THE REAL COHOMOLOGY RING OF A SPHERE BUNDLE OVER A DIFFERENTIABLE MANIFOLD

BY

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1. Introduction. The main result of this paper is already known; it follows from a theorem of A. Borel [2, paragraph 24] and has also been obtained by G. Hirsch [3]. However, it is believed that the proofs and descriptions of results given here are much simpler than in either of the above papers. The method used here is that of a previous paper [4] of the author; this paper could serve as an introduction to it.

We are concerned with the cohomology ring with real coefficients of a fiber space whose fiber is a \((k-1)\)-sphere, for \(k\) even. When given a cohomology theory whose cochains are anti-commutative, a description is given of the cohomology ring of the total space, in terms of the characteristic class \(W_k\) of the sphere space, which is practical for computations in many cases. In §5 these results are used to obtain, in a simplified form, the results of G. Hirsch [3] in terms of his “strict” triple products.

If the fibering sphere is of even dimension, then using real coefficients, \(W_k = 0\), and the cohomology ring is described in the paper [5] of W. S. Massey.

The author wishes to thank W. S. Massey for pointing out the simplification which occurs when given an anti-commutative ring of cochains, and for help in the preparation of this paper.

2. We suppose given a fiber space \((E, p, B, S^{k-1})\) with fiber a \((k-1)\)-sphere, \(k\) even. By “fiber space” is meant a “locally trivial fiber space”: For each \(x \in B\), there is a neighborhood \(V\) of \(x\) and a homeomorphism \(\phi\) mapping \(V \times S^{k-1}\) onto \(p^{-1}(V)\) such that \(p \phi(y, z) = y\) for \(y \in V\) and \(z \in S^{k-1}\). We will assume that the sphere space is orientable in the following sense: If \(S^{k-1}\) denotes the fiber over \(x \in B\), then the local system of groups defined by \(H^{k-1}(S^{k-1}, x)\), for \(x \in B\), is a simple system. We also assume that the base space \(B\) is compact, and that we are given a cohomology theory with real coefficients, whose cochains are anti-commutative. An example of this would be if \(B\) were a differentiable manifold, the sphere space had a differentiable structure, and the ring of cochains were the exterior differential forms. However, it is possible to have a ring of anti-commutative cochains under less restrictive assumptions.

We recall a result of Thom [7]: If \(A\) is the mapping cylinder of \(p : E \to B\), the Gysin sequence of \((E, p, B, S^{k-1})\) is isomorphic to the cohomology sequence of the pair \((A, E)\). In fact, there is an element \(u \in H^k(A, E)\) such that

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the homomorphism \( \theta: H^{q-k}(A) \to H^q(A, E) \) defined by \( \theta(x) = x \cup \eta \) (the cup product) is an isomorphism onto. The natural projection \( p_0: A \to B \) of the mapping cylinder onto the base space induces an isomorphism \( p_0^*: H^q(B) \to H^q(A) \). We then have the commutative diagram

\[
\begin{array}{c}
\cdots \to H^{q-k}(B) \xrightarrow{\mu} H^q(B) \xrightarrow{p_0^*} H^q(E) \xrightarrow{\psi} H^{q-k+1}(B) \to \cdots \\
\downarrow p_0^* \quad \downarrow \theta \quad \downarrow \theta \\
H^{q-k}(A) \quad \quad \quad H^q(A) \quad \quad \quad H^q(A, E) \quad \quad \quad H^q(E) \\
\downarrow \theta \quad \downarrow \theta \\
\cdots \to H^q(A, E) \xrightarrow{n^*} H^q(A) \xrightarrow{m^*} H^q(E) \xrightarrow{\delta^*} H^{q+1}(A, E) \cdots
\end{array}
\]

**Figure 1**

In this diagram, the top line is the Gysin sequence and the bottom line is the cohomology sequence of the pair \((A, E)\). According to the results of Thom, \( n^*(\eta) = p_0^*(\eta_k) \).

We will regard \( C^*(A, E) \) as a subgroup of \( C^*(A) \). It is actually an ideal in \( C^*(A) \) and \( C^*(E) \approx C^*(A)/C^*(A, E) \). Making this identification, \( n^* \) is induced by the inclusion \( C^*(A, E) \subset C^*(A) \), and \( m^* \) is induced by the natural map \( m^*: C^*(A) \to C^*(A)/C^*(A, E) \), where \( m: E \to A \) is the inclusion.

