VARIATIONAL METHODS FOR FUNCTIONS WITH POSITIVE REAL PART

BY

M. S. ROBERTSON

1. **Introduction.** M. M. Schiffer [8] has recently derived a formula for the variation of the Green's function of the most general plane domain $\mathfrak{D}$ with boundary $\mathfrak{C}$ due to a small shift

$$w^* = w + \rho^2\phi(w), \tag{1.1}$$

of the boundary. The variation $\delta g(\zeta, \eta)$ of the Green's function is given by the formula

$$\delta g(\zeta, \eta) = \mathrm{Re}\left\{\frac{\rho^2}{2\pi i}\oint_\Gamma p'(w, \eta)p'(w, \zeta)\phi(w)dw\right\} + o(\rho^2) \tag{1.2}$$

where $p(w, \eta)$ is an analytic function whose real part is the Green function $g(w, \eta)$ of $\mathfrak{D}$, and where Γ is a member of a curve system in $\mathfrak{D}$ homotopic to $\mathfrak{C}$. The function $\phi(w)$ is analytic on Γ and in the ring bounded by $\mathfrak{C}$ and Γ. If $\mathfrak{D}$ is simply-connected and if $z=\psi(w)$ maps $\mathfrak{D}$ on the interior of the unit circle $|z|<1$, then $g(w, \eta)$ is connected with $\psi(w)$ by the relation

$$g(w, \eta) = \log\left|\frac{1 - (\psi(\eta))^{-}\psi(w)}{\psi(w) - \psi(\eta)}\right|. \tag{1.3}$$

Here and throughout the paper $(\ )^{-}$ indicates the complex conjugate. With an appropriate choice of $\phi(w)$ one may then obtain variation formulas for univalent functions $w=f(z)$. J. A. Hummel [5] has recently used this method of interior variations to study the class of univalently star-like functions. The method may also be used to study those functions which are convex-in-one direction [2]. The choice of the shift function $\phi(w)$, however, is not always an obvious one for many special classes of univalent functions, in particular for the class of close-to-convex functions [6]. Many of these special classes, however, have representations of their member functions in terms of functions $P(z)$ with positive real part. It therefore becomes desirable to have a variational formula for $P(z)$ from which one may then easily obtain analogous variational formulas for the special classes of univalent functions.

It is the purpose of this paper first to derive a variational formula for the class $\mathcal{P}$ of normalized regular functions

$$P(z) = 1 + p_1z + p_2z^2 + \cdots + p_nz^n + \cdots, \qquad P(0) = 1, \tag{1.4}$$

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which have $\operatorname{Re} P(z)>0$ in $|z|<1$. Secondly, we shall apply the variational formula to $P(z)$ in order to solve extremal problems for the class $\mathcal{P}$ and in particular to obtain a characterization of the $(n-1)$ Euclidean coefficient space E_{n-1} for the extremal functions $P(z)$ for which $\operatorname{Re} p_n$ is a maximum. Although these results may also be deduced from the Carathéodory-Toeplitz theory [1; 3; 4; 7; 9; 10] it is interesting to see how simply they are derived by the variational method. Thirdly, we shall indicate how the variational formula for $P(z)$ leads to the variational formulas for bounded, regular functions $\omega(z)$ and also for regular functions $F(z)=f\{\omega(z)\}$ which are subordinate to a given univalent function $f(z)$ in $|z|<1$.

2. **A variational formula for the class $\mathcal{P}$.** The choice

$$\phi(w) = \frac{e^{i\theta}}{w - w_0}$$

in (1.2), where w_0 is interior to $\mathfrak{D}$, is the Schiffer case for univalent functions [8]. Hummel [5] chose

$$\phi(w) = wR[\psi(w)],$$

$$R(z) = e^{i\theta}\cdot\frac{1-\bar{z}_0 z}{z-z_0} + e^{-i\theta}\cdot\frac{z-z_0}{1-\bar{z}_0}, \qquad |z_0| < 1, \tag{2.1}$$

where $R(z)$ is real and bounded on $|z|=1$, to obtain the following variational formula for a normalized univalently star-like function $f(z)$ in $|z|<1, f(0)=0$, $f'(0)=1$:

$$f^*(z) = f(z)\left[1 - \rho^2(1 - |z_0|^2)\left(A(z) - \frac{zf'(z)}{f(z)}B(z)\right)\right] + o(\rho^2) \tag{2.2}$$

where the error term $o(\rho^2)$ is an analytic function in z and uniformly bounded in each interior region of $|z|<1$. $A(z)$ and $B(z)$ are defined by

$$A(z) = \frac{ze^{i\theta}}{z_0(z-z_0)} + \frac{ze^{-i\theta}}{1-\bar{z}_0 z} - \frac{e^{i\theta}f(z_0)}{z_0^2 f'(z_0)}, \qquad |z_0| < 1, \tag{2.3}$$

$$B(z) = \frac{e^{i\theta}f(z_0)}{z_0 f'(z_0)(z-z_0)} - e^{-i\theta}\left(\frac{f(z_0)}{z_0 f'(z_0)}\right)^{-}\frac{z}{1-\bar{z}_0 z}. \tag{2.4}$$

