

# A DECOMPOSITION FOR COMPLETE NORMED ABELIAN GROUPS WITH APPLICATIONS TO SPACES OF ADDITIVE SET FUNCTIONS

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**1. Introduction.** The purpose of this paper is twofold. Our principal objective is to present a Lebesgue type decomposition Theorem (Theorem 2.3) for a generalized complete normed abelian group  $G$ , where generalized means (1) that the norm ( $\|x\|$ ) of the nonzero elements  $x$  of  $G$  may be infinite (i.e. if  $x \in G$  and  $x \neq 0$ , then  $0 < \|x\| \leq \infty$ ) and (2) that only the subgroup of bounded elements  $x$  (i.e.  $[x; \|x\| < \infty]$ ) is required to be complete. In §3, we apply this decomposition theorem to the space of finitely additive set functions on an algebra  $S$  of subsets of a set  $X$  in order to generalize the Lebesgue decomposition for bounded and finitely set functions on  $S$  (cf. [2]).

The basic form of our decomposition depends on what we call an admissible algebra  $T$  of endomorphisms on  $G$  (Definition 2.3). It will be seen that  $T$  is a Boolean algebra of projection operators with a condition on the manner in which projection on disjoint subgroups effects the norm. It is this latter condition which will provide our principal analytic tool.

Throughout this paper,  $G$  will denote a generalized complete normed abelian group.

**2. Decompositions and examples.** We shall develop the notion of an admissible algebra  $T$  of endomorphisms on  $G$  in two stages: the first algebraic and the second analytic.

**DEFINITION 2.1.** A set  $T$  of endomorphisms on  $G$  is said to be an algebra of endomorphisms on  $G$  if whenever each of  $a$  and  $b$  is an element of  $T$ , then

- (1)  $ab = ba \in T$  where  $ab(x) = a(b(x))$  for  $x \in G$ ,
- (2)  $aa = a$ , and
- (3)  $a' = e - a \in T$  where  $e(x) = x$  for  $x \in G$ .

Moreover, for each element  $a$  of  $T$  we let  $P(a) = [x \in G; a(x) = x]$ .

We shall see that the mapping  $a \rightarrow P(a)$  is an isomorphism of  $T$  onto a Boolean algebra of subgroups of  $G$ . We have, from (2), that  $\|a\| = \|a^n\| \leq \|a\|^n$  and, hence, if  $a \neq 0$  then  $\|a\| \geq 1$  ( $\|a\|$  may be infinite). Moreover,  $T$  has the following properties:

- (i)  $0 \in T$  ( $aa' = 0$ ),
- (ii)  $e \in T$  ( $e = 0'$ ), and
- (iii)  $a + b - ab = (a'b')' \in T$  [note that  $a'b'(a + b - ab) = 0$  and  $a'b' + (a + b - ab) = a'b' + (ab + ab') + b - ab = (a'b' + ab') + b = b' + b = e$ ].

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**DEFINITION 2.2.** If  $T$  is an algebra of endomorphisms on  $G$  and each of  $a$  and  $b$  is an element of  $T$ , then  $a \leq b$  means  $ab = a$ .

**THEOREM 2.1.** *If each of  $a$  and  $b$  is an element of an algebra  $T$  of endomorphisms on  $G$ , then*

- (i)  $0 \leq a \leq e$ ,
- (ii)  $a \leq b \Leftrightarrow a = ab \Leftrightarrow ab' = 0 \Leftrightarrow b' = a'b' \Leftrightarrow a' \geq b' \Leftrightarrow b = a + a'b \Leftrightarrow$  there exists an element  $c$  of  $T$  such that  $a = bc$ ,
- (iii)  $ab = 0 \Leftrightarrow a = ab' \Leftrightarrow a \leq b'$ ,
- (iv) if  $ab = 0$ ,  $c \leq a$ , and  $d \leq b$ , then  $cd = 0$ ,
- (v)  $a \cap b = ab$  where  $a \cap b = \sup \{c \in T; c \leq a, c \leq b\}$ ,
- (vi)  $a \cup b = a + b - ab$  where  $a \cup b = \inf \{c \in T; c \geq a, c \geq b\}$ ,
- (vii)  $a \leq b \Leftrightarrow P(a) \subset P(b)$ ,
- (viii) if  $a \leq b$ , then  $P(b) = P(a) \oplus P(a'b)$ , and
- (ix)  $P(ab) = P(a) \cap P(b)$ .

**Proof.** Parts (i), (ii), (iii), (iv), (v), and (vi) follow readily from our definitions. (vii) If  $a \leq b$ , then  $b'a = ab' = 0$  and, hence, if  $ax = x$ , then  $b'(x) = b'a(x) = 0$ . (viii) It follows from (vii) that  $P(a) \oplus P(a'b) \subset P(b)$ . Suppose  $x \in P(b)$ . Then  $x = b(x) = (a + a'b)(x) = ax + a'b(x)$ ; however,  $a(x) \in P(a)$  ( $aa = a$ ) and  $a'b(x) \in P(a'b)$ . Thus,  $x \in P(a) \oplus P(a'b)$ . (ix) We have, by (vii), that  $P(ab) \subset P(a) \cap P(b)$ . Suppose  $x \in P(a) \cap P(b)$ . Then  $ab(x) = a(b(x)) = a(x) = x$  and, hence,  $x \in P(ab)$ .

We shall now introduce our analytic tool which we shall denote by Property A.

**DEFINITION 2.3.** If  $T$  is an algebra of endomorphisms on  $G$ , then  $T$  is said to be an admissible algebra of endomorphisms on  $G$ , if  $T$  has Property A: If  $x \in G$ ,  $\|x\| < \infty$ , and  $\delta > 0$ , then there exists  $\epsilon > 0$  such that if each of  $a$  and  $b$  is an element of  $T$  and  $\|a'b(x)\| > \delta$ , then  $\|(a + a'b)(x)\| > \|a(x)\| + \epsilon$ .