3. Let \( V \) be a representative cocycle for the characteristic class \( W_k \in H^k(B) \). Construct the algebraic mapping cylinder \( M \) of the map \( x \to xV \) for \( x \in C^*(B) \), that is, let

\[
M^p = C^p(B) \times C^{p+k+1}(B),
\]

and

\[
M = \sum_p M^p,
\]

\[
\delta(x, y) = (\delta x + yV, -\delta y) \quad \text{for } (x, y) \in M.
\]

It is easily seen that \((M, \delta)\) is a differential group, with addition defined componentwise.

We now introduce a multiplication in \( M \), by the formula

\[
(x, y) (v, w) = (xv, (-1)^p xw + yw)
\]

for \((x, y) \in M^p\) and \((v, w) \in M^q\). The product has degree \( p+q \), and is associative and anti-commutative. It is easily verified that \( \delta[(x, y)(v, w)] = [\delta(x, y)](v, w) + (-1)^p \delta(x, y) \delta(v, w) \) for \((x, y) \in M^p\). Consequently a product is induced in the derived group \( H^*(M) \).

**Theorem I.** Using this product, \( H^*(M) \) is isomorphic to \( H^*(E) \) as an algebra over the reals.
To prove this theorem, we consider the diagram of exact sequences

\[
0 \rightarrow C^p(B) \xrightarrow{i} M^p \xrightarrow{j} C^{p+k+1}(B) \rightarrow 0
\]

\[
\downarrow \delta \quad \downarrow \delta \quad \downarrow \delta
\]

\[
0 \rightarrow C^{p+1}(B) \xrightarrow{i} M^{p+1} \xrightarrow{j} C^{p+k+2}(B) \rightarrow 0
\]

where \(i(x) = (x, 0)\) for \(x \in C^p(B)\) and \(j(x, y) = y\) for \((x, y) \in M\). The left square commutes and the right square anti-commutes. From this diagram we obtain the exact cohomology sequence of the algebraic mapping cylinder:

\[
\cdots \rightarrow H^{p-k}(B) \xrightarrow{\mu} H^p(B) \xrightarrow{i^*} H^p(M) \xrightarrow{j^*} H^{p+k+1}(B) \rightarrow \cdots
\]

Here, \(\mu\) is induced by \(x \mapsto xV\), in other words \(\mu\) is the same map as in the Gysin sequence. We have the diagram

We will define a ring homomorphism \(\eta^*: H^p(M) \rightarrow H^p(E)\) for which \(\eta*i^* = p^*\) and \(-j^* = \psi\eta^*\); it will then follow by the five-lemma that \(\eta^*\) is an isomorphism onto.

To define \(\eta^*\), let \(V' \in C^k(A, E)\) be a representative cocycle for Thom's class \(\xi \in H^k(A, E)\). Then for some \(\alpha \in C^{k-1}(A)\), \(p_0^*(V) = V' + \delta \alpha\). Here, \(p_0^*\) is the cochain homomorphism induced by \(p_0: A \rightarrow B\). Define

\[
\eta: M^p \rightarrow C^p(A)/C^p(A, E) = C^p(E)
\]

by

\[
\eta(x, y) = m^\dagger(\rho_0^*(x) + (-1)^{p+1}(\rho_0^y)\alpha) \quad \text{for} \ (x, y) \in M^p.
\]

It is easily verified that \(\delta \eta = \eta \delta\) and that \(\eta\) is an additive homomorphism. Also, for \((x, y) \in M^p\) and \((u, v) \in M^q\), a straightforward computation shows that

\[
\eta\{ (x, y)(u, v) \} - \eta(x, y)\eta(u, v) = m^\dagger(\Gamma)
\]

where
\[ \Gamma = (-1)^{p+q} \beta(yw) \alpha + (-1)^p (\beta \alpha \gamma) (\beta \mu) + (-1)^{p+q+1} (\beta \gamma) \alpha (\beta \nu) \alpha. \]

\( \Gamma = 0 \) since the cochains are anti-commutative, and \( \alpha^2 = 0 \); therefore the induced map \( \eta^* : H^*(M) \to H^*(E) \) is a ring homomorphism.

