

THE REAL COHOMOLOGY RING OF A SPHERE BUNDLE OVER A DIFFERENTIABLE MANIFOLD

BY
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1. **Introduction.** The main result of this paper is already known; it follows from a theorem of A. Borel [2, paragraph 24] and has also been obtained by G. Hirsch [3]. However, it is believed that the proofs and descriptions of results given here are much simpler than in either of the above papers. The method used here is that of a previous paper [4] of the author; this paper could serve as an introduction to it.

We are concerned with the cohomology ring with real coefficients of a fiber space whose fiber is a $(k-1)$ -sphere, for k even. When given a cohomology theory whose cochains are anti-commutative, a description is given of the cohomology ring of the total space, in terms of the characteristic class W_k of the sphere space, which is practical for computations in many cases. In §5 these results are used to obtain, in a simplified form, the results of G. Hirsch [3] in terms of his "strict" triple products.

If the fibering sphere is of *even* dimension, then using real coefficients, $W_k=0$, and the cohomology ring is described in the paper [5] of W. S. Massey.

The author wishes to thank W. S. Massey for pointing out the simplification which occurs when given an anti-commutative ring of cochains, and for help in the preparation of this paper.

2. We suppose given a fiber space (E, p, B, S^{k-1}) with fiber a $(k-1)$ -sphere, k even. By "fiber space" is meant a "locally trivial fiber space": For each $x \in B$, there is a neighborhood V of x and a homeomorphism ϕ mapping $V \times S^{k-1}$ onto $p^{-1}(V)$ such that $p\phi(y, z) = y$ for $y \in V$ and $z \in S^{k-1}$. We will assume that the sphere space is orientable in the following sense: If S_x^{k-1} denotes the fiber over $x \in B$, then the local system of groups defined by $H^{k-1}(S_x^{k-1})$, for $x \in B$, is a simple system. We also assume that the base space B is compact, and that we are given a cohomology theory with real coefficients, whose cochains are anti-commutative. An example of this would be if B were a differentiable manifold, the sphere space had a differentiable structure, and the ring of cochains were the exterior differential forms. However, it is possible to have a ring of anti-commutative cochains under less restrictive assumptions.

We recall a result of Thom [7]: If A is the mapping cylinder of $p: E \rightarrow B$, the Gysin sequence of (E, p, B, S^{k-1}) is isomorphic to the cohomology sequence of the pair (A, E) . In fact, there is an element $\mathfrak{u} \in H^k(A, E)$ such that

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the homomorphism $\theta: H^{q-k}(A) \rightarrow H^q(A, E)$ defined by $\theta(x) = x \cup \mathfrak{U}$ (the cup product) is an isomorphism onto. The natural projection $p_0: A \rightarrow B$ of the mapping cylinder onto the base space induces an isomorphism $p_0^*: H^q(B) \rightarrow H^q(A)$. We then have the commutative diagram

$$\begin{array}{ccccccc}
 \dots & \rightarrow & H^{q-k}(B) & \xrightarrow{\mu} & H^q(B) & \xrightarrow{p^*} & H^q(E) & \xrightarrow{\psi} & H^{q-k+1}(B) & \rightarrow & \dots \\
 & & \downarrow p_0^* & & \downarrow p_0^* & & \downarrow \text{id.} & & \downarrow p_0^* & & \\
 & & H^{q-k}(A) & & & & & & H^{q-k+1}(A) & & \\
 & & \downarrow \theta & & \downarrow p_0^* & & \downarrow \text{id.} & & \downarrow \theta & & \\
 \dots & \rightarrow & H^q(A, E) & \xrightarrow{n^*} & H^q(A) & \xrightarrow{m^*} & H^q(E) & \xrightarrow{\delta^*} & H^{q+1}(A, E) & \rightarrow & \dots
 \end{array}$$

FIGURE 1

In this diagram, the top line is the Gysin sequence and the bottom line is the cohomology sequence of the pair (A, E) . According to the results of Thom, $n^*(\mathfrak{U}) = p_0^*(W_k)$.

We will regard $C^*(A, E)$ as a subgroup of $C^*(A)$. It is actually an ideal in $C^*(A)$ and $C^*(E) \approx C^*(A)/C^*(A, E)$. Making this identification, n^* is induced by the inclusion $C^*(A, E) \subset C^*(A)$, and m^* is induced by the natural map $m^#: C^*(A) \rightarrow C^*(A)/C^*(A, E)$, where $m: E \rightarrow A$ is the inclusion.

