GENERALIZED HOMOLOGY THEORIES(1)

BY

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1. **Introduction.** It is well known that the cohomology groups $H^n(X; \Pi)$ of a polyhedron X with coefficients in the abelian group Π can be characterized as the group of homotopy classes of maps of X into the Eilenberg-MacLane space $K(\Pi, n)$. Moreover, the cohomology *theory* with coefficients in Π can be described in this way; the existence of the coboundary homomorphism of the cohomology sequence of a pair is due to the fact that there are natural maps of the suspension $SK(\Pi, n)$ into $K(\Pi, n+1)$ for every n; in other words, the spaces $K(\Pi, n)$ are the components of a *spectrum*(2). In fact, if $\boldsymbol{E} = \{E_n\}$ is a spectrum, then the groups

$$H^n(X; \boldsymbol{E}) = \{X, \boldsymbol{E}\}_{-n} = \lim\, [S^k X, E_{n+k}]$$

are generalized cohomology groups of X, in the sense that they satisfy the Eilenberg-Steenrod axioms [8], except for the dimension axiom. These generalized cohomology groups are beginning to play a more important role in algebraic topology; for example, they may well furnish the correct setting in which to study cohomology operations of higher order. Moreover, E. H. Brown [4] has shown that, under a countability restriction on the coefficient groups, every generalized cohomology theory can be obtained in this way.

One may ask whether there is a corresponding situation for homology theory. The integral homology groups of a space X can be described by the Dold-Thom theorem [6], as the homotopy groups of the infinite symmetric product of X. However, the duality between homology and cohomology is not apparent from this description, nor is it clear how to generalize it. Examples of generalized homology theories are known; for instance, the stable homotopy groups. Like the homology and cohomology groups, the stable homotopy and cohomotopy groups satisfy Alexander duality [26]. Given a cohomology theory, one might then define the corresponding homology groups as the cohomology groups of the complement of X in a sphere in which X is imbedded. While this definition is perfectly satisfactory, it is awkward to work with because of the many choices involved. For practical as well as for aesthetic reasons, an intrinsic definition is to be preferred.

A rewording of the definition of the generalized cohomology groups de-

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(1) This research was sponsored by the Office of Ordnance Research, U. S. Army.

(2) The notion of spectrum was originated by Lima [16] and modified by Spanier [25]; our definition given in §4, differs from that of Spanier in that we impose no convergence condition.

fined above suggests a possible definition of the corresponding homology groups. Given a space X and a spectrum $\boldsymbol{E}$, the function spaces $F(X, E_n)$ of maps of X into E_n themselves form a spectrum $\boldsymbol{F}(X, \boldsymbol{E})$, and we have

$$H^n(X; \boldsymbol{E}) \approx \pi_{-n}(\boldsymbol{F}(X, \boldsymbol{E})).$$

The mapping functor F plays a role in topology analogous to that played by Hom in group theory. We seek an analogue of the tensor product; this is provided by the *reduced join* $\wedge$; in fact, the reduced join and mapping functors are *adjoint functors* in the sense of Kan [13] because of the well-known relation

$$F(X \wedge Y, Z) \equiv F(X, F(Y, Z)),$$

valid for well-behaved spaces. Now if X is a space and $\boldsymbol{E}$ a spectrum, then $\boldsymbol{E} \wedge X$ is again a spectrum and we define the generalized homology groups

$$H_n(X; \boldsymbol{E}) = \pi_n(\boldsymbol{E} \wedge X).$$

We prove that the generalized homology groups satisfy the Eilenberg-Steenrod axioms, except for the dimension axiom. With an appropriate notion of pairing of spectra, we can define cup- and cap-products. Using these we then prove an Alexander duality theorem. Moreover, we characterize the class of manifolds satisfying Poincaré duality for arbitrary spectra; it includes the Π-manifolds of J. H. C. Whitehead [35] and of Milnor [19].

The results of this paper were announced in [32].

§2 is devoted to general preliminaries, and §3 to homology and homotopy properties of the reduced join. Most of the results of these sections are well-known. §4 is devoted to properties of spectra. In §5 the generalized homology theories are introduced and the Eilenberg-Steenrod axioms proved. §6 is devoted to setting up the machinery of products, and in §7 the duality theorems are proved. In §8 we make use of Brown's theorem to prove the analogous result for homology theories.

2. **Preliminaries.** Let $\mathcal{W}_0$ be the category of spaces with base-point having the homotopy type of a CW-complex. More precisely, an object of $\mathcal{W}_0$ is a space X with base-point x_0, such that there exists a CW-complex K with base-point k_0 and a homotopy equivalence of the pairs $(X, \{x_0\})$ and $(K, \{k_0\})$; and a map of $\mathcal{W}_0$ is a continuous, base-point preserving map.

Let $\mathcal{W}$ be the category of spaces (without distinguished base-point) having the homotopy type of a CW-complex. Let P be a fixed space consisting of exactly one point p_0. If $X \in \mathcal{W}$, let X^+ be the topological sum of X and P; then (X^+, p_0) is an object of $\mathcal{W}_0$. If $X, Y \in \mathcal{W}$ and $f: X \to Y$, then f has a unique extension $f^+: X^+ \to Y^+$ such that $f^+(p_0) = p_0$, and f^+ is a map in $\mathcal{W}_0$. The correspondences $X \to X^+$, $f \to f^+$ define a functor $^+: \mathcal{W} \to \mathcal{W}_0$. Evidently we may regard $\mathcal{W}$ as a subcategory of $\mathcal{W}_0$.

In what follows, we shall use the terms "space" and "map" to refer to

objects and maps of $\mathcal{W}_0$; the terms "free space" and "free map" will refer to objects and maps of $\mathcal{W}$.

Let n be a positive integer. By "n-ad" we shall mean an n-ad $(X; X_1, \cdots, X_{n-1}; x_0)$ having the same homotopy type as some CW-n-ad $(K; K_1, \cdots, K_{n-1}; k_0)$. A 2-ad is also called a *pair*. We say that A is a subspace of X if and only if $(X; A; x_0)$ is a pair. The n-ads form a category $\mathcal{W}_0^n$. Similarly we define the category $\mathcal{W}^n$ of free n-ads.

The category $\mathcal{W}^n$ was introduced by Milnor [18].

If (X, A) is a pair, let X/A be the space obtained from X by collapsing A to a point, the base-point of X/A. Then X/A is a space, called the *quotient space* of X by A.

Let $X_1, \cdots, X_n$ be spaces with base-points $x_{10}, \cdots, x_{n0}$. Let $X = \prod_{i=1}^n X_i$ be the cartesian product of the X_i, with base-point $x_0 = (x_{10}, \cdots, x_{n0})$. Let T_i be the set of all points $(x_1, \cdots, x_n)$ such that $x_i = x_{i0}$, and let $T(X_1, \cdots, X_n) = \bigcup_{i=1}^n T_i$. It follows from [18, Proposition 3] that the $(n+1)$-ad $(X; T_1, \cdots, T_n; x_0)$ belongs to $\mathcal{W}_0^n$. The *n-fold reduced join* of the X_i is the space $\bigwedge_{i=1}^n X_i = X_1 \wedge \cdots \wedge X_n = X/T(X_1, \cdots, X_n)$; let $\bigwedge x_i = x_1 \wedge \cdots \wedge x_n$ be the image of $(x_1, \cdots, x_n)$ in $\bigwedge X_i$. If $f_i: X_i \to Y_i$ are maps $(i = 1, \cdots, n)$ and $f = \prod f_i$, then $f(T(X_1, \cdots, X_n)) \subset T(Y_1, \cdots, Y_n)$; hence f induces a map $\bigwedge f_i = f_1 \wedge \cdots \wedge f_n: \bigwedge X_i \to \bigwedge Y_i$. Clearly, if $f_i \simeq f_i'$ $(i = 1, \cdots, n)$ then $\bigwedge f_i \simeq \bigwedge f_i'$. Thus the n-fold reduced join is a covariant functor which preserves homotopy.

If X and Y are spaces, their *sum* is the space $X \vee Y = T(X, Y)$; if $f: X \to X'$, $g: Y \to Y'$, then the restriction of $f \times g$ to $X \vee Y$ is a map $f \vee g: X \vee Y \to X' \vee Y'$; and $f \simeq f'$, $g \simeq g'$ imply $f \vee g \simeq f' \vee g'$. Thus the sum is also a covariant functor preserving homotopy.

The sum is evidently commutative and associative and the n-fold reduced join symmetric in its arguments (up to natural homeomorphism). However, associative laws for the n-fold reduced join do not always hold.

Let $X_1, \cdots, X_n$ be CW-complexes. Then $X = \prod X_i$ is a closure-finite complex, but may fail to have the weak topology [35]. Let $w(X) = X$, retopologized with the weak topology; i.e., a subset A of X is closed in $w(X)$ if and only if, for every compact set C of X, $A \cap C$ is closed in X. Then $w(X)$ is a CW-complex, and the identity map $1: w(X) \to X$ is continuous.

LEMMA (2.1). *The identity map* $1: (w(X), x_0) \to (X, x_0)$ *is a homotopy equivalence.*

Proof. Let $x_1 = (x_{11}, \cdots, x_{n1})$ be a point of X. Then the map which sends the point $x \in X_k$ into the point $y \in X$ such that $y_i = x_{i1}$ $(i \neq k)$, $y_k = x$, is a continuous map

$$i_k: (X_k, x_{k1}) \to (X, x_1)$$

and also a continuous map

$$i'_k : (X_k, x_{k1}) \to (w(X), x_1).$$

It is well known that if $i>1$, the induced homomorphisms

$$i_{k*}: \pi_i(X_k, x_{k1}) \to \pi_i(X, x_1),$$
$$i'_{k*}: \pi_i(X_k, x_{k1}) \to \pi_i(w(X), x_1)$$

form injective direct sum representations of the groups $\pi_i(X, x_1)$, $\pi_i(w(X), x_1)$; while the obvious modification of the above statement holds for $i=0, 1$. Since $1 \circ i'_k = i_k$, it follows that $1: w(X) \to X$ is a singular homotopy equivalence. We have already observed that $X \in \mathcal{W}_0$; it follows from [18, Lemma 1] that 1 is a homotopy equivalence.

If σ is any subset of $\{1, \cdots, n\}$, let X_σ be the set of all points $x \in X$ such that $x_i = x_{i0}$ for all $i \notin \sigma$. Let X'_σ be the corresponding subspace of $w(X)$. Then X_σ is naturally homeomorphic with $\prod_{i\in\sigma} X_i$, and it is clear that X'_σ is homeomorphic with $w(\prod_{i\in\sigma} X_i)$. Since $X_\sigma \cap X_\tau = X_{\sigma\cap\tau}$, $X'_\sigma \cap X'_\tau = X'_{\sigma\cap\tau}$, it follows from Lemma (2.1) by induction on n that, if T'_i is the subspace corresponding to T_i, then

Corollary (2.2). *The identity map*

$$1: (w(X); T'_1, \cdots, T'_n; x_0) \to (X; T_1, \cdots, T_n; x_0)$$

is a homotopy equivalence.

Let $T'(X_1, \cdots, X_n) = \bigcup_{i=1}^n T'_i$, and let

$$w(X_1 \wedge \cdots \wedge X_n) = w(X)/T'(X_1, \cdots, X_n).$$

Then

Corollary (2.3). *The map*

$$w(X_1 \wedge \cdots \wedge X_n) \to X_1 \wedge \cdots \wedge X_n$$

induced by $1: w(X) \to X$ *is a homotopy equivalence.*

We now show that the n-fold reduced join satisfies associativity laws up to homotopy type. Any desired associative law can be obtained by iteration from (2.4) below.

Let X_i be spaces (not necessarily CW-complexes) for $i=1, \cdots, n$. Let $\{\sigma_1, \cdots, \sigma_r\}$ be a decomposition of $\{1, \cdots, n\}$ into disjoint subsets, and let $Y_k = \bigwedge_{i\in\sigma_k} X_i$. Then there is a natural map

$$p: \bigwedge_{i=1}^{n} X_i \to \bigwedge_{k=1}^{r} Y_k$$

such that, if $x_i \in X_i$ $(i=1, \cdots, n)$, and if $y_k = \bigwedge_{i\in\sigma_k} x_i$, then

$$p(\wedge x_i) = \wedge y_k;$$

p is one-to-one, continuous, and onto, but need not be a homeomorphism. However, we have

(2.4). *The map* $p: \wedge X_i \to \wedge Y_k$ *is a homotopy equivalence.*

Proof. Since the k-fold reduced join functors preserve homotopy, we may assume that the X_i are CW-complexes. The map p induces a map

$$p': w(\wedge X_i) \to w(\wedge w(Y_k))$$

which is one-to-one, continuous, and onto, and is an isomorphism of complexes. Since both spaces are CW-complexes, p' is a homeomorphism. The diagram

$$\begin{array}{ccc} w(\wedge X_i) & \xrightarrow{p'} & w(\wedge w(Y_k)) \\ 1\downarrow & & \downarrow 1 \\ \wedge X_i & \xrightarrow{p} & \wedge Y_k \end{array}$$

is commutative, and the identity map $w(\wedge X_i) \to \wedge X_i$ is a homotopy equivalence, by (2.3). The identity map $w(\wedge w(Y_k)) \to \wedge Y_k$ is the composite

$$w(\wedge w(Y_k)) \to \wedge w(Y_k) \to \wedge Y_k;$$

the first map is a homotopy equivalence, by (2.3); the second is a reduced join of maps which are again homotopy equivalences. Hence p is a homotopy equivalence.

The case $n=3$ of (2.4) was proved by Puppe [23, Satz 18].

We also note

(2.5). *If all but one of the spaces* $\prod_{k\in\sigma_i} X_k$ *is compact then* p *is a homeomorphism.*

This follows easily by several applications of [34, Lemma 4].

Let (X, A) be a pair and suppose that A is closed in X. Let $i: A \subset X$. Then, for any space Y,

(2.6). *The map* $i\wedge 1: A\wedge Y \to X\wedge Y$ *is an imbedding, and its image is a closed subspace of* $X\wedge Y$.

Under these conditions, we may consider $A\wedge Y$ as a closed subspace of $X\wedge Y$.

Suppose further that $p: X\to X/A$ is the identification map. Then

(2.7). *If* X *is compact, then* $p\wedge 1: X\wedge Y \to (X/A)\wedge Y$ *sends* $A\wedge Y$ *into the base-point and induces a homeomorphism of* $X\wedge Y/A\wedge Y$ *with* $(X/A)\wedge Y$.

Let X, Y, Z be spaces, and let $i_1: X\to X\vee Y$, $i_2: Y\to X\vee Y$ be the natural injections. Then $i_1\wedge 1: X\wedge Z\to (X\vee Y)\wedge Z$ and $i_2\wedge 1: Y\wedge Z \to (X\vee Y)\wedge Z$ induce a map

$$h: (X\wedge Z)\vee (Y\wedge Z) \to (X\vee Y)\wedge Z.$$

(2.8). *The spaces* $(X\wedge Z)\vee(Y\wedge Z)$ *and* $(X\vee Y)\wedge Z$ *are homeomorphic under* h.

If Y is a space, X a compact space, it follows from [18, Theorem 3] that the space $F(X, Y)$ of all maps of X into Y, with the compact-open topology, is a space in $\mathcal{W}_0$; the base-point of $F(X, Y)$ is the constant map of X into the base-point of Y. If $f: X' \to X$, $g: Y \to Y'$, the correspondence $h \to g \circ h \circ f$ is a map $F(f, g): F(X, Y) \to F(X', Y')$; and if $f \simeq f'$, $g \simeq g'$ then $F(f, g) \simeq F(f', g')$. Clearly F is a functor: $\mathcal{K}_0 \times \mathcal{W}_0 \to \mathcal{W}_0$, where $\mathcal{K}_0$ is the full subcategory of compact spaces in $\mathcal{W}_0$; F is covariant in its second argument and contravariant in its first.

If $f \in F(X \wedge Y, Z)$, let $\bar{f} \in F(X, F(Y, Z))$ be the map defined by

$$\bar{f}(x)(y) = f(x \wedge y).$$

It follows easily from standard work on the topology of function spaces (cf. [10; 12]) that

(2.9). *The correspondence $f \to \bar{f}$ is a homeomorphism of $F(X \wedge Y, Z)$ with $F(X, F(Y, Z))$.*

If $f \in F(X, Y)$, $x \in X$, let $e_0(f, x) = f(x)$. It follows from (2.9) that e_0 is continuous; since e_0 maps $F(X, Y) \vee X$ into y_0, we have

(2.10). *There is a continuous map $e: F(X, Y) \wedge X \to Y$ such that $e(f \wedge x) = f(x)$ for all $f \in F(X, Y)$, $x \in X$.*

The map e is called the *evaluation map*.

Let I be the free unit interval, and let T be the unit interval with base-point 0. Let $\dot{T} = S^0$ be the subspace $\{0, 1\}$ of T, and let $S = S^1 = T/\dot{T}$. The *cone* over X is the space $TX = T \wedge X$, and the *suspension* of X is the space $SX = S \wedge X$. The *loop-space* of X is the space $\Omega X = F(S, X)$. These operations are evidently covariant functors: $\mathcal{W}_0 \to \mathcal{W}_0$.

It follows from (2.4), (2.5), and commutativity of the reduced join that

(2.11). *The spaces $SX \wedge Y$ and $X \wedge SY$ have the same homotopy type; if X or Y is compact, they are homeomorphic.*

Furthermore, we have

(2.12). *If X is a compact space, then the spaces $F(SX, Y)$, $F(X, \Omega Y)$, and $\Omega F(X, Y)$ are naturally homeomorphic.*

(2.13). *$S^0 \wedge X$ is naturally homeomorphic with X.*

Let $[X, Y]$ be the set of homotopy classes of maps of X into Y; if $f: X \to Y$, let $[f]$ be the homotopy class of f. Then $[\ ,\ ]$ is a functor on $\mathcal{W}_0 \times \mathcal{W}_0$ to the category of sets with base-points. If $f: X' \to X$ and $g: Y \to Y'$, let

$$f^\# = [f, 1]: [X, Y] \to [X', Y],$$
$$g_\# = [1, g]: [X, Y] \to [X, Y'];$$

then

$$[f, g] = f^\# \circ g_\# = g_\# \circ f^\#.$$

(2.14). *If X and Y are compact, then $[X \wedge Y, Z]$ and $[X, F(Y, Z)]$ are in natural one-to-one correspondence.*

If X is compact, the sets $[SX, Y]$ and $[X, \Omega Y]$ have natural group structures, and

(2.15). *The groups* $[SX, Y]$ *and* $[X, \Omega Y]$ *are isomorphic.*

(2.16). *The two natural group structures on* $[SX, \Omega Y]$ *are identical and abelian.*

(2.17). *If* $n \geqq 2$, *the groups* $[S^n X, Y]$ *and* $[X, \Omega^n Y]$ *are abelian.*

(2.18). *If* $f: X' \to X$, $g: Y \to Y'$ *are maps, then*

$$\begin{aligned}(Sf)^\#&: [SX, Y] \to [SX', Y],\\ g_\#&: [SX, Y] \to [SX, Y'],\\ f^\#&: [X, \Omega Y] \to [X', \Omega Y],\\ (\Omega g)_\#&: [X, \Omega Y] \to [X, \Omega Y']\end{aligned}$$

are homomorphisms.

Since the reduced join functor preserves homotopy, so does the suspension functor. Hence the correspondence $f \to Sf$ induces a map $S_*: [X, Y] \to [SX, SY]$.

(2.19). *The map* $S_*: [SX, Y] \to [S^2X, SY]$ *is a homomorphism.*

Similarly the loop functor induces a map $\Omega_*: [X, Y] \to [\Omega X, \Omega Y]$, and

(2.20). *The map* $\Omega_*: [X, \Omega Y] \to [\Omega X, \Omega^2 Y]$ *is a homomorphism.*

3. **Homology and homotopy of the reduced join.** If (X, A) is a pair, G an abelian group, let $H_n(X, A; G)$, $H^n(X, A; G)$ be the singular homology and cohomology groups of (X, A) with coefficients in G. If X is a space with base-point x_0, let $\tilde{H}_q(X; G)$, $\tilde{H}^q(X; G)$ be the singular groups of the pair $(X, \{x_0\})$.

Let (X, A) be a pair, and let $p: X \to X/A$ be the identification map. Then p induces homomorphisms

$$\begin{aligned}p_*&: H_n(X, A; G) \to \tilde{H}_n(X/A; G),\\ p^*&: \tilde{H}^n(X/A; G) \to H^n(X, A; G),\end{aligned}$$

and we have

(3.1). *The above homomorphisms* p_*, p^* *are isomorphisms.*

In fact, let $f: (X, A) \to (K, L)$ be a homotopy equivalence, and let $g: (K, L) \to (X, A)$ be a homotopy inverse of f. Then f, g induce maps $\bar{f}: X/A \to K/L$, $\bar{g}: K/L \to X/A$, and it is easily verified that $\bar{f}$ is a homotopy equivalence with homotopy inverse $\bar{g}$. The diagram

$$\begin{array}{ccc} H_n(X, A; G) & \xrightarrow{f_*} & H_n(K, L; G) \\ \downarrow p_* & & \downarrow p_* \\ \tilde{H}_n(X/A; G) & \xrightarrow{\bar{f}_*} & \tilde{H}_n(K/L; G) \end{array}$$

is commutative. It is well known that $p_*: H_n(K, L; G) \approx \tilde{H}_n(K/L; G)$. Since f_* and $\bar{f}_*$ are isomorphisms, (3.1) follows for homology. The proof for cohomology is similar.

Let (X, A) be a pair and suppose A is a closed subset of X. By (2.6) and (2.13) we may regard X and TA as subspaces of TX. Moreover, $X \cup TA/TA$ is naturally homeomorphic with X/A. Let $p': X \cup TA \to X/A$ be the composition of this homeomorphism with the collapsing map of $X \cup TA$ into $X \cup TA/TA$. Furthermore, $X \cup TA/X$ is naturally homeomorphic with SA. Let $p'': X \cup TA \to SA$ be the composition of this homeomorphism with the collapsing map. Clearly p' is a homotopy equivalence. If $q: X/A \to X \cup TA$ is any homotopy inverse of p', the map $p'' \circ q: X/A \to SA$ will be called a *canonical map*.