Since there is a (1-1) correspondence between the functions $P(z)$ of the class $\mathcal{P}$, given by (1.4), and the univalently star-like functions $f(z)$, $f(0)=0$, $f'(0)=1$, by the relation

$$P(z) = \frac{zf'(z)}{f(z)}, \qquad \operatorname{Re} P(z) > 0, \qquad |z| < 1, \tag{2.5}$$

we therefore take the logarithmic derivative in (2.2) and replace $z(f^{*\prime}(z))/f^*(z)$ by $P^*(z)$. After some simplification we obtain

$$P^*(z) = P(z) - \rho^2(1 - |z_0|^2)z$$

$$\cdot\left[\frac{ze^{i\theta}}{z_0(z - z_0)} + \frac{ze^{-i\theta}}{1 - \bar{z}_0 z} - \frac{e^{i\theta}P(z)}{P(z_0)(z - z_0)} + \frac{e^{-i\theta}zP(z)}{\overline{(P(z_0))}^{-}(1 - \bar{z}_0 z)}\right]' + o(\rho^2), \tag{2.6}$$

where $[\ ]'$ denotes differentiation with respect to z. Let $\delta P(z) = P^*(z) - P(z)$, and write (2.6) in the form

$$\begin{aligned} \frac{-\delta P(z)}{\rho^2(1 - |z_0|^2)} &= \left(\frac{z_0 P'(z)}{P(z_0)} - 1\right)\frac{e^{i\theta}z}{z_0(z_0 - z)} + \left(\frac{z_0 P(z)}{P(z_0)} - z\right)\frac{e^{i\theta}z^4}{z_0(z_0 - z)^2} \\ &+ \frac{P'(z)}{(P(z_0))^{-}}\left(\frac{z^2 e^{-i\theta}}{1 - \bar{z}_0 z}\right) + \left(\frac{P(z)}{\overline{(P(z_0))}^{-}} + 1\right)\frac{e^{-i\theta}z}{(1 - \bar{z}_0 z)^2} + o(1), \end{aligned} \tag{2.7}$$

which is the required variational formula for functions of class $\mathcal{P}$.

If $P(z)$ has the power series expansion (1.4), and if we denote $\delta p_n = p_n^* - p_n$, (2.7) yields

$$\begin{aligned} \frac{-\delta p_n}{\rho^2(1 - |z_0|^2)} &= \frac{ne^{i\theta}}{P(z_0)}\sum_{k=0}^{n}\frac{p_k}{z_0^{n-k+1}} - \frac{ne^{i\theta}}{z_0^{n+1}} + \frac{ne^{-i\theta}}{\overline{(P(z_0))}^{-}}\sum_{k=0}^{n-1} p_k(\bar{z}_0)^{n-k-1}, \\ &+ ne^{-i\theta}(\bar{z}_0)^{n-1} + o(1) \end{aligned} \tag{2.8}$$

where $p_0 = 1$. Since for any complex number w, $\operatorname{Re} w = \operatorname{Re} \bar{w}$, we have

$$\begin{aligned} \frac{-\operatorname{Re}\delta p_n}{\rho^2(1 - |z_0|^2)} = n \operatorname{Re}\Bigg\{\frac{e^{i\theta}}{z_0}\Bigg[&\frac{1}{P(z_0)}\sum_{k=0}^{n-1}\left(\frac{p_k}{z_0^{n-k}} + \bar{p}_k z_0^{n-k}\right) \\ &+ \frac{p_n}{P(z_0)} + z_0^n - z_0^{-n}\Bigg]\Bigg\} + o(1). \end{aligned} \tag{2.9}$$