3. Let V be a representative cocycle for the characteristic class $W_k \in H^k(B)$. Construct the algebraic mapping cylinder M of the map $x \rightarrow xV$ for $x \in C^*(B)$, that is, let

$$\begin{aligned}
 M^p &= C^p(B) \times C^{p-k+1}(B), \\
 M &= \sum_p M^p,
 \end{aligned}$$

and

$$\delta(x, y) = (\delta x + yV, -\delta y) \quad \text{for } (x, y) \in M.$$

It is easily seen that (M, δ) is a differential group, with addition defined componentwise.

We now introduce a multiplication in M , by the formula

$$(x, y)(v, w) = (xv, (-1)^p xw + yv)$$

for $(x, y) \in M^p$ and $(v, w) \in M^q$. The product has degree $p+q$, and is associative and anti-commutative. It is easily verified that $\delta[(x, y)(v, w)] = [\delta(x, y)](v, w) + (-1)^p(x, y)\delta(v, w)$ for $(x, y) \in M^p$. Consequently a product is induced in the derived group $H^*(M)$.

THEOREM I. *Using this product, $H^*(M)$ is isomorphic to $H^*(E)$ as an algebra over the reals.*

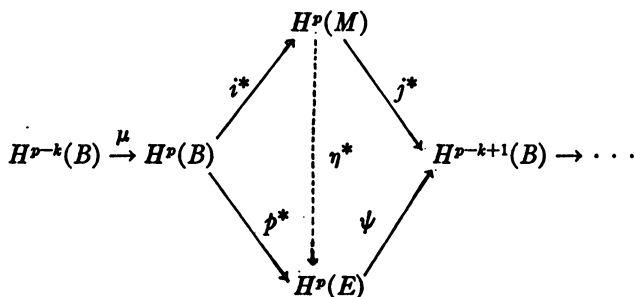
To prove this theorem, we consider the diagram of exact sequences

$$\begin{array}{ccccccc}
 0 & \rightarrow & C^p(B) & \xrightarrow{i} & M^p & \xrightarrow{j} & C^{p-k+1}(B) \rightarrow 0 \\
 & & \downarrow \delta & & \delta \downarrow & & \downarrow \delta \\
 0 & \rightarrow & C^{p+1}(B) & \xrightarrow{i} & M^{p+1} & \xrightarrow{j} & C^{p-k+2}(B) \rightarrow 0
 \end{array}$$

where $i(x) = (x, 0)$ for $x \in C^*(B)$ and $j(x, y) = y$ for $(x, y) \in M$. The left square commutes and the right square anti-commutes. From this diagram we obtain the exact cohomology sequence of the algebraic mapping cylinder:

$$\dots \rightarrow H^{p-k}(B) \xrightarrow{\mu} H^p(B) \xrightarrow{i^*} H^p(M) \xrightarrow{j^*} H^{p-k+1}(B) \rightarrow \dots$$

Here, μ is induced by $x \rightarrow xV$, in other words μ is the same map as in the Gysin sequence. We have the diagram



We will define a ring homomorphism $\eta^*: H^p(M) \rightarrow H^p(E)$ for which $\eta^*i^* = p^*$ and $-j^* = \psi\eta^*$; it will then follow by the five-lemma that η^* is an isomorphism onto.

To define η^* , let $V' \in C^k(A, E)$ be a representative cocycle for Thom's class $u \in H^k(A, E)$. Then for some $\alpha \in C^{k-1}(A)$, $p_0^\sharp(V) = V' + \delta\alpha$. Here, p_0^\sharp is the cochain homomorphism induced by $p_0: A \rightarrow B$. Define

$$\eta: M^p \rightarrow C^p(A)/C^p(A, E) = C^p(E)$$

by

$$\eta(x, y) = m^\sharp(p_0^\sharp(x) + (-1)^{p+1}(p_0^\sharp(y)\alpha)) \quad \text{for } (x, y) \in M^p.$$

It is easily verified that $\delta\eta = \eta\delta$ and that η is an additive homomorphism. Also, for $(x, y) \in M^p$ and $(u, v) \in M^q$, a straightforward computation shows that

$$\eta\{(x, y)(u, v)\} - \eta(x, y)\eta(u, v) = m^\sharp(\Gamma)$$

where

$$\Gamma = (-1)^{p+q+1}p_0^\sharp(yu)\alpha + (-1)^p(p_0^\sharp y)\alpha(p_0^\sharp u) + (-1)^{p+q+1}(p_0^\sharp y)\alpha(p_0^\sharp v)\alpha.$$

$\Gamma=0$ since the cochains are anti-commutative, and $\alpha^2=0$; therefore the induced map $\eta^*: H^*(M)\rightarrow H^*(E)$ is a ring homomorphism.

