VARIATIONAL METHODS FOR FUNCTIONS WITH
POSITIVE REAL PART

BY

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1. Introduction. M. M. Schiffer [8] has recently derived a formula for the variation of the Green's function of the most general plane domain \( \mathbb{D} \) with boundary \( \mathcal{C} \) due to a small shift

\[
\omega^* = \omega + \rho^2 \phi(\omega),
\]

of the boundary. The variation \( \delta g(\zeta, \eta) \) of the Green's function is given by the formula

\[
\delta g(\zeta, \eta) = \text{Re} \left\{ \frac{\rho^2}{2\pi i} \int_{\Gamma} \rho'(w, \eta) \phi(w, \zeta) \phi'(w) \, dw \right\} + o(\rho^2)
\]

where \( \rho(w, \eta) \) is an analytic function whose real part is the Green function \( g(w, \eta) \) of \( \mathbb{D} \), and where \( \Gamma \) is a member of a curve system in \( \mathbb{D} \) homotopic to \( \mathcal{C} \). The function \( \phi(w) \) is analytic on \( \Gamma \) and in the ring bounded by \( \mathcal{C} \) and \( \Gamma \). If \( \mathbb{D} \) is simply-connected and if \( z = \psi(w) \) maps \( \mathbb{D} \) on the interior of the unit circle \( |z| < 1 \), then \( g(w, \eta) \) is connected with \( \psi(w) \) by the relation

\[
g(w, \eta) = \log \left| \frac{1 - (\psi(\eta))^{-\psi(w)}}{\psi(w) - \psi(\eta)} \right|.
\]

Here and throughout the paper \( (\cdot)^{-} \) indicates the complex conjugate. With an appropriate choice of \( \phi(w) \) one may then obtain variation formulas for univalent functions \( w = f(z) \). J. A. Hummel [5] has recently used this method of interior variations to study the class of univalently star-like functions. The method may also be used to study those functions which are convex-in-one direction [2]. The choice of the shift function \( \phi(w) \), however, is not always an obvious one for many special classes of univalent functions, in particular for the class of close-to-convex functions [6]. Many of these special classes, however, have representations of their member functions in terms of functions \( P(z) \) with positive real part. It therefore becomes desirable to have a variational formula for \( P(z) \) from which one may then easily obtain analogous variational formulas for the special classes of univalent functions.

It is the purpose of this paper first to derive a variational formula for the class \( \mathcal{P} \) of normalized regular functions

\[
P(z) = 1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots, \quad P(0) = 1,
\]

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which have \( \text{Re} P(z) > 0 \) in \( |z| < 1 \). Secondly, we shall apply the variational formula to \( P(z) \) in order to solve extremal problems for the class \( \mathcal{P} \) and in particular to obtain a characterization of the \((n-1)\) Euclidean coefficient space \( E_{n-1} \) for the extremal functions \( P(z) \) for which \( \text{Re} p_n \) is a maximum. Although these results may also be deduced from the Carathéodory-Toeplitz theory \([1; 3; 4; 7; 9; 10]\) it is interesting to see how simply they are derived by the variational method. Thirdly, we shall indicate how the variational formula for \( P(z) \) leads to the variational formulas for bounded, regular functions \( \omega(z) \) and also for regular functions \( F(z) = f\{\omega(z)\} \) which are subordinate to a given univalent function \( f(z) \) in \( |z| < 1 \).

2. A variational formula for the class \( \mathcal{P} \). The choice

\[
\phi(w) = \frac{e^{i\theta}}{w - w_0}
\]

in (1.2), where \( w_0 \) is interior to \( \mathbb{D} \), is the Schiffer case for univalent functions \([8]\). Hummel \([5]\) chose

\[
\phi(w) = wR[\psi(w)],
\]

\[
R(z) = \frac{1 - \bar{z}_0z}{1 - \bar{z}_0}, \quad |z_0| < 1,
\]

where \( R(z) \) is real and bounded on \( |z| = 1 \), to obtain the following variational formula for a normalized univalently star-like function \( f(z) \) in \( |z| < 1, f(0) = 0, f'(0) = 1 \):

\[
f^*(z) = f(z) \left[ 1 - \rho^2(1 - |z_0|^2) \left( A(z) - \frac{zf'(z)}{f(z)} B(z) \right) \right] + o(\rho^2)
\]