**REMARK.** We note that Property A is a condition only on the bounded elements of  $G$ . At the end of this section, we shall give examples to show (1) that  $\epsilon$  may depend only on  $\delta$  (Example 2.2 with  $Q = 1$ ), (2) that  $\epsilon$  may depend on  $\delta$  and  $\|x\|$  but not on  $x$  (Example 2.2 with  $Q > 1$ ), and (3) that  $\epsilon$  may depend not only on  $\delta$  and  $\|x\|$  but also on  $x$  (Example 2.4).

Henceforth  $T$  shall denote an admissible algebra of endomorphisms on  $G$ .

**THEOREM 2.2.** *Suppose each of  $a$  and  $b$  is an element of  $T$ , then  $a \leq b$  if and only if  $\|a(x)\| \leq \|b(x)\|$  for all  $x \in G$ .*

**Proof.** If  $a \leq b$ , then  $b = a + a'b$  and, hence, if  $x \in G$ , then  $\|b(x)\| = \|(a + a'b)(x)\| \geq \|a(x)\|$ ; in fact, inequality holds unless  $\|a'b(x)\| = 0$ . If  $a \not\leq b$ , then  $ab' \neq 0$  and, hence, there exists an element  $x$  of  $G$  such that  $\|ab'(x)\| \neq 0$ . Thus,  $a(ab'(x)) = ab'(x) \neq 0$  while  $b(ab'(x)) = 0$ .

**COROLLARY 2.2.1.** *If  $a$  is an element of  $T$  and  $a \neq 0$ , then  $\|a\| = 1$ .*

**Proof.** We have remarked earlier that  $\|a\| = \|a^n\| \leq \|a\|^n$  and, hence,  $\|a\| \geq 1$ . By Theorem 2.2, we have that  $\|a\| \leq \|e\| = 1$ . Thus,  $\|a\| = 1$ .

**REMARK.** Later we shall give an example (Example 2.1) to show that the condition:  $a \leq b$  if and only if  $\|a(x)\| \leq \|b(x)\|$  for each  $x \in G$  is not sufficient to insure a decomposition. Property A is equivalent to: if  $x \in G$ ,  $\|x\| < \infty$ , and  $\delta > 0$ , then there exists  $\epsilon > 0$  such that if each of  $a$  and  $b$  is an element of  $T$ ,  $ab = 0$ , and  $\|b(x)\| > \delta$ , then  $\|(a+b)(x)\| > \|a(x)\| + \epsilon$ .

**LEMMA 2.3.1.** *If  $x \in G$ ,  $\{a_i\} \downarrow$  in  $T$ , and  $\lim_i \|a_i(x)\| < \infty$ , then  $\lim_i a_i(x)$  exists.*

**Proof.** Let  $L = \lim_i \|a_i(x)\|$  and let  $\delta > 0$ . There exists a positive integer  $k$  such that  $\|a_k(x)\| < \infty$ . There exists  $\epsilon > 0$  such that if each of  $c$  and  $d$  is an element of  $T$  and  $\|c'da_k(x)\| > \delta$ , then  $\|(c+c'd)a_k(x)\| > \|ca_k(x)\| + \epsilon$ . There exists a positive integer  $i$  such that  $i \geq k$  and  $\|a_i(x)\| < L + \epsilon$ . If  $j > i$ , then  $a_i = a_j + a'_j a_i$ . Thus,  $\|a_i(x)\| = \|a_j + a'_j a_i(x)\| < L + \epsilon \leq \|a_j(x)\| + \epsilon$  and, hence,  $\|a_i(x) - a_j(x)\| = \|a'_j a_i(x)\| \leq \delta$ .

**DEFINITION 2.4.** If each of  $x$  and  $y$  is an element of  $G$  and  $t > 0$ , then

- (1)  $Q(t, x) = [a \in T; \|a(x)\| < t]$ , and
- (2)  $r(t, x, y) = \sup [\|a(y)\|; a \in Q(t, x)]$ .

**LEMMA 2.3.2.** *Suppose each of  $x$  and  $y$  is an element of  $G$ ,  $\|y\| < \infty$ ,  $r(t) = r(t, x, y)$ ,  $r = \lim_{t \rightarrow 0+} r(t) < \infty$ , and  $\epsilon > 0$ . Then there exists a sequence  $\{b_i\} \downarrow$  in  $T$  such that*

- (1)  $\lim_i b_i(x) = 0$ ,
- (2)  $\lim_i \|b_i(y)\| > r - \epsilon$ , and
- (3)  $\lim_i b_i(y)$  exists.

**Proof.** If  $r = 0$ , it is sufficient to let  $b_i = 0$  for  $i \geq 1$ . Suppose  $r > 0$  and  $m$  is a positive integer such that  $2^{-m} < \epsilon$ . Let  $t_1 = 1$ . There exists  $\epsilon_1 > 0$  such that

- (1)  $\epsilon_1 < 2^{-(m+1)}$  and
- (2) if  $a, b \in T$  and  $\|a'b(y)\| > 2^{-(m+1)}$ , then  $\|(a+a'b)(y)\| > \|a(y)\| + \epsilon_1$ .