To prove the commutativity relations, let square brackets denote cohomology classes in the appropriate cohomology groups. For \([x] \in H^p(B)\),

\[ \eta^* \cdot [x] = \eta^*([x, 0]) = [m^* \rho^x_0] = \rho^*[x], \]

in view of the commutativity of Figure 1. For \([ (x, y) ] \in H^*(M) \),

\[ (\psi \eta^* ) [(x, y)] = m^* (\rho^x_0 + (-1)^{p+1} (\rho^y_0) \alpha) \]

\[ = \rho^0 \cdot (\rho^x_0 + (-1)^{p+1} (\rho^y_0) \alpha) \]

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We remark that the algebra \( H^*(M) \) is independent of the choice \( V \subseteq W_k \), as this theorem shows.

4. In this section we prove two propositions which should be useful in actual computations. Suppose \( C \) and \( C \) are graded, anti-commutative cochain rings, and \( V \) and \( V \) are \( C \) and \( C \)-cocycles, respectively. Let \( f : C \to C \) be an allowable homomorphism for which \( f(V) = V \). Let \( M \) and \( M \) be the algebraic mapping cylinders of \( x \to V \) and \( x \to x \) respectively.

**Proposition A.** There is an allowable homomorphism \( \phi : M \to M \), which preserves products, induced by \( f \). The homomorphism \( \phi^* : H^*(M) \to H^*(M) \) induced by \( \phi \) commutes with the homomorphisms of the algebraic mapping cylinders. Furthermore if \( f^* : H^*(C) \to H^*(C) \) is an isomorphism onto, then so is \( \phi^* \).

To prove this, define \( \phi \) by \( \phi(x, y) = (fx, fy) \) for \((x, y) \in M \). All the computations are straightforward. The last assertion follows from the five-lemma.

Now let \( M^* \) be a compact, connected, differentiable \( n \)-dimensional manifold, with \( H^*(M^*) = 0 \). Let \( C^* \) be the algebra of exterior differential forms, with differential operator \( d \).

**Proposition B(\(^\dagger\)).** It is possible to choose a finitely generated subring \( A^* \) of \( C^* \), for which the inclusion \( i : A^* \to C^* \) induces an isomorphism onto \( i^* : H^*(A^*) \to H^*(C^*) \).

**Proof.** Let 1, \( a_1, \ldots, a_k \) be a minimal set of generators of \( H^*(C^*) \), with representative cocycles \( 1, a_1', \ldots, a_k' \). These generate \( A \subseteq C^* \), and the inclusion \( i : A \to C^* \) induces \( i^* : H^*(A) \to H^*(C^*) \) which is onto. The kernel of \( i^* \) is generated as an ideal by \( b_1, \ldots, b_i \); choose \( b_1', \ldots, b_i' \subseteq C^* \) for which \( db_i = b_i \). Let \( A^* \) be generated by 1, \( a_1', \ldots, a_k', b_1', \ldots, b_i' \); make this set of

\(^\dagger\) This was suggested by W. S. Massey.
generators minimal. All of the $b_i$ have degree at least three, for they arise from cocycles $a_i^j$, $a_j^i$ for which $a_i^ja_j^i$ is a coboundary, and $a_i^ja_j^i$ has degree at least four. If $i^*: H^*(A') \to H^*(C^*)$ is not 1-1, repeat the process, adding $c_i^j$s which have degree at least four. The process must end, for $C^p = 0$ for $p > n$.

Using these two propositions, one can compute the cohomology rings in many cases.

5. We now suppose that the base space $B$ is a differentiable manifold, with a Riemannian metric, and we will use the exterior differential forms as cochains on $B$. Under these conditions, there are canonical additive homomorphisms $(\alpha$ and $\beta$ represent cocycles and coboundaries $)$

$$\alpha: H^p(B) \to Z^p(B)$$

and

$$\beta: \Theta^p(B) \to C^{p-1}(B).$$

$\alpha$ assigns to each cohomology class a representative cocycle (a harmonic form) and $\beta$ assigns to each $p$-coboundary a $(p-1)$-cochain such that $\delta\beta = \text{identity}$. (See [1] or [6]; an outline of the results is given in [3].)