Let $C(X)$ be the normalized singular chain-complex of the space X, and let $\tilde{C}(X)$ be the factor complex $C(X)/C(\{x_0\})$. If (X, X_0) is a pair and Y a space, the Eilenberg-Zilber map [9] is a chain map ζ of $C(X) \otimes C(Y)$ into $C(X \times Y)$; and ζ maps $C(X) \otimes C(\{y_0\}) + C(X_0) \otimes C(Y)$ into $C(X \times \{y_0\} \cup X_0 \times Y)$. Assume that X_0 is closed in X. Then the identification map of $X \times Y$ into $X \wedge Y$ sends the pair $(C(X \times Y), C(X \times \{y_0\} \cup X_0 \times Y))$ into $(C(X \wedge Y), C(X_0 \wedge Y))$. The composite map induces in turn a chain map

$$C(X)/C(X_0) \otimes \tilde{C}(Y) \to C(X \wedge Y)/C(X_0 \wedge Y),$$

and hence, for any pairing $A \otimes B \to C$ of coefficient groups, a homomorphism

$$\wedge: H_p(X, X_0; A) \otimes \tilde{H}_q(Y; B) \to H_{p+q}(X \wedge Y, X_0 \wedge Y; C),$$

called the *homology cross-product*. If $u \in H_p(X, X_0; A)$, $v \in \tilde{H}_q(Y; B)$, let $u \wedge v$ be the image of $u \otimes v$ under this map. Suppose $A = Z$, the additive group of integers, $B = C$, and the pairing is the natural one. Let $i \in H_1(T, \dot{T}; Z)$ be the homology class of the identity map of T into itself, regarded as a singular simplex, the vertices of T being taken in the order 0, 1; and let $s \in \tilde{H}_1(S; Z)$ be the image of i under the identification map $T \to S$. Then the $\wedge$-products with i and s are homomorphisms

$$i\wedge: \tilde{H}_q(Y; B) \to H_{q+1}(TY, Y; B),$$
$$\sigma_* = s\wedge: \tilde{H}_q(Y; B) \to \tilde{H}_{q+1}(SY; B).$$

It follows from the Künneth Theorem that $i\wedge$ and σ_* are isomorphisms; σ_* is called the *homology suspension.*

Let $1 \in \tilde{H}_0(S^0; Z)$ be the image of i under the boundary homomorphism of (T, S^0). The name is justified by the fact that, if $u \in \tilde{H}_q(Y; B)$ then

$$1 \wedge u = u \in \tilde{H}_q(S^0 \wedge Y; B) = \tilde{H}_q(Y; B)$$

under the identification $S_0 \wedge Y$ with Y. It follows that

(3.2). *If $u \in \tilde{H}_q(Y; B)$, and if $\partial_*: H_{q+1}(TY, Y; B) \to \tilde{H}_q(Y; B)$ is the boundary operator of the homology sequence of the pair (TY, Y), then*

$$\partial_*(i \wedge u) = u.$$

LEMMA (3.3). *If $h: X/A \to SA$ is a canonical map, the diagram*

$$\begin{array}{ccc} H_{n+1}(X, A) & \xrightarrow{\partial_*} & \tilde{H}_n(A) \\ \downarrow p_* & & \downarrow \sigma_* \\ \tilde{H}_{n+1}(X/A) & \xrightarrow[h_*]{} & \tilde{H}_{n+1}(SA) \end{array}$$

is anti-commutative.

$$\begin{tikzcd}
 & \tilde{H}_n(A) & & H_{n+1}(A', A) \arrow[ll, "\partial_2"'] \arrow[rr, "p_2''"] \arrow[dl, "j_2"'] \arrow[drr, "l_2"] & & \tilde{H}_{n+1}(SA) \\
H_{n+1}(X, A) \arrow[ur, "\partial_*"] \arrow[rr, "j_1"] \arrow[d, "p_*"'] \arrow[drr, "l_1"] & & H_{n+1}(X', A) \arrow[ul, "\partial_1"'] \arrow[rrr, "k_2"] \arrow[d, "k_1"] & & & H_{n+1}(X', X) \arrow[u, "p_1''"'] \\
\tilde{H}_{n+1}(X/A) & & H_{n+1}(X', A') \arrow[ll, "p_2'"] & & \tilde{H}_{n+1}(X') \arrow[ll, "i_1"] \arrow[ull, "i"'] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

FIGURE 1

Proof. Let $X' = X \cup TA$, $A' = TA$. Consider the diagram (Figure 1) in which all the homomorphisms are induced by inclusion maps, except that ∂_*, ∂_1, and ∂_2 are the boundary operators of the homology sequences of the appropriate pairs, p_* and p_2' are induced by the identification map $X \to X/A$, and p_1'' and p_2'' by the identification $A' \to SA$. The upper right and lower left corners are commutative; since A' is contractible, ∂_2 and i_1 are isomorphisms, and the remainder of the diagram satisfies the hypotheses of the "hexagon lemma" [8, I.15.1]. Hence

$$l_2 \circ \partial_2^{-1} \circ \partial_* = - i_2 \circ i_1^{-1} \circ l_1.$$

Since the triad $(X'; A', X)$ is proper, l_1 and l_2 are isomorphisms. Now $p_*' = p_2' \circ i_1$, and therefore $q_* = i_1^{-1} \circ p_2'^{-1}$; also $p_*'' = p_1'' \circ i_2$. Hence

$$\begin{aligned} h_* \circ p_* &= p_1'' \circ i_2 \circ i_1^{-1} \circ p_2'^{-1} \circ p_* = p_1'' \circ i_2 \circ i_1^{-1} \circ l_1 \\ &= - p_1'' \circ l_2 \circ \partial_2^{-1} \circ \partial_* = - p_2'' \circ \partial_2^{-1} \circ \partial_*. \end{aligned}$$

It remains to show that $p_2'' \circ \partial_2^{-1} = \sigma_*$.

Let $u \in \tilde{H}_n(A)$. Then, by (3.2),

$$\partial_2(i \wedge u) = u,$$

and therefore $\partial_2^{-1}(u) = i \wedge u$. Clearly $p_2''(i \wedge u) = s \wedge u$; hence $p_2'' \circ \partial_2^{-1} = \sigma_*$. This completes the proof.

Consider the maps $p: S \wedge X \wedge Y \to S \wedge (X \wedge Y)$, $p': S \wedge X \wedge Y \to (S \wedge X) \wedge Y$ of (2.5); they are homotopy equivalences, and therefore the composition

$$p_*' \circ p_*^{-1}: \tilde{H}_{n+1}(S(X \wedge Y); C) \to \tilde{H}_{n+1}(SX \wedge Y; C)$$

is an isomorphism for any group C. Hence the homomorphism $\sigma_L: \tilde{H}_n(X\wedge Y;C) \to \tilde{H}_{n+1}(SX\wedge Y;\, C)$ defined by

$$\sigma_L = p'_* \circ p_*^{-1} \circ \sigma_*$$

is an isomorphism. Similarly, the map $p'': S\wedge X\wedge Y \to X\wedge(S\wedge Y)$ defined by

$$p''(t \wedge x \wedge y) = x \wedge (t \wedge y)$$

is a homotopy equivalence. Hence the homomorphism $\sigma_R: \tilde{H}_n(X\wedge Y;\, C) \to \tilde{H}_{n+1}(X\wedge SY;\, C)$ defined by

$$\sigma_R = p''_* \circ p_*^{-1} \circ \sigma_*$$

is an isomorphism. Clearly

(3.4). *If* $u\in\tilde{H}_p(X;\, A)$, $v\in\tilde{H}_q(Y;\, B)$, *then*

$$\sigma_L(u \wedge v) = \sigma_* u \wedge v,$$
$$\sigma_R(u \wedge v) = (-1)^p u \wedge \sigma_* v.$$

The cohomology cross-product is a natural pairing

$$\wedge: \tilde{H}^p(X;\, A) \otimes \tilde{H}^q(Y;\, B) \to \tilde{H}^{p+q}(X \wedge Y;\, C),$$

defined in terms of a pairing $A \otimes B \to C$. Let $s^*\in\tilde{H}^1(S;\, Z)$ be the element such that the Kronecker index $\langle s^*, s\rangle = 1$. Then

$$s^*\wedge: \tilde{H}^q(Y;\, B) \to \tilde{H}^{q+1}(SY;\, B)$$

is an isomorphism; let

$$\sigma^*: \tilde{H}^{q+1}(SY;\, B) \to \tilde{H}^q(Y;\, B)$$

be the inverse of $s^*\wedge$.

The homomorphisms

$$\overset{*}{\sigma}_L: \tilde{H}^{n+1}(SX \wedge Y; C) \to \tilde{H}^n(X \wedge Y; C),$$

$$\overset{*}{\sigma}_R: \tilde{H}^{n+1}(X \wedge SY; C) \to \tilde{H}^n(X \wedge Y; C)$$

are defined analogously to σ_L, σ_R, and we have

(3.5). *If* $u\in\tilde{H}^p(SX;\, A)$, $v\in\tilde{H}^q(Y;\, B)$, *then*

$$\overset{*}{\sigma}_L(u \wedge v) = \overset{*}{\sigma} u \wedge v;$$

and if $u\in\tilde{H}^p(X;\, A)$, $v\in\tilde{H}^q(SY;\, B)$, *then*

$$\overset{*}{\sigma}_R(u \wedge v) = (-1)^p u \wedge \overset{*}{\sigma} v.$$

Suppose now that Y is a finite CW-complex. Let Y^p be the p-skeleton of Y if $p\geqq 0$, $Y^p=\{y_0\}$ if $p<0$; and $\tilde{C}_p(Y) = H_p(Y^p,\, Y^{p-1})$. The "filtration" of Y by the subspaces Y^p induces a "filtration" of $X\wedge Y$ by the subspaces

$X \wedge Y^p$; the latter "filtration" in turn gives rise to a bigraded exact couple [17]; cf. also [31]. Let $\{E^r\}$ be the associated spectral sequence. The following facts are immediate consequences of the corresponding facts for the spectral sequence of the fibration $X \times Y \to Y$:

(1) *The $\wedge$-product is an isomorphism*

$$\tilde{H}_q(X) \otimes \tilde{C}_p(Y) \approx E^1_{p,q};$$

(2) *under the above isomorphism, the boundary operator $d^1: E^1_{p,q} \to E^1_{p-1,q}$ corresponds to $(-1)^q(1 \otimes \partial)$, where $\partial: \tilde{C}_p(Y) \to \tilde{C}_{p-1}(Y)$ is the boundary operator of the reduced chain complex of Y;*

(3) $E^2_{p,q} \approx \tilde{H}_p(Y; \tilde{H}_q(X))$;

(4) *$d^r: E^r \to E^r$ is trivial for $r \geqq 2$;*

(5) *there is a filtration*

$$\tilde{H}_n(X \wedge Y) = J_n \supset \cdots \supset J_0 \supset J_{-1} = 0$$

of $\tilde{H}_n(X \wedge Y)$ such that

$$J_p/J_{p-1} \approx E^\infty_{p,n-p} \approx E^2_{p,n-p} \approx \tilde{H}_p(Y; \tilde{H}_{n-p}(X)).$$

Moreover, the spectral sequence from E^2 on is natural, and the filtration of (5) *is natural.*

Suppose moreover that $\tilde{H}_i(X) = 0$ for $0 \leqq i < r$. Then $J_n = J_{n-r}$; hence

(3.6). *There is a natural projection*

$$\pi: \tilde{H}_n(X \wedge Y) \to \tilde{H}_{n-r}(Y; \tilde{H}_r(X)).$$

The naturality of (3.6) will be used in §5.

We now consider the behavior of (3.6) under suspension. The suspension homomorphism $\sigma_*: \tilde{H}_r(X) \to \tilde{H}_{r+1}(SX)$ induces a homomorphism

$$\sigma_{**}: \tilde{H}_{n-r}(Y; \tilde{H}_r(X)) \to \tilde{H}_{n-r}(Y; \tilde{H}_{r+1}(SX)).$$

Consider the diagram

$$\begin{array}{ccc} \tilde{H}_n(X \wedge Y) & \xrightarrow{\pi} & \tilde{H}_{n-r}(Y; \tilde{H}_r(X)) \\ \downarrow \sigma_L & & \downarrow \sigma_{**} \\ \tilde{H}_{n+1}(SX \wedge Y) & \xrightarrow{\pi} & \tilde{H}_{n-r}(Y; \tilde{H}_{r+1}(SX)). \end{array} \tag{3.7}$$

(3.8). *The diagram* (3.7) *is commutative.*

Proof. Let $\langle D, E\rangle$ be the exact couple for $X \wedge Y$, $\langle 'D, 'E\rangle$ that for $SX \wedge Y = S(X \wedge Y)$. Then the homomorphism σ_L maps D into $'D$ and E into $'E$, and it is easily verified that the pair of homomorphisms (ϕ, ψ) defined by

$$\phi = (-1)^{p+q}\sigma_L: D_{p,q} \to 'D_{p,q+1},$$
$$\psi = (-1)^{p+q}\sigma_L: E_{p,q} \to 'E_{p,q+1}$$

is a map of couples of degree (0, 1). (Note that σ_L is not a map of couples; the diagram

$$\begin{array}{ccc} E_{p,q} & \longrightarrow & D_{p,q-1} \\ \downarrow \sigma_L & & \downarrow \sigma_L \\ E_{p,q+1} & \longrightarrow & D_{p,q} \end{array}$$

is anticommutative.) Moreover, the map θ: $\tilde{H}(X\wedge Y)\to\tilde{H}(SX\wedge Y)$ defined by

$$\theta = (-1)^n\sigma_L\colon \tilde{H}_n(X \wedge Y) \to \tilde{H}_{n+1}(SX \wedge Y)$$

sends J_p into $'J_p$, and induces homomorphisms θ_p of the successive quotients such that the diagram

$$\begin{array}{ccc} J_p/J_{p-1} & \to & E^2_{p,n-p} \\ \theta_p\downarrow & & \downarrow\psi \\ 'J_p/'J_{p-1} & \to & 'E^2_{p,n-p+1} \end{array} \tag{3.9}$$

in which the horizontal arrows represent the isomorphisms of (5) above, is commutative.

Consider the diagram

$$\begin{array}{ccc} \tilde{H}_r(X) \otimes \tilde{C}_{n-r}(Y) & \to & E^1_{n-r,r} \\ (-1)^n\sigma_* \otimes 1\downarrow & & \downarrow\psi_1 \\ \tilde{H}_{r+1}(SX) \otimes \tilde{C}_{n-r}(Y) & \to & 'E^1_{n-r,r+1} \end{array} \tag{3.10}$$

in which the horizontal arrows represent the isomorphisms of (1) above, given by the $\wedge$-product. This diagram is clearly commutative; under either route, an element $u\otimes c$ is mapped into $(-1)^n\sigma_*u\wedge c$. Passage to homology in (3.10) gives the commutative diagram

$$\begin{array}{ccc} \tilde{H}_{n-r}(Y; \tilde{H}_r(X)) & \longrightarrow & E^2_{n-r,r} \\ (-1)^n\sigma_{**}\downarrow & & \downarrow\psi_2 \\ \tilde{H}_{n-r}(Y; \tilde{H}_{r+1}(SX)) & \to & 'E^2_{n-r,r+1}. \end{array} \tag{3.11}$$

Commutativity of (3.9) and (3.11) and the definition of θ_p show that (3.7) is commutative.

(3.8) settles the behavior of π under suspension of X. We now consider its behavior under suspension of Y. Consider the diagram

$$\begin{array}{ccc} \tilde{H}_n(X \wedge Y) & \xrightarrow{\pi} & \tilde{H}_{n-r}(Y; \tilde{H}_r(X)) \\ \downarrow\sigma_R & & \downarrow(-1)^r\sigma_* \\ \tilde{H}_{n+1}(X \wedge SY) & \xrightarrow{\pi} & \tilde{H}_{n-r+1}(SY; \tilde{H}_r(X)). \end{array} \tag{3.12}$$

(3.13). *The diagram* (3.12) *is commutative.*

Proof. Let $\langle D, E\rangle$ be the couple for $X\wedge Y$, $\langle 'D, 'E\rangle$ that for $X\wedge SY$. Note that SY^p is the $(p+1)$-skeleton of SY. Define ϕ, ψ, θ, as in the proof of (3.8), to be $(-1)^n\sigma_R$, where n is the total degree. Then (ϕ, ψ) is a map of couples, and the diagram corresponding to (3.9) is commutative. The commutative diagram corresponding to (3.10) is

$$\begin{array}{ccc} \tilde{H}_r(X) \otimes \tilde{C}_{n-r}(Y) & \rightarrow & 'E^1_{n-r,r} \\ (-1)^{n-r}(1 \otimes \sigma_*) \downarrow & & \downarrow \psi_1 \\ \tilde{H}_r(X) \otimes \tilde{C}_{n-r+1}(SY) & \rightarrow & 'E^1_{n-r+1,r}. \end{array} \tag{3.14}$$

The commutativity of (3.12) now follows as before, passing to homology from (3.14).

Let $s_n\in\tilde{H}_n(S^n; Z)$ be defined inductively by

$$s_0 = \sigma_*^{-1}(s),$$
$$s_1 = s,$$
$$s_{n+1} = \sigma_* s_n.$$

The Hurewicz map $\eta\colon \pi_n(X) = [S^n, X]\rightarrow\tilde{H}_n(X)$ is defined by

$$\eta([f]) = f_*(s_n).$$

It is a homomorphism for $n\geqq 1$. Clearly $\eta\colon \pi_n\rightarrow\tilde{H}_n$ is a natural transformation of functors.

(3.15). *The homotopy and homology suspensions correspond under* η; *i.e., the diagram*

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{S_*} & \pi_{n+1}(SX) \\ \downarrow \eta & & \downarrow \eta \\ \tilde{H}_n(X) & \xrightarrow{\sigma_*} & \tilde{H}_{n+1}(SX) \end{array}$$

is commutative.

For if $f\colon S^n\rightarrow X$, then

$$\begin{aligned} \eta(S_*[f]) &= \eta(1 \wedge f) = (1 \wedge f)_*(s_{n+1}) \\ &= (1 \wedge f)_*(s \wedge s_n) = s \wedge f_*s_n \\ &= \sigma_*(f_*s_n) = \sigma_*\eta([f]). \end{aligned}$$

We now consider the homotopy groups.

(3.16). *If X is $(p-1)$-connected and Y is $(q-1)$-connected, then $X\wedge Y$ is $(p+q-1)$-connected. Moreover, if $p+q>1$,*

$$\pi_{p+q}(X \wedge Y) \approx \tilde{H}_p(X) \otimes \tilde{H}_q(Y).$$

Proof. We may assume that X and Y are CW-complexes, and that X^{p-1}

$=\{x_0\}$, $Y^{q-1}=\{y_0\}$. By Corollary (2.3), the natural map of $w(X\wedge Y)$ into $X\wedge Y$ is a homotopy equivalence. Hence it suffices to prove (3.16) for $w(X\wedge Y)$. But $w(X\wedge Y)$ is a CW-complex whose $(p+q-1)$-skeleton is a single point; hence $X\wedge Y$ is $(p+q-1)$-connected. The calculation of $\pi_{p+q}(X\wedge Y)$ follows from the Künneth and Hurewicz theorems.

There is also a $\wedge$-product in homotopy. Let $f: S^p\to X$, $g: S^q\to Y$; then $f\wedge g: S^p\wedge S^q\to X\wedge Y$. It is known (cf. [1]) that the correspondence $(f, g)\to f\wedge g$ induces a homomorphism

$$\wedge: \pi_p(X)\otimes\pi_q(Y)\to\pi_{p+q}(X\wedge Y).$$

We have the homotopy analogue of (3.5); if

$$S_L: \pi_n(X\wedge Y)\to\pi_{n+1}(SX\wedge Y),$$
$$S_R: \pi_n(S\wedge Y)\to\pi_{n+1}(X\wedge SY),$$

are defined by analogy with σ_L, σ_R, then

(3.17). *If* $u\in\pi_p(X)$, $v\in\pi_q(Y)$, *then*

$$S_L(u\wedge v) = S_*u\wedge v,$$
$$S_R(u\wedge v) = (-1)^p u\wedge S_*v.$$

We also have

(3.18). *The homology and homotopy $\wedge$-products correspond under the Hurewicz map, i.e., the diagram*

$$\begin{array}{ccc}
\pi_p(X)\otimes\pi_q(Y) & \xrightarrow{\wedge} & \pi_{p+q}(X\wedge Y) \\
\downarrow \eta\otimes\eta & & \downarrow \eta \\
\tilde{H}_p(X)\otimes\tilde{H}_q(Y) & \xrightarrow{\wedge} & \tilde{H}_{p+q}(X\wedge Y).
\end{array}$$

is commutative.

4. **Spectra**(2). A *spectrum* $\boldsymbol{E}$ is a sequence(3) $\{E_n \mid n\in Z\}$ of spaces together with a sequence of maps

$$\epsilon_n: SE_n\to E_{n+1}.$$

If $\boldsymbol{E}$, $\boldsymbol{E}'$ are spectra, a map $\boldsymbol{f}: \boldsymbol{E}\to\boldsymbol{E}'$ is a sequence of maps

$$f_n: E_n\to E_n'$$

such that the diagrams

$$\begin{array}{ccc}
SE_n & \xrightarrow{\epsilon_n} & E_{n+1} \\
Sf_n\downarrow & & \downarrow f_{n+1} \\
SE_n' & \xrightarrow{\epsilon_n'} & E_{n+1}'
\end{array}$$

(3) By a *sequence* we shall always mean a function on *all* the integers.

are *homotopy-commutative.* Two maps $\boldsymbol{f}$, $\boldsymbol{f}'$ are *homotopic* if and only if, for each $n, f_n \simeq f_n'$. Clearly, the spectra form a category $\mathcal{S}$.

REMARK 1. A spectrum can equally well be described by specifying the spaces E_n and maps $\tilde{\epsilon}_n: E_n \to \Omega E_{n+1}$. We say that $\boldsymbol{E}$ is an Ω-*spectrum* if and only if every map $\tilde{\epsilon}_n: E_n \to \Omega E_{n+1}$ is a homotopy equivalence.

REMARK 2. If E_n, $\epsilon_n: SE_n \to E_{n+1}$ (or $\tilde{\epsilon}_n: E_n \to \Omega E_{n+1}$) are defined for $n \geqq n_0$, we can specify a spectrum by defining E_n inductively for $n < n_0$ by

$$E_n = \Omega E_{n+1},$$
$$\tilde{\epsilon}_n: E_n \subset \Omega E_{n+1}.$$

EXAMPLE 1. Let S^n be the n-sphere, and let $\sigma_n: SS^n \to S^{n+1}$ be the identity map. Then $\boldsymbol{S} = \{S^n, \sigma_n\}$ is a spectrum, called the *spectrum of spheres.*

EXAMPLE 2. Let Π be an abelian group, and let $K(\Pi, n)$ be an Eilenberg-MacLane complex of type (Π, n). Let $i_{n+1} \in \tilde{H}^{n+1}(K(\Pi, n+1); \Pi)$ be the fundamental class. Let $\kappa_n: SK(\Pi, n) \to K(\Pi, n+1)$ be a map such that

$$\kappa_n^*(i_{n+1}) = \sigma^{*-1}(i_n) \in \tilde{H}^{n+1}(SK(\Pi, n); \Pi),$$

Let $\boldsymbol{K}(\Pi)$ be the spectrum so defined; $\boldsymbol{K}(\Pi)$ is called an *Eilenberg-MacLane spectrum.*

EXAMPLE 3. Let U be the infinite unitary group [3]. There is a canonical homotopy equivalence [28]

$$f: U \to \Omega^2 U.$$

Let $E_n = U$ if n is odd, $E_n = \Omega U$ if n is even. If n is odd, let $v_n: SU \to \Omega U$ be the map correspoding to f. If n is even, let $\tilde{v}_n: \Omega U \to \Omega U$ be the identity map. The resulting spectrum $\boldsymbol{U}$ is called the *unitary spectrum.*

EXAMPLE 4. Let $\boldsymbol{E}$ be a spectrum, X a compact space. Let $E_n' = E_n \wedge X$, and define $\epsilon_n': SE_n' \to E_{n+1}'$ to be the map

$$SE_n' = S \wedge E_n' = S \wedge E_n \wedge X \xrightarrow{\epsilon_n \wedge 1} E_{n+1} \wedge X = E_{n+1}'.$$

Let $\boldsymbol{E} \wedge X$ be the spectrum so defined.

EXAMPLE 5. Let $\boldsymbol{E}$ be a spectrum, X a compact space. Let $E_n' = X \wedge E_n$, and define $\epsilon_n': SE_n' \to E_{n+1}'$ to be the map

$$SE_n' = S \wedge X \wedge E_n \to X \wedge S \wedge E_n \xrightarrow{1 \wedge \epsilon_n} X \wedge E_{n+1},$$

where the map $S \wedge X \wedge E_n \to X \wedge S \wedge E_n$ interchanges the first and second factors. Let $X \wedge \boldsymbol{E}$ be the resulting spectrum. In particular, let $\boldsymbol{SE} = S \wedge \boldsymbol{E}$; $\boldsymbol{SE}$ is called the *suspension* of $\boldsymbol{E}$.

Note that the maps $x \wedge e \to e \wedge x$ of $X \wedge E_n$ into $E_n \wedge X$ defines a map of $X \wedge \boldsymbol{E}$ into $\boldsymbol{E} \wedge X$.

EXAMPLE 6. Let $\boldsymbol{E}$ be a spectrum, X a compact space. Let $F_n = F(X; E_n)$. Define $\phi_n: SF_n \to F_{n+1}$ as follows: if $f \in F_n$, $s \in S$, $x \in X$, then

$$\phi_n(s \wedge f)(x) = \epsilon_n(s \wedge f(x));$$

ϕ_n is continuous, since the corresponding map

$$(s \wedge f) \wedge x \to \epsilon_n(s \wedge f(x))$$

is the composition of ϵ_n with the map

$$s \wedge f \wedge x \xrightarrow{1 \wedge e} s \wedge f(x),$$

where e is the evaluation map, and e is continuous by (2.10). Then $\{F_n, \phi_n\}$ is a spectrum $\boldsymbol{F}(X, \boldsymbol{E})$.

In particular, if $\boldsymbol{E}$ is a spectrum, $\Omega\boldsymbol{E} = \boldsymbol{F}(S, \boldsymbol{E})$ is a spectrum, the *loop-spectrum* of $\boldsymbol{E}$.

By (2.12) there is a natural homeomorphism

$$\psi_k: F(SX, E_k) \to \Omega F(X, E_k).$$

A calculation shows that the ψ_k define an isomorphism

$$\boldsymbol{\psi}: \boldsymbol{F}(SX, \boldsymbol{E}) \to \Omega\boldsymbol{F}(X, \boldsymbol{E})$$

of spectra.

The spectrum $\boldsymbol{E}$ is said to be convergent if and only if there is an integer N such that E_{N+i} is i-connected for all $i \geqq 0$. Note that the spectra $\boldsymbol{S}$ and $\boldsymbol{K}(\Pi)$ are convergent, but $\boldsymbol{U}$ is not.

LEMMA (4.1). *Let $\boldsymbol{E}$ be a spectrum, N an integer. Then there exists a spectrum $\boldsymbol{E}'$ and a map $\boldsymbol{f}: \boldsymbol{E}' \to \boldsymbol{E}$ such that*

(1) *$E_i' = E_i$ and $f_i: E_i' \subset E_i$ for all $i \leqq N$;*

(2) *E_{N+i}' is $(i-1)$-connected for all $i \geqq 0$;*

(3) *$f_{i\#}: \pi_j(E_i') \approx \pi_j(E_i)$ for all $i \geqq N+1$, $j \geqq i-N$.*

Proof. If $i \geqq N+1$, let $E_i^{(0)}$ be the path-component of the base-point in E_i. Since SE_N is 0-connected, ϵ_N maps SE_N into $E_{N+1}^{(0)}$. Since $SE_i^{(0)}$ is 0-connected, ϵ_i maps $SE_i^{(0)}$ into $E_{i+1}^{(0)}$ for all $i \geqq N+1$. Then the $E_i^{(0)}$ $(i \geqq N+1)$ and the E_i $(i \leqq N)$, together with the maps ϵ_i or $\epsilon_i | E_i^{(0)}$, form a spectrum $\boldsymbol{E}^{(0)}$, which is mapped into $\boldsymbol{E}$ by the inclusion maps. Clearly it suffices to prove the theorem for $\boldsymbol{E}^{(0)}$; i.e., we may assume that E_i is 0-connected for $i \geqq N+1$.

