.** The lemma can be proved by a simplified version of what we are going to give for the proof of a similar statement about cells of higher dimensions.

Let \(Q^n\) denote, hereafter, a space satisfying the three conditions of the characterization theorem. Then by Lemmas 6 and 7, we have

**Lemma 8.** \(Q^1\) and \(Q^2\) are 1-sphere and 2-sphere, respectively.
4. The proof of the characterization theorem. In this section, $K_i$ will denote a covering complex of $Q^n$ in the statement of the characterization theorem.

**Theorem 9.** (1) For each $r$-cell $\sigma^r$ of any $K_i$, $Fr \sigma^r$ is a topological $(r-1)$-sphere.

(2) If $\sigma^r = \sigma^r_1 + \sigma^r_2 + \sigma^r_{r-1}$ is the simple subdivision of $\sigma^r$, then $Fr \sigma^{r-1}$ is a tame $(r-2)$-sphere in $Fr \sigma^r$.

(3) If $\tau^r$ is a derived cell of $\sigma^r$ such that $Fr \sigma^r \cdot Fr \tau^r$ is not empty, then $Fr \sigma^r \cdot Fr \tau^r$ is a euclidean $(r-1)$-cell tame in $Fr \sigma^r$.

(4) For each $r$-cell $\sigma^r$ of any $K_i$, $Cl \sigma^r$ is a euclidean $r$-cell and $Fr \sigma^r = Bd(Cl \sigma^r)$.

**Proof.** Since the theorem is certainly true for $r=2$, we apply induction on $r$. Suppose the theorem has been proved for all cells of dimension less than $s$.

The proof of Statement 1. Let $\sigma^s$ be a given $s$-cell of given $K_i$. By Condition 2 of the characterization theorem, $\sigma^s$ will be simply subdivided sooner or later. Let $\sigma^s = \sigma^s_1 + \sigma^s_2 + \sigma^s_{s-1}$ be the simple subdivision of $\sigma^s$. Then $Fr \sigma^s = A + B$ with $A \cdot B = Fr \sigma^{s-1}$, where $A = Fr \sigma^s \cdot Fr \sigma^s_1$ and $B = Fr \sigma^s \cdot Fr \sigma^s_2$. We first claim that $Fr \sigma^s$ is an $(s-1)$-manifold with boundary. Note that $A$ and $B$ are, by the definition of simple subdivision, homeomorphic to $Cl \sigma^{s-1}$, which in turn is, by induction hypothesis, a euclidean $(s-1)$-cell. To justify our claim, it suffices to show that each point of $A \cdot B$ has a neighborhood whose closure is a euclidean $(s-1)$-cell in $A + B$, since each point of $A + B - A \cdot B$ has this property already. We show this by showing that there exists an $(s-1)$-manifold $X$ with boundary containing $A \cdot B$ such that there is an open subset of $X$ containing $A \cdot B$ which is at the same time an open set of $A + B$. Let $q$ be a point on $Fr \sigma^s - A \cdot B$. There is an integer $k$ such that a derived cell $\tau^r$ of $\tau^s$ in $K_k$ has the closure containing $q$ and not meeting $A \cdot B$. By Condition (3) of the characterization theorem and induction hypothesis, $Cl(Fr \sigma^s - Fr \tau^s)$ is a euclidean $(s-1)$-cell $X$. The set $X - Fr \tau^s$ equals $Fr \sigma^s - Fr \tau^s$. This set contains $A \cdot B$ and is open in $X$ and in $Fr \sigma^s$. We have proved that $A + B$ is an $(s-1)$-manifold with boundary. By Lemma 4, $A \cdot B \subset Bd A \cdot Bd B$. Since $A \cdot B = Fr \sigma^{s-1}$ is an $(s-2)$-sphere by induction hypothesis, it follows that $Bd A = A \cdot Bd B$. On the other hand, given two euclidean $k$-cells $C$ and $D$, any homeomorphism of $Bd C$ into $Bd D$ can be extended to a homeomorphism of $C$ onto $D$. Therefore, $Fr \sigma^s(= A + B)$ has a homeomorphism onto $S^{s-1}$.

The proof of Statement 2. Let $\sigma^s = \sigma^s_1 + \sigma^s_2 + \sigma^s_{s-1}$ be the simple subdivision of $\sigma^s$. Since $Fr \sigma^{s-1} = Bd A = Bd B = A \cdot B$ (see the proof of Statement 1), any homeomorphism of $Fr \sigma^{s-1}$ into $S^{s-2}$ can be extended to that of $A + B$ into $S^{s-1}$.
The proof of Statement 3. Entirely similar to the proof of Statement 2 after one goes through an argument similar to the proof of Statement 1.