There exists  $a_1 \in Q(t_1, x)$  such that  $r(t_1) - \|a_1(y)\| < \epsilon_1$ . Let  $t_2 = 2^{-1}(t_1 - \|a_1(x)\|)$ . If  $a \in Q(t_2, x)$ , then  $\|(a_1 + a'_1 a)(x)\| \leq \|a_1(x)\| + \|a(x)\| < t_1$  and, hence,  $\|(a_1 + a'_1 a)(y)\| \leq r(t_1) < \|a_1(y)\| + \epsilon_1$ . Thus,  $\|a'_1 a(y)\| \leq 2^{-(m+1)}$ . There exists  $\epsilon_2 > 0$  such that if  $\|a'b(y)\| > 2^{-(m+2)}$ , then  $\|(a+a'b)(y)\| > \|a(y)\| + \epsilon_2$ . There exists  $a_2 \in Q(t_2, x)$  such that  $r(t_2) < \|a_2(y)\| + \epsilon_2$ . If we repeat the preceding process inductively, we obtain a sequence  $\{a_i\}$  of elements of  $T$ , a sequence  $\{e_i\}$  of positive numbers, and a sequence  $\{t_i\}$  of positive numbers such that

- (1)  $t_1 = 1$  and  $t_{i+1} = 2^{-1}(t_i - \|a_i(x)\|)$  for  $i > 1$ ,
- (2)  $0 < \epsilon_i < 2^{-(m+i)}$ ,
- (3) if  $a, b \in T$  and  $\|a'b(y)\| > 2^{-(m+i)}$ , then  $\|(a+a'b)(y)\| > \|a(y)\| + \epsilon_i$ ,
- (4)  $a_i \in Q(t_i, x)$ ,
- (5)  $r(t_i) < \|a_i(y)\| + \epsilon_i$ , and

(6) if  $a \in Q(t_{i+1}, x)$ , then  $(a_i + a'_i a) \in Q(t_i, x)$  which implies  $\|(a_i + a'_i a)(y)\| \leq r(t_i) < \|a_i(y)\| + \epsilon_i$  and hence,  $\|a'_i a(y)\| \leq 2^{-(m+i)}$ .

For each positive integer  $i$ ,  $a_i = a_i a'_{i-1} + a_i a_{i-1} a'_{i-2} + \dots + \prod_{j \leq i} a_j$ . Let  $b_i = \prod_{j \leq i} a_j$ . Then  $\{b_i\} \downarrow$  in  $T$ . Moreover,

$$\begin{aligned}
 (1) \quad & \|b_i(x)\| \leq \|a_i(x)\| \leq 2^{-(i-1)}, \\
 & \|(a_i - b_i)(y)\| \leq \|a_i a'_{i-1}(y)\| + \|a_i a_{i-1} a'_{i-2}(y)\| \\
 (2) \quad & + \dots + \left\| \left( \prod_{1 \leq j \leq i} a_j \right) a'_i(y) \right\| \leq \sum_{j < i} 2^{-(m+j)}, \text{ and} \\
 (3) \quad & r(t_i) - \|b_i(y)\| \leq r(t_i) - \|a_i(y)\| + \|(a_i - b_i)(y)\| \\
 & < \epsilon_i + \sum_{j < i} 2^{-(m+j)} < 2^{-(m+i)} + \sum_{j < i} 2^{-(m+j)} < 2^{-(m)} < \epsilon.
 \end{aligned}$$

Hence,  $\lim_i \|b_i(y)\| \geq r - \epsilon$ . However,  $\lim_i b_i(y) \leq \lim_i r(t_i) < \infty$  which implies (Lemma 2.3.1) that  $\lim_i b_i$  exists.

DEFINITION 2.5. If each of  $x$  and  $y$  is an element of  $G$ , then  $y$  is said to be

(1) absolutely continuous with respect to  $x \pmod T$  if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $a$  is an element of  $T$  and  $\|a(x)\| < \delta$ , then  $\|a(y)\| < \epsilon$ , and

(2) singular with respect to  $x \pmod T$  if for each  $\epsilon > 0$ , there exists an element  $a$  of  $T$  such that  $\|a(x)\| < \epsilon$  and  $\|a'(y)\| < \epsilon$ . Moreover, we denote by  $G_a(x, T)$  the set of elements  $h$  of  $G$  which are absolutely continuous with respect to  $x \pmod T$  and we denote by  $G_s(x, T)$  the set of elements  $u$  of  $G$  which are singular with respect to  $x \pmod T$ .

LEMMA 2.3.3. If  $x \in G$ , then each of  $G_a(x, T)$  and  $G_s(x, T)$  is a subgroup of  $G$  and  $G_a(x, T) \cap G_s(x, T) = 0$ . Moreover, if  $h \in G_a(x, T)$ , then  $G_s(h, T) \supset G_s(x, T)$ .

Proof. Suppose each of  $y$  and  $z$  is an element of  $G_a(x, T)$  and  $\epsilon > 0$ , then there exists  $\delta > 0$  such that if  $a \in T$  and  $\|a(x)\| < \delta$ , then each of  $\|a(y)\|$  and  $\|a(z)\| < \epsilon/2$  and, hence,  $\|a(y+z)\| < \epsilon$ . Thus,  $G_a(x, T)$  is an algebraic subgroup of  $G$ . Suppose  $\{y_i\}$  is a sequence of elements of  $G_a(x, T)$ ,  $\lim_i y_i = y$ , and  $\epsilon > 0$ . Then there exists a positive integer  $i$  such that  $\|y_i - y\| < \epsilon/2$  and there exists  $\delta > 0$  such that if  $a \in T$  and  $\|a(x)\| < \delta$ , then  $\|a(y_i)\| < \epsilon/2$  and, hence,  $\|a(y)\| \leq \|a(y_i)\| + \|a(y - y_i)\| \leq \|a(y_i)\| + \|y - y_i\| < \epsilon$ . Thus,  $y \in G_a(x, T)$ . Suppose each of  $y$  and  $z$  is an element of  $G_s(x, T)$  and  $\epsilon > 0$ , then there exists  $a$  and  $b \in T$  such that  $\|a(x)\| < \epsilon/2$ ,  $\|b(x)\| < \epsilon/2$ ,  $\|a'(y)\| < \epsilon/2$  and  $\|b'(z)\| < \epsilon/2$  and, hence,  $\|(a+b-ab)(x)\| \leq \|a(x)\| + \|(b-ab)(x)\| \leq \|a(x)\| + \|b(x)\| < \epsilon$  and  $\|(a+b-ab)'(y+z)\| = \|a'b'(y+z)\| \leq \|a'(y)\| + \|b'(z)\| < \epsilon$ . Suppose  $\{y_i\}$  is a sequence of elements of  $G_s(x, T)$ ,  $\lim_i y_i = y$ , and  $\epsilon > 0$ . Then there exists a positive integer  $i$  such that  $\|y_i - y\| < \epsilon/2$  and there exists  $a \in T$  such that  $\|a(x)\| < \epsilon/2$  and  $\|a'(y_i)\| < \epsilon/2$  and, hence,  $\|a'(y)\| \leq \|a'(y_i)\| + \|a'(y - y_i)\| < \epsilon$ . Therefore,  $G_s(x, T)$  is a subgroup of  $G$ . Suppose  $h \in G_a(x, T)$ ,  $s \in G_s(x, T)$ , and  $\epsilon > 0$ . Then there exists  $\delta > 0$  such that if  $a \in T$  and  $\|a(x)\| < \delta$ , then