We now construct E_i' as an $(i-N-1)$-connected fibre space over E_i by the method of [5; 29]. Let $i \geqq N+1$, and let E_i^* be a space containing E_i such that

(1) (E_i^*, E_i) is a relative CW-complex [30];

(2) the inclusion map induces isomorphisms

$$\pi_j(E_i) \approx \pi_j(E_i^*) \qquad (j \leqq i - N - 1);$$

(3) $$\pi_j(E_i^*) = 0 \qquad (j \geqq i - N).$$

It follows that

$$\pi_j(E_i^*, E_i) = 0 \qquad (j \leqq i - N),$$
$$\pi_j(E_i^*, E_i) \approx \pi_{j-1}(E_i) \qquad (j \geqq i - N + 1).$$

Let E_i' be the space of paths in E_i^* which start at the base-point and end in E_i, and let $f_i: E_i' \to E_i$ be the end-point map. Then $\pi_j(E_i') \approx \pi_{j+1}(E_i^*, E_i)$; hence E_i' is $(i-N-1)$-connected. Consider the composition of $\epsilon_i: SE_i \to E_{i+1}$ with the inclusion map $E_{i+1} \to E_{i+1}^*$. Since (SE_i^*, SE_i) is a relative CW-complex, obstruction theory applies [7; 30]; the obstructions to extending the above map lie in the groups

$$H^{j+1}(SE_i^*, SE_i; \pi_j(E_{i+1}^*)) \approx H^j(E_i^*, E_i; \pi_j(E_{i+1}^*)).$$

Since (E_i^*, E_i) is $(i-N)$-connected, the cohomology groups of (E_i^*, E_i) with any coefficients vanish if $j \leqq i-N$. If $j \geqq i-N+1$, then $\pi_j(E_{i+1}^*) = 0$. Hence there exists an extension $\epsilon_i^*: SE_i^* \to E_{i+1}^*$ of the above map, and another obstruction argument shows that the homotopy class of ϵ_i^* rel. SE_i is uniquely determined. Let F_i be the space of paths in SE_i^* which start at the base-point and end in SE_i. Composition with ϵ_i^* is a map $\bar{\epsilon}_i: F_i \to E_{i+1}'$. Define $k_i: SE_i' \to F_i$ by

$$k_i(t \wedge u)(s) = t \wedge u(s) \qquad (u \in E_i', t \in S, s \in I).$$

Finally, let $\epsilon_i' = \bar{\epsilon}_i \circ k_i: SE_i' \to E_{i+1}'$. Clearly the diagrams

$$\begin{array}{ccc} SE_i' & \xrightarrow{\epsilon_i'} & E_{i+1}' \\ Sf_i \downarrow & & \downarrow f_{i+1} \\ SE_i & \xrightarrow[\epsilon_i]{} & E_{i+1} \end{array}$$

are commutative.

To complete the definition of $\boldsymbol{E}'$, it remains to define a map $\epsilon_N': SE_N \to E_{N+1}'$. Let K be a CW-complex, $f: SE_N \to K$ a homotopy equivalence, $g: K \to SE_N$ a homotopy inverse of f. The homotopy groups of E_{N+1}^* vanish in all dimensions; hence the fibre of $f_{N+1}: E_{N+1}'$ has vanishing homotopy groups. Therefore the map $\epsilon_N \circ g: K \to SE_N$ can be lifted to a map $h: K \to E_{N+1}'$ such that $f_{n+1} \circ h = \epsilon_N \circ g$. Let $\epsilon_N' = h \circ f: SE_N' \to E_{N+1}$; then

$$f_{N+1} \circ \epsilon_N' = f_{N+1} \circ h \circ f = \epsilon_N \circ g \circ f \simeq \epsilon_N.$$

REMARK. It follows from the results of [18] that all the spaces constructed in the proof have the homotopy type of CW-complexes.

Let $\boldsymbol{E}$ be a spectrum, n an integer (not necessarily $\geqq 0$). Let $\epsilon_k^\flat: \pi_{n+k}(E_k) \to \pi_{n+k+1}(E_{k+1})$ be the composite

$$\pi_{n+k}(E_k) \xrightarrow{S_*} \pi_{n+k+1}(SE_k) \xrightarrow{\epsilon_{k\#}} \pi_{n+k+1}(E_{k+1}), \tag{4.2}$$

whenever $n \geqq -k$. Then the groups $\pi_{n+k}(E_k)$, together with the homomor-

phisms $\epsilon_k^\flat$, form a direct system. The nth homotopy group $\pi_n(\boldsymbol{E})$ is defined to be the direct limit of this system. If $\boldsymbol{f}: \boldsymbol{E}\to\boldsymbol{E}'$ is a map of spectra then the diagrams

$$\begin{array}{ccccc}\pi_{n+k}(E_k) & \xrightarrow{S_*} & \pi_{n+k+1}(SE_k) & \xrightarrow{\epsilon_{k\flat}} & \pi_{n+k+1}(E_{k+1}) \\ \downarrow f_{k\flat} & & \downarrow (Sf_k)_\flat & , & \downarrow f_{k+1\flat} \\ \pi_{n+k}(E_k') & \xrightarrow{S_*} & \pi_{n+k+1}(SE_k') & \xrightarrow{\epsilon_{k\flat}'} & \pi_{n+k+1}(E_{k+1}')\end{array}$$

are commutative; hence the $f_{k\flat}$ induce a homorphism $\boldsymbol{f}_\flat: \pi_n(\boldsymbol{E})\to\pi_n(\boldsymbol{E}')$.

For example,

(1) $\pi_n(\boldsymbol{S})$ is the stable homotopy group of the n-stem;

(2) $\pi_0(\boldsymbol{K}(\Pi))\approx\Pi$, $\pi_n(\boldsymbol{K}(\Pi))=0$ for $n\neq 0$;

(3) $\pi_n(\boldsymbol{U})=0$ (n odd), $\pi_n(\boldsymbol{U})\approx Z$ (n even).

Consider the spectrum $\boldsymbol{SE}$ of Example 5 above, and the groups $\pi_{n+1}(\boldsymbol{SE})$. The suspension homomorphism $S_*: \pi_{n+k}(E_k)\to\pi_{n+k+1}(SE_k)$ map the groups of the direct system for $\pi_n(\boldsymbol{E})$ into those for $\pi_{n+1}(\boldsymbol{SE})$; however, they do not define a map of the direct systems. For consider the diagram

$$\begin{array}{ccccc}\pi_{n+k}(E_k) & \xrightarrow{S_*} & \pi_{n+k+1}(SE_k) & \xrightarrow{\epsilon_{k\flat}} & \pi_{n+k+1}(E_{k+1}) \\ \downarrow S_* & & & , & \downarrow S_* \\ \pi_{n+k+1}(SE_k) & \xrightarrow{S_*} & \pi_{n+k+2}(S^2E_k) & \xrightarrow{\epsilon_{k\flat}'} & \pi_{n+k+2}(SE_{k+1})\end{array}$$

in which the top row is (4.2) and the bottom is (4.2) for the spectrum $\boldsymbol{SE}$. Let $f: S^{n+k}\to E_k$ represent $\alpha\in\pi_{n+k}(E_k)$. Then the element $\alpha'=S_*\epsilon_{k\flat}S_*(\alpha)$ is represented by the map $f': S^{n+k+2}\to SE_{k+1}$ defined by

$$f'(t_1\wedge t_2\wedge s) = t_1\wedge\epsilon_k(t_2\wedge f(s)) \qquad (s\in S^{n+k},\ t_1,\ t_2\in S),$$

while the element $\alpha''=\epsilon_{k\flat}'S_*S_*(\alpha)$ is represented by f'', where

$$f''(t_1\wedge t_2\wedge s) = t_2\wedge\epsilon_k(t_1\wedge f(s)).$$

The map $t_1\wedge t_2\wedge s\to t_2\wedge t_1\wedge s$ of S^{n+k+2} into itself has degree -1. Hence $\alpha''=-\alpha'$.

It follows that the maps $(-1)^kS_*: \pi_{n+k}(E_k)\to\pi_{n+k+1}(SE_k)$ define a homomorphism of the direct system for $\pi_n(\boldsymbol{E})$ into that for $\pi_{n+1}(\boldsymbol{SE})$ and therefore define a homomorphism

$$\boldsymbol{S}_*: \pi_n(\boldsymbol{E})\to\pi_{n+1}(\boldsymbol{SE}).$$

THEOREM (4.3). *For any spectrum* $\boldsymbol{E}$, $\boldsymbol{S}_*$ *is an isomorphism.*

Proof. Let $\alpha\in\text{Kernel } \boldsymbol{S}_*$, and let $\alpha'\in\pi_{n+k}(E_k)$ be a representative of α. Then, for some $l\geqq 0$, the image of $S_*\alpha'$ in $\pi_{n+k+l+1}(SE_{k+l})$ is zero. Replacing α' by its image in $\pi_{n+k+l}(E_{k+l})$, we may assume that $S_*\alpha'=0$.

Suppose that $\boldsymbol{E}$ is convergent. Choose N so that E_{N+i} is i-connected for

all $i \geqq 0$. We may assume that $k > n+2N+1$. Then E_k is $(k-N)$-connected, and $n+k<2k-2N-1=2(k-N)-1$; hence $S_*: \pi_{n+k}(E_k) \to \pi_{n+k+1}(SE_k)$ is an isomorphism [2] and therefore $\alpha'=0$. Hence $\alpha=0$.

In the general case, by Lemma (4.1), we can choose a spectrum $\boldsymbol{E}'$ and map $\boldsymbol{f}: \boldsymbol{E}' \to \boldsymbol{E}$ such that $E_i' = E_i$, $f_i: E_i' \subset E_i$ for $i \leqq k$ and E_{k+i} is $(i-1)$-connected for $i \geqq 1$. Then α' represents an element $\alpha^* \in \pi_n(\boldsymbol{E}')$ and $\boldsymbol{f}_\#(\alpha^*)=\alpha$. Since $S_*\alpha'=0$, it follows that $\boldsymbol{S}_*(\alpha^*)=0$. Since $\boldsymbol{E}'$ is convergent, $\alpha^*=0$. Hence $\alpha=0$ and therefore $\boldsymbol{S}_*$ is a monomorphism.

The proof that $\boldsymbol{S}_*$ is an epimorphism is similar; it is first proved for convergent spectra, and therefore arbitrary spectra by means of Lemma (4.1).

The correspondence $[S^{n+k}, \Omega E_k] \equiv [S^{n+k+1}, E_k]$ of (2.15) is an isomorphism $\omega_k: \pi_{n+k}(\Omega E_k) \approx \pi_{n+k+1}(E_k)$. Let $\phi_k: S\Omega E_k \to \Omega E_{k+1}$ be the maps of the spectrum $\Omega \boldsymbol{E}$ (see Example 6, above). Then the diagrams

$$\begin{array}{ccc} \pi_{n+k}(\Omega E_k) & \xrightarrow{\phi_k^\flat} & \pi_{n+k+1}(\Omega E_{k+1}) \\ \downarrow \omega_k & & \downarrow \omega_{k+1} \\ \pi_{n+k+1}(E_k) & \xrightarrow{\epsilon_k^\flat} & \pi_{n+k+2}(E_{k+1}) \end{array}$$

are *anti-commutative*. Hence the isomorphisms

$$(-1)^k \omega_k: \pi_{n+k}(\Omega E_k) \to \pi_{n+k+1}(E_{k+1})$$

induce an isomorphism

$$\boldsymbol{\omega}: \pi_n(\Omega \boldsymbol{E}) \to \pi_{n+1}(\boldsymbol{E}).$$

Let $\boldsymbol{E}$ be a spectrum. The suspension in homology is an isomorphism

$$\sigma_*: \tilde{H}_{n+k}(E_k) \to \tilde{H}_{n+k+1}(SE_k);$$

the groups $\tilde{H}_{n+k}(E_k)$ form a direct system under the composite homomorphisms $\epsilon_{k*} \circ \sigma_*: \tilde{H}_{n+k}(E_k) \to \tilde{H}_{n+k+1}(E_{k+1})$. The nth *homology group of* $\boldsymbol{E}$ is the direct limit

$$\tilde{H}_n(\boldsymbol{E}) = \lim_{k} \tilde{H}_{n+k}(E_k)$$

of this direct system. Let $\eta: \pi_{n+k}(E_k) \to \tilde{H}_{n+k}(E_k)$ be the Hurewicz homomorphism. Then the diagram

$$\begin{array}{ccccc} \pi_{n+k}(E_k) & \xrightarrow{S_*} & \pi_{n+k+1}(SE_k) & \xrightarrow{\epsilon_{k\#}} & \pi_{n+k+1}(E_{k+1}) \\ \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\ \tilde{H}_{n+k}(E_k) & \xrightarrow{\sigma_*} & \tilde{H}_{n+k+1}(SE_k) & \xrightarrow{\epsilon_{k*}} & \tilde{H}_{n+k+1}(E_{k+1}) \end{array}$$

is commutative by (3.15) and naturality. Hence η induces a homomorphism

$$\boldsymbol{\eta}: \pi_n(\boldsymbol{E}) \to \tilde{H}_n(\boldsymbol{E})$$

of the direct limits.

D. M. Kan (unpublished) has proved a Hurewicz theorem for spectra in the semi-simplicial setting. Since we do not need the Hurewicz theorem in what follows except in a special case, we content ourselves with the trivial observation that

(4.4). *For any abelian group* Π,

$$\mathfrak{n}: \pi_0(\boldsymbol{K}(\Pi)) \approx \tilde{H}_0(\boldsymbol{K}(\Pi)).$$

Just as in the case of the homotopy suspension, the homomorphisms $\sigma_k: \tilde{H}_{n+k}(E_k) \to \tilde{H}_{n+k+1}(SE_k)$ do not induce a homomorphism of $\tilde{H}_n(\boldsymbol{E})$ into $\tilde{H}_{n+1}(S\boldsymbol{E})$; the diagram

$$\begin{array}{ccccc}
\tilde{H}_{n+k}(E_k) & \xrightarrow{\sigma_*} & \tilde{H}_{n+k+1}(SE_k) & \xrightarrow{\epsilon_{k*}} & \tilde{H}_{n+k+1}(E_{k+1}) \\
\downarrow \sigma_* & & & & \downarrow \sigma_* \\
\tilde{H}_{n+k+1}(SE_k) & \xrightarrow[\sigma_*]{} & \tilde{H}_{n+k+2}(S^2E_k) & \xrightarrow[\epsilon'_{k*}]{} & \tilde{H}_{n+k+2}(SE_{k+1})
\end{array}$$

is anti-commutative (because the map ϵ'_k involves the "twisting map" $t_1 \wedge t_2 \to t_2 \wedge t_1$ of $S^2 = S \wedge S$ into itself, and this map sends $s \wedge s$ into $-s \wedge s$). Hence a homomorphism

$$\mathfrak{d}_*: \tilde{H}_n(\boldsymbol{E}) \to \tilde{H}_{n+1}(S\boldsymbol{E}) \tag{4.5}$$

is defined by the homomorphisms

$$(-1)^k \sigma_*: \tilde{H}_{n+k}(E_k) \to \tilde{H}_{n+k+1}(SE_k).$$

Clearly, in view of the definition of $\mathbf{S}_*$ and (3.15), we have

(4.6). *The homomorphisms* $\mathbf{S}_*$ *and* $\mathfrak{d}_*$ *correspond under the Hurewicz map; i.e., the diagram*

$$\begin{array}{ccc}
\pi_n(\boldsymbol{E}) & \xrightarrow{\mathbf{S}_*} & \pi_{n+1}(S\boldsymbol{E}) \\
\downarrow \mathfrak{n} & & \downarrow \mathfrak{n} \\
\tilde{H}_n(\boldsymbol{E}) & \xrightarrow[\mathfrak{d}_*]{} & \tilde{H}_{n+1}(S\boldsymbol{E})
\end{array}$$

is commutative.

5. **Generalized homology theories.** Let $\mathcal{P}$ be the category whose objects are finite free CW-complexes and whose maps are arbitrary continuous free maps. Let $\mathcal{P}_0$ be the category whose objects are finite CW-complexes with base-vertex and whose maps are all base-point preserving maps. Let $\mathcal{P}^2$ be the category of pairs in $\mathcal{P}$; i.e., the objects of $\mathcal{P}^2$ are pairs (X, A), where X is a free CW-complex and A is a subcomplex of X, and whose maps are all continuous maps of pairs. Let $\mathcal{A}$ be the category of abelian groups and homomorphisms.

The suspension operation is a covariant functor $S: \mathcal{P}_0 \to \mathcal{P}_0$. Let $T: \mathcal{P}^2 \to \mathcal{P}^2$ be the covariant functor defined by

$T(X, A) = (A, \varnothing)$ for any $(X, A) \in \mathcal{P}^2$,

$T(f) = f \mid (A, \varnothing): (A, \varnothing) \to (B, \varnothing)$ for any map $f:(X, A) \to (Y, B)$ in $\mathcal{P}^2$.

A *generalized homology theory* $\mathfrak{H}$ on $\mathcal{P}$ is a sequence of covariant functors

$$H_n: \mathcal{P}^2 \to \mathfrak{A},$$

together with a sequence of natural transformations

$$\partial_n: H_n \to H_{n-1} \circ T$$

satisfying the following conditions:

(1) *If $f_0, f_1 \in \mathcal{P}^2$ are homotopic maps, then $H_n(f_0) = H_n(f_1)$ for all n.*

(2) *If $(X; A, B)$ is a triad in $\mathcal{P}$ such that $X = A \cup B$, and if $k: (A, A \cap B) \subset (X, B)$, then*

$$H_n(k): H_n(A, A \cap B) \approx H_n(X, B)$$

for all n.

(3) *If $(X, A) \in \mathcal{P}^2$, and if $i: (A, \varnothing) \subset (X, \varnothing)$, $j: (X, \varnothing) \subset (X, A)$, then the homology sequence*

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial_{n+1}(X, A)} H_n(A, \varnothing) \xrightarrow{H_n(i)} H_n(X, \varnothing) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\partial_n(X, A)} H_{n-1}(A, \varnothing) \to \cdots$$

of (X, A) is exact.

In other words, a generalized homology theory satisfies the Eilenberg-Steenrod axioms [8], except for the dimension axiom.

A *generalized homology theory* $\tilde{\mathfrak{H}}$ on $\mathcal{P}_0$ is a sequence of covariant functors

$$\tilde{H}_n: \mathcal{P}_0 \to \mathfrak{A},$$

together with a sequence of natural transformations

$$\sigma_n: \tilde{H}_n \to \tilde{H}_{n+1} \circ S$$

satisfying the following conditions:

(1) *If $f_0, f_1 \in \mathcal{P}_0$ are homotopic maps, then $\tilde{H}_n(f_0) = \tilde{H}_n(f_1)$.*

(2) *If $X \in \mathcal{P}_0$, then*

$$\sigma_n(X): \tilde{H}_n(X) \approx \tilde{H}_{n+1}(SX).$$

(3) *If (X, A) is a pair in $\mathcal{P}_0$, $i: A \subset X$, and if $p: X \to X/A$ is the identification map, then the sequence*

$$\tilde{H}_n(A) \xrightarrow{\tilde{H}_n(i)} \tilde{H}_n(X) \xrightarrow{\tilde{H}_n(p)} \tilde{H}_n(X/A)$$

is exact.

Suppose $\mathfrak{H}$ is a generalized homology theory on $\mathcal{P}$. If $(X, x_0)\in\mathcal{P}_0$, then $(X, \{x_0\})\in\mathcal{P}^2$; let

$$\tilde{H}_n(X, x_0) = H_n(X, \{x_0\}).$$

If f: $(X, x_0)\to(Y, y_0)$ is a map in $\mathcal{P}$, we may regard f as a map in $\mathcal{P}^2$; let

$$\tilde{H}_n(f) = H_n(f)\colon H_n(X, \{x_0\}) \to H_n(Y, \{Y_0\}).$$

If $(X, x_0)\in\mathcal{P}_0$, define $\sigma_n(X)\colon \tilde{H}_n(X)\to\tilde{H}_{n+1}(SX)$ to be the composition

$$-H_{n+1}(p) \circ \partial^{-1},$$

where

$$\partial\colon H_{n+1}(TX, X) \to H_n(X, \{x_0\})$$

is the boundary operator of the homology sequence of the triple $(TX, X, \{x_0\})$ (as in ordinary homology theory the homology sequence of a triple in $\mathcal{P}$ is exact, and $H_q(TX, \{x_0\})=0$ for all q; cf. [8, I, 8.1, 10.2, 11.8] and note that the dimension axiom is not used), and where p: $(TX, X)\to(SX, x_0)$ is the identification map.

The standard argument (cf. [6] or [14]) shows that the $\tilde{\mathfrak{H}}=\{\tilde{H}_n, \sigma_n\}$ so defined is a generalized homology theory on $\mathcal{P}_0$, and that the correspondence $\mathfrak{H}\to\tilde{\mathfrak{H}}$ is a one-to-one correspondence between generalized homologies on $\mathcal{P}$ and on $\mathcal{P}_0$. The inverse correspondence can be described as follows. Let $\tilde{\mathfrak{H}}$ be a generalized homology theory on $\mathcal{P}_0$. Then if $(X, A)\in\mathcal{P}^2$, let

$$H_n(X, A) = \tilde{H}_n(X/A).$$

To define $\partial_n(X, A)\colon H_n(X, A)\to H_{n-1}(A, \varnothing)$, let $h\colon X/A\to SA$ be a canonical map as in §3, and let $\partial_n(X, A)$ be minus the composite

$$H_n(X, A) = \tilde{H}_n(X/A) \xrightarrow{\tilde{H}_n(h)} \tilde{H}_n(SA) \xrightarrow{\sigma_{n-1}(A)^{-1}} \tilde{H}_{n-1}(A) = H_{n-1}(A,\varnothing).$$

Let $\boldsymbol{E}=\{E_n, \epsilon_n\}$ be a spectrum. For any $X\in\mathcal{P}_0$, let

$$\tilde{H}_n(X; \boldsymbol{E}) = \pi_n(\boldsymbol{E} \wedge X).$$

If $f\colon X\to Y$ is a map in $\mathcal{P}_0$, then the maps $1\wedge f\colon E_k\wedge X\to E_k\wedge Y$ are the components of a map

$$1 \wedge f\colon \boldsymbol{E} \wedge X \to \boldsymbol{E} \wedge Y;$$

let

$$\tilde{H}_n(f; \boldsymbol{E}) = (1 \wedge f)_\#\colon \pi_n(\boldsymbol{E} \wedge X) \to \pi_n(\boldsymbol{E} \wedge Y).$$

Clearly $\tilde{H}_n(\ ; \boldsymbol{E})$ is a covariant functor.

Let $X\in\mathcal{P}_0$. Then $\sigma_n(X; \boldsymbol{E})\colon \tilde{H}_n(X; \boldsymbol{E})\to\tilde{H}_{n+1}(SX; \boldsymbol{E})$ is defined to be the composition

$\tilde{H}_n(X; \boldsymbol{E})$

$$= \pi_n(\boldsymbol{E} \wedge X) \xrightarrow{S_*} \pi_{n+1}(S \wedge \boldsymbol{E} \wedge X) \xrightarrow{f_\#} \pi_{n+1}(\boldsymbol{E} \wedge S \wedge X) = \tilde{H}_{n+1}(SX; \boldsymbol{E}),$$

where $\boldsymbol{f}: S\wedge\boldsymbol{E}\wedge X \to \boldsymbol{E}\wedge S\wedge X$ is induced by the map $s\wedge e\wedge x \to e\wedge s\wedge x$ of $S\wedge E_k\wedge X$ into $E_k\wedge S\wedge X$ (note that $\boldsymbol{f}$ is an isomorphism of spectra because of (2.5) and the remark after Example 5 of §4). Clearly $\sigma_n(\ ;\boldsymbol{E})$ is a natural transformation.

Note that

(5.1). *$\sigma_n(X; \boldsymbol{E})$ is induced by the homomorphisms*

$$(-1)^k S_R: \pi_{n+k}(E_k \wedge X) \to \pi_{n+k+1}(E_k \wedge SX).$$

Let $\tilde{\mathfrak{H}}(\boldsymbol{E})$ be the pair of sequences $\{\tilde{H}_n(\ ;\boldsymbol{E}), \sigma_n(\ ;\boldsymbol{E})\}$.