We construct the algebraic mapping cylinder of $x \to x\alpha(W_k)$ for $x \in C^*(B)$. This gives the exact sequence

$$0 \to \frac{H^p(B)}{W_k \cdot H^{p-k}(B)} \overset{i}{\to} H^p(M) \overset{j}{\to} (\text{kernel } \mu)^{p-k+1} \to 0$$

where $i$ and $j$ are induced by $i^*$ and $j^*$ in the obvious ways, and

$$(\text{kernel } \mu)^{p-k+1} = (\text{kernel } \mu) \cap H^{p-k+1}(B).$$

Note that a multiplication

$$\frac{H^p(B)}{W_k \cdot H^{p-k}(B)} \times \text{kernel } \mu \to \text{kernel } \mu$$

is defined. This sequence splits; we define $\theta$: (kernel $\mu)^{p-k+1} \to H^p(M)$ by requiring $\theta(y)$ to be the cohomology class in $H^p(M)$ of $(-\beta(\alpha y \cdot \alpha W_k), \alpha y)$. It is easily verified that $\theta$ is additive and $j\theta = \text{identity}$. Thus every $x \in H^p(M)$ may be written uniquely in the form $x = i(x_1) + \theta(x_2)$, where $x_2 = j(x)$. The product of two elements may be computed as follows ($p$ is the degree of $x_1$):

$$(i(x_1) + \theta(x_2))(i(y_1) + \theta(y_2)) = i(x_1y_1) + \theta(x_2i(y_1)) + i(x_1)\theta(y_2) + \theta(x_2)\theta(y_2).$$

Applying $j$ to both sides, we get

$$j\left\{ (i(x_1) + \theta(x_2))(i(y_1) + \theta(y_2)) \right\} = j(\theta(x_2)i(y_1)) + j(i(x_1)\theta(y_2)) + j(\theta(x_2)\theta(y_2))$$

$$= x_2y_1 + (-1)^p x_1y_2 + j(\theta(x_2)\theta(y_2)),$$

since $j(\theta(x_2)i(y_1)) = x_2y_1$ and $j(i(x_1)\theta(y_2)) = (-1)^p x_1y_2$. To prove this last rela-
tion, let $x' \in H^p(B)$ represent $x_1$; then $i(x_1)$ is represented by $(ax', 0) \in M^p$. $\theta(y_2)$ is represented by $(-\beta[(a y_2)(a W_k)], a y_2)$. Using the product formula in $M^p$, we obtain $j(i(x_1) \theta(y_2)) = (-1)^p x_1 y_2$. The other formula is proved similarly.

From the above equation (*) we obtain

$$(i(x_1) + \theta(x_2))(i(y_1) + \theta(y_2)) = i(x_1 y_1) + \theta(x_2 y_1 + (-1)^p x_1 y_2 + j(\theta x_2 \cdot \theta y_2)).$$

We may compute $j(\theta x_2 \cdot \theta y_2)$; it is the cohomology class of

$$-(a x_2) \beta(a y_2 \cdot a W_k) + (-1)^{p+1} \beta(a y_2 \cdot a W_k) a(y_2).$$

However, this is just $(-1)^{p+1} (x_2, W_k, y_2)$, the triple product in the strict sense of Hirsch [3]. We thus have the following theorem.

**Theorem II.** For the base space $B$ a differentiable manifold with Riemannian metric, we have

$$H^p(E) \cong \frac{H^p(B)}{W_k \cdot H^{p-k}(B)} \oplus \text{(kernel } \mu)_{p-k+1}.$$

The multiplication is given in terms of the strict triple product by

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_1 + (-1)^p x_1 y_2 + (-1)^{p+1}(x_2, W_k, y_2)).$$

Theorem II is simpler than the corresponding result of Hirsch [3]; Hirsch uses several cohomology operations. In the above formulation, $x_1$ and $y_1$ are elements of $H^*(B)/W_k \cdot H^*(B)$. The author believes that the extra operations that Hirsch uses only lead to different representative elements for $x_1$ and $y_1$, and thus give the same product.

**References**


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