where the error term \( o(\rho^2) \) is an analytic function in \( z \) and uniformly bounded in each interior region of \( |z| < 1 \). \( A(z) \) and \( B(z) \) are defined by

\[
A(z) = \frac{ze^{i\theta}}{z_0(z - z_0)} + \frac{ze^{-i\theta}}{1 - \bar{z}_0z} - \frac{e^{i\theta}f(z_0)}{zf'(z_0)}, \quad |z_0| < 1,
\]

\[
B(z) = \frac{e^{i\theta}f(z_0)}{zf'(z_0)(z - z_0)} - e^{-i\theta} \left( \frac{f(z_0)}{zf'(z_0)} \right) - \frac{z}{1 - \bar{z}_0z}.
\]

Since there is a \((1,1)\) correspondence between the functions \( P(z) \) of the class \( \mathcal{P} \), given by (1.4), and the univalently star-like functions \( f(z), f(0) = 0, f'(0) = 1 \), by the relation

\[
P(z) = \frac{zf'(z)}{f(z)}, \quad \text{Re} P(z) > 0, \quad |z| < 1,
\]

we therefore take the logarithmic derivative in (2.2) and replace \( z(f^*(z))/f^*(z) \) by \( P^*(z) \). After some simplification we obtain
\[ P^*(z) = P(z) - p^2(1 - |z_0|^2)z \]

(2.6)

\[ \left[ \frac{ze^{i\theta}}{z_0(z - z_0)} + \frac{ze^{-i\theta}}{1 - z_0z} - \frac{e^{i\theta}P(z)}{P(z_0)(z - z_0)} + \frac{e^{-i\theta}zP(z)}{(P(z_0))(1 - z_0z)} \right]' + o(p^2), \]

where \([ ]\)' denotes differentiation with respect to \(z\). Let \(\delta P(z) = P^*(z) - P(z)\), and write (2.6) in the form

\[ \frac{-\delta P(z)}{p^2(1 - |z_0|^2)} = \left( \frac{z_0P'(z)}{P(z_0)} - 1 \right) \frac{e^{i\theta}z}{z_0(z - z_0)} + \left( \frac{z_0P(z)}{P(z_0)} - z \right) \frac{e^{-i\theta}z}{z_0(z - z_0)} \]

\[ + \frac{P'(z)}{(P(z_0))^2} \left( \frac{z^2e^{-i\theta}}{1 - z_0z} \right) + \left( \frac{P(z)}{(P(z_0))^2} + 1 \right) \frac{e^{-i\theta}z}{z_0(z - z_0)} + o(1), \]

(2.7)

which is the required variational formula for functions of class \(\mathcal{P}\).

If \(P(z)\) has the power series expansion (1.4), and if we denote \(\delta p_n = p^*_n - p_n\), (2.7) yields

\[ \frac{-\delta p_n}{p^2(1 - |z_0|^2)} = \frac{ne^{i\theta}}{P(z_0)} \sum_{k=0}^{n} \frac{p_k}{z_0^{n-k+1}} - \frac{ne^{i\theta}}{P(z_0)} \sum_{k=0}^{n-1} \frac{p_k(z_0)^{n-k-1}}{(P(z_0))^2} \]

\[ + ne^{-i\theta}(z_0)^{n-1} + o(1), \]

(2.8)

where \(p_0 = 1\). Since for any complex number \(w\), \(\text{Re } w = \text{Re } \bar{w}\), we have

\[ \frac{-\text{Re } \delta p_n}{p^2(1 - |z_0|^2)} = n \text{Re} \left\{ \frac{e^{i\theta}}{z_0} \left[ \frac{1}{P(z_0)} \sum_{k=0}^{n-1} \left( \frac{p_k}{z_0^{n-k}} + \frac{\bar{p}_k}{z_0^{n-k}} \right) \right] \right\} + o(1). \]

(2.9)