THEOREM (5.2). *For any spectrum $\boldsymbol{E}$, $\tilde{\mathfrak{H}}(\boldsymbol{E})$ is a generalized homology theory on $\mathcal{P}_0$.*

Proof. If $f, f': X \to Y$ are homotopic maps, then, for every k, $(1\wedge f)_\# = (1\wedge f')_\#: \pi_{n+k}(E_k\wedge X) \to \pi_{n+k}(E_k\wedge Y)$. Hence $(1\wedge \boldsymbol{f})_\# = (1\wedge \boldsymbol{f}')_\#$, i.e., $\tilde{H}_n(f; \boldsymbol{E}) = \tilde{H}_n(f'; \boldsymbol{E})$.

By Theorem (4.3), $\boldsymbol{S}_*$ is an isomorphism; hence $\sigma_n(X; \boldsymbol{E})$ is an isomorphism.

Let (X, A) be a pair in $\mathcal{P}_0$, let $i: A \subset X$, and let $p: X \to X/A$ be the identification map. For each k, we have the commutative diagram

$$\text{(5.3)}\qquad \begin{array}{ccccc} \pi_{n+k}(E_k \wedge A) & \xrightarrow{(1\wedge i)_\#} & \pi_{n+k}(E_k \wedge X) & \xrightarrow{(1\wedge j)_\#} & \pi_{n+k}(E_k \wedge X, E_k \wedge A) \\ & & & \searrow^{(1\wedge p)_\#} & \downarrow (1\wedge p')_\# \\ & & & & \pi_{n+k}(E_k \wedge (X/A)) \end{array}$$

in which the top line is exact. (We have identified $E_k\wedge X/E_k\wedge A$ with $E_k\wedge(X/A)$ by (2.7).)

Suppose that $\boldsymbol{E}$ is convergent and choose N so that E_{N+i} is i-connected for all $i>0$; we may assume $n+N \geqq 2$. By (3.16), $E_{N+i}\wedge X$ and $E_{N+i}\wedge A$ are i-connected, and therefore $(E_{N+i}\wedge X, E_{N+i}\wedge A)$ is i-connected. Let $A^* = T(E_{N+i}\wedge A)$, $X^* = (E_{N+i}\wedge X)\cup T(E_{N+i}\wedge A)$. By the Blakers-Massey triad theorem [21, Theorem 3.4], the triad

$$(X^*; A^*, E_{N+i} \wedge X)$$

is $(2i+1)$-connected, at least if $i \geqq 2$. Then the inclusion map induces isomorphisms

$$\pi_j(E_{N+i} \wedge X, E_{N+i} \wedge A) \approx \pi_j(X^*, A^*)$$

for $j \leqq 2i$. It follows that

$$\text{(5.4)}\qquad (1 \wedge p')_\#: \pi_j(E_{N+i} \wedge X, E_{N+i} \wedge A) \approx \pi_j(E_{N+i} \wedge (X/A))$$

for $j \leqq 2i$.

Suppose that $k \geqq n+2N$. Then E_k is $(k-N)$-connected and $(k-N) \geqq n+N \geqq 2$; hence $n+k \leqq 2(k-N)$. Thus (5.4) holds with $i=k-N$, $j=n+k$. Thus, in the diagram (5.3),

$$\text{Kernel } (1 \wedge p)_\# = \text{Image } (1 \wedge i)_\#$$

for k sufficiently large. Since the direct limit of exact sequences is exact [8, VIII, 5.4], the sequence

$$(5.5) \qquad \pi_n(\boldsymbol{E} \wedge A) \xrightarrow{(1 \wedge i)_\#} \pi_n(\boldsymbol{E} \wedge X) \xrightarrow{(1 \wedge p)_\#} \pi_n(\boldsymbol{E} \wedge (X/A))$$

is exact, provided that the spectrum $\boldsymbol{E}$ is convergent.

Let $\boldsymbol{E}$ be an arbitrary spectrum, and let $\alpha \in \operatorname{Ker}(1 \wedge p)_\#$. Choose a representative $\alpha' \in \pi_{n+k}(E_k \wedge X)$ of α; increasing k if necessary, we may assume

$$\alpha' \in \operatorname{Ker}(1 \wedge p)_\#: \pi_{n+k}(E_k \wedge X) \to \pi_{n+k}(E_k \wedge (X/A)).$$

By Lemma (4.1), there is a convergent spectrum $\boldsymbol{E}'$ and a map $\boldsymbol{f}: \boldsymbol{E}' \to \boldsymbol{E}$ such that $E_i' = E_i$, $f_i: E_i' \subset E_i$, for $i \leqq k$. Then α' represents an element $\alpha^* \in \pi_n(\boldsymbol{E}' \wedge X)$ such that $\boldsymbol{f}_\#(\alpha^*) = \alpha$; and $(1 \wedge p)_\#(\alpha^*) = 0$. Since $\boldsymbol{E}'$ is convergent, the sequence (5.5) for $\boldsymbol{E}'$ is exact; hence there is an element $\beta \in \pi_n(\boldsymbol{E}' \wedge A)$ such that $(1 \wedge i)_\#(\beta) = \alpha^*$. Then $(1 \wedge i)_\#(\boldsymbol{f}_\# \beta) = \alpha$. Hence, for any $\boldsymbol{E}$, $\operatorname{Ker}(1 \wedge p)_\# \subset \operatorname{Im}(1 \wedge i)_\#$. Since the opposite inclusion is trivial, the exactness of (5.5) is established for any arbitrary spectrum $\boldsymbol{E}$, and the proof of Theorem (5.2) is complete.

Let P be a free space consisting of just one point. If $\mathfrak{H}$ is a generalized homology theory on $\mathcal{P}$, the *coefficient groups* of $\mathfrak{H}$ are the groups $H_n(P)$. If $\tilde{\mathfrak{H}}$ is a generalized homology theory on $\mathcal{P}_0$, the *coefficient groups* of $\tilde{\mathfrak{H}}$ are the groups $\tilde{H}_n(S^0)$. Since P^+ is naturally homeomorphic with S^0, the coefficient groups of corresponding theories are naturally isomorphic.

Let $\boldsymbol{E}$ be a spectrum. Then $\boldsymbol{E} \wedge S^0$ can be identified with $\boldsymbol{E}$. Hence

(5.6). *The coefficient groups of $\tilde{\mathfrak{H}}(\boldsymbol{E})$ (or of $\mathfrak{H}(\boldsymbol{E})$) are the homotopy groups of $\boldsymbol{E}$; specifically $H_n(P; \boldsymbol{E}) \approx \tilde{H}_n(S^0; \boldsymbol{E}) \approx \pi_n(\boldsymbol{E})$.*

COROLLARY (5.7)(4). *If Π is an abelian group, then $\mathfrak{H}(\boldsymbol{K}(\Pi))$ is the (unique) homology theory on $\mathcal{P}$ with coefficients in Π.*

We now make explicit the isomorphism of Corollary (5.7). Let

$$(5.8) \qquad \pi_k: \tilde{H}_{n+k}(K(\Pi, k) \wedge X) \to \tilde{H}_n(X; \tilde{H}_k(K(\Pi, k)))$$

be the projections of (3.6). By (3.8) and naturality, the diagrams

$$\begin{array}{ccccc}
\tilde{H}_{n+k}(K(\Pi, k)) \wedge X) & \xrightarrow{\sigma_L} & \tilde{H}_{n+k+1}(SK(\Pi, k) \wedge X) & \xrightarrow{(\epsilon_k \wedge 1)_*} & \tilde{H}_{n+k+1}(K(\Pi, k+1) \wedge X) \\
\downarrow \pi_k & & \downarrow \pi_{k+1} & & \downarrow \pi_{k+1} \\
\tilde{H}_n(X; \tilde{H}_k(K(\Pi, k))) & \xrightarrow[\sigma_{**}]{} & \tilde{H}_n(X; \tilde{H}_{k+1}(SK(\Pi, k))) & \xrightarrow[\epsilon_{k**}]{} & \tilde{H}_n(X; \tilde{H}_{k+1}(K(\Pi, k+1)))
\end{array}$$

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(4) A weaker version of this result was proved by the author in [31].

are commutative (ϵ_{k**} is the coefficient group homomorphism induced by $\epsilon_{k*}: \tilde{H}_{k+1}(SK(\Pi,\ k)) \to \tilde{H}_{k+1}(K(\Pi,\ k+1))$), and therefore define a homomorphism

$$\pi: \tilde{H}_n(\boldsymbol{K}(\Pi) \wedge X) \to \tilde{H}_n(X; \tilde{H}_0(\boldsymbol{K}(\Pi))) \approx \tilde{H}_n(X; \Pi).$$

of the direct limits. Composition of π with the Hurewicz map

$$\mathfrak{n}: \pi_n(\boldsymbol{K}(\Pi) \wedge X) \to \tilde{H}_n(\boldsymbol{K}(\Pi) \wedge X)$$

is a homomorphism

$$\rho_n(X): \tilde{H}_n(X; \boldsymbol{K}(\Pi)) \to \tilde{H}_n(X; \Pi).$$

Clearly ρ_n is a natural transformation of functors. Because of (3.13) and (5.1), the diagram

$$\begin{array}{ccc} \tilde{H}_n(X; \boldsymbol{K}(\Pi)) & \xrightarrow{\rho_n(X)} & \tilde{H}_n(X; \Pi) \\ \sigma_n(X; \boldsymbol{K}(\Pi)) \downarrow & & \downarrow \sigma_* \\ \tilde{H}_{n+1}(SX; \boldsymbol{K}(\Pi)) & \xrightarrow[\rho_{n+1}(SX)]{} & \tilde{H}_{n+1}(SX; \Pi) \end{array}$$

is commutative. Hence ρ is a map of homology theories.

THEOREM (5.9). *The natural transformation*

$$\rho: \tilde{\mathfrak{H}}(\boldsymbol{K}(\Pi)) \to \mathfrak{H}(\Pi)$$

is an isomorphism.

Proof. By the uniqueness theorem of Eilenberg-Steenrod [8, III, 10.1], suitably modified in accordance with the fact that we are dealing with a homology theory on $\mathcal{P}_0$, it suffices to prove that

$$\rho_0(S^0): \tilde{H}_0(S^0; \boldsymbol{K}(\Pi)) \approx \tilde{H}_0(S^0; \Pi),$$

i.e., that the composite

$$\begin{array}{ccccc} \pi_0(\boldsymbol{K}(\Pi) \wedge S^0) & \xrightarrow{\mathfrak{n}} & \tilde{H}_0(\boldsymbol{K}(\Pi) \wedge S^0) & \xrightarrow{\pi} & \tilde{H}_0(S^0; \Pi) \\ \| & & \| & & \\ \pi_0(\boldsymbol{K}(\Pi)) & \xrightarrow[\mathfrak{n}]{} & \tilde{H}_0(\boldsymbol{K}(\Pi)) & & \end{array}$$

is an isomorphism. But $\mathfrak{n}$ is an isomorphism, by (4.4), and (5.8) is an isomorphism for every k; hence ρ is an isomorphism.

A *generalized cohomology theory* $\mathfrak{H}^*$ *on* $\mathcal{P}$ is a sequence of contravariant functors

$$H^n: \mathcal{P}^2 \to \mathfrak{A}$$

together with a sequence of natural transformations

$$\delta^n: H^{n-1} \circ T \to H^n$$

satisfying axioms analogous to those for a generalized homology theory. Similarly, a generalized cohomology theory $\tilde{\mathfrak{H}}^*$ on $\mathcal{P}_0$ is a sequence of contravariant functors

$$\tilde{H}^n: \mathcal{P}_0 \to \mathcal{A}$$

and a sequence of natural transformations

$$\sigma^n: \tilde{H}^{n+1} \circ S \to \tilde{H}^n$$

satisfying the appropriate axioms. The two kinds of theories are again in one-to-one correspondence $\mathfrak{H}^* \to \tilde{\mathfrak{H}}^*$ via:

$$H^n(X, A) = \tilde{H}^n(X/A), \qquad \tilde{H}^n(X, x_0) = H^n(X, \{x_0\});$$
$$\delta^n(X, A) = -\tilde{H}^n(h) \circ \sigma^n(A)^{-1},$$

where $h: X/A \to SA$ is canonical;

$$\sigma^n(X) = -\delta^{-1} \circ H^{n+1}(p),$$

where $\delta: H^n(X, \{x_0\}) \to H^{n+1}(TX, X)$ is the coboundary operator of the cohomology sequence of $(TX, X, \{x_0\})$, and $p: (TX, X) \to (SX, \{x_0\})$ is the identification map.

Let $\boldsymbol{E}$ be a spectrum, $X \in \mathcal{P}_0$. Let

$$\tilde{H}^n(X; \boldsymbol{E}) = \pi_{-n}(\boldsymbol{F}(X, \boldsymbol{E})).$$

If $f: X \to Y$ is a map, then the maps $F(f, 1): F(Y, E_k) \to F(X, E_k)$ are easily verified to define a map $\boldsymbol{f}': \boldsymbol{F}(Y, \boldsymbol{E}) \to \boldsymbol{F}(X, \boldsymbol{E})$. Let

$$\tilde{H}^n(f) = \boldsymbol{f}_\#: \pi_{-n}(\boldsymbol{F}(Y, \boldsymbol{E})) \to \pi_{-n}(\boldsymbol{F}(X, \boldsymbol{E})).$$

Clearly $\tilde{H}^n$ is a contravariant functor.

Let $X \in \mathcal{P}_0$. Define $\sigma^n(X): \tilde{H}^{n+1}(SX; \boldsymbol{E}) \to \tilde{H}^n(X; \boldsymbol{E})$ to be the composition

$$\pi_{-n-1}(\boldsymbol{F}(SX, \boldsymbol{E})) \xrightarrow{\boldsymbol{\psi}_\#} \pi_{-n-1}(\Omega\boldsymbol{F}(X, \boldsymbol{E})) \xrightarrow{\boldsymbol{\omega}} \pi_{-n}(\boldsymbol{F}(X, \boldsymbol{E}))$$

where $\boldsymbol{\psi}: \boldsymbol{F}(SX, \boldsymbol{E}) \to \Omega\boldsymbol{F}(X, \boldsymbol{E})$ is the isomorphism given in §4, Example 6, and $\boldsymbol{\omega}$ is the isomorphism of §4. Clearly $\boldsymbol{\sigma}^n$ is a natural isomorphism.

We omit the proofs of the following statements, which are analogous to (5.2), (5.6), and (5.7).

(5.10). *For any spectrum* $\boldsymbol{E}$, $\tilde{\mathfrak{H}}^*(\boldsymbol{E}) = \{\tilde{H}^n, \sigma^n\}$ *is a generalized cohomology theory on* $\mathcal{P}_0$.

(5.11). *The coefficient groups of* $\tilde{\mathfrak{H}}^*(\boldsymbol{E})$ (*and of* $\mathfrak{H}^*(\boldsymbol{E})$) *are given by*

$$H^n(P; \boldsymbol{E}) \approx \tilde{H}^n(S^0; \boldsymbol{E}) \approx \pi_{-n}(\boldsymbol{E}).$$

(5.12). *If* Π *is an abelian group, then* $\mathfrak{H}^*(\boldsymbol{K}(\Pi))$ *is the* (*unique*) *cohomology theory on* $\mathcal{P}$ *with coefficients in* Π.

Of course, (5.12) is well known.

We now give an alternative description of the generalized cohomology groups which will be useful in subsequent calculations. We have

$$\tilde{H}^n(X; \boldsymbol{E}) = \pi_{-n}(\boldsymbol{F}(X, \boldsymbol{E})) = \lim_k \pi_{k-n}(F(X, E_k)) = \lim_q \pi_q(F(X, E_{q+n})).$$

Now

$$\pi_q(F(X, E_{q+n})) = [S^q, F(X, E_{q+n})] \approx [X \wedge S^q, E_{q+n}] \approx [S^qX, E_{q+n}],$$

so that

$$\tilde{H}^n(X; \boldsymbol{E}) \approx \lim_q [S^qX, E_{q+n}].$$

Thus an element of $\tilde{H}_n(X; \boldsymbol{E})$ is represented by a map of S^qX into E_{n+q}. One can verify that the homomorphism $[S^qX, E_{q+n}] \to [S^{q+1}X, E_{q+n+1}]$ which corresponds to the homomorphism $\pi_q(F(X, E_{q+n})) \to \pi_{q+1}(F(X, E_{q+n+1}))$ of the direct system for $\pi_{-n}(\boldsymbol{F}(X, \boldsymbol{E}))$ is the composition

$$[S^qX, E_{q+n}] \xrightarrow{S_*} [S^{q+1}X, SE_{q+n}] \xrightarrow{\epsilon_{q+n\#}} [S^{q+1}X, E_{q+n+1}];$$

i.e., if $f: S^qX \to E_{q+n}$ represents $\alpha \in \tilde{H}^n(X; \boldsymbol{E})$, then so does $\epsilon_{q+n} \circ Sf: S^{q+1}X \to E_{q+n+1}$. Moreover, if $g: Y \to X$ is a map, then $\tilde{H}^n(g; \boldsymbol{E})(\alpha)$ is represented by $f \circ S^q g: S^qY \to E_{q+n}$. Finally, if $f: S^q(SX) \to E_{n+q+1}$ represents $\alpha \in \tilde{H}^{n+1}(SX; \boldsymbol{E})$, then f represents $(-1)^{n+1}\sigma^n(X; \boldsymbol{E})(\alpha) \in \tilde{H}^n(X; \boldsymbol{E})$.

As the notation suggests, $\tilde{H}_n$ and $\tilde{H}^n$ are really functors of two variables. For example, let X be a fixed space $\in \mathcal{P}_0$ and let $\boldsymbol{g}: \boldsymbol{E} \to \boldsymbol{F}$ be a map of spectra; then the maps $g_k \wedge 1: E_k \wedge X \to F_k \wedge X$ are the components of a map $\boldsymbol{g} \wedge 1: \boldsymbol{E} \wedge X \to \boldsymbol{F} \wedge X$, and thereby induce a homomorphism

$$\tilde{H}_n(X; \boldsymbol{g}): \tilde{H}_n(X; \boldsymbol{E}) \to \tilde{H}_n(X; \boldsymbol{F}).$$

The reader may verify that

(5.13). *If $f: X \to Y$ is a map in $\mathcal{P}_0$ and $\boldsymbol{g}: \boldsymbol{E} \to \boldsymbol{F}$ is a map of spectra, then the diagram*

$$\begin{array}{ccc}
\tilde{H}_n(X; \boldsymbol{E}) & \xrightarrow{\tilde{H}_n(X; \boldsymbol{g})} & \tilde{H}_n(X; \boldsymbol{F}) \\
\tilde{H}_n(f; \boldsymbol{E}) \downarrow & & \downarrow \tilde{H}_n(f; \boldsymbol{F}) \\
\tilde{H}_n(Y; \boldsymbol{E}) & \xrightarrow{\tilde{H}_n(Y; \boldsymbol{g})} & \tilde{H}_n(Y; \boldsymbol{F})
\end{array}$$

is commutative.

(5.14). *If $g: \boldsymbol{E} \to \boldsymbol{F}$ is a map of spectra, then the diagram*

$$\begin{array}{ccc}
\tilde{H}_n(X; \boldsymbol{E}) & \xrightarrow{\tilde{H}_n(X; \boldsymbol{g})} & \tilde{H}_n(X; \boldsymbol{F}) \\
\sigma_n(X; \boldsymbol{E}) \downarrow & & \downarrow \sigma_n(X; \boldsymbol{F}) \\
\tilde{H}_{n+1}(SX; \boldsymbol{E}) & \xrightarrow{\tilde{H}_{n+1}(SX; \boldsymbol{g})} & \tilde{H}_{n+1}(SX; \boldsymbol{F})
\end{array}$$

is commutative.

Similar results hold for cohomology. But we shall not pursue this subject further.

In what follows, if $\boldsymbol{E}$ is a spectrum, $\mathfrak{H}(\boldsymbol{E})$ and $\mathfrak{H}^*(\boldsymbol{E})$ will be the generalized homology and cohomology theories on $\mathcal{P}$ which correspond to the ones we have defined on $\mathcal{P}_0$. We shall make the customary abbreviations: e.g., f_* for $\tilde{H}_n(f; \boldsymbol{E})$, ∂ for $\partial_n(X, A)$, etc.

Remark. Homology and cohomology theories are really defined on the suspension category $\mathcal{S}\mathcal{P}_0$. We have preferred to work in $\mathcal{P}_0$ (or $\mathcal{P}$) since the axioms are normally formulated there.

(5.15). *If $X \in \mathcal{P}_0$ and X is acyclic (over the integers), then, for any spectrum $\boldsymbol{E}$, $\tilde{H}_q(X; \boldsymbol{E}) = \tilde{H}^q(X; \boldsymbol{E}) = 0$.*

For SX is simply connected and therefore contractible; hence $\tilde{H}_{q+1}(SX; \boldsymbol{E}) = \tilde{H}^{q+1}(SX; \boldsymbol{E}) = 0$ for all q. Since σ_* and σ^* are isomorphisms, the conclusion follows.

Corollary (5.16). *If X, Y, and $f: X \to Y$ are in $\mathcal{P}_0$, and if $f_*: \tilde{H}_q(X; Z) \approx \tilde{H}_q(Y; Z)$ for all q, then, for any spectrum $\boldsymbol{E}$, $f_*: \tilde{H}_q(X; \boldsymbol{E}) \approx \tilde{H}_q(Y; \boldsymbol{E})$.*

6. **Products.** In ordinary homology theory, a pairing $A \otimes B \to C$ of coefficient groups gives rise to various products. If one studies these products, one sees that they can be described in terms of a family of mappings $f_{p,q}: K(A, p) \wedge K(B, q) \to K(C, p+q)$. A study of the relationships between the maps $f_{p,q}$ and the spectral maps suggests a notion of *pairing* of two arbitrary spectra to a third. In this section we define the notion of pairing and study the resulting products.

More specifically, suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are spectra, and that $\mathfrak{f}: (\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{C}$ is a pairing. The pairing $\mathfrak{f}$ gives rise to pairings $\mathfrak{f}_*: \pi_p(\boldsymbol{A}) \otimes \pi_q(\boldsymbol{B}) \to \pi_{p+q}(\boldsymbol{C})$ of their homotopy groups. Now homology and cohomology groups of a space X are homotopy groups of certain spectra associated with X; accordingly, we shall associate with $\mathfrak{f}$ four pairings of such associated spectra. Thus we obtain four types of products; these are "external" products, analogous to the cross- and slant-products in ordinary homology theory. "Internal" products are then constructed by the Lefschetz method [15] (cf. also Steenrod's account of Lefschetz' work in [11]) with the aid of a diagonal map.

Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be spectra, and let $f_{p,q}: A_p \wedge B_q \to C_{p+q}$ be a double sequence of maps. Consider the diagram (Figure 2) in which the maps λ, μ are defined by

$$\lambda(s \wedge (a \wedge b)) = (s \wedge a) \wedge b,$$

$$\mu(s \wedge (a \wedge b)) = a \wedge (s \wedge b).$$

Because of (2.5) we may identify $S(A_p \wedge B_q)$ with $S \wedge A_p \wedge B_q$; it follows that λ and μ are continuous. The maps α' and β' are defined to make the two left-hand triangles commutative; and α_p, β_q, γ_{p+q} are the maps of the spectra

$$\begin{tikzcd}
SA_p \wedge B_q \arrow[r, "\alpha_p \wedge 1"] & A_{p+1} \wedge B_q \arrow[dr, "f_{p+1,q}"] & \\
S(A_p \wedge B_q) \arrow[u, "\lambda"] \arrow[ur, "\alpha'"] \arrow[r, "Sf_{p,q}"] \arrow[d, "\mu"'] \arrow[dr, "\beta'"'] & SC_{p+q} \arrow[r, "\gamma_{p+q}"] & C_{p+q+1} \\
A_p \wedge SB_q \arrow[r, "1 \wedge \beta_q"'] & A_p \wedge B_{q+1} \arrow[ur, "f_{p,q+1}"'] &
\end{tikzcd}$$

FIGURE 2

$\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$. The three maps $f_{p+1,q} \circ \alpha'$, $\gamma_{p+q} \circ Sf_{p,q}$, $f_{p,q+1} \circ \beta'$ represent elements θ', θ, θ'' of the group $[S(A_p \wedge B_q), C_{p+q+1}]$. We say that the $f_{p,q}$ form a *pairing* $\mathfrak{f}$: $(\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{C}$ of $\boldsymbol{A}$ with $\boldsymbol{B}$ to $\boldsymbol{C}$, if and only if, for each (p, q),

(6.2) $$\theta' = \theta = (-1)^p \theta''.$$

EXAMPLE 1. Let A^i $(i = 1, 2, 3)$ be abelian groups, and let $\boldsymbol{A}^i = \boldsymbol{K}(A^i)$. Let h: $A^1 \otimes A^2 \to A^3$ be a pairing. Let $i_p^k \in H^p(A_p^k; A^k)$ be the fundamental classes. Let $i_p^1 \wedge i_q^2 \in H^{p+q}(A_p^1 \wedge A_q^2; A^3)$ be the $\wedge$-product of i_p^1 and i_q^2 defined by the pairing h. Let $f_{p,q}$: $A_p^1 \wedge A_q^2 \to A_{p+q}^3$ be the map such that $f_{p,q}^*(i_{p+q}^3) = i_p^1 \wedge i_q^2$. Then the maps $f_{p,q}$ define a pairing $\mathfrak{f}$: $(\boldsymbol{A}^1, \boldsymbol{A}^2) \to \boldsymbol{A}^3$.