3. **Extremal functions for the class** $\mathcal{P}$. For any positive integer n let $P(z)$ be an extremal function of the class $\mathcal{P}$ for which the coefficient p_n in (1.4) is real, positive and a maximum over all functions of the class. Since $\mathcal{P}$ is compact p_n attains its maximum. Its value is 2 as is well known from the Carathéodory theory, although we do not assume this fact here. Since $\operatorname{Re} \delta p_n \leqq 0$ in (2.9) and θ is arbitrary we have from (2.9) on replacing z_0 by z

$$\frac{1}{P(z)}\left\{p_n + \sum_{k=0}^{n-1}\left(\frac{p_k}{z^{n-k}} + \bar{p}_k z^{n-k}\right)\right\} + z^n - z^{-n} = 0, \qquad p_n > 0,\ p_0 = 1, \tag{3.1}$$

$$P(z) = \frac{1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots + p_{2n-1}z^{2n-1} + z^{2n}}{1 - z^{2n}} = \frac{Q_{2n}(z)}{1 - z^{2n}} \tag{3.2}$$

where $p_{2n-k} = \bar{p}_k$, $k = 1, 2, \cdots, 2n-1$.

Although ordinarily the variational formulas for extremal functions of various classes of functions lead to differential equations we find here for the class $\mathcal{P}$ that we are led directly to the extremal functions (3.2) without encounting a differential equation. However, the formula (3.2) may be simplified further. We shall presently show that whenever $P(z)$ has the form (3.2) and has a positive real part in $|z|<1$ then $p_n \leqq 2$, and if $p_n=2$ then $(1+z^n)$ is a factor of both numerator and denominator of (3.2). In this case (3.2) becomes

$$P(z) = \frac{1 + p_1 z + p_2 z^2 + \cdots + p_{n-1}z^{n-1} + z^n}{1 - z^n}, \tag{3.3}$$

where $p_{n-k} = \bar{p}_k$, $k=1, 2, \cdots, n-1$.

To obtain (3.3) we place $z = re^{k\pi i/n}$, $0<r<1$, $k=$odd integer, in (3.2). Then

$$P(re^{k\pi i/n}) = \frac{Q_{2n}(re^{k\pi i/n})}{1 - r^{2n}}.$$

Since $\operatorname{Re} P(z)>0$ it follows that $\operatorname{Re} Q_{2n}(re^{k\pi i/n})>0$. Letting $r\to 1$ we have $\operatorname{Re} Q_{2n}(e^{k\pi i/n}) \geqq 0$. However, $Q_{2n}(e^{k\pi i/n})$ is real. This follows since $p_\nu z^\nu + p_{2n-\nu}z^{2n-\nu} = p_\nu z^\nu + \bar{p}_\nu z^{-\nu} =$ a real number when $z=e^{k\pi i/n}$, and because $p_n z^n$ and z^{2n} are also real for this choice of z. Thus

$$Q_{2n}(e^{k\pi i/n}) \geqq 0. \tag{3.4}$$

Let

$$P_{n-1}(z) = \sum_{s=1}^{n-1} p_s z^s, \tag{3.5}$$

$$Q_{2n}(z) = 1 + p_n z^n + z^{2n} + P_{n-1}(z) + z^{2n}\cdot\left(P_{n-1}\left(\frac{1}{\bar{z}}\right)\right)^{-}. \tag{3.6}$$

For k odd, $z=e^{k\pi i/n}$, we have $z^{2n}=1$, $z^n=-1$, so that

$$0 \leqq Q_{2n}(e^{k\pi i/n}) = (2 - p_n) + P_{n-1}(e^{k\pi i/n}) + (P_{n-1}(e^{k\pi i/n}))^{-}, \tag{3.7}$$

$$0 \leqq (2 - p_n) + 2 \operatorname{Re} P_{n-1}(e^{k\pi i/n}) = (2 - p_n) + 2 \operatorname{Re}\left(\sum_{s=1}^{n-1} p_s e^{ks\pi i/n}\right). \tag{3.8}$$

By virtue of the identity

$$\sum_{\nu=1}^{n} e^{(2\nu-1)s\pi i/n} = 0, \qquad s = 1, 2, \cdots, n-1, \tag{3.9}$$

it follows that

$$\operatorname{Re}\sum_{\nu=1}^{n} P_{n-1}(e^{(2\nu-1)\pi i/n}) = 0. \tag{3.10}$$

From (3.8) and (3.10) we then have

$$0 \leqq \sum_{\nu=1}^{n} [2 - p_n + 2 \operatorname{Re} P_{n-1}(e^{(2\nu-1)\pi i/n})] = n(2 - p_n). \tag{3.11}$$

Thus $p_n \leqq 2$. But since $(1+2\sum_1^\infty z^n)$ is a member of class $\mathcal{P}$ and p_n is maximal we must have $p_n = 2$. In this case (3.8) reduces to

$$\operatorname{Re} P_{n-1}(e^{k\pi i/n}) \geqq 0, \qquad k \text{ odd}. \tag{3.12}$$

However, only equality can hold in (3.12) since otherwise (3.10) would be contradicted. (3.7) now becomes

$$Q_{2n}(e^{k\pi i/n}) = 2 \operatorname{Re} P_{n-1}(e^{k\pi i/n}) = 0, \qquad k \text{ odd}. \tag{3.13}$$

It follows at once that $(1+z^n)$ is a factor of $Q_{2n}(z)$.