3. Extremal functions for the class \(\mathcal{P}\). For any positive integer \(n\) let \(P(z)\) be an extremal function of the class \(\mathcal{P}\) for which the coefficient \(p_n\) in (1.4) is real, positive and a maximum over all functions of the class. Since \(\mathcal{P}\) is compact \(p_n\) attains its maximum. Its value is 2 as is well known from the Carathéodory theory, although we do not assume this fact here. Since \(\text{Re } \delta p_n \leq 0\) in (2.9) and \(\theta\) is arbitrary we have from (2.9) on replacing \(z_0\) by \(z\)

\[ \left\{ \frac{p_n + \sum_{k=0}^{n-1} \left( \frac{p_k}{z^{n-k}} + \bar{p}_kz^{n-k} \right)}{1 - z^{2n}} \right\} + z^n - z^{-n} = 0, \quad p_n > 0, \quad p_0 = 1, \]

(3.1)

\[ P(z) = \frac{1 + p_1z + p_2z^2 + \cdots + p_nz^n + \cdots + p_{2n-1}z^{2n-1} + z^{2n}}{1 - z^{2n}} = \frac{Q_{2n}(z)}{1 - z^{2n}} \]

(3.2)

where \(p_{2n-k} = \bar{p}_k, k = 1, 2, \cdots, 2n - 1\).

Although ordinarily the variational formulas for extremal functions of various classes of functions lead to differential equations we find here for the
class $\varphi$ that we are led directly to the extremal functions (3.2) without encountering a differential equation. However, the formula (3.2) may be simplified further. We shall presently show that whenever $P(z)$ has the form (3.2) and has a positive real part in $|z| < 1$ then $p_n \leq 2$, and if $p_n = 2$ then $(1 + z^n)$ is a factor of both numerator and denominator of (3.2). In this case (3.2) becomes

$$P(z) = \frac{1 + p_1 z + p_2 z^2 + \cdots + p_{n-1} z^{n-1} + z^n}{1 - z^n},$$

where $p_{n-k} = \bar{p}_k$, $k = 1, 2, \ldots, n-1$.

To obtain (3.3) we place $z = re^{\frac{\pi}{n}}$, $0 < r < 1$, $k = \text{odd integer}$, in (3.2). Then

$$Q_{2n}(e^{\frac{k\pi}{n}}) = \frac{Q_{2n}(re^{\frac{k\pi}{n}})}{1 - r^{2n}}.$$ 

Since $\text{Re} P(z) > 0$ it follows that $\text{Re} Q_{2n}(re^{\frac{k\pi}{n}}) > 0$. Letting $r \to 1$ we have $\text{Re} Q_{2n}(e^{\frac{k\pi}{n}}) \geq 0$. However, $Q_{2n}(e^{\frac{k\pi}{n}})$ is real. This follows since $p_n z^n + p_{2n-k} z^{2n-k} = p_n z^n + \bar{p}_k z^k$ is a real number when $z = e^{\frac{k\pi}{n}}$, and because $p_n z^n$ and $z^{2n}$ are also real for this choice of $z$. Thus

$$Q_{2n}(e^{\frac{k\pi}{n}}) \geq 0.$$ 

Let

$$P_{n-1}(z) = \sum_{s=1}^{n-1} p_s z^s,$$

$$Q_{2n}(z) = 1 + p_n z^n + z^{2n} + P_{n-1}(z) + z^{2n} \left( P_{n-1} \left( \frac{1}{z} \right) \right).$$

For $k$ odd, $z = e^{\frac{k\pi}{n}}$, we have $z^{2n} = 1$, $z^n = -1$, so that

$$Q_{2n}(e^{\frac{k\pi}{n}}) = (2 - p_n) + P_{n-1}(e^{\frac{k\pi}{n}}) + (P_{n-1}(e^{\frac{k\pi}{n}}))^*,$$

$$Q_{2n}(e^{\frac{k\pi}{n}}) = (2 - p_n) + 2 \Re P_{n-1}(e^{\frac{k\pi}{n}}) + 2 \Re \left( \sum_{s=1}^{n-1} p_s e^{\frac{s\pi}{n}} \right).$$

By virtue of the identity

$$\sum_{s=1}^{n} e^{(2s-1)\frac{s\pi}{n}} = 0,$$

it follows that

$$\Re \sum_{s=1}^{n} P_{n-1}(e^{(2s-1)\frac{\pi}{n}}) = 0.$$ 

From (3.8) and (3.10) we then have
Thus $p_n \leq 2$. But since $(1 + 2 \sum_1^n z^s)$ is a member of class $\Phi$ and $p_n$ is maximal we must have $p_n = 2$. In this case (3.8) reduces to

(3.12) $\Re P_{n-1}(e^{i \pi/n}) \geq 0$, $k$ odd.