EXAMPLE 2. Let $\boldsymbol{E}$ be a spectrum, and let $\boldsymbol{S}$ be the spectrum of spheres. Let $f_{p,q}$: $S^p \wedge E_q \to E_{p+q}$ be the composite

$$S^p E_q \xrightarrow{S^{p-1}\epsilon_q} S^{p-1} E_{q+1} \xrightarrow{S^{p-2}\epsilon_{q+1}} \cdots \to SE_{p+q-1} \xrightarrow{\epsilon_{p+q-1}} E_{p+q}.$$

Again the $f_{p,q}$ are the components of a pairing $\mathfrak{f}$: $(\boldsymbol{S}, \boldsymbol{E}) \to \boldsymbol{E}$; we call $\mathfrak{f}$ the *natural pairing.*

Let $f'_{p,q}$: $E_p \wedge S^q \to E_{p+q}$ be a map representing $(-1)^{pq}$ times the element of the group $[E_p \wedge S^q, E_{p+q}]$ represented by the map

$$E_p \wedge S^q \to S^q \wedge E_p \xrightarrow{f_{q,p}} E_{p+q},$$

where $\mathfrak{f}$: $(\boldsymbol{S}, \boldsymbol{E}) \to \boldsymbol{E}$ is the natural pairing (if $q = 0$, the set $[E_p \wedge S^q, E_{p+q}]$ may fail to have a group structure, but $(-1)^{pq} = 1$; thus the homotopy class of $f'_{p,q}$ is uniquely defined for all p, q). Then the $f'_{p,q}$ define a pairing $\mathfrak{f}'$: $(\boldsymbol{E}, \boldsymbol{S}) \to \boldsymbol{E}$, which is also called the *natural pairing.* (Note that, if $\boldsymbol{E} = \boldsymbol{S}$, the two natural pairings coincide, up to homotopy.)

Let $\mathfrak{f}$: $(\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{C}$ be a pairing. Define

$$f'_{k,l}: \pi_{p+k}(A_k) \otimes \pi_{q+l}(B_l) \to \pi_{p+q+k+l}(C_{k+l})$$

by

$$f'_{k,l}(u \otimes v) = f_{k,l\#}(u \wedge v),$$

where $u \wedge v \in \pi_{p+q+k+l}(A_k \wedge B_l)$ is the homotopy $\wedge$-product of §3. Then

$$\begin{aligned} f'_{k,l+1}(u \otimes \overset{\flat}{\beta_1} v) &= f_{k,l+1\#}(u \wedge \beta_{l\#} S_* v) \\ &= (f_{k,l+1} \circ (1 \wedge \beta_l))_\#(u \wedge S_* v) \\ &= (-1)^{p+k}(f_{k,l+1} \circ \beta')(S_L(u \wedge v)) \text{ by (3.17)} \\ &= (-1)^p(\gamma_{k+l} \circ Sf_{k,l})_\#(S_L(u \wedge v)) \text{ by (6.2)} \\ &= (-1)^p \overset{\flat}{\gamma}_{k+l}(f_{k,l\#}(u \wedge v)) \\ &= (-1)^p \overset{\flat}{\gamma}_{k+l}(f'_{k,l}(u \otimes v)). \end{aligned}$$

Similarly,

$$f'_{k+1,l}(\overset{\flat}{\alpha_k} u \otimes v) = \overset{\flat}{\gamma}_{k+1}(f'_{k,l}(u \otimes v)).$$

Therefore the homomorphisms

$$(-1)^{pl} f'_{k,l}\colon \pi_{p+k}(A_k) \otimes \pi_{q+l}(B_l) \to \pi_{p+q+k+l}(C_{k+l})$$

commute with the homomorphisms of the direct systems for $\pi_p(\boldsymbol{A})$, $\pi_q(\boldsymbol{B})$, $\pi_{p+q}(\boldsymbol{C})$ and therefore define a pairing

$$\mathfrak{f}_*\colon \pi_p(\boldsymbol{A}) \otimes \pi_q(\boldsymbol{B}) \to \pi_{p+q}(\boldsymbol{C}).$$

For example, let $\mathfrak{f}\colon (\boldsymbol{S}, \boldsymbol{A}) \to \boldsymbol{A}$, $\mathfrak{f}'\colon (\boldsymbol{A}, \boldsymbol{S}) \to \boldsymbol{A}$ be the natural pairings, and let $i \in \pi_0(\boldsymbol{S})$ be the element represented by the identity map $S^0 \to S^0$. Then it is clear that

(6.3). *If* $v \in \pi_q(\boldsymbol{A})$, *then* $\mathfrak{f}_*(i \otimes v) = v = \mathfrak{f}'_*(v \otimes i)$.

Let X, $Y \in \mathcal{P}_0$, and let $\mathfrak{f}\colon (\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{C}$ be a pairing of spectra. By (2.9), (2.5), and (2.10), the operation of forming the $\wedge$-product of maps is a map

$$F(X, A_p) \wedge F(Y, B_q) \to F(X \wedge Y, A_p \wedge B_q);$$

composing this map with

$$f_{p,q\#}\colon F(X \wedge Y, A_p \wedge B_q) \to F(X \wedge Y, C_{p+q}),$$

we obtain a map

$$f^1_{p,q}\colon F(X, A_p) \wedge F(Y, B_q) \to F(X \wedge Y, C_{p+q}).$$

Lemma (6.4). *The maps* $f^1_{p,q}$ *define a pairing*

$$\mathfrak{f}^1\colon (\boldsymbol{F}(X, \boldsymbol{A}), \boldsymbol{F}(Y, \boldsymbol{B})) \to \boldsymbol{F}(X \wedge Y, \boldsymbol{C}).$$

Proof. We must prove that the diagram analogous to (6.1) satisfies the relations analogous to (6.2); i.e., we must compare the three routes from $S(F(X, A_p) \wedge F(Y, B_q))$ into $F(X \wedge Y, C_{p+q+1})$ suggested by (6.1). Let $s \in S$,

$f \in F(X, A_p)$, $g \in F(Y, B_q)$. Then the upper route sends $s \wedge (f \wedge g)$ into the map $h_1: X \wedge Y \to C_{p+q+1}$ defined by

$$h_1(x \wedge y) = f_{p+1,q}(\alpha_p(s \wedge f(x)) \wedge g(y));$$

the middle route sends it to the map h defined by

$$h(x \wedge y) = \gamma_{p+q}(s \wedge (f(x) \wedge g(y)));$$

the bottom to the map h_2 given by

$$h_2(x \wedge y) = f_{p,q+1}(f(x) \wedge \beta_q(s \wedge g(y))).$$

Let $j: S \to S$ be a map of degree $(-1)^p$, and let $H_2: S(A_p \wedge B_q) \wedge I \to C_{p+q+1}$ be a map (which exists because of the second of the relations (6.2)) such that

$$H_2(u \wedge 0) = f_{p+1,q}(\beta'(u)),$$
$$H_2((s \wedge (a \wedge b)) \wedge 1) = \gamma_{p+q}(j(s) \wedge f_{p,q}(a \wedge b)).$$

Define

$$\tilde{H}_2: S(F(X, A_p) \wedge F(Y, B_q)) \wedge I \to F(X \wedge Y, C_{p+q+1})$$

as follows; if $s \in S$, $f \in F(X, A_p)$, $g \in F(Y, B_q)$, $t \in I$, $x \in X$, $y \in Y$, then

$$(6.5) \qquad \tilde{H}_2((s \wedge (f \wedge g)) \wedge t)(x \wedge y) = H_2((s \wedge (f(x) \wedge f(y))) \wedge t).$$

Evidently $\tilde{H}_2$, if continuous, is the desired homotopy. By (2.9), $\tilde{H}_2$ is continuous if the right side of (6.5) depends continuously on the point $((s \wedge (f \wedge g)) \wedge t) \wedge (x \wedge y)$. Since X, Y, I, S are compact Hausdorff spaces, it follows from three applications of Lemma 4 of [34] that

$$((S \wedge (F(X, A_p) \wedge F(Y, B_q))) \wedge I) \wedge (X \wedge Y)$$

is an identification space of

$$S \times F(X, A_p) \times F(Y, B_q) \times I \times X \times Y.$$

Hence, it suffices to prove that the right side of (6.5) is a continuous function of the six variables (s, f, g, t, x, y). By (2.10), the evaluation maps $(f, x) \to f(x)$ and $(g, y) \to g(y)$ are continuous. Since H_2 is continuous, the result follows.

The proof of the first relation of (6.2) is similar.

We now define three further sets of maps

$$\overset{2}{f}_{p,q}: (A_p \wedge X \wedge Y) \wedge F(X, B_q) \to C_{p+q} \wedge Y,$$
$$\overset{3}{f}_{p,q}: (A_p \wedge X) \wedge (B_q \wedge Y) \to C_{p+q} \wedge X \wedge Y,$$
$$\overset{4}{f}_{p,q}: F(X \wedge Y, A_p) \wedge (B_q \wedge Y) \to F(X, C_{p+q})$$

as follows.

First, let $F_q = F(X, B_q)$, and consider the map

$$A_p \wedge X \wedge Y \wedge F_q \to A_p \wedge F_q \wedge X \wedge Y \to A_p \wedge (F_q \wedge X) \wedge Y$$
$$\xrightarrow{1 \wedge e \wedge 1} A_p \wedge B_q \wedge Y \to (A_p \wedge B_q) \wedge Y \xrightarrow{f_{p,q} \wedge 1} C_{p+q} \wedge Y$$

in which the first map permutes cyclically the last three factors, the second and fourth are the natural maps, and $e\colon F_q \wedge X \to B_q$ is the evaluation map. Let $h\colon (A_p \wedge X \wedge Y) \wedge F_q \to A_p \wedge X \wedge Y \wedge F_q$ be a homotopy inverse of the natural identification. The composition of the above map with h is the map $f^2_{p,q}$.

Next, consider the map

$$A_p \wedge X \wedge B_q \wedge Y \to A_p \wedge B_q \wedge X \wedge Y \to (A_p \wedge B_q) \wedge X \wedge Y$$
$$\xrightarrow{f_{p,q} \wedge 1} C_{p+q} \wedge X \wedge Y$$

where the first map interchanges the second and third factors and the second is the natural map. Let $h'\colon (A_p \wedge X) \wedge (B_q \wedge Y) \to A_p \wedge X \wedge B_q \wedge Y$ be a homotopy inverse of the natural identification. Then $f^3_{p,q}$ is the composition of the above map with h'.

Finally, consider the map

$$F(X \wedge Y, A_p) \wedge B_q \wedge Y \to (F(X \wedge Y, A_p) \wedge B_q) \wedge Y$$
$$\to F(X \wedge Y, A_p \wedge B_q) \wedge Y \to F(Y, F(X, A_p \wedge B_q)) \wedge Y$$
$$\xrightarrow{e} F(X, A_p \wedge B_q) \xrightarrow{F(1, f_{p,q})} F(X, C_{p+q}),$$

where the first map is the natural identification, the second is induced by the map $f \wedge b \to g$, where $g(x) = f(x) \wedge b$, the third is induced by the homeomorphism of (2.9), composed with the map induced by the "twist" $X \wedge Y \to Y \wedge X$, and e is the evaluation map. As before, let $h''\colon F(X \wedge Y, A_p) \wedge (B_q \wedge Y) \to F(X \wedge Y, A_p) \wedge B_q \wedge Y$ be a homotopy inverse of the natural map. Then $f^4_{p,q}$ is the composition of h'' and the above map.

LEMMA (6.6). *The maps $f^i_{p,q}$ $(i = 2, 3, 4)$ define pairings*

$$\mathfrak{f}^2\colon (\boldsymbol{A} \wedge X \wedge Y, \boldsymbol{F}(X, \boldsymbol{B})) \to \boldsymbol{C} \wedge Y,$$
$$\mathfrak{f}^3\colon (\boldsymbol{A} \wedge X, \boldsymbol{B} \wedge Y) \to \boldsymbol{C} \wedge X \wedge Y,$$
$$\mathfrak{f}^4\colon (\boldsymbol{F}(X \wedge Y, \boldsymbol{A}), \boldsymbol{B} \wedge Y) \to \boldsymbol{F}(X, \boldsymbol{C}).$$

The pairings $\mathfrak{f}^i$ give rise, in turn, to pairings

$$\wedge\colon \tilde{H}^p(X; \boldsymbol{A}) \otimes \tilde{H}^q(Y; \boldsymbol{B}) \to \tilde{H}^{p+q}(X \wedge Y; \boldsymbol{C}),$$
$$\backslash\colon \tilde{H}_n(X \wedge Y; \boldsymbol{A}) \otimes \tilde{H}^q(X; \boldsymbol{B}) \to \tilde{H}_{n-q}(Y; \boldsymbol{C}),$$
$$\wedge\colon \tilde{H}_p(X; \boldsymbol{A}) \otimes \tilde{H}_q(Y; \boldsymbol{B}) \to \tilde{H}_{p+q}(X \wedge Y; \boldsymbol{C}),$$
$$/\colon \tilde{H}^n(X \wedge Y; \boldsymbol{A}) \otimes \tilde{H}_q(Y; \boldsymbol{B}) \to \tilde{H}^{n-q}(X; \boldsymbol{C}),$$

which are analogous to the four standard external products in the usual

homology theory. Specifically, if $u \in \tilde{H}^p(X; \boldsymbol{A}) = \pi_{-p}(\boldsymbol{F}(X, \boldsymbol{A}))$, $v \in \tilde{H}^q(Y; \boldsymbol{B}) = \pi_{-q}(\boldsymbol{F}(Y, \boldsymbol{B}))$, then

$$u \wedge v = \overset{1}{\mathfrak{f}}_*(u \otimes v) \in \pi_{-p-q}(\boldsymbol{F}(X \wedge Y, \boldsymbol{C})) = \tilde{H}^{p+q}(X \wedge Y; \boldsymbol{C}).$$

If $z \in \tilde{H}_n(X \wedge Y; \boldsymbol{A}) = \pi_n(\boldsymbol{A} \wedge X \wedge Y)$, $w \in \tilde{H}^q(X; \boldsymbol{B}) = \pi_{-q}(\boldsymbol{F}(X; \boldsymbol{B}))$, then

$$z \backslash w = \overset{2}{\mathfrak{f}}_*(z \otimes w) \in \pi_{n-q}(\boldsymbol{C} \wedge Y) = \tilde{H}_{n-q}(Y; \boldsymbol{C}).$$

If $z \in \tilde{H}_p(X; \boldsymbol{A}) = \pi_p(\boldsymbol{A} \wedge X)$, $w \in \tilde{H}_q(Y; \boldsymbol{B}) = \pi_q(\boldsymbol{B} \wedge Y)$, then

$$z \wedge w = \overset{3}{\mathfrak{f}}_*(z \otimes w) \in \pi_{p+q}(\boldsymbol{C} \wedge X \wedge Y) = \tilde{H}_{p+q}(X \wedge Y; \boldsymbol{C}).$$

Finally, if $u \in \tilde{H}^n(X \wedge Y; \boldsymbol{A}) = \pi_{-n}(\boldsymbol{F}(X \wedge Y, \boldsymbol{A}))$, $z \in \tilde{H}_q(Y; \boldsymbol{B}) = \pi_q(\boldsymbol{B} \wedge Y)$, then

$$u/z = \overset{4}{\mathfrak{f}}_*(u \otimes z) \in \pi_{-n+q}(\boldsymbol{F}(X, \boldsymbol{C})) = \tilde{H}^{n-q}(X; \boldsymbol{C}).$$

We now describe these products in terms of representative maps. First, let $g: S^kX \to A_{p+k}$, $h: S^lY \to B_{q+l}$ represent $u \in \tilde{H}^p(X; \boldsymbol{A})$, $v \in \tilde{H}^q(Y; \boldsymbol{B})$. Then it follows from the discussion of §5 and the relevant definitions that $u \wedge v$ is $(-1)^{p(q+l)}$ times the element represented by the map

$$\begin{aligned} S^{k+l}(X \wedge Y) = S^k \wedge S^l \wedge X \wedge Y \to S^k \wedge X \wedge S^l \wedge Y \\ \xrightarrow{g \wedge h} A_{p+k} \wedge B_{q+l} \xrightarrow{f_{p+k,q+l}} C_{p+q+k+l}, \end{aligned} \tag{6.7}$$

where the first map interchanges the second and third factors.

Let $g: S^{n+k} \to A_k \wedge X \wedge Y$, $h: S^lX \to B_{q+l}$ represent $z \in \tilde{H}_n(X \wedge Y; \boldsymbol{A})$, $w \in \tilde{H}^q(X; \boldsymbol{B})$. Then $z \backslash w$ is $(-1)^{n(q+l)}$ times the element represented by the map

$$\begin{aligned} S^{n+k+l} = S^{n+k} \wedge S^l \xrightarrow{g \wedge 1} A_k \wedge X \wedge Y \wedge S^l \to A_k \wedge S^l \wedge X \wedge Y \\ \xrightarrow{1 \wedge h \wedge 1} A_k \wedge B_{q+l} \wedge Y \xrightarrow{f_{k,q+l} \wedge 1} C_{k+q+l} \wedge Y, \end{aligned} \tag{6.8}$$

where the second map permutes cyclically the last three factors.

Let $g: S^{p+k} \to A_k \wedge X$, $h: S^{q+l} \to B_l \wedge Y$ represent $z \in \tilde{H}_p(X; \boldsymbol{A})$, $w \in \tilde{H}_q(Y; \boldsymbol{B})$. Then $z \wedge w$ is $(-1)^{pl}$ times the element represented by the map

$$\begin{aligned} S^{p+q+k+l} = S^{p+k} \wedge S^{q+l} \xrightarrow{g \wedge h} A_k \wedge X \wedge B_l \wedge Y \to A_k \wedge B_l \wedge X \wedge Y \\ \xrightarrow{f_{k,l} \wedge 1} C_{k+l} \wedge X \wedge Y, \end{aligned} \tag{6.7a}$$

where the second map interchanges the second and third factors.

Finally, let $g: S^k(X \wedge Y) \to A_{n+k}$ and $h: S^{q+l} \to B_l \wedge Y$ represent $u \in \tilde{H}^n(X \wedge Y; \boldsymbol{A})$, $z \in \tilde{H}_q(Y; \boldsymbol{B})$; then u/z is $(-1)^{nl}$ times the element represented by the map

$$\begin{aligned}(6.8\text{a})\quad S^{k+q+l} \wedge X = S^k \wedge S^{q+l} \wedge X &\xrightarrow{1 \wedge h \wedge 1} S^k \wedge B_l \wedge Y \wedge X \\ &\to S^k \wedge X \wedge Y \wedge B_l \xrightarrow{g \wedge 1} A_{n+k} \wedge B_l \xrightarrow{f_{n+k,l}} C_{n+k+l},\end{aligned}$$

where the second map interchanges the second and fourth factors.

It is clear that:

(6.9). *If* $\phi: X \to X'$, $\psi: Y \to Y'$, $u \in \tilde{H}^p(X'; \boldsymbol{A})$, $v \in \tilde{H}^q(Y'; \boldsymbol{B})$, *then*

$$(\phi \wedge \psi)^*(u \wedge v) = \phi^* u \wedge \psi^* v \in \tilde{H}^{p+q}(X \wedge Y; \boldsymbol{C}).$$

(6.10). *If* $\phi: X' \to X$, $\psi: Y \to Y'$, $z \in \tilde{H}_n(X' \wedge Y; \boldsymbol{A})$, $w \in \tilde{H}^q(X; \boldsymbol{B})$, *then*

$$((\phi \wedge \psi)_* z) \backslash w = \psi_*(z \backslash \phi^* w) \in \tilde{H}_{n-q}(Y'; \boldsymbol{C}).$$

(6.9a). *If* $\phi: X \to X'$, $\psi: Y \to Y'$, $z \in \tilde{H}_p(X; \boldsymbol{A})$, $w \in \tilde{H}_q(Y; \boldsymbol{B})$, *then*

$$(\phi \wedge \psi)_*(z \wedge w) = \phi_* z \wedge \psi_* w \in \tilde{H}_{p+q}(X \wedge Y; \boldsymbol{C}).$$

(6.10a). *If* $\phi: X' \to X$, $\psi: Y \to Y'$, $u \in \tilde{H}^n(X \wedge Y'; \boldsymbol{A})$, $z \in \tilde{H}_q(Y; \boldsymbol{B})$, *then*

$$((\phi \wedge \psi)^* u)/z = \phi^*(u/\psi_* z) \in \tilde{H}^{n-q}(X'; \boldsymbol{C}).$$

We next consider the behavior of the above products under suspension. Let

$$\begin{aligned}\overset{*}{\sigma}_L &: \tilde{H}^n(SX \wedge Y; \boldsymbol{E}) \to \tilde{H}^{n-1}(X \wedge Y; \boldsymbol{E}),\\ \overset{*}{\sigma}_R &: \tilde{H}^n(X \wedge SY; \boldsymbol{E}) \to \tilde{H}^{n-1}(X \wedge Y; \boldsymbol{E}),\\ \sigma_L &: \tilde{H}_n(X \wedge Y; \boldsymbol{E}) \to \tilde{H}_{n+1}(SX \wedge Y; \boldsymbol{E}),\\ \sigma_R &: \tilde{H}_n(X \wedge Y; \boldsymbol{E}) \to \tilde{H}_{n+1}(X \wedge SY; \boldsymbol{E})\end{aligned}$$

be the homomorphisms induced by the suspension operations in cohomology and homology, defined in the same way as the corresponding operations of §3.