The proof of Statement 4. In this section, if $K$ is a complex, we denote by $K'$ the sum of all cells of $K$ except those cells of the highest dimension.

Let $\sigma^r$ be an $s$-cell of any $K_i$. Then for any integer $j \geq i$, the cells of $K_j$ on $\text{Cl}\, \sigma^r$ form a complex which we might call the subcomplex of $K_j$ on $\text{Cl}\, \sigma^r$. Let the subcomplex of $K_i$ on $\text{Cl}\, \sigma^r$ be denoted by $L_i$. The sequence $K_i, K_{i+1}, \ldots$, induces a sequence $L_1, L_2, \ldots$, of covering complexes of $\text{Cl}\, \sigma^r$ which satisfies Conditions (2) and (3) of the characterization theorem. Let $C$ be the unit sphere and $S$ the sum of $C$ and its interior in $E^s$. We will show that $\text{Cl}\, \sigma^r$ is homeomorphic to $S$. For this purpose, we first construct a sequence $M_1, M_2, \ldots$ of covering complexes of $S$.

Since $L_i^r = \text{Fr}\, \sigma^r$ is a topological $(s-1)$-sphere by Statement 1, there exists a homeomorphism $h_i$ of $\text{Fr}\, \sigma^r$ into $C$. Denote by $\pi^r$ the set $\text{Int}\, S$. If $\sigma^r, \tau \neq \sigma^r$, is an element of $L_i$, we denote by $\pi^r$ the set $h_i(\sigma^r)$. In other words, the image of a cell on $\text{Fr}\, \sigma^r$ under $h_i$ will be expressed by changing the letter $\sigma$ to the letter $\pi$. The $\pi$'s form a covering complex of $S$ which we denote by $M_i$. Let $f_i$ be a 1-1 map of $L_i$ onto $M_i$ defined by $f_i(\sigma) = \pi$. The map $f_i$ has the property that given two cells $\sigma$ and $\sigma'$ of $L_i$, $\sigma$ is on $\text{Cl}\, \sigma'$ if and only if $f_i(\sigma)$ is on $\text{Cl}(f_i(\sigma'))$.

Suppose we have constructed $M_1, M_2, \ldots, M_{p-1}$ together with $h_1, h_2, \ldots, h_{p-1}$ and $f_1, f_2, \ldots, f_{p-1}$, where each $M_{i+1}$ is a simple subdivision of $M_i$, $h_i$ is a homeomorphism of $L_i^r$ onto $M_i^r$ and $f_i$ is a 1-1 map of $L_i$ onto $M_i$ such that

1. $\sigma$ is on $\text{Cl}\, \sigma'$ if and only if $f_i(\sigma)$ is on $\text{Cl}(f_i(\sigma'))$ for any $\sigma$ and $\sigma'$ of $L_i$,
2. $f_i(\sigma) = h_i(\sigma)$ for any cell $\sigma$ on $L_i^r$,
3. $h_{i+1}$ is an extension of $h_i$,
4. for $\sigma \in L_i$ and $\sigma' \in L_j$, $\sigma \subset \sigma'$ if and only if $f_i(\sigma) \subset f_j(\sigma')$ and
5. for each $s$-cell $\pi \in M_i$, $\text{Cl}\, \pi$ is a euclidean $s$-cell with $\pi = \text{Int}(\text{Cl}\, \pi)$.

We now construct $M_p$ from $M_{p-1}$ as follows. Let $\sigma_0 = \sigma_1' + \sigma_2' + \cdots + \sigma_{r-1}'$ be the simple subdivision of $L_{p-1}$ to $L_p$.

**CASE 1.** $r < s$. Since $L_p' = L_{p-1}'$, we let $h_p = h_{p-1}$ and $M_p$ and $f_p$ be determined by means of $h_{p-1}$ in the obvious way.