However, only equality can hold in (3.12) since otherwise (3.10) would be contradicted. (3.7) now becomes

(3.13) $Q_{2n}(e^{i \pi/n}) = 2 \Re P_{n-1}(e^{i \pi/n}) = 0$, $k$ odd.

It follows at once that $(1 + z^n)$ is a factor of $Q_{2n}(z)$.

If we set

(3.14) $1 + z^n + \sum_{s=1}^{2n-1} p_s z^s = Q_{2n}(z) = (1 + z^n) \sum_0^n q_s z^s$

where $p_{2n-s} = \tilde{p}_s$, $s = 1, \ldots , 2n-1$, and $p_n = 2$, we find that

$q_s = p_s$, $s = 1, 2, \ldots , n - 1$;
$q_0 = q_n = 1$;
$p_{n-s} = q_{n-s} = \tilde{p}_{n-s}$, $s = 1, 2, \ldots , n - 1$.

Thus we have shown that (3.3) follows from (3.2).

It is interesting to observe also that the extremal functions of (3.3) satisfy the identity

(3.15) $P(z) + \left( P \left( \frac{1}{\bar{z}} \right) \right)^{-1} = 0$.

Consequently, the real part of $P(z)$ vanishes identically on $|z| = 1$.

If we let $\omega_k = e^{i \pi k/n}$, $k = 1, 2, \ldots , n$, we may write $P(z)$ of (3.3) in the form

(3.16) $P(z) = \sum_{k=1}^n \lambda_k \left( \frac{1 + \omega_k z}{1 - \omega_k z} \right)$, $0 \leq \lambda_k \leq 1$, $\sum_1^n \lambda_k = 1$.

Re $P(z) = 0$ on $|z| = 1$. But if we let $z = e^{i \theta} \omega_s$, we find that the real part of the right-hand side of equation (3.16) is unbounded unless $\lambda_s$ is real. Moreover $\lambda_s \geq 0$. For if we assume $\lambda_s \neq 0$ and let $z = r \omega_s$, $0 < r < 1$, as $r$ approaches 1 we find that Re $P(z)$ must coincide in sign with that of $\lambda_s$. Furthermore

$\sum_1^n \lambda_k = P(0) = 1$.

It follows from (3.3) and (3.16) that the coefficients $p_n$ in (3.3) must be expressible in terms of the barycentric coordinates $\lambda_k$ as follows:
\[ \begin{align*}
(3.17) & \quad \phi_\nu = 2 \sum_{k=1}^{n} \lambda_k e^{2 \pi k \nu i / n}, \quad 0 \leq \lambda_k \leq 1, \quad \sum_{k=1}^{n} \lambda_k = 1, \quad \nu = 1, 2, \cdots, n; \\
\phi_{n-\nu} &= \bar{\phi}_\nu, \quad \nu \leq n - \nu, \quad 1 \leq \nu < n.
\end{align*} \]

Conversely, if \( 0 \leq \lambda_k \leq 1, \sum \lambda_k = 1, \) then \( P(z) \) given by (3.16) has \( \text{Re} \ P(z) > 0, \ |z| < 1. \)

Let \( p_k = x_k + iy_k. \) Since \( p_{n-k} = \bar{p}_k, \) it is seen that the coefficient space \( E_{n-1} \) of the extremal functions \( P(z), \) for which \( p_n = 2, \) depends upon \( (n-1) \) real variables \( x_k, \ y_k. \)

\[ E_{n-1} = E_{n-1}(p_1, p_2, \cdots, p_{(n-1)/2}) = E_{n-1}(x_1, y_1, x_2, y_2, \cdots, x_{(n-1)/2}, y_{(n-1)/2}) \]

for \( n \) odd \( >1, \) and

\[ E_{n-1} = E_{n-1}(p_1, p_2, \cdots, p_{n/2}) = E_{n-1}(x_1, y_1, \cdots, x_{(n-2)/2}, y_{(n-2)/2}, x_{n/2}) \]

for \( n \) even. We define \( E_0 \) to be the point \( p_1 = 2, \) corresponding to the extremal \( P(z) = (1+z)/(1-z). \)