(6.11). *If* $u \in \tilde{H}^p(SX; \boldsymbol{A})$, $v \in \tilde{H}^q(Y; \boldsymbol{B})$, *then*

$$\overset{*}{\sigma} u \wedge v = \overset{*}{\sigma}_L(u \wedge v) \in \tilde{H}^{p+q-1}(X \wedge Y; \boldsymbol{C}).$$

Proof. Let $g: S^k(SX) \to A_{p+k}$, $h: S^l Y \to B_{q+l}$ represent u, v. Then $g: S^{k+1}X \to A_{p+k}$ represents $(-1)^p \sigma^* u$. By (6.7), a representative of $(-1)^{p(q+l)} u \wedge v$ is the composite of two maps; the map $S^{k+l+1} \wedge X \wedge Y$ into $S^k \wedge SX \wedge S^l \wedge Y$ which sends the point $s \wedge t \wedge s' \wedge t' \wedge x \wedge y \in S^k \wedge S \wedge S^{l-1} \wedge S \wedge X \wedge Y$ into $(s \wedge t' \wedge x) \wedge (t \wedge s' \wedge y)$, followed by the map $f_{p+k,q+l} \circ (g \wedge h)$. From the description of σ^* given in §5, the same map represents $(-1)^{p+q+p(q+l)} \sigma_L^*(u \wedge v)$. On the other hand, a representative of $(-1)^p(-1)^{(p-1)(q+l)} \sigma^* u \wedge v$ is again the composite of two maps; the first is the map of $S^{k+l+1} \wedge X \wedge Y$ into $S^{k+1}X \wedge S^l Y$ which sends $s \wedge t \wedge s' \wedge t' \wedge x \wedge y$ into $(s \wedge t \wedge x) \wedge (s' \wedge t' \wedge y)$, and the second is $f_{p+k,q+l} \circ (g \wedge h)$. The map

$$s \wedge t \wedge s' \wedge t' \to s \wedge t' \wedge t \wedge s'$$

of S^{k+l+1} into itself has degree $(-1)^l$. Hence $\sigma^* u \wedge v$ and $\sigma_L^*(u \wedge v)$ differ only in the sign

$$(-1)^{l+p+(p-1)(q+l)+p+q+p(q+l)} = 1.$$

By similar arguments, we have

(6.12). *If* $u \in \tilde{H}^p(X; \boldsymbol{A})$, $v \in \tilde{H}^q(SY; \boldsymbol{B})$, *then*

$$u \wedge \overset{*}{\sigma} v = (-1)^p \overset{*}{\sigma}_R(u \wedge v) \in \tilde{H}^{p+q-1}(X \wedge Y; \boldsymbol{C}).$$

(6.13). *If* $z \in \tilde{H}_n(X \wedge Y; \boldsymbol{A})$, $w \in \tilde{H}^q(SX; \boldsymbol{B})$, *then*

$$z \backslash \sigma^* w = (-1)^n \sigma_L z \backslash w \in \tilde{H}_{n-q+1}(Y; \boldsymbol{C}).$$

(6.14). *If* $z \in \tilde{H}_n(X \wedge Y; \boldsymbol{A})$, $w \in \tilde{H}^q(X; \boldsymbol{B})$, *then*

$$\sigma_*(z \backslash w) = \sigma_R z \backslash w \in \tilde{H}_{n-q+1}(SY; \boldsymbol{C}).$$

(6.11a). *If* $z \in \tilde{H}_p(X; \boldsymbol{A})$, $w \in \tilde{H}_q(Y; \boldsymbol{B})$, *then*

$$\sigma_* z \wedge w = \sigma_L(z \wedge w) \in \tilde{H}_{p+q+1}(SX \wedge Y; \boldsymbol{C}).$$

(6.12a). *If* $z \in \tilde{H}_p(X; \boldsymbol{A})$, $w \in \tilde{H}_q(Y; \boldsymbol{B})$, *then*

$$z \wedge \sigma_* w = (-1)^p \sigma_R(z \wedge w) \in \tilde{H}_{p+q+1}(X \wedge SY; \boldsymbol{C}).$$

(6.13a). *If* $u \in \tilde{H}^n(X \wedge SY; \boldsymbol{A})$, $z \in \tilde{H}_q(Y; \boldsymbol{B})$, *then*

$$u/\sigma_* z = (-1)^n \overset{*}{\sigma}_R u/z \in \tilde{H}^{n-q-1}(X; \boldsymbol{C}).$$

(6.14a). *If* $u \in \tilde{H}^n(SX \wedge Y; \boldsymbol{A})$, $z \in \tilde{H}_q(Y; \boldsymbol{B})$, *then*

$$\overset{*}{\sigma}(u/z) = \overset{*}{\sigma}_L u/z \in \tilde{H}^{n-q-1}(X; \boldsymbol{C}).$$

As a special case of the \-product, we have the Kronecker index, defined as follows. The \-product defines a pairing

$$\tilde{H}_n(X \wedge S^0; \boldsymbol{A}) \otimes \tilde{H}^q(X; \boldsymbol{B}) \to \tilde{H}_{n-q}(S^0; \boldsymbol{C}) = \pi_{n-q}(\boldsymbol{C}).$$

Identifying X with $X \wedge S^0$ as in (2.13), we obtain a pairing

$$\tilde{H}_n(X; \boldsymbol{A}) \otimes \tilde{H}^q(X; \boldsymbol{B}) \to \pi_{n-q}(\boldsymbol{C}).$$

If $u \in \tilde{H}_n(X; \boldsymbol{A})$, $u' \in \tilde{H}^q(X; \boldsymbol{B})$, let $\langle u, u' \rangle$ be the image of $u \otimes u'$ under the above map. Evidently:

(6.10b). *If* $\phi: X' \to X$, $u \in \tilde{H}_n(X'; \boldsymbol{A})$, $u' \in \tilde{H}^q(X; \boldsymbol{B})$, *then*

$$\langle \phi_* u, u' \rangle = \langle u, \phi^* u' \rangle \in \pi_{n-q}(\boldsymbol{C}).$$

(6.13b). *If* $u \in \tilde{H}_n(X; \boldsymbol{A})$, $u' \in \tilde{H}^q(SX; \boldsymbol{B})$, *then*

$$\langle u, \sigma^* u' \rangle = (-1)^n \langle \sigma_* u, u' \rangle \in \pi_{n-q+1}(\boldsymbol{C}).$$

Suppose, in particular, that $X = S^0$; then $\tilde{H}_n(X; \boldsymbol{A}) = \pi_n(\boldsymbol{A})$, $\tilde{H}^q(X; \boldsymbol{B}) = \pi_{-q}(\boldsymbol{B})$, and it is clear that, with these identifications, $\langle u, u' \rangle = \mathfrak{f}_*(u \otimes u')$.

We now define the "internal" products. It will be more convenient to phrase the definitions in terms of the "non-reduced" theory. Let $(X; X_1, X_2)$ be a triad in $\mathcal{P}$. Then the diagonal map $\Delta: X \to X \wedge X$, followed by the $\wedge$-product of the natural projections $\pi_1: X \to X/X_1$, $\pi_2: X \to X/X_2$, sends $X_1 \cup X_2$ into the base-point, and thereby induces the map

$$\bar{\Delta}: X/X_1 \cup X_2 \to X/X_1 \wedge X/X_2.$$

If $\mathfrak{f}: (\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{C}$ is a pairing of spectra, $u \in H^p(X, X_1; \boldsymbol{A})$, $v \in H^q(X, X_2; \boldsymbol{B})$, let

$$u \smile v = \bar{\Delta}^*(u \wedge v) \in H^{p+q}(X, X_1 \cup X_2; \boldsymbol{C}).$$

Similarly, if $z \in H_n(X, X_1 \cup X_2; \boldsymbol{A})$, $w \in H^q(X, X_1; \boldsymbol{B})$, let

$$z \frown w = \bar{\Delta}_* z \backslash w \in H_{n-q}(X, X_2; \boldsymbol{C}).$$

Let $\phi: (X; X_1, X_2) \to (Y; Y_1, Y_2)$ be a map of triads. Then ϕ induces maps $\phi_i: X/X_i \to Y/Y_i$ and $\phi_{12}: X/X_1 \cup X_2 \to Y/Y_1 \cup Y_2$. Moreover, the diagram

$$\begin{array}{ccc} X/X_1 \cup X_2 & \xrightarrow{\bar{\Delta}} & X/X_1 \wedge X/X_2 \\ \downarrow \phi_{12} & & \downarrow \phi_1 \wedge \phi_2 \\ Y/Y_1 \cup Y_2 & \xrightarrow[\bar{\Delta}]{} & Y/Y_1 \wedge Y/Y_2 \end{array}$$

is commutative. Hence it follows from (6.9) and (6.10) that

(6.15). *If $u \in H^p(Y, Y_1; \boldsymbol{A})$ and $v \in H^q(Y, Y_2; \boldsymbol{B})$, then*

$$\phi^*(u \smile v) = \phi^* u \smile \phi^* v \in H^{p+q}(X, X_1 \cup X_2; \boldsymbol{C}).$$

(6.16). *If $z \in H_n(X, X_1 \cup X_2; \boldsymbol{A})$ and $v \in H^q(Y, Y_1; \boldsymbol{B})$, then*

$$\phi_* z \frown v = \phi_*(z \frown \phi^* v) \in H_{n-q}(Y, Y_2; \boldsymbol{C}).$$

We now apply (6.11)–(6.14) to obtain formulas for the behavior of the cup- and cap-products under boundary and coboundary operators. We first prove a lemma. Let (X, A) be a pair in $\mathcal{P}$, and let $\lambda: X/A \to X \cup TA$ be a homotopy inverse of the natural map $p_1: X \cup TA \to X/A$. Let $p_2: X \cup TA \to SA$ be the natural map. Let $h = p_2 \circ \lambda: X/A \to SA$, so that h is a canonical map. Let $i: A \subset X$, and let $\bar{\Delta}: X/A \to X/A \wedge X$, $\Delta: A \to A \wedge A$ be the diagonal maps. Let $\phi_1, \phi_2: X/A \to SA \wedge X$ be the maps defined by

$$(6.17) \qquad \begin{aligned} \phi_1 &= (h \wedge 1) \circ \bar{\Delta}, \\ \phi_2 &= (1 \wedge i) \circ S\Delta \circ h \end{aligned}$$

(we are identifying $SA \wedge X$ and $S(A \wedge X)$ under the natural equivalence). Then

LEMMA (6.18). *The maps ϕ_1 and ϕ_2 are homotopic.*

Proof. Since p_1 is a homotopy equivalence, it suffices to prove that $\phi_1 \circ p_1$ and $\phi_2 \circ p_1$ are homotopic. Define $d: X \cup TA \to (X \cup TA) \wedge X$ by

$$d(x) = x \wedge x,$$
$$d(t \wedge a) = t \wedge a \wedge i(a).$$

The diagram

$$\begin{tikzcd}
X \cup TA \arrow[r, "d"] \arrow[dd, "p_1"'] & (X \cup TA) \wedge X \arrow[dd, "p_1 \wedge 1"'] \arrow[dr, "p_2 \wedge 1"] & \\
 & & SA \wedge X \\
X/A \arrow[r, "\bar{\Delta}"'] & X/A \wedge X \arrow[ur, "h \wedge 1"'] &
\end{tikzcd}$$

is homotopy-commutative; in fact, the square is strictly commutative, and

$$(h \wedge 1) \circ (p_1 \wedge 1) = (h \circ p_1) \wedge 1 = (p_2 \circ \lambda \circ p_1) \wedge 1 \simeq p_2 \wedge 1$$

since λ is a homotopy inverse of p_1. Thus we have shown that $\phi_1 \circ p_1$ is homotopic to $(p_2 \wedge 1) \circ d$.

On the other hand,

$$\phi_2 \circ p_1 = (1 \wedge i) \circ S\Delta \circ h \circ p_1 \simeq (1 \wedge i) \circ S\Delta \circ p_2$$

and one verifies immediately that the latter map is equal to $(p_2 \wedge 1) \circ d$.

Let $\tau: S(A \wedge A) \to A \wedge SA$ be the map which sends $s \wedge (a \wedge a')$ into $a \wedge (s \wedge a')$, and let $\bar{\Delta}': X/A \to X \wedge X/A$ be the diagonal map.

Lemma (6.18a). *The maps $\phi_1, \phi_2: X/A \to X \wedge SA$ defined by*

$$\phi_1 = (1 \wedge h) \circ \bar{\Delta}',$$
$$\phi_2 = (i \wedge 1) \circ \tau \circ (S\Delta) \circ h$$

are homotopic.

(6.19). *Let (X, X_1) be a pair in $\mathcal{P}$, $i: X_1 \subset X$, and let $u \in H^{p-1}(X_1; \mathbf{A})$, $v \in H^q(X; \mathbf{B})$. Then*

$$\delta(u \smile i^* v) = \delta u \smile v \in H^{p+q}(X, X_1; \mathbf{C}).$$

Proof. Consider the diagram (Figure 3; for brevity we have omitted the names of the coefficient spectra). The diagram is commutative, by (6.9), (6.11), naturality of σ^*, and finally (6.18), which is used to prove commutativity of the lower right hand corner (we have again taken the liberty of identifying $SX_1 \wedge X_1$ with $S(X_1 \wedge X_1)$). The image of $u \otimes v$ by the route which goes along the top and down the right-hand side is $-\delta(u \smile i^* v)$; its image under

$$\begin{tikzcd}
H^{p-1}(X_1)\otimes H^q(X) \arrow[r,"1\otimes i^*"] & H^{p-1}(X_1)\otimes H^q(X_1) \arrow[r,"\wedge"] & \tilde{H}^{p+q-1}(X_1\wedge X_1) \arrow[r,"\Delta^*"] & H^{p+q-1}(X_1) \\
\tilde{H}^p(SX_1)\otimes H^q(X) \arrow[u,"\sigma^*\otimes 1"] \arrow[r,"1\otimes i^*"] \arrow[dr,"\wedge"'] \arrow[dd,"h^*\otimes 1"'] & \tilde{H}^p(SX_1)\otimes H^q(X_1) \arrow[u,"\sigma^*\otimes 1"'] \arrow[r,"\wedge"] & \tilde{H}^{p+q}(SX_1\wedge X_1) \arrow[u,"\sigma^*=\sigma_L^*"'] \arrow[r,"(S\Delta)^*"] & \tilde{H}^{p+q}(SX_1) \arrow[u,"\sigma^*"'] \arrow[dd,"h^*"] \\
 & \tilde{H}^{p+q}(SX_1\wedge X) \arrow[ur,"(1\wedge i)^*"'] \arrow[d,"(h\wedge 1)^*"] & & \\
\tilde{H}^p(X/X_1)\otimes H^q(X) \arrow[r,"\wedge"] & \tilde{H}^{p+q}(X/X_1\wedge X) \arrow[rr,"\bar{\Delta}^*"] & & \tilde{H}^{p+q}(X/X_1)
\end{tikzcd}$$

FIGURE 3

the route which goes down the left-hand side and then along the bottom is $-\delta u \smile v$. By commutativity, they are equal.

Similarly, using (6.18a) instead of (6.18), we have

(6.20). *Let* (X, X_2) *be a pair in* $\mathcal{P}$, $i: X_2 \subset X$, *and let* $u \in H^p(X; \boldsymbol{A})$, $v \in H^{q-1}(X_2; \boldsymbol{B})$. *Then*

$$\delta(i^*u \smile v) = (-1)^p u \smile \delta v \in H^{p+q}(X, X_2; \boldsymbol{C}).$$

The analogous results for the cap-product are proved similarly.

(6.21). *Let* (X, X_1) *be a pair in* $\mathcal{P}$, $i: X_1 \subset X$, *and let* $z \in H_n(X, X_1; \boldsymbol{A})$, $w \in H^q(X_1; \boldsymbol{B})$. *Then*

$$i_*(\partial z \frown w) = (-1)^n z \frown \delta w \in H_{n-q-1}(X; \boldsymbol{C}).$$

(6.22). *Let* (X, X_2) *be a pair in* $\mathcal{P}$, $i: X_2 \subset X$, *and let* $z \in H_n(X, X_2; \boldsymbol{A})$, $w \in H^q(X; \boldsymbol{B})$. *Then*

$$\partial(z \frown w) = \partial z \frown i^*w \in H_{n-q-1}(X_2; \boldsymbol{C}).$$

Steenrod [27] has given systems of axioms which characterize cup- and cap-products in ordinary homology theory; full details were to have appeared in vol. 2 of [8]. Suppose $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are Eilenberg-MacLane spectra, and the pairing $(\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{C}$ is that induced, as in Example 1, above, by a pairing of the coefficient groups $A \otimes B \to C$. Then (6.19), (6.20), and (6.15) become (3.2), (3.3) and (3.4) of [27]. For the cap-product, (6.21), (6.22), and (6.16) become (3.7), (3.8), and (3.9) of [27], except for sign; however, if we change the definition of the cap-product given above by inserting the sign $(-1)^{nq+q}$, our signs agree with Steenrod's. Hence

THEOREM (6.23). *If* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ *are Eilenberg-MacLane spectra and* $\mathfrak{f}: (\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{C}$ *is induced by a pairing* $A \otimes B \to C$ *of their coefficient groups, then, under the isomorphisms of* (5.7) *and* (5.12), *the cup-product defined above reduces to the usual cup-product, while the cap-product defined above reduces to* $(-1)^{nq+q}$ *times the usual cap-product.*

It should be noted that the definition of the ordinary cap-product is not absolutely standard; for example, Whitney's original definition [36] differs from ours by the sign $(-1)^k$, where k is the binomial coefficient $C_{q+1,2}$.

We now define a second kind of cup- and cap-product. These products will be used in §7.

Let K be a finite simplicial complex, and let K' be the first barycentric subdivision of K. Let L be a subcomplex of K. The *supplement* [2] L^* *of* L *in* K is the subcomplex of K' consisting of all simplexes, none of whose vertices are in L'. There is a natural imbedding of K into the join of L and L^*; the point $x \in K$ is mapped into the point $(1-t)y+tz$ of the join if and only if x lies on the line-segment yz and divides it in the ratio $t: 1-t$; we shall identify K with its image in the join. Let

$$N(L) = \{(1-t)y + tz \in K \mid t \leqq 1/2\},$$
$$N(L^*) = \{(1-t)y + tz \mid t \geqq 1/2\};$$

then $N(L)$ is a closed neighborhood of L and $N(L^*)$ is a closed neighborhood of L^*, $K = N(L) \cup N(L^*)$. Moreover,

L is a deformation retract of $N(L)$,
L^* is a deformation retract of $N(L^*)$,
$N(L^*)$ is a deformation retract of $K - L$.

Let M be a subcomplex of L. Then $N(M) \subset N(L)$, $M^* \supset L^*$, and $N(M^*) \supset N(L^*)$. Hence K is the union of the three sets $N(M)$, $N(L^*)$, and $N(L) \cap N(M^*)$. Moreover, $L \cap M^*$ is the supplement of M in L. Hence we can define a map $\Delta': K \to N(L)/N(M) \wedge N(M^*)/N(L^*)$, called the *reduced diagonal map*, by

$$\Delta'(x) = \begin{cases} \pi(x) \wedge \pi'(x) & (x \in N(L) \cap N(M^*)), \\ * & (x \in N(M) \cup N(L^*)), \end{cases}$$

where $\pi: N(L) \to N(L)/N(M)$, $\pi': N(M^*) \to N(M^*)/N(L^*)$ are the natural maps, and $*$ is the base-point. Combining Δ' with the $\wedge$- and slant-product pairings, we define cup- and cap-product pairings

$$H^p(N(L), N(M); \boldsymbol{A}) \otimes H^q(N(M^*), N(L^*); \boldsymbol{B}) \to H^{p+q}(K; \boldsymbol{C}),$$
$$H_n(K; \boldsymbol{A}) \otimes H^q(N(L), N(M); \boldsymbol{B}) \to H_{n-q}(N(M^*), N(L^*); \boldsymbol{C})$$

for any pairing $\mathfrak{f}: (\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{C}$ of spectra. Specifically, if $u \in H^p(N(L), N(M); \boldsymbol{A})$, $v \in H^q(N(M^*), N(L^*); \boldsymbol{B})$, $z \in H_n(K; \boldsymbol{A})$, $w \in H^q(N(L), N(M); \boldsymbol{B})$, let

$$u \cup v = \Delta'^*(u \wedge v),$$
$$z \cap w = \Delta'_* z \backslash w.$$

Since the inclusions $(L, M) \subset (N(L), N(M))$ and $(M^*, L^*) \subset (N(M^*), N(L^*))$

are homotopy equivalences, we may regard these products as pairings

$$
\begin{aligned}
&H^p(L, M; A) \otimes H^q(M^*, L^*; B) \to H^{p+q}(K; C), \\
&H_n(K; A) \otimes H^q(L, M; B) \to H_{n-q}(M^*, L^*; C).
\end{aligned} \tag{6.24}
$$

Note that, if $K = L$ and $M = \emptyset$, these products agree with the earlier ones.

We now prove three lemmas which will establish properties of the cap-product useful in §7. The first two are easily verified; we prove only the third in detail.

LEMMA (6.25). *Let $L_1 \supset L_2 \supset L_3$ be subcomplexes of K. Let*

$$
\begin{aligned}
i_1 &: N(L_2)/N(L_3) \to N(L_1)/N(L_3), \\
i_1^* &: N(L_3^*)/N(L_1^*) \to N(L_3^*)/N(L_2^*)
\end{aligned}
$$

be the maps induced by the appropriate inclusions. Then the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\quad\Delta'\quad} & N(L_2)/N(L_3) \wedge N(L_3^*)/N(L_2^*) \\
\Delta' \downarrow & & \downarrow i_1 \wedge 1 \\
N(L_1)/N(L_3) \wedge N(L_3^*)/N(L_1^*) & \xrightarrow[1 \wedge i_1']{} & N(L_1)/N(L_3) \wedge N(L_3^*)/N(L_2^*)
\end{array}
$$

is commutative.

LEMMA (6.26). *Let $L_1 \supset L_2 \supset L_3$ be subcomplexes of K, and let*

$$
\begin{aligned}
i_2 &: N(L_1)/N(L_3) \to N(L_1)/N(L_2), \\
i_2' &: N(L_2^*)/N(L_1^*) \to N(L_3^*)/N(L_1^*)
\end{aligned}
$$

be the maps induced by the appropriate inclusions. Then the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\quad\Delta'\quad} & N(L_1)/N(L_3) \wedge N(L_3^*)/N(L_1^*) \\
\Delta' \downarrow & & \downarrow i_2 \wedge 1 \\
N(L_1)/N(L_2) \wedge N(L_2^*)/N(L_1^*) & \xrightarrow[1 \wedge i_2']{} & N(L_1)/N(L_2) \wedge N(L_3^*)/N(L_1^*)
\end{array}
$$

is commutative.

LEMMA (6.27). *Let $L_1 \supset L_2$ be subcomplexes of K. Let*

$$
\begin{aligned}
k &: N(L_1)/N(L_2) \to SN(L_2), \\
k' &: K/N(L_2^*) \to SN(L_2^*)
\end{aligned}
$$

be canonical maps, and let $\pi_2' : N(L_2^*) \to N(L_2^*)/N(L_1^*)$ be the identification map. Then the diagram (Figure 4) in which τ interchanges the first and second factors, is homotopy-commutative.

$$\begin{tikzcd}
K \arrow[r, "\Delta'"] \arrow[dd, "\Delta'"'] & N(L_2) \wedge K/N(L_2^*) \arrow[r, "1 \wedge k'"] & N(L_2) \wedge SN(L_2^*) \arrow[d, "1 \wedge S\pi_2'"] \\
 & & N(L_2) \wedge S \wedge N(L_2^*)/N(L_1^*) \arrow[d, "\tau"] \\
N(L_1)/N(L_2) \wedge N(L_2^*)/N(L_1^*) \arrow[rr, "k \wedge 1"'] & & S \wedge N(L_2) \wedge N(L_2^*)/N(L_1^*)
\end{tikzcd}$$

FIGURE 4

Proof. It suffices to prove that this is so for particular canonical maps, which we now construct. We first describe a homotopy

$$Q: K \times I \rightarrow K \cup T(N(L_2))$$

of the inclusion map which shrinks $N(L_2)$ to the vertex of its cone. Let $x=(1-t)y+tz$ with $y\in L_2$, $z\in L_2^*$; if $0<t<1$, let $\bar{x}=1/2y+1/2z$. Then

$$Q(x,s) = \begin{cases} (1-s)\wedge x & (0 \leqq t \leqq 1/2), \\ (4t-1-s)\wedge \bar{x} & (1/2 \leqq t \leqq (s+2)/4), \\ (1-t+s/4)y+(t-s/4)z & ((s+2)/4 \leqq t \leqq 3/4), \\ (1+s-t-st)y+(st+t-s)z & (3/4 \leqq t \leqq 1). \end{cases}$$

Note that the points of L_2^* are stationary. Let Q_1 be the end-value of the homotopy, so that $Q_1: (K,\ N(L_2)) \rightarrow (K\cup T(N(L_2)),*)$. The induced map $Q_1': K/N(L_2)\rightarrow K\cup T(N(L_2))$ is a homotopy inverse of $p_1: K\cup T(N(L_2)) \rightarrow K/N(L_2)$, and therefore the composition of Q_1' with the projection $p_2: K\cup T(N(L_2))\rightarrow SN(L_2)$ is a canonical map k_0. Explicitly, if $\pi: K\rightarrow K/N(L_2)$ and p: $T\rightarrow S$ are the identifications, then

$$k_0(\pi(x)) = \begin{cases} * & \text{if } t \leqq 1/2, \text{ or } t \geqq 3/4, \\ p(4t-2)\wedge x & \text{if } 1/2 \leqq t \leqq 3/4. \end{cases}$$

Note that, since $L_1 \supset L_2$, the above deformation leaves $N(L_1)$ in $N(L_1) \cup TN(L_2)$. Hence $k=k_0 | N(L_1)/N(L_2)$ is also canonical.

A similar construction, with the roles of L_2 and L_2^* interchanged, shows that a canonical map $k': K/N(L_2^*)\rightarrow SN(L_2^*)$ is given by

$$k'(\pi'(x)) = \begin{cases} * & \text{if } t \leqq 1/4, \text{ or } t \geqq 1/2, \\ p(4t-1)\wedge x & \text{if } 1/4 \leqq t \leqq 1/2, \end{cases}$$

where $\pi': K\rightarrow K/N(L_2^*)$ is the identification map.

Now consider the two maps ϕ_1, ϕ_2: $K\rightarrow SN(L_2)\wedge N(L_2^*)/N(L_1^*)$ which are to be proved homotopic. If $x\in N(L_2)$, then

$$(6.28)\qquad \phi_1(x) = \begin{cases} * & (t \leqq 1/4, \text{ or } t \geqq 1/2), \\ p(4t-1) \wedge x \wedge \pi_2'(\bar{x}) & (1/4 \leqq t \leqq 1/2); \end{cases}$$

if $x \in N(L_2^*)$, then $\phi_2(x) = *$ and $t \geqq 1/2$; hence (6.28) holds for all $x \in K$. Similarly, if $x \in N(L_1) \cap N(L_2^*)$,

$$(6.29)\qquad \phi_2(x) = \begin{cases} * & (t \leqq 1/2, \text{ or } t \geqq 3/4), \\ p(4t-2) \wedge \bar{x} \wedge \pi_2'(x) & (1/2 \leqq t \leqq 3/4); \end{cases}$$

if $x \in N(L_2)$, then $\phi_2(x) = *$ and $t \leqq 1/2$; if $x \in N(L_1^*)$, then $\phi_2(x) = *$ and $\pi_2'(x) = *$; hence (6.29) likewise holds for all $x \in K$.

Define $\phi: K \to SN(L_2) \wedge N(L_2^*)/N(L_1^*)$ by

$$\phi(x) = \begin{cases} * & (t \leqq 1/4, \text{ or } t \geqq 3/4), \\ p(2t - 1/2) \wedge \{(5/4 - t)y + (t - 1/4)z\} & \\ \qquad \wedge\, \pi_2'\{(3/4 - t)y + (t + 1/4)z\} & (1/4 \leqq t \leqq 3/4). \end{cases}$$

The homotopy Φ defined by

$$\Phi(x, s) = \begin{cases} * & (t \leqq 1/4, \text{ or } t \geqq (s+3)/4(s+1)), \\ p((4t-1)(s+1)/2) \wedge \{(1 - t + (1-s)/4)y + (t - (1-s)/4)z\} & \\ \quad \wedge\, \pi'\{(1/2 - (t - 1/4)(1-s))y + (1/2 + (t - 1/4)(1-s))z\} & \\ & (1/4 \leqq t \leqq (s+3)/4(s+1), \end{cases}$$

deforms ϕ into ϕ_1. A similar homotopy deforms ϕ into ϕ_2.