If we set

$$1 + z^{2n} + \sum_{s=1}^{2n-1} p_s z^s = Q_{2n}(z) = (1 + z^n) \sum_{0}^{n} q_s z^s \tag{3.14}$$

where $p_{2n-s} = \bar{p}_s$, $s = 1, \cdots, 2n-1$, and $p_n = 2$, we find that

$$\begin{aligned} q_s &= p_s, & s &= 1, 2, \cdots, n-1; \\ q_0 &= q_n = 1; \\ \bar{p}_s &= p_{2n-s} = q_{n-s} = p_{n-s}, & s &= 1, 2, \cdots, n-1. \end{aligned}$$

Thus we have shown that (3.3) follows from (3.2).

It is interesting to observe also that the extremal functions of (3.3) satisfy the identity

$$P(z) + \left(P\left(\frac{1}{\bar{z}}\right)\right)^- = 0. \tag{3.15}$$

Consequently, the real part of $P(z)$ vanishes identically on $|z| = 1$.

If we let $\omega_k = e^{2k\pi i/n}$, $k = 1, 2, \cdots, n$, we may write $P(z)$ of (3.3) in the form

$$P(z) = \sum_{k=1}^{n} \lambda_k \left(\frac{1 + \omega_k z}{1 - \omega_k z}\right), \quad 0 \leqq \lambda_k \leqq 1, \ \sum_{1}^{n} \lambda_k = 1. \tag{3.16}$$

$\operatorname{Re} P(z) \equiv 0$ on $|z| = 1$. But if we let $z = e^{i\theta} \to \bar{\omega}_\nu$ we find that the real part of the right-hand side of equation (3.16) is unbounded unless λ_ν is real. Moreover $\lambda_\nu \geqq 0$. For if we assume $\lambda_\nu \neq 0$ and let $z = r\bar{\omega}_\nu$, $0 < r < 1$, as r approaches 1 we find that $\operatorname{Re} P(z)$ must coincide in sign with that of λ_ν. Furthermore

$$\sum_{1}^{n} \lambda_k = P(0) = 1.$$

It follows from (3.3) and (3.16) that the coefficients p_n in (3.3) must be expressible in terms of the barycentric coordinates λ_k as follows:

$$(3.17)\quad \begin{cases} p_\nu = 2\sum_{k=1}^{n} \lambda_k e^{2\nu k\pi i/n}, & 0 \leqq \lambda_k \leqq 1,\ \sum_1^n \lambda_k = 1, \quad \nu = 1, 2, \cdots, n; \\ p_{n-\nu} = \bar{p}_\nu, & \nu \leqq n-\nu,\ 1 \leqq \nu < n. \end{cases}$$

Conversely, if $0 \leqq \lambda_k \leqq 1$, $\sum_1^n \lambda_k = 1$, then $P(z)$ given by (3.16) has Re $P(z) > 0$, $|z| < 1$.

Let $p_k = x_k + iy_k$. Since $p_{n-k} = \bar{p}_k$, it is seen that the coefficient space E_{n-1} of the extremal functions $P(z)$, for which $p_n = 2$, depends upon $(n-1)$ real variables x_k, y_k.

$$(3.18)\quad \begin{aligned} E_{n-1} &= E_{n-1}(p_1, p_2, \cdots, p_{(n-1)/2}) \\ &= E_{n-1}(x_1, y_1, x_2, y_2, \cdots, x_{(n-1)/2}, y_{(n-1)/2}) \end{aligned}$$

for n odd > 1, and

$$(3.19)\quad \begin{aligned} E_{n-1} &= E_{n-1}(p_1, p_2, \cdots, p_{n/2}) \\ &= E_{n-1}(x_1, y_1, \cdots, x_{(n-2)/2}, y_{(n-2)/2}, x_{n/2}) \end{aligned}$$

for n even. We define E_0 to be the point $p_1 = 2$ corresponding to the extremal $P(z) = (1+z)/(1-z)$.