From (3.17) it is readily seen that \( E_1(p_1) \) is the closed 1-simplex, or line segment \( -2 \leq p_1 \leq 2, \) corresponding to

\[ P(z) = \frac{1 + p_1z + z^2}{1 - z^2}, \quad p_1 \text{ real}, \quad -2 \leq p_1 \leq 2. \]

\( E_2(p_1) \) is a closed 2-simplex consisting of an equilateral triangle with vertices \((2, 0), (-1, 3^{1/2})\) and \((-1, -3^{1/2})\) and \( E_2 \) corresponds to the extremal functions of the form

\[ P(z) = \frac{1 + p_1z + p_2z^2 + z^3}{1 - z^3}, \quad p_1 \subset E_2. \]

\( E_3(p_1, p_2) \) is a tetrahedron with vertices \((0, 2, -2), (0, -2, -2), (-2, 0, 2)\) and \((2, 0, 2). \) \( E_3 \) corresponds to the extremal function

\[ P(z) = \frac{1 + p_1z + p_2z^2 + p_3z^3 + z^4}{1 - z^4}, \]

where \( p_3 \) is real.

In general \( E_{n-1} \) is the closed \((n-1)\)-simplex with the \( n \) vertices:

\[ \left( \begin{array}{cccc}
2 \cos \frac{2\pi}{n}, & 2 \sin \frac{2\pi}{n}, & 2 \cos \frac{4\pi}{n}, & 2 \sin \frac{4\pi}{n}, \cdots, \\
2 \cos \frac{4\nu \pi}{n}, & 2 \sin \frac{4\nu \pi}{n}, & 2 \cos \frac{4\nu \pi}{n}, & 2 \sin \frac{4\nu \pi}{n}, \cdots,
\end{array} \right) \]

\[ \left( \begin{array}{cccc}
2 \cos \frac{2\pi}{n}, & 2 \sin \frac{2\pi}{n}, & 2 \cos \frac{4\pi}{n}, & 2 \sin \frac{4\pi}{n}, \cdots, \\
2 \cos \frac{2\nu \pi}{n}, & 2 \sin \frac{2\nu \pi}{n}, & 2 \cos \frac{2\nu \pi}{n}, & 2 \sin \frac{2\nu \pi}{n}, \cdots,
\end{array} \right), \]

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\[ \nu = 1, 2, \ldots, n, \text{ when } n \text{ is even, or } \]
\[ \left( 2 \cos \frac{2\nu\pi}{n}, 2 \sin \frac{2\nu\pi}{n}, 2 \cos \frac{4\nu\pi}{n}, 2 \sin \frac{4\nu\pi}{n}, \ldots, \right) \]
(3.24)
\[ 2 \cos \left( \frac{\nu\pi}{n} - 1 \right), 2 \sin \left( \frac{\nu\pi}{n} - 1 \right) \]
if \( n \) is odd.

The boundary hyperplanes of \( E_{n-1} \) (corresponding to a \( \lambda_k = 0 \)) for \( n \) odd, \( n \geq 1 \), have the equations, for \( k = 0, 1, \ldots, n - 1 \),
\[ 1 + \sum_{m=1}^{(n-1)/2} \left( x_m \cos \frac{2km\pi}{n} - y_m \sin \frac{2km\pi}{n} \right) = 0; \]
(3.25)
for \( n \) even, \( n > 2 \), the equations of the hyperplanes are
\[ 1 + \frac{(-1)^k}{2} x_{n/2} + \sum_{m=1}^{(n-2)/2} \left( x_m \cos \frac{2km\pi}{n} - y_m \sin \frac{2km\pi}{n} \right) = 0 \]
where \( p_m = x_m + iy_m \).

It is seen that the hyperplanes (3.25) and (3.26) are tangent to the spheres
\[ \sum_{m=1}^{(n-1)/2} (x_m + y_m)^2 = \frac{4}{2n - 2}, \quad \sum_{m=1}^{(n-2)/2} (x_m + y_m) = \frac{4}{2n - 3}, \]
(3.27)
respectively.

We summarize these results in the following theorem and corollaries.