The basic properties of the cap-product needed in §7 can now be established. Let $L_1 \supset L_2 \supset L_3$ be subcomplexes of K. Let $\mathfrak{f}: (\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{C}$ be a pairing of spectra, and let $z \in H_n(K; \boldsymbol{A})$. We consider the cap-product as a pairing of the form (6.24). Consider the diagram

$$(6.30)\qquad \begin{array}{ccccccccc} \cdots \to H^q(L_1, L_2) & \xrightarrow{i_2^*} & H^q(L_1, L_3) & \xrightarrow{i_1^*} & H^q(L_2, L_3) & \xrightarrow{\delta} & H^{q+1}(L_1, L_2) & \longrightarrow \cdots \\ \downarrow z\frown & & \downarrow z\frown & & \downarrow z\frown & & \downarrow z\frown & \\ \cdots \to H_{n-q}(L_2^*, L_1^*) & \xrightarrow{i_{2*}'} & H_{n-q}(L_3^*, L_1^*) & \xrightarrow{i_{1*}'} & H_{n-q}(L_3^*, L_2^*) & \xrightarrow{\partial} & H_{n-q-1}(L_2^*, L_1^*) & \to \cdots \end{array}$$

in which the upper row is the cohomology sequence of the triple (L_1, L_2, L_3) with respect to $\boldsymbol{B}$, while the lower row is the homology sequence of the triple (L_3^*, L_2^*, L_1^*) with respect to $\boldsymbol{C}$.

THEOREM (6.31). *The two left-hand squares of the diagram* (6.30) *are commutative; the third is commutative up to the sign* $(-1)^{n+1}$.

Proof. Let $w \in H^q(N(L_1), N(L_2); \boldsymbol{B})$. Then

$$z \frown i_2^* w = \Delta_*' z \backslash i_2^* w = (i_2 \wedge 1)_* \Delta_*' z \backslash w \qquad \text{by (6.10)},$$
$$i_{2*}'(z \frown w) = i_{2*}'(\Delta_*' z \backslash w) = (1 \wedge i_2')_* \Delta_*' z \backslash w \qquad \text{by (6.10)};$$

the equality of these elements follows from Lemma (6.26).

The commutativity of the middle square follows by a similar argument from Lemma (6.25).

The third square breaks up into two parts:

$$\text{(6.32)}\qquad \begin{array}{ccccc} H^q(L_2, L_3) & \xrightarrow{j^*} & H^q(L_2) & \xrightarrow{\delta} & H^{q+1}(L_1, L_2) \\ \downarrow z\frown & & \downarrow z\frown & & \downarrow z\frown \\ H_{n-q}(L_3^*, L_2^*) & \xrightarrow{j'_*} & H_{n-q}(K, L_2^*) & \xrightarrow{\partial} & H_{n-q-1}(L_2^*, L_1^*) \end{array}$$

where $j\colon N(L_2)\to N(L_2)/N(L_3)$ and $j'\colon N(L_3^*)/N(L_2^*)\to K/N(L_2^*)$ are induced by inclusion maps. The left-hand square in (6.32) is the left-hand square in the diagram analogous to (6.30) for the triple $(L_2, L_3, \emptyset)$ and is therefore commutative. The right-hand square in (6.32) is the right-hand square in the diagram analogous to (6.30) for the triple $(L_1, L_2, \emptyset)$. Hence we may assume $L_3=\emptyset$. Let $w\in H^q(N(L_2); \boldsymbol{B})$. Then

$$\begin{aligned} -z\frown \delta w &= z\frown k^*\sigma^{*-1}w = \Delta'_* z\backslash k^*\sigma^{*-1}w \\ &= (k\wedge 1)_*\Delta'_* z\backslash \sigma^{*-1}w && \text{by (6.10)} \\ &= (-1)^{n+1}\overset{-1}{\sigma_L}\,(k\wedge 1)_*\Delta'_* z\backslash w && \text{by (6.13)} \end{aligned}$$

where k is a canonical map as in Lemma (6.27). Also

$$\begin{aligned} -\partial(z\frown w) &= -\partial(\Delta'_*z\backslash w) = \pi'_{2*}\sigma^{-1}k_*(\Delta'_*z\backslash w) \\ &= \pi'_{2*}\sigma^{-1}((1\wedge k')_*\Delta'_*z\backslash w) && \text{by (6.10)} \\ &= \pi'_{2*}(\overset{-1}{\sigma_R}(1\wedge k')_*\Delta'_*z\backslash w) && \text{by (6.14)} \\ &= (1\wedge \pi'_2)_*\overset{-1}{\sigma_R}(1\wedge k')_*\Delta'_* z\backslash w && \text{by (6.10)} \\ &= \overset{-1}{\sigma_R}(1\wedge S\pi'_2)_*(1\wedge k')_*\Delta'_* z\backslash w \\ &= \overset{-1}{\sigma_L}\tau_*(1\wedge S\pi'_2)_*(1\wedge k')_*\Delta'_* z\backslash w. \end{aligned}$$

By Lemma (6.27), we have

$$\partial(z\frown w) = (-1)^{n+1}z\frown \delta w.$$

COROLLARY (6.33). *If L_1, M_1, L_2, M_2 are subcomplexes of K such that $L_1\supset M_1\cup L_2$, $M_1\cap L_2\supset M_2$, and if $i\colon (L_2, M_2)\subset(L_1, M_1)$, $i'\colon (M_1^*, L_1^*)\subset(M_2^*, L_2^*)$, then the diagram*

$$\begin{array}{ccc} H^q(L_1, M_1) & \xrightarrow{i^*} & H^q(L_2, M_2) \\ \downarrow z\frown & & \downarrow z\frown \\ H_{n-q}(M_1^*, L_1^*) & \xrightarrow{i'_*} & H_{n-q}(M_2^*, L_2^*) \end{array}$$

is commutative.

For i can be decomposed as the composite $(L_2, M_2) \subset (L_1, M_2) \subset (L_1, M_1)$ and the diagram is the composite of two diagrams, each of which is commutative, by Theorem (6.31).

7. **Duality theorems.** Our objective in this section is to prove duality theorems of Poincaré and Alexander type. The Alexander duality theorem is found to hold for subcomplexes of a sphere, without restriction on the spectrum involved. On the other hand, the Poincaré duality requires some hypotheses on the manifold or on the spectra; the class of manifolds for which Poincaré duality holds for arbitrary spectra is characterized; it properly contains the class of Π-manifolds.

Throughout this section we fix a spectrum $\boldsymbol{A}$, together with a pairing $\mathfrak{f}\colon (\boldsymbol{A}, \boldsymbol{A}) \to \boldsymbol{A}$ and a map $\boldsymbol{g}\colon \boldsymbol{S} \to \boldsymbol{A}$. Let $\mathfrak{h}\colon (\boldsymbol{S}, \boldsymbol{A}) \to \boldsymbol{A}$, $\mathfrak{h}'\colon (\boldsymbol{A}, \boldsymbol{S}) \to \boldsymbol{A}$ be the natural pairings. Let $i_0 \in \pi_0(\boldsymbol{S})$ be the element represented by the identity map $S^0 \to S^0$, and let $i = \boldsymbol{g}_\#(i_0) \in \pi_0(\boldsymbol{A})$. We assume

(7.1). *The diagram*

$$\begin{tikzcd} S^p \wedge A_q \arrow[r, "g_p \wedge 1"] \arrow[dr, "h_{p,q}"'] & A_p \wedge A_q \arrow[d, "f_{p,q}"] & A_p \wedge S_q \arrow[l, "1 \wedge g_q"'] \arrow[dl, "h'_{p,q}"] \\ & A_{p+q} & \end{tikzcd}$$

is commutative.

It follows easily that

(7.2). *If* $u \in \pi_q(\boldsymbol{A})$, *then* $\mathfrak{f}_*(i \otimes u) = \mathfrak{f}_*(u \otimes i) = u$.

Examples of such spectra are $\boldsymbol{S}$, $\boldsymbol{U}$, and $\boldsymbol{K}(A)$, where A is a ring with unit.

A spectrum $\boldsymbol{B}$, together with a pairing $\mathfrak{g}\colon (\boldsymbol{A}, \boldsymbol{B}) \to \boldsymbol{B}$, will be called an ***A**-module* if and only if the diagrams

$$\begin{tikzcd} S^p \wedge B_q \arrow[rr, "g_p \wedge 1"] \arrow[dr, "k_{p,q}"'] & & A_p \wedge B_q \arrow[dl, "g_{p,q}"] \\ & B_{p+q} & \end{tikzcd}$$

are commutative, $\mathfrak{k} = \{k_{p,q}\}$ being the natural pairing $(\boldsymbol{S}, \boldsymbol{B}) \to \boldsymbol{B}$.

As before, we have easily

(7.3). *If* $u \in \pi_p(B)$, *then* $\mathfrak{g}_*(i \otimes u) = u$.

Note that every spectrum may be considered as an $\boldsymbol{S}$-module. If A is a ring with unit and B is a left A-module, then $\boldsymbol{K}(B)$ is a $\boldsymbol{K}(A)$-module.

Remark. We may assume that $\boldsymbol{S}$ is a subspectrum of $\boldsymbol{K}(Z)$. If $\boldsymbol{B}$ is a spectrum, one may ask whether the natural pairing $(\boldsymbol{S}, \boldsymbol{B})$ can be extended to a pairing $(\boldsymbol{K}(Z), \boldsymbol{B}) \to \boldsymbol{B}$; if this is so, then $\boldsymbol{B}$ is a $\boldsymbol{K}(Z)$-module. Suppose

$\boldsymbol{B}$ is a $\boldsymbol{K}(Z)$-module which is a convergent Ω-spectrum; then, if k is sufficiently large and l is fixed, the homomorphism

$$\tilde{H}_{n+l}(B_l; Z) \approx \pi_{n+k+l}(K(Z, k) \wedge B_l) \xrightarrow{g_{k,l}} \pi_{n+k+l}(B_{k+l}) \approx \pi_{n+l}(B_l),$$

is a left inverse of the Hurewicz homomorphism $\eta: \pi_{n+l}(B_l) \to \tilde{H}_{n+l}(B_l; Z)$. It follows from an argument due to Moore [22, Theorem 3.29] that the k-invariants of B_l vanish; hence $\boldsymbol{B}$ is essentially a product of Eilenberg-MacLane spectra.

By *n-manifold* we shall mean a compact connected triangulated space K which is a homology n-manifold in the sense that K has the same local homology groups, at each point, as an n-sphere. Let K be an n-manifold.

By the Hopf theorem, $H^n(K; \boldsymbol{S}) \approx H^n(K; Z)$; let $z_0' \in H^n(K; \boldsymbol{S})$ be a generator, and let $z' = H^n(K, \boldsymbol{g})(z_0') \in H^n(K; \boldsymbol{A})$.

We say that K is *$\boldsymbol{A}$-orientable* if and only if there is a class $z \in H_n(K; \boldsymbol{A})$ such that

$$\langle z, z' \rangle = i \in \pi_0(\boldsymbol{A}).$$

Such a class z is called a *fundamental class.*

THEOREM (7.4). *Let K be an $\boldsymbol{A}$-orientable n-manifold and let $z \in H_n(K; \boldsymbol{A})$ be a fundamental class. Let L and M be subcomplexes of K such that $L \supset M$. Then, for any $\boldsymbol{A}$-module $\boldsymbol{B}$,*

$$Z\frown: H^q(L, M; \boldsymbol{B}) \approx H_{n-q}(M^*, L^*; \boldsymbol{B}) \qquad \textit{for all } q. \tag{7.5}$$

Proof. The proof proceeds in four steps. In the first, we show that (7.5) holds when L is a vertex and $M = \emptyset$. In the second, we prove (7.5) when L is a simplex of K with boundary M. We then show that (7.5) holds when (L, M) is replaced by (L_p, L_{p-1}), L_p being the union of M with the p-skeleton of L. The general result is then achieved by a standard kind of spectral sequence argument.

Step 1. Suppose that L is a vertex, $M = \emptyset$. Then $M^* = K$, and L^* is the complement of the barycentric star of L. It follows that K/L^*, and therefore also $K/N(L^*)$, is a homology n-sphere. The constant map $h: N(L) \to L$ is a homotopy equivalence; composing $\Delta': K \to N(L) \wedge K/N(L^*)$ with $h \wedge 1$, we obtain a map $\Delta'': K \to L \wedge K/N(L^*)$ homotopically equivalent to Δ'. Now $L \wedge K/N(L^*)$ can be identified with $K/N(L^*)$; under this identification, Δ'' becomes the identification map $p: K \to K/N(L^*)$.

Since K is a manifold, $K/N(L^*)$ is a homology n-sphere, and it follows that, for any spectrum $\boldsymbol{C}$,

$$\tilde{H}_q(K/N(L^*); \boldsymbol{C}) \approx \tilde{H}_q(S^n; \boldsymbol{C}) \approx \tilde{H}_{q-n}(S^0; \boldsymbol{C}) = \pi_{q-n}(\boldsymbol{C}).$$

We proceed to make this isomorphism explicit.

Let $j: \boldsymbol{S} \to \boldsymbol{K}(Z)$ be the natural map. Then the diagram

$$\begin{array}{ccc}\tilde{H}^n(K/N(L^*);\mathbf{S}) & \xrightarrow{j_*} & \tilde{H}^n(K/N(L^*);Z)\\ \downarrow p^* & & \downarrow p^*\\ H^n(K;\mathbf{S}) & \xrightarrow{j_*} & H^n(K;Z)\end{array}$$

is commutative and, by the Hopf theorem, both homomorphisms j_* are isomorphisms. Since p^*: $\tilde{H}^n(K/N(L^*);\ Z)\to H^n(K;Z)$ is an epimorphism, so is p^*: $\tilde{H}^n(K/N(L^*);\ \mathbf{S})\to H^n(K;\mathbf{S})$. Choose a class $z_1'\in\tilde{H}^n(K/N(L^*);\ \mathbf{S})$ such that $p^*(z_1')=z_0'$.

Let $h: K/N(L^*)\to S^n$ be a representative of z_1'. Then it is clear that h_*: $\tilde{H}_q(K/N(L^*);Z)\approx\tilde{H}_q(S^n;Z)$ for all q. By (5.16),

$$h_*: \tilde{H}_q(K/N(L^*);\mathbf{C}) \approx \tilde{H}_q(S^n;\mathbf{C})$$

for all q and for every spectrum $\mathbf{C}$.

The iterated suspension σ^n: $\tilde{H}_{q-n}(S^0;\ \mathbf{C})\to\tilde{H}_q(S^n;\ \mathbf{C})$ is an isomorphism for all q, $\mathbf{C}$. Hence

$$\phi = \sigma_*^{-n}\circ h_*: \tilde{H}_q(K/N(L^*);\mathbf{C}) \approx \tilde{H}_{q-n}(S^0;\mathbf{C}) = \pi_{q-n}(\mathbf{C}).$$

Consider the diagram

$$\begin{array}{ccc}\tilde{H}_n(L\wedge K/N(L^*);\mathbf{A})\otimes H^q(L;\mathbf{B}) & \xrightarrow{\ \backslash\ } & \tilde{H}_{n-q}(K/N(L^*);\mathbf{B})\\ \downarrow & & \\ \tilde{H}_n(K/N(L^*);\mathbf{A}\otimes H^q(L;\mathbf{B}) & & \Big\downarrow \phi\\ \phi\otimes 1\downarrow & & \\ \pi_0(\mathbf{A})\otimes\pi_{-q}(\mathbf{B}) & \xrightarrow[\mathfrak{g}_*]{} & \pi_{-q}(\mathbf{B}).\end{array}$$

(The unlabelled arrow is induced by the natural identification $L\wedge K/N(L^*)=K/N(L^*)$.) We claim that this diagram is commutative. In fact, the diagram can be enlarged to the diagram (Figure 5). Using (6.10) and (6.14), we see that the upper parts of the diagram are all commutative, and it suffices to verify that the lower region is commutative. But this is immediate, from the definitions of the slant product and $\mathfrak{g}_*$.

This being so, it remains to verify that $\phi(p_*(z))=\pm i$. For then, if $u'\in H^q(L,\ \mathbf{B})$, we have

$$\begin{aligned}\phi(z\frown u') &= \phi(\Delta_*'z\backslash u') = \mathfrak{g}_*(\phi(p_*z)\otimes u')\\ &= \pm\,\mathfrak{g}_*(i\otimes u') = \pm\,u',\end{aligned}$$

and therefore $z\frown$ is an isomorphism with inverse $\pm\phi$.

Now $\phi(p_*(z))=\sigma_*^{-n}h_*p_*(z)$, and

$$i = \langle z, z'\rangle = \langle z, g_*(z_0')\rangle = \langle z, g_*p^*z_1'\rangle.$$

$$\begin{tikzcd}
\tilde{H}_n(K/N(L^*); A) \otimes H^q(L; B) \arrow[d, "h_* \otimes 1"] & \tilde{H}_n(L \wedge K/N(L^*); A) \otimes H^q(L; B) \arrow[l] \arrow[r, "\backslash"] \arrow[d, "(1 \wedge h)_* \otimes 1"] & \tilde{H}_{n-q}(K/N(L^*); B) \arrow[d, "h_*"] \\
\tilde{H}_n(S^n; A) \otimes H^q(L; B) & \tilde{H}_n(L \wedge S^n; A) \otimes H^q(L; B) \arrow[l] \arrow[r, "\backslash"] & \tilde{H}_{n-q}(S^n; B) \\
\tilde{H}_0(S^0; A) \otimes H^q(L; B) \arrow[u, "\sigma_*^n \otimes 1"] \arrow[dr, equal] & \tilde{H}_0(L \wedge S^0; A) \otimes H^q(L; B) \arrow[l] \arrow[r, "\backslash"] \arrow[u, "\sigma_R^n \otimes 1"] & \tilde{H}_{-q}(S^0; B) \arrow[u, "\sigma_*^n"] \arrow[d, equal] \\
 & \pi_0(A) \otimes \pi_{-q}(B) \arrow[r, "\mathfrak{g}_*"'] & \pi_{-q}(B)
\end{tikzcd}$$

FIGURE 5

Let $z_2' = h^{*-1}(z_1') \in \tilde{H}^n(S^n; \mathbf{S})$; thus

$$i = \langle z, \mathfrak{g}_* p^* h^* z_2' \rangle = \langle z, p^* h^* \mathfrak{g}_* z_2' \rangle = \langle h_* p_* z, \mathfrak{g}_* z_2' \rangle$$

by (6.10a) and the analogue of (5.13) for cohomology. Now z_2' generates the infinite cyclic group $\tilde{H}^n(S^n; \mathbf{S})$ and therefore $z_3' = \sigma^{*n} z_2'$ generates $\tilde{H}^0(S^0; \mathbf{S}) = \pi_0(\mathbf{S})$; thus $z_3' = \pm i_0$. Hence

$$\begin{aligned} i &= \pm \langle h_* p_* z, \mathfrak{g}_* \sigma^{*-n} i_0 \rangle = \pm \langle h_* p_* z, \sigma^{*-n} \mathfrak{g}_* i_0 \rangle \\ &= \pm \langle \sigma_*^{-n} h_* p_* z, i \rangle = \pm \langle \phi(p_*(z)), i \rangle \end{aligned}$$

by (6.13a) and the analogue of (5.14) for cohomology. But clearly

$$\langle \phi(p_*(z)), i \rangle = \mathfrak{f}_*(\phi(p_*(z)) \otimes i) = \phi(p_*(z))$$

by (7.2). This completes Step 1.

Step 2. We next prove, by induction on p, that (7.5) holds whenever L is a p-simplex of K and $M = \dot{L}$. We have already proved the case $p = 0$. Assume that (7.5) holds for all simplexes of dimension less than p, and let E be a p-simplex. Choose a $(p-1)$-face F of E and let F' be the union of the remaining $(p-1)$-faces, so that $\dot{E} = F \cup F'$, $\dot{F} = F \cap F'$. It follows[5] that $\dot{E}^* = F^* \cap F'^*$, $\dot{F}^* = F^* \cup F'^*$. Let $k: (F, \dot{F}) \subset (\dot{E}, F')$ and $k': (F'^*, \dot{E}^*) \subset (\dot{F}^*, F^*)$ be the inclusions. The diagram

$$\begin{array}{ccccc} H^q(E, E; \mathbf{B}) & \xleftarrow{\delta} & H^{q-1}(E, F'; \mathbf{B}) & \xrightarrow{k^*} & H^{q-1}(F, \dot{F}; \mathbf{B}) \\ -z\cap \downarrow & & \downarrow z\cap & & \downarrow z\cap \\ H_{n-q}(E^*, E^*; \mathbf{B}) & \xleftarrow{\partial} & H_{n-q+1}(F'^*, E^*; \mathbf{B}) & \xrightarrow{k'_*} & H_{n-q+1}(\dot{F}^*, F^*; \mathbf{B}) \end{array}$$

is commutative, by Theorem (6.31) and Corollary (6.33). Now δ is an isomorphism, since F' is a deformation retract of E, and k^*, k'_* are isomorphisms

(5) It is easily verified that if L, M are any subcomplexes of K, then $(L \cup M)^* = L^* \cap M^*$ and $(L \cap M)^* = L^* \cup M^*$.

by the excision axiom. The right-hand $z\frown$ is an isomorphism by induction hypothesis. Therefore, it suffices to prove that ∂ is an isomorphism, i.e., that $H_k(F'^*, E^*; \boldsymbol{B})=0$ for all k. It suffices in turn, by (5.15), to prove that F'^*/E^* is acyclic (over the integers). Now $F'^*=E^*\cup D(F)$, where $D(F)$ is the dual cell of F; and $E^*\cap D(F)$ is the closure of the complement of $D(E)$ in the boundary $\dot{D}(F)$ of $D(F)$. But $D(F)$ and $D(E)$ are acyclic, and $\dot{D}(F)$ is a combinatorial manifold which is a homology sphere; by the Alexander duality theorem, $E^*\cap D(F)$ is acyclic. Hence $D(F)/E^*\cap D(F)=F'^*/E^*$ is acyclic.

Step 3. Let L, M be subcomplexes of K such that $L\supset M$. Let $E_1, \cdots, E_s$ be the p-simplexes of $L-M$; then

$$
\begin{aligned}
L_p &= L_{p-1}\cup \bigcup_{i=0}^{s} E_i, \\
E_i\cap L_{p-1} &= \dot{E}_i, \qquad\qquad (7.6)\\
E_i\cap E_j &\subset L_{p-1} \qquad\qquad \text{if } i\neq j.
\end{aligned}
$$

Hence

$$
\begin{aligned}
L_p^* &= L_{p-1}^*\cap \bigcap_{i=1}^{s} E_i^*, \\
E_i^*\cup L_{p-1}^* &= \dot{E}_i^*, \qquad\qquad (7.7)\\
E_i^*\cup E_j^* &\supset L_{p-1}^* \qquad\qquad \text{if } i\neq j.
\end{aligned}
$$

Let $j_i\colon (E_i, \dot{E}_i)\subset(L_p, L_{p-1})$, and let $j_i'\colon (L_{p-1}^*, L_p^*)\subset(\dot{E}_i^*, E_i^*)$. Since (7.6) holds, it follows from the direct sum theorem [8, III, 2.3c] that the homomorphisms

$$j_i^*\colon H^q(L_p, L_{p-1}; \boldsymbol{B}) \to H^q(E_i, \dot{E}_i; \boldsymbol{B})$$

form a projective representation of $H^q(L_p, L_{p-1}; \boldsymbol{B})$ as a direct sum. (Observe that the proof of the direct sum theorem does not use the dimension axiom.) A similar argument shows that the homomorphisms

$$j_{i*}'\colon H_{n-q}(L_{p-1}^*, L_p^*; \boldsymbol{B}) \to H_{n-q}(\dot{E}_i^*, E_i^*; \boldsymbol{B})$$

form a projective representation of $H_{n-q}(L_{p-1}^*, L_p^*; \boldsymbol{B})$ as a direct sum. By Corollary (6.33), the diagrams

$$
\begin{array}{ccc}
H^q(L_p, L_{p-1}; \boldsymbol{B}) & \xrightarrow{j_i^*} & H^q(E_i, \dot{E}_i; \boldsymbol{B}) \\
\downarrow z\frown & & \downarrow z\frown \\
H_{n-q}(L_{p-1}^*, L_p^*; \boldsymbol{B}) & \xrightarrow{j_{i*}'} & H_{n-q}(\dot{E}_i^*, E_i^*; \boldsymbol{B})
\end{array}
$$

are commutative, and the truth of (7.5) for the pair (L_p, L_{p-1}) now follows from Step 2.