From (3.17) it is readily seen that $E_1(p_1)$ is the closed 1-simplex, or line segment $-2 \leqq p_1 \leqq 2$, corresponding to

$$(3.20)\quad P(z) = \frac{1 + p_1 z + z^2}{1 - z^2}, \qquad p_1 \text{ real}, -2 \leqq p_1 \leqq 2.$$

$E_2(p_1)$ is a closed 2-simplex consisting of an equilateral triangle with vertices $(2, 0)$, $(-1, 3^{1/2})$ and $(-1, -3^{1/2})$ and E_2 corresponds to the extremal functions of the form

$$(3.21)\quad P(z) = \frac{1 + p_1 z + \bar{p}_1 z^2 + z^3}{1 - z^3}, \qquad p_1 \subset E_2.$$

$E_3(p_1, p_2)$ is a tetrahedron with vertices $(0, 2, -2)$, $(0, -2, -2)$, $(-2, 0, 2)$ and $(2, 0, 2)$. E_3 corresponds to the extremal function

$$(3.22)\quad P(z) = \frac{1 + p_1 z + p_2 z^2 + \bar{p}_1 z^3 + z^4}{1 - z^4},$$

where p_2 is real.

In general E_{n-1} is the closed $(n-1)$-simplex with the n vertices:

$$(3.23)\quad \begin{aligned} \Big(&2\cos\frac{2\nu\pi}{n},\ 2\sin\frac{2\nu\pi}{n},\ 2\cos\frac{4\nu\pi}{n},\ 2\sin\frac{4\nu\pi}{n},\ \cdots, \\ &\qquad 2\cos(n-2)\frac{\nu\pi}{n},\ 2\sin(n-2)\frac{\nu\pi}{n},\ 2\cos\nu\pi\Big), \end{aligned}$$

$\nu=1, 2, \cdots, n$, when n is even, or

$$\left(2\cos\frac{2\nu\pi}{n}, 2\sin\frac{2\nu\pi}{n}, 2\cos\frac{4\nu\pi}{n}, 2\sin\frac{4\nu\pi}{n}, \cdots, 2\cos(n-1)\frac{\nu\pi}{n}, 2\sin(n-1)\frac{\nu\pi}{n}\right) \tag{3.24}$$

if n is odd.

The boundary hyperplanes of E_{n-1} (corresponding to a $\lambda_k=0$) for n odd, $n>1$, have the equations, for $k=0, 1, \cdots, n-1$,

$$1+\sum_{m=1}^{(n-1)/2}\left(x_m\cos\frac{2km\pi}{n}-y_m\sin\frac{2km\pi}{n}\right)=0; \tag{3.25}$$

for n even, $n>2$, the equations of the hyperplanes are

$$1+\frac{(-1)^k}{2}x_{n/2}+\sum_{m=1}^{(n-2)/2}\left(x_m\cos\frac{2km\pi}{n}-y_m\sin\frac{2km\pi}{n}\right)=0 \tag{3.26}$$

where $p_m=x_m+iy_m$.

It is seen that the hyperplanes (3.25) and (3.26) are tangent to the spheres

$$\sum_{m=1}^{(n-1)/2}(x_m^2+y_m^2)=\frac{4}{2n-2}, \qquad x_{n/2}^2+\sum_{m=1}^{(n-2)/2}(x_m^2+y_m^2)=\frac{4}{2n-3}, \tag{3.27}$$

respectively.

We summarize these results in the following theorem and corollaries.

THEOREM 1. *Let the function*

$$P(z)=1+p_1z+\cdots+p_nz^n+\cdots$$

be regular and have a positive real part in $|z|<1$. *Then* $|p_n|\leqq 2$, *and* $p_n=2$ *for a given n when, and only when,* $P(z)$ *is of the form*

$$P(z)=\frac{1+p_1z+p_2z^2+\cdots+p_{n-1}z^{n-1}+z^n}{1-z^n}, \qquad p_{n-k}=\bar{p}_k, 0<k<n,$$

and the coefficient space E_{n-1} *of* $P(z)$ *is the closed* $(n-1)$*-simplex determined by the equations* (3.17). *The vertices of the* $(n-1)$*-simplex are given by* (3.23) *and* (3.24) *and the boundary hyperplanes by the equations* (3.25) *and* (3.26).