$\|a(h)\| < \epsilon$  and, since  $s \in G_s(x, T)$ , there exists  $a \in T$  such that  $\|a(x)\| < \min[\epsilon, \delta]$  and  $\|a'(s)\| < \min[\epsilon, \delta]$ . Thus,  $\|a(h)\| < \epsilon$  and  $\|a'(s)\| < \epsilon$ . Hence,  $s \in G_s(h, T)$ .

REMARK. We shall give two examples (Examples 3.1 and 3.2) to show that, in general, one can not assert that  $G$  is the direct sum of  $G_a(x, T)$  and  $G_s(x, T)$ ; however, Theorem 2.3 shows that  $[y \in G; \|y\| < \infty] \subset G_a(x, T) \oplus G_s(x, T)$  for each  $x \in G$ .

LEMMA 2.3.4. *If each of  $x$  and  $y$  is a nonzero element of  $G$ , each of  $\|x\|$  and  $\|y\|$  is finite, and  $y$  is singular with respect to  $x$ , then  $\|x+y\| > \max[\|x\|, \|y\|]$ .*

Proof. Since the relation of being singular is symmetric, it is sufficient to show that  $\|x+y\| > \|x\|$ . There exists a sequence  $\{a_i\}$  of elements of  $T$  such that  $a_i x \rightarrow x$  and  $a'_i(y) \rightarrow y$ . There exists  $\epsilon > 0$  such that if  $a \in T$  and  $\|a'(x+y)\| > \|y\|/2$ , then  $\|x+y\| = \|a(x+y) + a'(x+y)\| > \|a(x+y)\| + \epsilon$ . Thus, since  $a_i(x+y) \rightarrow x$  and  $\|a'_i(x+y)\| \rightarrow y$ ,  $\|x+y\| = \lim_i \|a_i(x+y) + a'_i(x+y)\| \geq \|x\| + \epsilon$ .

LEMMA 2.3.5. *Suppose each of  $x$  and  $y$  is an element of  $G$ ,  $\{a_i\} \downarrow$  in  $T$ ,  $z = \lim_i a_i(y)$ ,  $r = \lim_{t \rightarrow 0+} r(t, x, y)$ , and  $\lim_i a_i(x) = 0$ . Then  $\|z\| \leq r$ .*

Proof. It is sufficient to suppose  $r < \infty$ . If  $\epsilon > 0$ , then there exists  $t > 0$  such that if  $a \in T$  and  $\|a(x)\| < t$ , then  $\|a(y)\| < r + \epsilon/2$  and there exists a positive integer  $i$  such that  $\|a_i(x)\| < t$  and  $\|a_i(y) - z\| < \epsilon/2$ . Thus,  $\|z\| \leq \|z - a_i(y)\| + \|a_i(y)\| < r + \epsilon$ .

THEOREM 2.3. *Suppose each of  $x$  and  $y$  is an element of  $G$  and  $\|y\| < \infty$ . Then there exists uniquely an element  $h$  of  $G$  and an element  $s$  of  $G$  such that*

- (1)  $y = h + s$ ,
- (2)  $h$  is absolutely continuous with respect to  $x \pmod T$ , and
- (3)  $s$  is singular with respect to  $x \pmod T$ .

Proof. Uniqueness follows from Lemma 2.3.3; the problem is to show existence. Let  $r = \lim_{t \rightarrow 0+} r(t, x, y)$ . If  $r = 0$ , let  $h = y$  and  $s = 0$  ( $y \in G_a(x, T)$  if and only if  $r = 0$ ). Suppose  $r > 0$ . For each positive integer  $i$ , there exists  $\epsilon_i > 0$  such that if each of  $a$  and  $b$  is an element of  $T$  and  $\|a'b(y)\| > 2^{-i}$ , then  $\|(a+a'b)(y)\| > \|a(y)\| + \epsilon_i$ . There exists (Lemma 2.3.2) a sequence  $\{a(1, i)\} \downarrow$  in  $T$  and an element  $z_1$  of  $G$  such that (1)  $\lim_i a(1, i)(x) = 0$ , (2)  $z_1 = \lim_i a(1, i)(y)$ , and (3)  $r - \|z_1\| < \epsilon_1$ . Let  $y_1 = \lim_i a(1, i)'(y) = y - z_1$  and let  $r_1 = \lim_{t \rightarrow 0+} r(t, x, y_1)$ . We assert that  $r_1 \leq 2^{-1}$ . Suppose, on the contrary, that  $r_1 > 2^{-1}$ . Then there exists a sequence  $\{b_i\} \downarrow$  in  $T$  and an element  $w$  of  $G$  such that (1)  $\lim_i b_i(x) = 0$ , (2)  $w = \lim_i b_i(y_1)$  and (3)  $\|w\| > 2^{-1}$  (Lemma 2.3.2 again). However,  $\lim_i \|b_i(y_1)\| = \lim_i \|b_i a(1, i)'(y)\|$  and, hence,

$$\begin{aligned} \|z_1 + w\| &= \lim_i \|a(1, i)(y) + a(1, i)'b_i(y)\| \geq \lim_i \|a(1, i)(y)\| + \epsilon_1 \\ &= \|z_1\| + \epsilon_1 > r; \end{aligned}$$

but,

$$\lim_i \|(a(1, i) + a(1, i)'b_i)(x)\| \leq \lim_i \|a(1, i)(x)\| + \|b_i(x)\| = 0.$$