Step 4. We can now prove (7.5) in general. We have

$$
\begin{aligned}
L = L_n \supset L_{n-1} \supset \cdots \supset L_0 \supset L_{-1} = M, \\
M^* = L_{-1}^* \supset L_0^* \supset \cdots \supset L_n^* = L^*,
\end{aligned}
$$

while $L_p = L$ if $p > n$, $L_p = M$ if $p < 0$, $L_p^* = L^*$ if $p > n$, $L_p^* = M^*$ ($p < 0$). The above filtrations give rise to two bigraded exact couples

$$\mathfrak{E} = \langle D, E; i, j, \delta \rangle, \qquad {}^*\mathfrak{E} = \langle {}^*D, {}^*E; {}^*i, {}^*j, \partial \rangle,$$

where

$$\begin{aligned} D^{p,q} &= H^{p+q}(L, L_{p-1}; B), & {}^*D^{p,q} &= H_{n-p-q}(L_{p-1}^*, L^*; B), \\ E^{p,q} &= H^{p+q}(L_p, L_{p-1}; B), & {}^*E^{p,q} &= H_{n-p-q}(L_{p-1}^*, L_p^*; B) \end{aligned}$$

while the sequences

$$\cdots \longrightarrow D^{p+1,q-1} \xrightarrow{i} D^{p,q} \xrightarrow{j} E^{p,q} \xrightarrow{\delta} D^{p+1,q} \longrightarrow \cdots$$

$$\cdots \longrightarrow {}^*D^{p+1,q-1} \xrightarrow{{}^*i} {}^*D^{p,q} \xrightarrow{{}^*j} {}^*E^{p,q} \xrightarrow{\partial} {}^*D^{p+1,q} \longrightarrow \cdots$$

are the cohomology and homology exact sequences of the triples (L, L_p, L_{p-1}) and (L_{p-1}^*, L_p^*, L^*) respectively. Let $\mathfrak{E}_r$, ${}^*\mathfrak{E}_r$ be the $(r-1)$st derived couples, and let d_r, ${}^*d_r$ be their derived operators.

Note that $D^{p,q} = {}^*D^{p,q} = 0$ if $p > n$, while $E^{p,q} = {}^*E^{p,q} = 0$ if $p < 0$ or $p > n$. It follows by standard arguments (cf. [31, §4]) that $E_r^{p,q} = E_{r+1}^{p,q}$ and ${}^*E_r^{p,q} = {}^*E_{r+1}^{p,q}$ provided that $r \geqq \max(p+1, n-p+1)$. Moreover, let $J^{p,q}$ be the image of the injection

$$H^{p+q}(L, L_{p-1}; B) \rightarrow H^{p+q}(L, M; B)$$

and let ${}^*J^{p,q}$ be the image of the injection

$$H_{n-p-q}(L_{p-1}^*, L^*; B) \rightarrow H_{n-p-q}(M^*, L^*; B).$$

Then

$$\begin{aligned} D^{r,0} &= H^r(L, M; B) = J^{0,r} \supset J^{1,r-1} \supset \cdots \supset J^{n,r-n} \supset J^{n+1,r-n-1} = 0, \\ {}^*D^{r,0} &= H_{n-r}(M^*, L^*; B) = {}^*J^{0,r} \supset {}^*J^{1,r-1} \supset \cdots \supset {}^*J^{n,r-n} \supset {}^*J^{n+1,r-n-1} \\ &= 0, \end{aligned}$$

and

$$E_\infty^{p,q} \approx J^{p,q}/J^{p+1,q-1}, \qquad {}^*E_\infty^{p,q} \approx {}^*J^{p,q}/{}^*J^{p+1,q-1}.$$

Thus the spectral sequences associated with the couples $\mathfrak{E}$, ${}^*\mathfrak{E}$ converge in a strong sense.

Define $\phi: D \rightarrow {}^*D$, $\psi: E \rightarrow {}^*E$, by

$$\begin{aligned} \phi(w) &= (-1)^{(n+1)(p+q)} z \frown w & (w &\in D^{p,q}), \\ \psi(w) &= (-1)^{(n+1)(p+q)} z \frown w & (w &\in E^{p,q}); \end{aligned}$$

it follows from Theorem (6.31) that (ϕ, ψ) is a map of $\mathfrak{E}$ into ${}^*\mathfrak{E}$.

By Step 3 of the proof, $\psi: E^{p,q} \rightarrow {}^*E^{p,q}$ is an isomorphism. It follows that

$\psi_r: E_r^{p,q} \approx {}^*E_r^{p,q}$ for all r, and therefore that $\psi_\infty: E_\infty^{p,q} \approx {}^*E_\infty^{p,q}$.

The map $\phi: D^{r,0} \to {}^*D^{r,0}$ maps $J^{p,q}$ into ${}^*J^{p,q}$ and the induced homomorphism of $J^{p,q}/J^{p+1,q-1}$ into ${}^*J^{p,q}/{}^*J^{p+1,q-1}$ is ψ_∞. Hence it follows by an inductive argument, starting with $\phi: J^{n+1,r-n-1} \to {}^*J^{n+1,r-n-1}$, that $\phi: D^{r,0} = H^r(L, M; \boldsymbol{B}) \approx H_{n-r}(M^*, L^*; \boldsymbol{B}) = {}^*D^{r,0}$. This completes the proof.

Corollary (7.8) (Poincaré duality theorem). *If K is an $\boldsymbol{A}$-orientable n-manifold, and $z \in H_n(K; \boldsymbol{A})$ is a fundamental class, then, for any $\boldsymbol{A}$-module $\boldsymbol{B}$,*

$$z \frown: H^q(K; \boldsymbol{B}) \approx H_{n-q}(K; \boldsymbol{B}).$$

Suppose now that $\boldsymbol{A} = \boldsymbol{S}$. Let $\boldsymbol{f}: \boldsymbol{S} \to \boldsymbol{K}(Z)$ be the natural map. If $z \in H_n(K; \boldsymbol{S})$ is a fundamental class, so that $\langle z, z' \rangle = i_0 \in \pi_0(\boldsymbol{S})$, then

$$\boldsymbol{f}_*(i_0) = \boldsymbol{f}_*\langle z, z' \rangle = \langle \boldsymbol{f}_* z, \boldsymbol{f}_* z' \rangle = \langle \boldsymbol{f}_* z, z_0' \rangle \in \pi_0(\boldsymbol{K}(Z)) \approx Z,$$

and since $\boldsymbol{f}_*(i_0)$ is a generator and the Kronecker index reduces to the usual one, we see that K is orientable and $\boldsymbol{f}_* z$ is a fundamental class with integer coefficients. Hence the homomorphism

$$\boldsymbol{f}_*: H_n(K; \boldsymbol{S}) \to H_n(K; Z)$$

is an epimorphism. Conversely, if K is orientable and $\boldsymbol{f}_*$ is an epimorphism, it is easy to see that K is $\boldsymbol{S}$-orientable.

Let x_0 be a base-point in K. Then $\tilde{H}_n(K; \boldsymbol{S})$ is the nth stable homotopy group $\Sigma_n(K)$, and it follows easily that K is $\boldsymbol{S}$-orientable if and only if the Hurewicz homomorphism $\Sigma_n(K) \to \tilde{H}_n(K)$ is an epimorphism. Suppose that K is differentiable; then a recent result of Milnor and Spanier [20] shows that K is $\boldsymbol{S}$-orientable if and only if its stable normal bundle is fibre-homotopically trivial. It follows that every Π-manifold in the sense of Milnor [19] is $\boldsymbol{S}$-orientable. Clearly every Π-manifold in the sense of J. H. C. Whitehead [33] is $\boldsymbol{S}$-orientable.

If K is $\boldsymbol{S}$-orientable, we have seen that Poincaré duality holds for arbitrary spectra. Moreover, once a fundamental class has been chosen, the duality is natural (for maps of spectra), i.e., if $\boldsymbol{h}: \boldsymbol{B} \to \boldsymbol{C}$ is a map of spectra, then

$$\boldsymbol{h}_*(z \frown u) = z \frown \boldsymbol{h}_*(u)$$

for all $u \in H^q(K; \boldsymbol{B})$.

Conversely, we have

(7.9). *If K satisfies Poincaré duality naturally, i.e., if there exist natural isomorphisms*

$$\rho_q: H^q(K;\ ) \approx H_{n-q}(K;\ )$$

over the category of spectra, then K is $\boldsymbol{S}$-orientable.

Proof. The diagram

$$\begin{array}{ccc} H^0(K;\mathbf{S}) & \xrightarrow{\rho_0(\mathbf{S})} & H_n(K;\mathbf{S}) \\ \downarrow f_* & & \downarrow f_* \\ H^0(K;Z) & \xrightarrow{\rho_0(\mathbf{K}(Z))} & H_n(K;Z) \end{array}$$

is commutative. To show that $f_*: H_n(K; \mathbf{S}) \to H_n(K; Z)$ is an epimorphism, it suffices to show that $f_*: H^0(K; \mathbf{S}) \to H^0(K; Z)$ is an epimorphism. Let P be a point, $g: K \to P$. Then the diagram

$$\begin{array}{ccccc} \pi_0(\mathbf{S}) & = & H^0(P;\mathbf{S}) & \xrightarrow{g^*} & H^0(K;\mathbf{S}) \\ & & \downarrow f_* & & \downarrow f_* \\ \pi_0(\mathbf{K}(Z)) & = & H^0(P;Z) & \xrightarrow{g^*} & H^0(K;Z) \end{array}$$

is commutative. But $f_*: \pi_0(\mathbf{S}) \approx \pi_0(\mathbf{K}(Z))$ and $g^*: H^0(P; Z) \approx H^0(K; Z)$. Hence $f_* \circ g^*: \pi_0(\mathbf{S}) \approx H^0(K; Z)$ and it follows that f_* is an epimorphism.

Another consequence of Theorem (7.4) is

COROLLARY (7.10). *If K is a proper, nonempty subpolyhedron of S^n and if $K' \subset S^n - K$ is an $(n-1)$-dual*(6) *of K, and x_0, x_0' are base-points of K, K', then*

$$\tilde{H}^q(K; \mathbf{B}) \approx \tilde{H}_{n-q-1}(K'; \mathbf{B})$$

for any spectrum $\mathbf{B}$.

Proof. We can find a triangulation of (S^n, K) so fine that K' is contained in the supplement K^* of K in S^n relative to this triangulation. If $i: K' \subset K^*$, then

$$i_*: \tilde{H}_k(K') \approx \tilde{H}_k(K^*)$$

for all k; it follows by (5.16) that

$$i_*: \tilde{H}_k(K'; \mathbf{B}) \approx \tilde{H}_k(K^*; \mathbf{B})$$

for any $\mathbf{B}$. By Theorem (7.4), with K replaced by S^n (note that S^n is $\mathbf{S}$-orientable)

$$\begin{aligned} H^q(K, x_0; \mathbf{B}) &\approx H_{n-q}(\{x_0\}^*, K^*; \mathbf{B}) \\ &\approx H_{n-q-1}(K^*, x_0^*; \mathbf{B}) \end{aligned}$$

where x_0^* is a base-point of K^*.

We conclude with a remark which was suggested to us by J. H. C. Whitehead. It is known(7) that the Hurewicz map $\Sigma_n(X) \to \tilde{H}_n(X)$ is a $\mathcal{C}$-isomorphism [24], where $\mathcal{C}$ is the class of torsion groups. Hence if K is an arbitrary

(6) n-dual in the sense of [26].

(7) A proof of this fact can be found in the mimeographed notes *Lectures on characteristic classes* by John Milnor (Princeton, 1957), p. 108, Lemma 9.

orientable manifold and if $z \in H_n(K;\, \mathbf{S})$ does not belong to the kernel of f_*, the entire proof of Theorem (7.4) can be reworded in terms of $\mathcal{C}$-isomorphisms and we have

(7.11). *If K is a compact connected orientable triangulable n-manifold, then there is a class $z \in H_n(K;\, \mathbf{S})$ such that*

$$z\frown:\ H^q(K;\, \mathbf{B}) \to H_{n-q}(K;\, \mathbf{B})$$

is an isomorphism modulo the class of torsion groups.

8. **Brown's theorem.** In this section we outline a proof of the following theorem.

THEOREM 8.1. *Let $\tilde{\mathfrak{H}}$ be a homology theory on $\mathcal{P}_0$, and suppose that the groups $\tilde{H}_q(S^0)$ are all countable. Then there is an Ω-spectrum $\mathbf{E}$ and a natural isomorphism*

$$T\colon\ \tilde{\mathfrak{H}}(\mathbf{E}) \approx \tilde{\mathfrak{H}}.$$

Moreover, $\mathbf{E}$ is unique in the sense that, if $\mathbf{E}'$ is an Ω-spectrum and

$$T'\colon\ \tilde{\mathfrak{H}}(\mathbf{E}') \approx \tilde{\mathfrak{H}}$$

is a natural isomorphism then there is a map

$$f\colon\ \mathbf{E} \to \mathbf{E}'$$

such that each $f_q\colon E_q \to E_q'$ is a homotopy equivalence.

This theorem is the analogue of one proved by E. H. Brown [4] for cohomology theories. The proof consists of constructing a cohomology theory $\tilde{\mathfrak{H}}^*$, using the duality theory developed by Spanier in [25]. Brown's theorem then provides a natural isomorphism $T^*\colon \tilde{\mathfrak{H}}^*(\mathbf{E}) \approx \tilde{\mathfrak{H}}^*$. Using duality and the slant-product of §6, we then construct $T\colon \tilde{\mathfrak{H}}(\mathbf{E}) \to \tilde{\mathfrak{H}}$. We assume familiarity with Spanier's paper. (Caution: Spanier defines SX as $X \wedge S$, rather than $S \wedge X$, but this does not affect the arguments.)

Let $X \in \mathcal{P}_0$, and let $u\colon Y \wedge X \to S^n$ be a duality map [25, §5]. Let $\Gamma_q(u) = \tilde{H}_{n-q}(Y)$. If $u'\colon Y' \wedge X \to S^m$ is also a duality map then there is a unique S-map $\alpha \in \{Y,\ Y'\}_{m-n}$ such that, if k is sufficiently large and $f\colon S^kY \to S^{k+n-m}Y'$ is a representative of α, then the diagram

$$\begin{tikzcd}
S^kY \wedge X \arrow[d, "f \wedge 1"'] \arrow[dr, "S^k u"] & \\
S^{k+n-m}Y' \wedge X \arrow[r, "S^{k+n-m}u'"'] & S^{n+k}
\end{tikzcd}$$

is homotopy-commutative ([25, (5.11)]; if $k=m-n\geqq 0$, then $\alpha=D_m(u', S^k u)\{1\}\in\{S^kY, Y'\}=\{Y, Y'\}_k$, and if $k<0$, then $\alpha=D_n(S^{-k}u', u)\{1\}$). Define $\gamma(u, u'):\Gamma(u)\to\Gamma(u')$ to be the homomorphism $\alpha_*: \tilde{H}_{n-q}(Y)\to\tilde{H}_{m-q}(Y')$ induced by α. (Note: α_* is the composition

$$\tilde{H}_{n-q}(Y)\xrightarrow{\sigma_*^k}\tilde{H}_{n-q+k}(S^kY)\xrightarrow{f_*}\tilde{H}_{n-q+k}(S^{k+n-m}Y')\xrightarrow{\sigma_*^{m-n-k}}\tilde{H}_{m-q}(Y').)$$

Then it is easy to verify that

$$\gamma(u, u) = \text{identity},$$

$$\gamma(u', u'')\circ\gamma(u, u') = \gamma(u, u'')$$

for any three duality maps u, u', u''. Thus the groups $\Gamma_q(u)$ and homomorphisms $\gamma(u, u')$ form a *transitive system* in the sense of [8, p. 17]; accordingly we may define $\tilde{H}^q(X)$ to be the unique group associated with this transitive system.

Let $f: X\to X'$ be a map in $\mathcal{P}_0$. Choose duality maps

$$u: Y\wedge X\to S^n,$$

$$u': Y'\wedge X'\to S^n.$$

If $\alpha\in\{X, X'\}$ is the S-class of f, let $\beta=D_n(u, u')\alpha\in\{Y', Y\}$. It is easily verified, using [25, (6.3)] that the homomorphism

$$\tilde{H}^q(f): \tilde{H}^q(X')\to\tilde{H}^q(X)$$

corresponding to $\beta_*: \tilde{H}_{n-q}(Y')\to\tilde{H}_{n-q}(Y)$ is independent of the choices of the duality maps, and that $\tilde{H}^q: \mathcal{P}_0\to\mathcal{A}$ is a contravariant functor. Since β depends only on the S-class of f, the homotopy axiom is satisfied.

Let $X\in\mathcal{P}_0$, and let $u: Y\wedge X\to S^n$ be a duality map. Then the map

$$Y\wedge S\wedge X\to S\wedge Y\wedge X\xrightarrow{Su} S^{n+1},$$

in which the first map interchanges the first two factors, is a duality map

$$v: Y\wedge SX\to S^{n+1}.$$

It follows from [25, (6.2)] that the homomorphism of $\tilde{H}^q(X)$ into $\tilde{H}^{q+1}(SX)$ induced by the identity map of $\Gamma_q(u)=\tilde{H}_{n-q}(Y)=\Gamma_{q+1}(v)$ induces an isomorphism $\theta_u: \tilde{H}^q(X)\approx\tilde{H}^{q+1}(SX)$. If $u': Y'\wedge X\to S^m$ is another duality map, then

$$\theta_{u'} = (-1)^{m-n}\theta_u.$$

Therefore the isomorphism $(-1)^n\theta_u$ is independent of u. Let $\sigma^*: \tilde{H}^{q+1}(SX)\approx\tilde{H}^q(X)$ be the inverse of this isomorphism. Clearly σ^* is a natural transformation.

Finally, if (X, A) is a pair in $\mathcal{P}_0$, then the exactness of the sequence

$$\tilde{H}^q(X/A) \xrightarrow{\tilde{H}^q(p)} \tilde{H}^q(X) \xrightarrow{\tilde{H}^q(i)} \tilde{H}^q(A)$$

follows from [25, (6.10)].

We have thus defined a cohomology theory $\tilde{\mathfrak{H}}^* = \{\tilde{H}^q, \sigma^*\}$ on $\mathcal{P}_0$. Evidently $\tilde{H}^q(S^0) \approx \tilde{H}_{-q}(S^0)$; hence these groups are all countable. By Brown's theorem [4, Theorem II] there is an Ω-spectrum $\boldsymbol{E}$ and a natural isomorphism

$$T^*: \tilde{\mathfrak{H}}^*(\boldsymbol{E}) \approx \tilde{\mathfrak{H}}^*.$$

We now construct a natural transformation

$$T: \tilde{\mathfrak{H}}(\boldsymbol{E}) \to \tilde{\mathfrak{H}}.$$

Let $X \in \mathcal{P}_0$, and let $u: Y \wedge X \to S^n$ be a duality map. Let s be the natural generator of $\tilde{H}^n(S^n; \boldsymbol{S})$ and let $x = u^*(s) \in \tilde{H}^n(Y \wedge X; \boldsymbol{S})$. By means of the natural pairing $(\boldsymbol{S}, \boldsymbol{E}) \to \boldsymbol{E}$, the slant product by x is a homomorphism

$$x/: \tilde{H}_q(X; \boldsymbol{E}) \to \tilde{H}^{n-q}(Y; \boldsymbol{E}).$$

Brown's mapping is an isomorphism

$$T^*: \tilde{H}^{n-q}(Y; \boldsymbol{E}) \to \tilde{H}^{n-q}(Y).$$

Using the duality map $u': X \wedge Y \to S^n$ which is the transpose of u, we obtain an isomorphism

$$\tilde{H}^{n-q}(Y) \approx \tilde{H}_q(X).$$

The composite of the above homomorphisms is a homomorphism

$$T_u: \tilde{H}_q(X; \boldsymbol{E}) \to \tilde{H}_q(X).$$

The following facts are easily verified, using the properties of the /-product given in §6 and the results of [25]:

(1) $T_{Su} = (-1)^n T_u: \tilde{H}_q(X; \boldsymbol{E}) \to \tilde{H}_q(X)$.

(2) If $u_1: Y_1 \wedge X \to S^n$ is a duality map and $f: Y \to Y_1$ is a map such that the diagram

$$\begin{tikzcd}
Y \wedge X \arrow[dd, "f \wedge 1"'] \arrow[dr, "u"] & \\
& S^n \\
Y_1 \wedge X \arrow[ur, "u_1"'] &
\end{tikzcd}$$

is commutative, then $T_u = T_{u_1}: \tilde{H}_q(X; \boldsymbol{E}) \to \tilde{H}_q(X)$.

(3) If u_1: $Y_1 \wedge X_1 \to S^n$ is a duality map and f: $X \to X_1$, g: $Y_1 \to Y$ are maps such that the diagram

$$\begin{array}{ccc} Y_1 \wedge X & \xrightarrow{g \wedge 1} & Y \wedge X \\ 1 \wedge f \downarrow & & \downarrow u \\ Y_1 \wedge X_1 & \xrightarrow[u_1]{} & S^n \end{array}$$

is commutative, then the diagram

$$\begin{array}{ccc} \tilde{H}_q(X; \boldsymbol{E}) & \xrightarrow{T_u} & \tilde{H}_q(X) \\ f_* \downarrow & & \downarrow f_* \\ \tilde{H}_q(X_1; \boldsymbol{E}) & \xrightarrow[T_{u_1}]{} & \tilde{H}_q(X_1) \end{array}$$

is also commutative.

(4) Let v: $Y \wedge SX \to S^{n+1}$ be the duality map used in the definition of σ^*. Then the diagram

$$\begin{array}{ccc} \tilde{H}_q(X; \boldsymbol{E}) & \xrightarrow{T_u} & \tilde{H}_q(X) \\ \sigma_* \downarrow & & \downarrow \sigma_* \\ \tilde{H}_{q+1}(SX; \boldsymbol{E}) & \xrightarrow[T_v]{} & \tilde{H}_{q+1}(SX) \end{array}$$

commutes except for the sign $(-1)^{n+1}$.

It follows from (1) and (2) that the homomorphism

$$T = (-1)^{c_{n,q}} T_u: \tilde{H}_q(X; \boldsymbol{E}) \to \tilde{H}_q(X)$$

is independent of the duality map u; from (3) that T is a natural transformation of functors, and from (4) that T commutes with suspension. Hence T: $\tilde{\mathfrak{H}}(\boldsymbol{E}) \to \tilde{\mathfrak{H}}$ is a natural transformation of homology theories.

It remains to prove that T is an isomorphism. If $X = S^0$, we may choose $Y = S^0$ and u: $S^0 \wedge S^0 \to S^0$ to be the obvious homeomorphism; then $x/$ is the identity map of $\tilde{H}_q(S^0; \boldsymbol{E}) = \pi_q(\boldsymbol{E}) = \tilde{H}^{-q}(S^0; \boldsymbol{E})$. Hence T: $\tilde{H}_q(S^0; \boldsymbol{E}) \approx \tilde{H}_q(S^0)$. The fact that T is an isomorphism on $\mathcal{P}_0$ now follows by standard methods.

Finally, suppose that T: $\tilde{\mathfrak{H}}(\boldsymbol{E}) \to \tilde{\mathfrak{H}}$, T': $\tilde{\mathfrak{H}}(\boldsymbol{E}') \to \tilde{\mathfrak{H}}$ are natural isomorphisms. Then $T^{-1} \circ T'$: $\tilde{\mathfrak{H}}(\boldsymbol{E}') \to \tilde{\mathfrak{H}}(\boldsymbol{E})$ is a natural isomorphism. We can then turn the above proof "inside out" to construct a natural isomorphism of $\tilde{\mathfrak{H}}^*(\boldsymbol{E}')$ with $\tilde{\mathfrak{H}}^*(\boldsymbol{E})$. Application of Brown's theorem then proves the uniqueness.

Corollary (8.2). *Let X, $Y \in \mathcal{P}_0$, and let u: $Y \wedge X \to S^n$ be a duality map. Then, for any spectrum* $\boldsymbol{E}$,

$$u^*s/: \tilde{H}_q(X; E) \approx H^{n-q}(Y; E).$$

Thus we have again proved the Alexander duality theorem, in the more general form suggested by [25]. It is likely that a theorem, similar to (7.4), containing both Alexander and Poincaré duality, could be proved in a similar way. We prefer the present version of (7.4) because the version of Poincaré duality given there is parallel to the usual one in the sense that the isomorphism is given by a cap-product. On the other hand, (8.2) has the advantage of being more general than (7.11), as well as being parallel to the standard version of Alexander duality. For these reasons, as well as because we obtain (8.2) "free of charge" from the proof of Theorem (8.1), we have included both versions of Alexander duality.

REMARK. It would be desirable to have a proof of Theorem 8.1 which does not depend on Brown's theorem. It is not known whether the countability hypothesis is necessary for Brown's theorem. It is not inconceivable that a direct proof of Theorem 8.1 without the countability hypothesis could be found. If so, the above procedure could be reversed to prove Brown's theorem without the countability hypothesis.

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