COROLLARY 1. *If n is an odd positive integer* >1 *and*

$$|p_1|^2+\cdots+|p_{(n-1)/2}|^2\leqq\frac{2}{n-1},$$

or if n is an even integer $\geqq 2$ *and*

$$|p_1|^2 + \cdots + |p_{n/2}|^2 \leqq \frac{4}{2n-3},$$

then the function

$$P(z) = (1 + p_1 z + p_2 z^2 + \cdots + p_{n-1} z^{n-1} + z^n) \div (1 - z^n),$$

$p_{n-k} = \bar{p}_k$, $2k \leqq n$, *has* $\operatorname{Re} P(z) > 0$ *in* $|z| < 1$.

COROLLARY 2. *With the notation of Theorem* 1 *the boundary of the* $(n-1)$-*simplex* E_{n-1} *given by* (3.17) *is determined from the equation* $\Delta_n = 0$, $n > 1$, *where* Δ_n *is the determinant*

$$\Delta_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_{n-1} \\ \bar{p}_1 & 2 & p_1 & \cdots & p_{n-2} \\ \bar{p}_2 & \bar{p}_1 & 2 & \cdots & p_{n-3} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \bar{p}_{n-1} & \bar{p}_{n-2} & \bar{p}_{n-3} & \cdots & 2 \end{vmatrix} = 2^n (n!)^2 \sin^2 \frac{\pi}{n} \cdot \prod_{k=1}^{n} \lambda_k.$$

Proof of Corollary 2. Using the representation (3.17) for the coefficients p_ν we write Δ_n as the product of two determinants, $\Delta_n = 2^n A_n B_n$, where

$$A_n = \begin{vmatrix} \lambda_1, & \lambda_1 e^{-2\pi i/n}, & \lambda_1 e^{-4\pi i/n}, & \cdots & \lambda_1 e^{-2(n-1)\pi i/n} \\ \lambda_2, & \lambda_2 e^{-4\pi i/n}, & \lambda_2 e^{-8\pi i/n}, & \cdots & \lambda_2 e^{-4(n-1)\pi i/n} \\ \lambda_3, & \lambda_3 e^{-6\pi i/n}, & \lambda_3 e^{-12\pi i/n}, & \cdots & \lambda_3 e^{-6(n-1)\pi i/n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \lambda_n, & \lambda_n e^{-2n\pi i/n}, & \lambda_n e^{-4n\pi i/n}, & \cdots & \lambda_n e^{-2n(n-1)\pi i/n} \end{vmatrix}$$

and B_n is the determinant A_n with the λ's replaced by 1's and i replaced by $-i$. Thus $\prod_1^n \lambda_k$ is a factor of A_n and we obtain

$$\Delta_n = 2^n \left(\prod_1^n \lambda_k \right) |B_n|^2 \geqq 0.$$

In the determinant B_n except for the first column, the elements in the columns add up to zero, so that the order of B_n is easily reduced a step at a time. Thus

$$\begin{aligned} \Delta_n &= 2^n \left(\prod_1^n \lambda_k \right) n^2 (n-1)^2 \cdots 3^2 \left\| \begin{matrix} 1 & 1 \\ 1 & e^{2\pi i/n} \end{matrix} \right\|^2 \\ &= 2^n (n!)^2 \sin^2 \frac{\pi}{n} \left(\prod_1^n \lambda_k \right) \qquad n > 1,\ 0 < \lambda_k < 1. \\ &> 0, \end{aligned} \tag{3.28}$$

Since a boundary point of E_{n-1} corresponds to a λ_k having the values 0 or 1,

$\Delta_n = 0$ defines the boundary of E_{n-1} for $n > 1$. When $n = 1$, $\Delta_1 = 2$ and E_0 is the point $p_1 = 2$.

Although we have confined our attention to extremal functions for which Re p_n is a maximum, the formula (2.8) for δp_n may be used to determine the extremal functions $P(z)$ which maximize Re $F(p_1, p_2, \cdots, p_n)$ where $F(p_1, \cdots, p_n)$ is any continuous function having continuous partial derivatives in an open set containing the coefficient space $V_n(p_1, \cdots, p_n)$ of the class $\mathcal{P}$ and for which the partial derivatives $\lambda_k = \partial F/\partial p_k$, $k = 1, \cdots, n$, are not all zero at the point $(p_1, \cdots, p_n)$ determined by the extremal function. We find that the extremal function maximizes $\mathrm{Re}(\sum_1^n \lambda_k p_k)$ and is the function

$$P(z) = \frac{\sum_{k=1}^{n} k\left(\lambda_k \sum_{s=0}^{k} \frac{p_s}{z^{k-s}} + \bar{\lambda}_k \sum_{s=0}^{k} \bar{p}_s z^{k-s}\right)}{\sum_{k=1}^{n} k\left(\frac{\lambda_k}{z^k} - \bar{\lambda}_k z^k\right)}, \qquad p_0 = 1. \tag{3.29}$$

It is seen from (3.29) that the extremal functions $P(z)$ maximizing Re F have the property that the real part of $P(z)$ vanishes identically on $|z| = 1$.