**Theorem 1.** Let the function
\[ P(z) = 1 + p_1z + \ldots + p_nz^n + \ldots \]
be regular and have a positive real part in \( |z| < 1 \). Then \( |p_n| \leq 2 \), and \( p_n = 2 \) for a given \( n \) when, and only when, \( P(z) \) is of the form
\[ P(z) = \frac{1 + p_1z + p_2z^2 + \cdots + p_{n-1}z^{n-1} + z^n}{1 - z^n}, \quad p_{n-k} = p_k, 0 < k < n, \]
and the coefficient space \( E_{n-1} \) of \( P(z) \) is the closed \((n-1)\)-simplex determined by the equations (3.17). The vertices of the \((n-1)\)-simplex are given by (3.23) and (3.24) and the boundary hyperplanes by the equations (3.25) and (3.26).

**Corollary 1.** If \( n \) is an odd positive integer \( > 1 \) and
\[ |p_1|^2 + \cdots + |p_{(n-1)/2}|^2 \leq \frac{2}{n - 1}, \]
or if \( n \) is an even integer \( \geq 2 \) and
\[ |p_1|^2 + \cdots + |p_{n/2}|^2 \leq \frac{4}{2n - 3}, \]

then the function
\[ P(z) = (1 + p_1z + p_2z^2 + \cdots + p_{n-1}z^{n-1} + z^n) \div (1 - z^n), \]
\[ p_{n-k} = \bar{p}_k, \ 2k \leq n, \text{ has } \Re P(z) > 0 \text{ in } |z| < 1. \]

**Corollary 2.** With the notation of Theorem 1 the boundary of the \((n - 1)\)-simplex \(E_{n-1}\) given by (3.17) is determined from the equation \(\Delta_n = 0, \ n > 1\), where \(\Delta_n\) is the determinant
\[
\Delta_n = \begin{vmatrix}
p_1 & p_2 & \cdots & p_{n-1} \\
p_2 & p_1 & \cdots & p_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n-1} & p_{n-2} & \cdots & 2
\end{vmatrix} = 2^n(n!)^2 \sin^2 \frac{\pi}{n} \prod_{k=1}^{n} \lambda_k.
\]

**Proof of Corollary 2.** Using the representation (3.17) for the coefficients \(p\), we write \(\Delta_n\) as the product of two determinants, \(\Delta_n = 2^nA_nB_n\), where
\[
A_n = \begin{vmatrix}
\lambda_1, \lambda_1 e^{-2\pi i/n}, \lambda_1 e^{-4\pi i/n}, \cdots \lambda_1 e^{-(n-1)\pi i/n} \\
\lambda_2, \lambda_2 e^{-2\pi i/n}, \lambda_2 e^{-4\pi i/n}, \cdots \lambda_2 e^{-(n-1)\pi i/n} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_n, \lambda_n e^{-2\pi i/n}, \lambda_n e^{-4\pi i/n}, \cdots \lambda_n e^{-(n-1)\pi i/n}
\end{vmatrix}
\]
and \(B_n\) is the determinant \(A_n\) with the \(\lambda\)'s replaced by 1's and \(i\) replaced by \(-i\). Thus \(\prod_1^n \lambda_k\) is a factor of \(A_n\) and we obtain
\[ \Delta_n = 2^n \left( \prod_{k=1}^{n} \lambda_k \right) |B_n|^2 \geq 0. \]

In the determinant \(B_n\) except for the first column, the elements in the columns add up to zero, so that the order of \(B_n\) is easily reduced a step at a time. Thus
\[ \Delta_n = 2^n \left( \prod_{k=1}^{n} \lambda_k \right) n^2(n - 1)^2 \cdots 3^2 \left\| \begin{array}{cc}
1 & 1 \\
1 & e^{2\pi i/n}
\end{array} \right\|^2 \\
= 2^n(n!)^2 \sin^2 \frac{\pi}{n} \left( \prod_{k=1}^{n} \lambda_k \right) n > 1, \ 0 < \lambda_k < 1. \]

Since a boundary point of \(E_{n-1}\) corresponds to a \(\lambda_k\) having the values 0 or 1,
$\Delta_n = 0$ defines the boundary of $E_{n-1}$ for $n > 1$. When $n = 1$, $\Delta_1 = 2$ and $E_0$ is the point $p_1 = 2$.