To prove the commutativity relations, let square brackets denote cohomology classes in the appropriate cohomology groups. For $[x]\in H^p(B)$, $\eta^*i^*[x]=\eta^*[(x, 0)]=[m^\sharp p_0^\sharp x]=p^*[x]$, in view of the commutativity of Figure 1. For $[(x, y)]\in H^p(M)$,

$$\begin{aligned} (\psi\eta^*)[(x, y)] &= \psi[m^\sharp(p_0^\sharp x + (-1)^{p+1}(p_0^\sharp y)\alpha)] \\ &= p_0^{*-1}\theta^{-1}\delta^*[m^\sharp(p_0^\sharp x + (-1)^{p+1}(p_0^\sharp y)\alpha)] \\ &= p_0^{*-1}\theta^{-1}[p_0^\sharp(\delta x) + (p_0^\sharp y)(p_0^\sharp V - V')] \\ &= p_0^{*-1}\theta^{-1}[-(p_0^\sharp y)V'] = -[y] = -j^*[(x, y)]. \end{aligned}$$

We remark that the algebra $H^*(M)$ is independent of the choice $V\in W_k$, as this theorem shows.

4. In this section we prove two propositions which should be useful in actual computations. Suppose C and \bar{C} are graded, anti-commutative cochain rings, and V and \bar{V} are C and \bar{C} k -cocycles, respectively. Let $f: C\rightarrow\bar{C}$ be an allowable homomorphism for which $f(V)=\bar{V}$. Let M and \bar{M} be the algebraic mapping cylinders of $x\rightarrow xV$ and $x\rightarrow x\bar{V}$ respectively.

PROPOSITION A. *There is an allowable homomorphism $\phi: M\rightarrow\bar{M}$, which preserves products, induced by f . The homomorphism $\phi^*: H^*(M)\rightarrow H^*(\bar{M})$ induced by ϕ commutes with the homomorphisms of the algebraic mapping cylinders. Furthermore if $f^*: H^*(C)\rightarrow H^*(\bar{C})$ is an isomorphism onto, then so is ϕ^* .*

To prove this, define ϕ by $\phi(x, y)=(fx, fy)$ for $(x, y)\in M$. All the computations are straightforward. The last assertion follows from the five-lemma.

Now let M^n be a compact, connected, differentiable n -dimensional manifold, with $H^1(M^n)=0$. Let C^* be the algebra of exterior differential forms, with differential operator d .

PROPOSITION B⁽¹⁾. *It is possible to choose a finitely generated subring A^* of C^* , for which the inclusion $i: A^*\rightarrow C^*$ induces an isomorphism onto $i^*: H^*(A^*)\rightarrow H^*(C^*)$.*

Proof. Let $1, a_1, \dots, a_k$ be a minimal set of generators of $H^*(C^*)$, with representative cocycles $1', a'_1, \dots, a'_k$. These generate $A\subset C^*$, and the inclusion $i: A\rightarrow C^*$ induces $i^*: H^*(A)\rightarrow H^*(C^*)$ which is onto. The kernel of i^* is generated as an ideal by b_1, \dots, b_l ; choose $b'_1, \dots, b'_l\in C^*$ for which $db'_i = b_i$. Let A' be generated by $1, a'_1, \dots, a'_k, b'_1, \dots, b'_l$; make this set of

(¹) This was suggested by W. S. Massey.

generators minimal. All of the b_i have degree at least three, for they arise from cocycles a'_i, a'_j for which $a'_i a'_j$ is a coboundary, and $a'_i a'_j$ has degree at least four. If $i^{*'}: H^*(A') \rightarrow H^*(C^*)$ is not 1-1, repeat the process, adding c'_i 's which have degree at least four. The process must end, for $C^p = 0$ for $p > n$.

Using these two propositions, one can compute the cohomology rings in many cases.

5. We now suppose that the base space B is a differentiable manifold, with a Riemannian metric, and we will use the exterior differential forms as cochains on B . Under these conditions, there are canonical additive homomorphisms (Z and \mathfrak{B} represent cocycles and coboundaries)

$$\alpha: H^p(B) \rightarrow Z^p(B)$$

and

$$\beta: \mathfrak{B}^p(B) \rightarrow C^{p-1}(B).$$

α assigns to each cohomology class a representative cocycle (a harmonic form) and β assigns to each p -coboundary a $(p-1)$ -cochain such that $\delta\beta = \text{identity}$. (See [1] or [6]; an outline of the results is given in [3].)