In particular, one may obtain the extremal functions which minimize the Toeplitz form

$$F = \sum p_{\mu-\nu} X_\nu \bar{X}_\mu.$$

F then turns out to be non-negative in accordance with the Carathéodory-Toeplitz theory. We omit the details.

Turning to another problem we shall now see how the variational technique easily leads to the well-known inequality

$$\mathrm{Re}\, P(z) \geqq \frac{1-r}{1+r}, \qquad |z| = r < 1, \tag{3.30}$$

for functions of class $\mathcal{P}$.

Let $P_0(z)$ be an extremal function for which, when z is fixed in the unit circle, Re $P_0(z)$ is a minimum for the class $\mathcal{P}$. By a rotation in the z-plane we may assume z to be a positive number r. Since Re $\delta P_0(r) \geqq 0$ in (2.7) we have Re $e^{i\theta}A \leqq 0$ for all θ where

$$\begin{aligned} A = A(r) = &\left(\frac{z_0 P_0'(r)}{P_0(z_0)} - 1\right)\frac{r}{z_0(z_0 - r)} + \left(\frac{z_0 P_0(r)}{P_0(z_0)} - r\right)\frac{r}{z_0(z_0 - r)^2} \\ &+ \frac{(P_0'(r))^-}{P_0(z_0)} \cdot \frac{r^2}{1 - z_0 r} + \left(\frac{(P_0(r))^-}{P_0(z_0)} + 1\right)\frac{r}{(1 - z_0 r)^2}. \end{aligned} \tag{3.31}$$

Since θ is arbitrary it follows that $A = 0$. Replacing z_0 by z and solving the equation $A = 0$ for $P_0(z)$ we obtain

$$
(3.32)\quad
\begin{aligned}
&(1-r^2)(1-z^2)P_0(z) = A_3z^3 + A_2z^2 + A_1z + A_0,\\
&A_0 = -rP_0'(r) + r^3(P_0'(r))^- + P_0(r) + r^2(P_0(r))^- = (1-r^2)P_0(0)\\
&\quad = 1-r^2,\\
&A_1 = (1+2r^2)P_0'(r) - (2r^2+r^4)(P_0'(r))^- - 2r\{P_0(r) + (P_0(r))^-\},\\
&A_2 = (2r^3+r)(P_0'(r))^- - (2r+r^3)P_0'(r) + r^2P_0(r) + (P_0(r))^-,\\
&A_3 = r^2\{P'(r) - (P_0'(r))^-\}.
\end{aligned}
$$

Since $\operatorname{Re} P(r) \geqq \operatorname{Re} P_0(r)$ for all $P(z) \in \mathcal{P}$, and since $P(z) = P_0(ze^{i\theta}) \in \mathcal{P}$, we have $\operatorname{Re} P_0(re^{i\theta}) \geqq \operatorname{Re} P_0(r)$ for all θ. Thus $\operatorname{Re} P_0(r)$ is a minimum value of $\operatorname{Re} P_0(re^{i\theta})$ as a function of θ. It follows from the Cauchy-Riemann equations that

$$
IP_0'(r) = \frac{\partial}{\partial r} IP_0(re^{i\theta})\bigg|_{\theta=0} = -\frac{1}{r}\frac{\partial}{\partial\theta} \operatorname{Re} P_0(re^{i\theta})\bigg|_{\theta=0} = 0.
$$

Thus $P_0'(r)$ is real. Then $A_3 = 0$. Since A_0 is real it follows that $\{P_0(r)+r^2(P_0(r))^-\}$ and $\{P_0(r)+(P_0(r))^-\}$ are both real. This implies that $P_0(r)$ is real. Since $A_2 - A_0 = (1-r^2)\{(P_0(r))^- - P_0(r)\}$, we have $A_2 = A_0 = 1-r^2$. Also A_1 is seen to be real. We have now seen that $P_0(z)$ is of the form

$$
(3.33)\qquad P_0(z) = \frac{1+kz+z^2}{1-z^2}, \qquad k \text{ real}.
$$

Since $P_0(r) \geqq 0$, we have $k \geqq -2$. Since $P_0(r)$ is minimal for the class $\mathcal{P}$ we must take $k = -2$. In this case

$$
(3.34)\qquad P_0(z) = \frac{1-z}{1+z}
$$

so that (3.30) follows with equality holding only for the function $P_0(\epsilon z)$, $|\epsilon| = 1$.