**CASE 2.** $r = s$. By Statement 2, $h_{p-1} (\text{Fr}\, \sigma^{r-1})$ is a tame $(s-2)$-sphere in $h_{p-1} (\text{Fr}\, \sigma_0')$. There exists a euclidean $(s-1)$-cell $P$ in $\pi_0'$, where $\pi_0' = f_{p-1} (\sigma_0')$, such that $P$ is the intersection of two euclidean $s$-cells $Q$ and $R$ that make up $\text{Cl}\, \pi_0'$, $\text{Bd}\, P$ is tame on $\text{Bd}\, Q$ and $\text{Bd}\, R$ and $P$ meets $\text{Bd}(\text{Cl}\, \pi_0')$ exactly at $\text{Bd}\, P$. Of $Q$ and $R$, suppose $Q$ meets $h_{p-1} (\text{Fr}\, \sigma_1' - \text{Fr}\, \sigma_2')$. Then we let $\pi_1' = \text{Int}\, Q$, $\pi_2' = \text{Int}\, R$ and finally $\pi^{r-1} = \text{Int}\, P$. Then these $\pi$'s form a covering complex $M_p$ of $S$ which is a simple subdivision of $M_{p-1}$. The map $f_p$ is nothing but changing the letter $\sigma$ to the letter $\pi$. Since any homeomorphism of $\text{Fr}\, \sigma^{r-1}$ into $\text{Bd}\, P$ has an extension to a homeomorphism of $\text{Cl}\, \sigma^{r-1}$ onto $P$, we obtain a homeomorphism $h_p$ of $L_p'$ onto $M_p'$ which is an extension of $h_{p-1}$.
Thus-constructed \( M_1, M_2, \ldots, M_p \) together with \( h_1, h_2, \ldots, h_p \) and \( f_1, f_2, \ldots, f_p \) satisfy Conditions (1)-(5) in the preceding paragraph. Thus we have inductively constructed \( M' \)'s, \( h' \)'s and \( f' \)'s subject to the four conditions cited.

We note that mesh \( M_i \) does not necessarily converge to zero. We observe that, however, given any \( s \)-cell \( \sigma \) of any \( M_i \), mesh \( 0_j, j = i, i+1, \ldots \), converges to zero, where \( 0_j \) is the subcomplex of \( M_j \) on \( \text{Fr} \sigma \). This is true because \( h_i = h_{i+1} = \cdots \) on \( \text{Fr} \sigma \) and this homeomorphism is uniformly continuous, \( \text{Fr} \sigma \) being compact.

For any \( M_i \), the collection of the \( s \)-cells of \( M_i \) is a partitioning \( G_i \) of \( S \). These \( G_i, G_2, \ldots \) satisfy all conditions of Lemmas 1 and 2. Therefore, by Lemma 2, we find an integer \( m_1 \) and a homeomorphism \( T_1 \) of \( S \) onto itself such that \( T_1 \) leaves invariant each point of \( M'_1 \) and \( T_1 \) carries each element of \( M_{m_1} \) into a set of diameter less than \( 1/2 \). Let \( N_1 = T_1(M_{m_1}) \). We now apply Lemma 2 to the sequence \( N_1 = T_1(M_{m_1}), T_1(M_{m_1+1}), \ldots \) to find an integer \( m_2 \) and a homeomorphism \( T_2 \) of \( S \) onto itself such that \( T_2 \) leaves each point of \( N'_1 \) and carries each element of \( T_1(M_{m_2}) \) into a set of diameter less than \( 1/4 \). We let \( N_2 = T_2T_1(M_{m_2}) \). Now we apply Lemma 2 to \( T_2T_1(M_{m_2}), T_2T_1(M_{m_2+1}), \ldots \), and so forth.

There are integers \( m_1, m_2, \ldots \), complexes \( N_1, N_2, \ldots \) and homeomorphisms \( T_1, T_2, \ldots \) of \( S \) onto itself such that \( N_p = T_p(M_{m_p}) \), where \( T_p = T_pT_{p-1} \cdots T_2T_1 \). \( T_p \) leaves each point of \( N_{p-1} \) and carries each element of \( T_{p-1}(M_{m_p}) \) into a set of diameter less than \( 1/2^p \). This shows that each \( N_{p+1} \) is obtainable from \( N_p \) by a finite number of simple subdivisions.

We now consider a map \( g_i \) of \( L_{m_i} \) onto \( N_i \) defined by \( g_i(\sigma) = T_i(f_{m_i}(\sigma)) \). We note that a cell \( \sigma \) in \( L_{m_i} \) is contained in a cell \( \sigma' \) in \( L_{m_j} \) if and only if \( g_i(\sigma) \) is contained in \( g_j(\sigma') \). Using these \( g \)'s, a homeomorphism \( g \) of \( \text{Cl} \sigma' \) onto \( S \) can be readily constructed. Under the map \( g \), \( \text{Fr} \sigma \) is mapped into \( C \). This proves Statement 4.

The proof of the characterization theorem. The necessity can be proved by a suitable construction and the sufficiency follows from Statement 4 of Theorem 9.

Added in proof. It can be shown that all the \( n \)-cells of any \( K_i \) in the characterization theorem are tame in the \( n \)-sphere \( Q^n \).

Reference


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