This contradicts the supposition that  $r_1 > 2^{-1}$ . There exists a sequence  $\{a(2, i)\} \downarrow$  in  $T$  and an element  $z_2$  of  $G$  such that (1)  $\lim_i a(2, i)(x) = 0$ , (2)  $z_2 = \lim_i a(2, i)(y_1) = \lim_i a(2, i)a(1, i)'(y)$ , and (3)  $r_1 - \|z_2\| < \epsilon_2$ . Let  $y_2 = \lim_i a(2, i)'(y_1) = \lim_i a(2, i)'a(1, i)'(y)$  and let  $r_2 = \lim_{t \rightarrow 0^+} r(t, x, y_2)$ . Then  $r_2 \leq 2^{-2}$ . Proceeding by induction, either there exists a smallest positive integer  $i$  such that  $r_i = 0$  or  $r_i > 0$  for each positive integer  $i$ . In the former case we let  $h = y_i$  and  $s = \sum_{j \leq i} z_j$  while in the latter case we let  $h = \lim_i y_i$  and  $s = \sum z_i$ —of course, we must first show that each of  $\lim_i y_i$  and  $\sum z_i$  exists. Since  $y_i = y - \sum_{j \leq i} z_j$ , it is sufficient to show that  $\sum z_i$  exists and this is done as follows. Let  $s_i = \sum_{j \leq i} z_j$ . If  $j > i$ , then  $\|s_j - s_i\| = \|\sum_{k \leq j} z_k - \sum_{k \leq i} z_k\| = \|\sum_{i < k \leq j} z_k\| \leq \sum_{i < k \leq j} \|z_k\| \leq$  (Lemma 2.3.5)  $\sum_{i < k \leq j} r_{k-1} \leq \sum_{i < k \leq j} 2^{-(k-1)} < 2^{-(i-1)}$  and hence,  $\lim_i s_i = \sum z_i$  exists. By our construction, each  $z_i \in G_s(x)$  and, by Lemma 2.3.3,  $G_s(x, T)$  is a subgroup of  $G$ . Thus,  $s \in G_s(x, T)$ . In order to complete a proof of Theorem 2.3, it is sufficient to show that  $h \in G_a(x, T)$ . To this end, suppose  $\epsilon > 0$  and  $2^{-(i-1)} < \epsilon/2$ . Then  $\|h - y_i\| = \|s - s_i\| \leq 2^{-(i-1)} < \epsilon/2$  and  $r_i = \lim_{t \rightarrow 0^+} r(t, x, y_i) \leq 2^{-i} < \epsilon/2$  which implies that there exists  $t > 0$  such that  $r(t, x, y_i) < \epsilon/2$ . If  $a \in T$  and  $\|a(x)\| < t$ , then  $\|a(h)\| = \|a(h - y_i) + a(y_i)\| \leq \|h - y_i\| + \|a(y_i)\| < \epsilon/2 + \epsilon/2 = \epsilon$ . Therefore,  $h \in G_a(x, T)$ .

**DEFINITION 2.6.** The statement that a finite subset  $\{a_i; i \leq n\}$  of  $T$  is a finite partition of  $e$  in  $T$  means that  $a_i a_j = 0$  if  $i \neq j$  and  $\sum_{i \leq n} a_i = e$ .

**THEOREM 2.4.** Suppose  $x \in G$ ,  $\|x\| < \infty$ , and  $\epsilon > 0$ . Then there exists a finite partition  $P = \{a_i; i \leq n\}$  of  $e$  in  $T$  such that if  $a \in T$  and  $i \leq n$ , then at least one of  $\|a a_i(x)\|$  and  $\|a' a_i(x)\| < \epsilon$ .

**Proof.** Suppose, on the contrary, that Theorem 2.4 is false. Then there exists a pair  $(x, \epsilon)$  which contradicts Theorem 2.4:  $\|x\| < \infty$ ,  $\epsilon > 0$ , and if  $\{a_i; i \leq n\}$  is a finite partition of  $e$  in  $T$  then there exists an element  $a$  of  $T$  and a positive integer  $i \leq n$  such that each of  $\|a a_i(x)\|$  and  $\|a' a_i(x)\| \geq \epsilon$ . Moreover, since the pair  $(x, \epsilon)$  contradicts Theorem 2.4, for each element  $a$  of  $T$  at least one of the pairs  $(a(x), \epsilon)$  and  $(a'(x), \epsilon)$  contradicts Theorem 2.4, i.e., if  $P = \{a_i; i \leq m\}$  works for  $a(x)$  (i.e., if  $b \in T$  and  $i \leq m$  imply at least one of  $\|b a_i a(x)\|$  and  $\|b' a_i a(x)\| < \epsilon$ ) and  $Q = \{b_j; j \leq n\}$  works for  $a'(x)$ , then  $R = \{a_i a; i \leq m\} \cup \{b_j a'; j \leq n\}$  works for  $x$ . Hence, there exists  $a_1 \in T$  such that (1)  $\|a_1(x)\| \geq \epsilon$  and (2) the pair  $(a_1'(x), \epsilon)$  contradicts Theorem 2.4;  $\dots$ ; there exists  $a_{i+1} \in T$  such that (1)  $\|a_{i+1} \prod_{j \leq i} a_j'(x)\| \geq \epsilon$  and (2) the pair  $(\prod_{j \leq i+1} a_j'(x), \epsilon)$  contradicts Theorem 2.4. Let  $b_i = \sum_{j \leq i} a_j$ . But, by Lemma 2.3.1,  $\lim_i b_i'(x)$  exists and, hence,  $\lim_i b_i(x)$  exists. This contradiction ( $\|x\| < \infty$ ) establishes Theorem 2.4.