Although we have confined our attention to extremal functions for which $\text{Re } p_n$ is a maximum, the formula (2.8) for $\delta p_n$ may be used to determine the extremal functions $P(z)$ which maximize $\text{Re } F(p_1, p_2, \cdots, p_n)$ where $F(p_1, \cdots, p_n)$ is any continuous function having continuous partial derivatives in an open set containing the coefficient space $V_n(p_1, \cdots, p_n)$ of the class $\Theta$ and for which the partial derivatives $\lambda_k = \partial F / \partial p_k$, $k = 1, \cdots, n$, are not all zero at the point $(p_1, \cdots, p_n)$ determined by the extremal function. We find that the extremal function maximizes $\text{Re} \left( \sum_n \lambda_k p_k \right)$ and is the function

$$P(z) = \frac{\sum_{k=1}^n k \left( \lambda_k \sum_{s=0}^k \frac{p_z z^{k-s}}{z^{k-s}} + \bar{\lambda}_k \sum_{s=0}^k \bar{p}_z z^{k-s} \right)}{\sum_{k=1}^n k \left( \frac{\lambda_k}{z^k} - \frac{\bar{\lambda}_k}{z^k} \right)}, \quad p_0 = 1. \tag{3.29}$$

It is seen from (3.29) that the extremal functions $P(z)$ maximizing $\text{Re } F$ have the property that the real part of $P(z)$ vanishes identically on $|z| = 1$.

In particular, one may obtain the extremal functions which minimize the Toeplitz form

$$F = \sum \lambda_r x_r \bar{x}_r.$$ 

$F$ then turns out to be non-negative in accordance with the Carathéodory-Toeplitz theory. We omit the details.

Turning to another problem we shall now see how the variational technique easily leads to the well-known inequality

$$(3.30) \quad \text{Re } P(z) \geq \frac{1 - r}{1 + r}, \quad |z| = r < 1,$$

for functions of class $\Theta$.

Let $P_0(z)$ be an extremal function for which, when $z$ is fixed in the unit circle, $\text{Re } P_0(z)$ is a minimum for the class $\Theta$. By a rotation in the $z$-plane we may assume $z$ to be a positive number $r$. Since $\text{Re } \delta P_0(r) \geq 0$ in (2.7) we have $\text{Re } e^{i \theta} A \leq 0$ for all $\theta$ where

$$(3.31) \quad A = A(r) = \left( \frac{z_o P_0'(r)}{P_0(z_0)} - 1 \right) \frac{r}{z_0(z_0 - r)} + \left( \frac{z_0 P_0(r)}{P_0(z_0)} - r \right) \frac{r}{z_0(z_0 - r)^2}$$

$$+ \left( \frac{P_0'(r)}{P_0(z_0)} - \frac{r^2}{1 - z_0 r} + \frac{(P_0(r))_-}{P_0(z_0)} - 1 \right) \frac{r}{(1 - z_0 r)^2}.$$ 

Since $\theta$ is arbitrary it follows that $A = 0$. Replacing $z_0$ by $z$ and solving the equation $A = 0$ for $P_0(z)$ we obtain
(1 - r^2)(1 - z^2)P_o(z) = A_3z^3 + A_2z^2 + A_1z + A_0,
A_0 = - rP'_o(r) + r^2(P'_o(r))^2 + P_o(r) + r^2(P_o(r))^2 = (1 - r^2)P_o(0)
= 1 - r^2,
(3.32)
A_1 = (1 + 2r^2)P'_o(r) - (2r^2 + r^3)(P'_o(r))^2 - 2r\{P_o(r) + (P_o(r))^2\},
A_2 = (2r^3 + r)(P'_o(r))^2 - (2r + r^3)P'_o(r) + r^2P_o(r) + (P_o(r))^2,
A_3 = r^2\{P'_o(r) - (P'_o(r))^2\}.

Since Re P(r) \geq Re P_0(r) for all P(z) \in \Phi, and since P(z) = P_0(ze^{i\theta}) \in \Phi, we have Re P_0(re^{i\theta}) \geq Re P_0(r) for all \theta. Thus Re P_0(r) is a minimum value of
Re P_0(re^{i\theta}) as a function of \theta. It follows from the Cauchy-Riemann equations that

\frac{\partial P'_o(r)}{\partial r} = - \frac{1}{r} \frac{\partial}{\partial \theta} Re P_0(re^{i\theta}) \bigg|_{\theta=0} = 0.