It should be noticed that the equation $A_0 = 1 - r^2$ may not be treated as a differential equation for finding $P_0(r)$ unless it is first shown that the extremal function P_0 does not vary with r.

4. **Interior variations for subordinate functions.** Let the analytic function

$$
(4.1)\qquad f(z) = A_1z + A_2z^2 + \cdots + A_nz^n + \cdots, \qquad f(0) = 0,\ A_1 \neq 0,
$$

be regular and univalent in $|z| < 1$. Let

$$
(4.2)\qquad F(z) = a_1z + a_2z^2 + \cdots + a_nz^n + \cdots, \qquad F(0) = 0,
$$

be regular and subordinate to $f(z)$ in $|z| < 1$. Then

$$
(4.3)\qquad F(z) = f(\omega(z))
$$

where $\omega(z)$ is regular in $|z| < 1$, $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$. We may write $\omega(z)$ in the form

$$\omega(z) = \frac{P(z) - 1}{P(z) + 1} \tag{4.4}$$

where $P(z)$ is a member of class $\mathcal{P}$. We also denote by $z=\phi(w)$ the inverse function of $w=f(z)$. If $P^*(z)$ is given by (2.6) upon varying $P(z)$, and if $\omega^*(z)$ corresponds by (4.4) to $P^*(z)$, we easily obtain

$$\begin{aligned} \omega^*(z) &= \frac{P^*(z) - 1}{P^*(z) + 1} = \frac{P(z) - 1}{P(z) + 1} + \rho^2\lambda(z) + o(\rho^2) \\ &= \omega(z) + \rho^2\lambda(z) + o(\rho^2), \end{aligned} \tag{4.5}$$

where

$$\lambda(z) = -\frac{1}{2}(1 - |z_0|^2)(1 - \phi(F))^2 A(z) \tag{4.6}$$

and where $A(z)$ is defined as the right-hand side of equation (2.7) omitting the term $o(1)$. We also have

$$P(z) = \frac{1 + \phi(F)}{1 - \phi(F)}, \qquad (P(z) + 1)^2 = 4(1 - \phi(F))^{-2}, \tag{4.7}$$

$$P'(z) = \frac{2\phi'(F)F'(z)}{(1 - \phi(F))^2}. \tag{4.8}$$

We now write $A(z)$ in the form

$$\begin{aligned} A(z) = &\frac{2\phi'(F)F'(z)}{(1 - \phi(F))^2}\left\{\frac{e^{i\theta}z}{P(z_0)(z_0 - z)} + \frac{z^2e^{-i\theta}}{(P(z_0))^-(1 - \bar{z}_0z)}\right\} \\ &+ \frac{1 + \phi(F)}{1 - \phi(F)}\left\{\frac{e^{i\theta}z}{P(z_0)(z_0 - z)^2} + \frac{e^{-i\theta}z}{(P(z_0))^-(1 - \bar{z}_0z)^2}\right\} \\ &+ \left\{\frac{e^{-i\theta}z}{(1 - \bar{z}_0z)^2} - \frac{e^{i\theta}z^2}{z_0(z_0 - z)^2} - \frac{e^{i\theta}z}{z_0(z_0 - z)}\right\} \end{aligned} \tag{4.9}$$

where

$$P(z_0) = \frac{1 + \phi(F(z_0))}{1 - \phi(F(z_0))}, \qquad |z_0| < 1.$$

$$\begin{aligned} F^*(z) &= f(\omega(z) + \rho^2\lambda(z) + o(\rho^2)) \\ &= F(z) + f'(\omega(z))\lambda(z)\rho^2 + o(\rho^2) \\ &= F(z) + \frac{\lambda(z)}{\phi'(F)}\rho^2 + o(\rho^2). \end{aligned} \tag{4.10}$$

Thus the variational formula for analytic functions $F(z)$ subordinate to a given univalent function $f(z)$ in $|z| < 1$ is given by

$$F^*(z) = F(z) - \rho^2(1 - |z_0|^2)\frac{(1 - \phi(F))^2}{2\phi'(F)} A(z) + o(\rho^2) \tag{4.11}$$

where $A(z)$ is defined by (4.9) and ϕ is the inverse of f.

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Rutgers, The State University,
New Brunswick, New Jersey