We shall apply Theorem 2.4 in §3. However, we shall first conclude this

section by giving four examples. Our first example sheds some light on the question: How strong an analytic condition is needed on an algebra  $U$  of endomorphisms on  $G$  in order to assure that Theorem 2.3 will hold (mod  $U$ )?

**EXAMPLE 2.1.** In this example,  $T$  will be an algebra of endomorphisms on  $G$  for which Theorem 2.3 does not hold; however,  $T$  will have the property that if  $a, b \in T$ , then  $a \leq b$  if and only if  $\|a(x)\| \leq \|b(x)\|$  for all  $x \in G$ .

Let  $S$  be an algebra of subsets of a set  $X$ ,  $S$  contain an infinite number of elements,  $G = [x; x \text{ is a real valued function on } X, \|x\| = \sup [ |x(t)| ; t \in X ]]$ , and  $T = [P_E; P_E(x) = x \cdot C(E) \text{ where } C(E)(t) = 1 \text{ if } t \in E \text{ and } C(E)(t) = 0 \text{ if } t \notin E]$ . Then there exist bounded elements  $x$  and  $y$  of  $G$  such that if each of  $h$  and  $s$  is an element of  $G$ ,  $y = h + s$ , and  $h$  is absolutely continuous with respect to  $x$  (mod  $T$ ), then  $s$  is not singular with respect to  $x$  (mod  $T$ ).

**Proof.** Since  $S$  is infinite, there exists a sequence  $\{E_i\}$  of non-null pairwise disjoint elements of  $S$ . Let  $y = C(X)$  and  $x = \sum 2^{-i} C(E_i)$ . Suppose  $y = h + s$  and  $h \in G_a(x, T)$ . Then there exists  $\delta > 0$  such that if  $E \in S$  and  $\|P_E(x)\| < \delta$ , then  $\|P_E(h)\| < 2^{-1}$  and, hence, there exists a positive integer  $i$  such that

$$\left\| h \cdot C \left( \bigcup_{j>i} E_j \right) \right\| < 2^{-1}. \text{ Thus, for all } j > i, \inf [s(t); t \in E_j] \geq 2^{-1}$$

and, hence,  $s$  is not singular with respect to  $x$  (mod  $T$ ).

**EXAMPLE 2.2.** Let  $X, S, G$ , and  $T$  be defined as in Example 2.1 except that if  $x \in G$ , then  $\|x\| = (\sum_{t \in X} |x(t)|^Q)^{1/Q}$ , where  $Q$  is a real number  $\geq 1$ . Then  $T$  is an admissible algebra of endomorphisms on  $G$ .

**EXAMPLE 2.3.** Let  $G$  be a Hilbert space, let  $[E_\lambda; -\infty \leq \lambda \leq \infty]$  be a resolution of the identity, and let  $T$  be the algebra of projection operators generated by projections of the form  $E_{\lambda+\mu} - E_\lambda, \mu \geq 0$ . Then  $T$  is an admissible algebra of endomorphisms on  $G$ .

**EXAMPLE 2.4.** Let  $X, S, G$ , and  $T$  be defined as in Example 2.1 except that  $X$  is the set of positive integers, if  $x \in G$ , then  $\|x\| = |x(1)| + \sum_{i \geq 1} (|x(2i)|^i + |x(2i+1)|^i)^{1/i}$ , and each one element subset  $[i]$  of  $X$  is an element of  $S$ . For each positive integer  $i$  we let  $x_i = C([2i, 2i+1])$  and we let  $a_i = P_{[2i]}$ . Then  $\|x_i\| = 2^{1/i}, \|a_i(x_i)\| = 1$  and  $\|a_i'(x_i)\| = 1$ . Thus, in this example, while  $T$  is admissible, the  $\epsilon$  we get in satisfying Property A depends not only on  $\delta$  and  $\|x\|$  but also on  $x$ .

**3. Spaces of finitely additive set functions.** Throughout this section,  $X$  will denote a set,  $S$  will denote an algebra of subsets of  $X$ ,  $G$  will denote the generalized complete normed abelian group of finitely additive set functions on  $S$  where the norm ( $\|x\|$ ) of the elements  $x$  of  $G$  is the total variation ( $V(x, X)$ ) of  $x$  on  $X$ , and  $T$  will denote the admissible algebra of projection operators induced by  $S$ , i.e.,  $T = [P_E; P_E(x)(F) = x(E \cap F) \text{ for } E, F \in S \text{ and } x \in G]$ .

Let us recall that if  $S$  is an infinite set, then there exist unbounded finitely additive set functions  $x$  on  $S$  (i.e., elements  $x$  of  $G$  such that  $\|x\| = \infty$ ).

We shall extend the Lebesgue type decomposition for bounded and finitely additive set functions on a set algebra  $S$  which was presented in [2]. The definitions of absolute continuity and singularity which we use here are equivalent to those which were used in [2]. In order to make this paper self-contained with respect to notation and terminology, it is necessary to observe the following:

- (1)  $\|P_E(x)\| = V(x, E)$  for  $E \in S$  and  $x \in G$ ,
- (2)  $P_E P_F = P_{E \cap F}$ ,
- (3)  $P_{E'} = P_{(E^c)}$ , where  $E' = X - E$ ,
- (4) if  $E \cap F = \emptyset$ :  $P_E P_F = 0$ , then  $\|P_E(x) + P_F(x)\| = \|P_{E \cup F}(x)\| = \|P_E(x)\| + \|P_F(x)\|$  for all  $x \in G$ , and
- (5)  $P_E \leq P_F$  if and only if  $E \subset F$ . Our first extension is the following consequence of Theorem 2.3.