Thus P'_o(r) is real. Then A_3 = 0. Since A_0 is real it follows that \{P_o(r) + r^2(P_o(r))^2\} and \{P_o(r) + (P_o(r))^2\} are both real. This implies that P_0(r) is real. Since A_2 - A_0 = (1 - r^2)\{(P_o(r))^2 - P_o(r)\}, we have A_2 = A_0 = 1 - r^2. Also A_1 is seen to be real. We have now seen that P_o(z) is of the form

P_o(z) = \frac{1 + kz + z^2}{1 - z^2},
k real.

Since P_o(r) \geq 0, we have k \geq -2. Since P_o(r) is minimal for the class \Phi we must take k = -2. In this case

P_o(z) = \frac{1 - z}{1 + z}

so that (3.30) follows with equality holding only for the function P_o(ez), |e| = 1.

It should be noticed that the equation A_0 = 1 - r^2 may not be treated as a
differential equation for finding P_o(r) unless it is first shown that the extremal
function P_o does not vary with r.

4. Interior variations for subordinate functions. Let the analytic function

f(z) = A_1z + A_2z^2 + \cdots + A_nz^n + \cdots, \quad f(0) = 0, A_1 \neq 0,

be regular and univalent in |z| < 1. Let

F(z) = a_1z + a_2z^2 + \cdots + a_nz^n + \cdots, \quad F(0) = 0,

be regular and subordinate to f(z) in |z| < 1. Then

F(z) = f(\omega(z))

where \omega(z) is regular in |z| < 1, \omega(0) = 0, |\omega(z)| < 1 for |z| < 1. We may write \omega(z) in the form
where \( P(z) \) is a member of class \( \Phi \). We also denote by \( z = \phi(w) \) the inverse function of \( w = f(z) \). If \( P^*(z) \) is given by (2.6) upon varying \( P(z) \), and if \( \omega^*(z) \) corresponds by (4.4) to \( P^*(z) \), we easily obtain

\[
\omega^*(z) = \frac{P(z) - 1}{P(z) + 1} = \frac{P(z) - 1}{P(z) + 1} + \rho^3 \lambda(z) + o(\rho^3)
\]

where

\[
\lambda(z) = -\frac{1}{2} \left(1 - |z_0|^2\right) (1 - \phi(F))^2 A(z)
\]

and where \( A(z) \) is defined as the right-hand side of equation (2.7) omitting the term \( o(1) \). We also have

\[
P(z) = \frac{1 + \phi(F)}{1 - \phi(F)}, \quad (P(z) + 1)^2 = 4(1 - \phi(F) - 2),
\]

\[
P'(z) = \frac{2\phi'(F)F'(z)}{(1 - \phi(F))^2}.
\]

We now write \( A(z) \) in the form

\[
A(z) = \frac{2\phi'(F)F'(z)}{(1 - \phi(F))^2} \left\{ \frac{e^{i\theta} z}{P(z_0)(z_0 - z)} + \frac{z^2 e^{-i\theta}}{(P(z_0))^{-1}(1 - \bar{z}_0 z)} \right\} + \frac{1 + \phi(F)}{1 - \phi(F)} \left\{ \frac{e^{i\theta} z}{P(z_0)(z_0 - z)^2} + \frac{e^{-i\theta}}{(P(z_0))^{-1}(1 - \bar{z}_0 z)^2} \right\} + \left\{ \frac{e^{-i\theta} z}{(1 - \bar{z}_0 z)^2} - \frac{e^{i\theta} z^2}{z_0(z_0 - z)^2} - \frac{e^{2i\theta} z}{z_0(z_0 - z)} \right\}
\]

where

\[
P(z_0) = \frac{1 + \phi(F(z_0))}{1 - \phi(F(z_0))}, \quad |z_0| < 1.
\]

\[
F^*(z) = f(\omega(z) + \rho^3 \lambda(z) + o(\rho^3)) = F(z) + f'(\omega(z))\lambda(z)\rho^3 + o(\rho^3)
\]

\[
= F(z) + \frac{\lambda(z)}{\phi'(F)}\rho^3 + o(\rho^3).
\]
Thus the variational formula for analytic functions $F(z)$ subordinate to a given univalent function $f(z)$ in $|z| < 1$ is given by

$$F^*(z) = F(z) - \rho^2 (1 - \left| z_0 \right|^2) \frac{(1 - \phi(F))^2}{2\phi'(F)} A(z) + o(\rho^2)$$

where $A(z)$ is defined by (4.9) and $\phi$ is the inverse of $f$.

References


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