We construct the algebraic mapping cylinder of $x \rightarrow x\alpha(W_k)$ for $x \in C^*(B)$. This gives the exact sequence

$$0 \rightarrow \frac{H^p(B)}{W_k \cdot H^{p-k}(B)} \xrightarrow{i} H^p(M) \xrightarrow{j} (\text{kernel } \mu)^{p-k+1} \rightarrow 0$$

where i and j are induced by i^* and j^* in the obvious ways, and

$$(\text{kernel } \mu)^{p-k+1} = (\text{kernel } \mu) \cap H^{p-k+1}(B).$$

Note that a multiplication

$$\frac{H^p(B)}{W_k \cdot H^{p-k}(B)} \times \text{kernel } \mu \rightarrow \text{kernel } \mu$$

is defined. This sequence splits; we define $\theta: (\text{kernel } \mu)^{p-k+1} \rightarrow H^p(M)$ by requiring $\theta(y)$ to be the cohomology class in $H^*(M)$ of $(-\beta(\alpha y \cdot \alpha W_k), \alpha y)$. It is easily verified that θ is additive and $j\theta = \text{identity}$. Thus every $x \in H^*(M)$ may be written uniquely in the form $x = i(x_1) + \theta(x_2)$, where $x_2 = j(x)$. The product of two elements may be computed as follows (p is the degree of x_1):

$$(i(x_1) + \theta(x_2))(i(y_1) + \theta(y_2)) = i(x_1 y_1) + \theta(x_2) i(y_1) + i(x_1) \theta(y_2) + \theta(x_2) \theta(y_2).$$

Applying j to both sides, we get

$$\begin{aligned} & j\{(i(x_1) + \theta(x_2))(i(y_1) + \theta(y_2))\} \\ (*) \quad & = j(\theta(x_2) i(y_1)) + j(i(x_1) \theta(y_2)) + j(\theta(x_2) \theta(y_2)) \\ & = x_2 y_1 + (-1)^p x_1 y_2 + j(\theta(x_2) \theta(y_2)), \end{aligned}$$

since $j(\theta(x_2) i(y_1)) = x_2 y_1$ and $j(i(x_1) \theta(y_2)) = (-1)^p x_1 y_2$. To prove this last rela-

tion, let $x'_1 \in H^p(B)$ represent x_1 ; then $i(x_1)$ is represented by $(\alpha x'_1, 0) \in M^p$. $\theta(y_2)$ is represented by $(-\beta[(\alpha y_2)(\alpha W_k)], \alpha y_2)$. Using the product formula in M^p , we obtain $j(i(x_1)\theta(y_2)) = (-1)^p x_1 y_2$. The other formula is proved similarly.

From the above equation (*) we obtain

$$(i(x_1) + \theta(x_2))(i(y_1) + \theta(y_2)) = i(x_1 y_1) + \theta(x_2 y_1) + (-1)^p x_1 y_2 + j(\theta x_2 \cdot \theta y_2).$$

We may compute $j(\theta x_2 \cdot \theta y_2)$; it is the cohomology class of

$$-(\alpha x_2)\beta(\alpha y_2 \cdot \alpha W_k) + (-1)^{p+1}\beta(\alpha x_2 \cdot \alpha W_k)\alpha(y_2).$$

However, this is just $(-1)^{p+1}\langle x_2, W_k, y_2 \rangle$, the triple product in the strict sense of Hirsch [3]. We thus have the following theorem.

THEOREM II. *For the base space B a differentiable manifold with Riemannian metric, we have*

$$H^p(E) \approx \frac{H^p(B)}{W_k \cdot H^{p-k}(B)} \oplus (\text{kernel } \mu)^{p-k+1}.$$

The multiplication is given in terms of the strict triple product by

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_1) + (-1)^p x_1 y_2 + (-1)^{p+1}\langle x_2, W_k, y_2 \rangle.$$

Theorem II is simpler than the corresponding result of Hirsch [3]; Hirsch uses several cohomology operations. In the above formulation, x_1 and y_1 are elements of $H^*(B)/W_k \cdot H^*(B)$. The author believes that the extra operations that Hirsch uses only lead to different representative elements for x_1 and y_1 , and thus give the same product.

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