**THEOREM 3.1.** *If  $x$  is a finitely additive set function on  $S$  and  $y$  is a bounded and finitely additive set function on  $S$ , then there exists uniquely an element  $h$  of  $G$  and an element  $s$  of  $G$  such that*

- (1)  $y = h + s$ ,
- (2)  $h$  is absolutely continuous with respect to  $x \pmod{T}$ , and
- (3)  $s$  is singular with respect to  $x \pmod{T}$ .

**THEOREM 3.2.** *If  $x$  is a bounded finitely additive set function on  $S$  and  $y$  is absolutely continuous with respect to  $x \pmod{T}$ , then  $y$  is bounded.*

**Proof.** Since  $y$  is absolutely continuous with respect to  $x \pmod{T}$ , there exists  $\delta > 0$  such that if  $E \in S$  and  $V(x, E) < \delta$ , then  $V(y, E) < 1$ . By Theorem 2.4, there exists a finite partition  $[P_{E_i}; i \leq n]$  of  $P_x$  in  $T$  such that if  $E \in S$  and  $i \leq n$ , then at least one of  $V(x, E \cap E_i)$  and  $V(x, E' \cap E_i) < \delta$  and, hence, at least one of  $V(y, E \cap E_i)$  and  $V(y, E' \cap E_i) < 1$ . For each positive integer  $i \leq n$ ,  $|y(E \cap E_i) - y(E_i)| = |y(E' \cap E_i)|$ . Hence  $|y(E \cap E_i)| < |y(E_i)| + 1$  for all  $E \in S$ . Thus,  $V(y, E_i) \leq 2(\sup\{|y(E \cap E_i)|; E \in S\}) \leq 2(|y(E_i)| + 1) < \infty$  for  $i \leq n$  and, hence,  $\|y\| = V(y, X) = \sum_{i \leq n} V(y, E_i) < \infty$ .

In the general setting of §2, the analog of Theorem 3.2 is not, in general, true. For example, let  $S$  be infinite and let  $T'$  be the subalgebra of  $T$  which consists of 0 and  $e$ . Then any two nonzero elements of  $G$  are absolutely continuous with respect to each other  $(\pmod{T'})$ ; but, there exist unbounded, as well as nonzero bounded, elements of  $G$ .

**THEOREM 3.3.** *If each of  $x$  and  $y$  is a finitely additive set function on  $S$  and at least one of  $x$  and  $y$  is bounded, then  $y$  is decomposable with respect to  $x \pmod{T}$  if and only if there exists a sequence  $\{E_i\} \downarrow$  in  $S$  such that  $\lim_i V(x, E_i) = 0$  and  $\lim_i V(y, E_i) < \infty$ .*



**Proof.** If  $\|y\| < \infty$  a decomposition exists; moreover, it is sufficient to let  $E_i = \theta$  for  $i \geq 1$ . Suppose  $\|y\| = \infty$  and  $\|x\| < \infty$ .

*Necessity.* Suppose  $y = h + s$ ,  $h \in G_a(x, T)$ , and  $s \in G_s(x, T)$ . Then, by Theorem 3.3,  $\|h\| < \infty$  and, by the definition of singularity, there exists a sequence  $\{F_i\}$  of elements of  $S$  such that  $V(x, F_i) < 2^{-i}$  and  $V(y, F_i) < 2^{-i}$ . Let  $E_i = \prod_{j \leq i} F_j$ . Then  $\{E_i\} \downarrow$  in  $S$ ,  $V(x, E_i) < 2^{-i}$ , and  $V(y, E_i) = V(h, E_i) + V(s, E_i) \leq V(h, X) + \sum_{j \leq i} V(s, F_j) < \|h\| + 1 < \infty$ .

*Sufficiency.* Let  $y_i = P'_{E_i}(y)$ . Then  $\|y_{i+j} - y_i\| = |V(y, E'_{i+j}) - V(y, E_i)| = V(y, E_i \cap E'_{i+j}) = \|y_{i+j}\| - \|y_i\|$ . Hence  $z = \lim_i y_i$  exists and  $\|z\| < \infty$ ; moreover,  $y - z \in G_s(x; T)$ . By Theorem 3.1, there exist  $h$  and  $s_1$  such that  $z = h + s_1$ ,  $h \in G_a(x, T)$ , and  $s_1 \in G_s(x, T)$ . Finally,  $s = y - h = (y - z) + s_1 \in G_s(x, T)$ .

**EXAMPLE 3.1.** Let  $X$  be the set of positive integers and let  $S$  be the algebra of all subsets of  $X$ . Let  $x \in G$  such that if  $E \subset X$  and  $E \neq \theta$ , then  $x(E) = \sum_{i \in E} 2^{-i}$ . Let  $y \in G$  such that  $y(X) = 0$  and  $y(\{i\}) = 1$  for all  $i \in X$ . Then  $y$  is not decomposable with respect to  $x \pmod T$ .

**EXAMPLE 3.2.** Let  $X$  be the half open interval  $[0, 1)$ . Let  $S$  be the algebra of subsets of  $X$  generated by elements of the form  $E(m, n) = [m/2^n, (m+1)/2^n, 0 \leq m < 2^n]$ , i.e.,  $S = [\cup_{i \leq k} E(m_i, n_i); 0 \leq m_i < 2^{n_i}]$ . We shall define  $y$  inductively as follows. Let  $y(X) = 1$ ,  $y(E(2m, n+1)) = 2y(E(m, n))$ , and  $y(E(2m+1, n+1)) = -y(E(m, n))$ . Then  $y$  is unbounded on each nonempty element of  $S$ . Hence  $y$  is decomposable with respect to no bounded finitely additive set function on  $S$  except the constant function 0; but, every finitely additive set function on  $S$  is absolutely continuous with respect to  $y$ . Cameron (cf. [1]) has shown that a complex Wiener measure is unbounded on every nonempty set of the algebra on which it is defined.

**COROLLARY 3.3.1.** *If  $y$  is an unbounded finitely additive set function on  $S$  (i.e.,  $y \in G$  and  $\|y\| = \infty$ ), then there exists a bounded finitely additive set function  $x$  on  $S$  such that  $y$  is not decomposable with respect to  $x$ .*

**Proof.** Let  $K = [E \in S; V(y, E) < \infty]$ . Then  $K$  is a proper ideal in  $S$ . There exists a maximal proper ideal  $J$  in  $S$  such that  $K \subset J$ . There exists, uniquely,  $x \in G$  such that  $x(E) = 0$  if  $E \in J$  and  $x(E) = 1$  if  $E \notin J$ . It is impossible to decompose  $y$  with respect to  $x$ : if  $V(x, E) < 1$ , then  $E' \notin K$  and, hence,  $V(y, E') = \infty$ .

**THEOREM 3.4.** *Suppose  $S$  is a sigma algebra,  $y$  is a countably additive set function on  $S$ , and  $x$  is a finitely additive set function on  $S$ . Then  $y$  is  $(\epsilon - \delta)$  absolutely continuous with respect to  $x \pmod T$  if and only if  $y$  is 0-0 absolutely continuous with respect to  $x$ , i.e., if and only if  $E \in S$  and  $V(x, E) = 0$  imply  $V(y, E) = 0$ .*

**Proof.** *Sufficiency.* Suppose  $y \notin G_a(x, T)$ . Then  $\lim_{t \rightarrow 0^+} r(t, x, y) > 0$ , and, by Lemma 3.2, there exists a sequence  $\{E_i\} \downarrow$  in  $S$  such that  $\lim_i V(x, E_i) = 0$

and  $\lim_i V(y, E_i) > 0$ . Since  $S$  is a sigma algebra,  $E = \bigcap E_i \in S$ ; moreover,  $V(x, E) \leq \lim_i V(x, E_i) = 0$ . Finally, since  $\nu$  is countably additive on  $S$ ,  $V(y, E) = \lim_i V(y, E_i) > 0$ .

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2. R. B. Darst, *A decomposition of finitely additive set functions*, J. Math. Reine Angew. **210** (1962), 31-37.

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## ERRATA TO VOLUME 98

C. C. Elgot. *Decision problems of finite automata design and related arithmetics*

Page 23, Lines 10, 11. Replace each  $\hat{f}$  by  $\hat{p}$ .

Page 23, 3.6(b), Line 2. The words "by a finite number . . ." should start a new line.

Page 24, Line 9 (second display formula). Replace " $(a, b)$ " by " $(b, a)$ ".

Page 46, 8.6.2, Line 5. Replace "let  $n$  be the maximum" by "let  $n$  be one more than the maximum".

Line 7. Replace "for some  $n$ -ary  $R$ " by "for some  $R$  which is  $n$ -ary".

The third sentence (beginning on the sixth line) of §8.6.2 on page 46 is in error but is readily correctable. "It may be seen that  $T_{m+m'+r}^\infty(\Lambda_x M) = S_1 \cup S_2 \cup \dots \cup S_k$ , where  $S_j$ ,  $j=1, 2, \dots, k$ , is the set of all infinite  $R_j$ -sequences  $f$  such that  $(f \upharpoonright n) \in E_j$ , for appropriate  $R_j$ ,  $E_j$ , and that  $k$  need not be 1. For example, let  $M$  be

$$0 \in F_1 \wedge 0 \notin F_2 \wedge (x \in F_1 \wedge x \notin F_2 \cdot \vee \cdot x \in F_1 \wedge x \in F_2) : \vee :$$

$$0 \notin F_1 \wedge 0 \in F_2 \wedge (x \in F_1 \wedge x \in F_2 \cdot \vee \cdot x \notin F_1 \wedge x \in F_2).$$

Then  $T_2^\infty(\Lambda_x M)$  is the union of the set of all infinite sequences in  $\langle 1, 0 \rangle$  and  $\langle 1, 1 \rangle$  which begin with  $\langle 1, 0 \rangle$  and the set of all infinite sequences in  $\langle 0, 1 \rangle$  and  $\langle 1, 1 \rangle$  which begin with  $\langle 0, 1 \rangle$ . Thus, in this case,  $k=2$ . Let  $Q$  be

$$(0 \in F_1 \wedge 0 \notin F_2 \cdot \vee \cdot 0 \notin F_1 \wedge 0 \in F_2)$$

$$: \wedge : (x \in F_1 \wedge x \notin F_2 \wedge x \in F_3 \wedge x' \in F_3 \cdot \vee \cdot x \in F_1 \wedge x \in F_2 \wedge x \in F_3 \wedge x' \in F_3$$

$$\cdot \vee \cdot x \in F_1 \wedge x \in F_2 \wedge x \notin F_3 \wedge x' \in F_3 \cdot \vee \cdot x \notin F_1 \wedge x \in F_2 \wedge x \notin F_3 \wedge x' \notin F_3).$$

Then  $\Lambda_x M \equiv \vee_{F_3} \Lambda_x Q$  and  $T_3^\infty \Lambda_x Q$  is a set of  $R$ -sequences, for the binary  $R$  indicated by the formula, beginning in a designated way and  $T_2^\infty(\Lambda_x M)$  is a projection of  $T_3^\infty(\Lambda_x Q)$ . Quite generally it is the case that  $S_1 \cup S_2 \cup \dots \cup S_k$  is the projection of a set of  $R$ -sequences beginning in a designated way so