KILLING THE MIDDLE HOMOTOPY GROUPS OF ODD DIMENSIONAL MANIFOLDS

BY

C. T. C. WALL

The main object of this paper is to prove the theorem: If W is an m-parallelisable $(2m+1)$-manifold, whose boundary has no homology in dimensions m, $m+1$; then W is χ-equivalent to an m-connected manifold.

This is written as a sequel to Milnor's paper *A procedure for killing homotopy groups of differentiable manifolds.* We attempt to preserve the notations of this paper, and refer to it as [M].

Milnor proves in [M] that W is χ-equivalent to an $(m-1)$-connected manifold, and we show in §1 that we can reduce $H_m(W)$ to a finite group. §2 is devoted to the definition and study of a nonsingular bilinear form on this group, symmetric if m is odd, and skew if m is even. §3 applies these results to prove the theorem above. It follows, in the notation of [8], that $\Theta_{2m}(\partial\pi)=0$ (this has also been proved by Milnor and Kervaire). In §4 we prove a more precise version of Milnor's reduction of $(m-1)$-parallelisable to $(m-1)$-connected manifolds; this is applied in §5 to obtain results about the topology of certain $(m-1)$-parallelisable $(2m+1)$-manifolds. Our results are complete for a class of 5-manifolds, and yield an interesting test for cobordism.

Throughout this paper, "manifold" shall mean "compact connected differential manifold." Here, "differential" means "endowed with differential structure"; it seems a more suitable word for this concept than "differentiable," which ought to mean "admitting at least one differential structure."

1. **Preliminaries.** We consider manifolds W of dimension $2m+1$ (where $1<m$). We suppose that W is m-parallelisable, and that we have already killed the homotopy groups $\pi_i(W)$ for $i<m$; we will study the possibility of killing $\pi_m(W)$. Since $m<(1/2)$ dim W, every element of $\pi_m(W)$ is representable by an imbedding $f_0\colon S^m\to W$. The induced bundle $f_0^*(\tau^{2m+1})$ is trivial, so by Lemma 3 of [M] there exists an imbedding $f_1\colon S^m\times D^{m+1}\to W$ extending f_0. In this case we can carry out surgery without trouble; the only snag is that we are not sure of simplifying $\pi_m(W)$ when we do it.

We reconsider the proof of Lemma 2 of [M]. It is convenient to give it a somewhat different form. We first pass from W to the manifold W'' obtained by removing the interior of $f_1(S^m\times D^{m+1})$ from it, and then to the manifold W' obtained by glueing $D^{m+1}\times S^m$ in its place. It is easy to see that $\pi_m(W'')\to\pi_m(W)$ is onto, and its kernel is generated by the class of $f_2(e\times S^m)$, where $f_2\colon S^m\times S^m\to W''$ is induced by f_1 (and we use e indiscriminately to denote an

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unspecified base point). For the same reason, $\pi_m(W'') \to \pi_m(W')$ is onto, and its kernel is generated by the class of $f_2(S^m \times e)$. We will usually denote these two classes in $\pi_m(W'')$, or rather, the corresponding ones in $H_m(W'')$, by z, x.

We suppose, as in §6 of [M], that $H_{m+1}(\text{Bd } W)$ and $H_m(\text{Bd } W)$ vanish, hence so also does $H_{m-1}(\text{Bd } W)$. The homology sequence for the pair $(W, \text{Bd } W)$ and the Universal Coefficient Theorem, imply that $H_m(W) = H_m(W, \text{Bd } W)$ and $H_{m+1}(W) = H_{m+1}(W, \text{Bd } W)$ with any coefficient group, and similarly for cohomology, so in these dimensions Poincaré duality for W has the same form as for a closed manifold.

We first consider the case when $H_m(W)$ has elements of infinite order.

LEMMA 1. *Let x be of infinite order in $H_m(W)$ and be indivisible. Then if we perform surgery on W starting from x, the new class z in $H_m(W'')$ vanishes, so $H_m(W')$ is obtained from $H_m(W)$ by killing x.*

Proof. Denote the chain $f_1(e \times D^{m+1})$ by $\bar{q}$. Since x is indivisible, by Poincaré duality there is a class $p \in H_{m+1}(W)$ with unit intersection number with x. Since $f_1((S^m - e) \times D^{m+1})$ is a cell, we may choose a representative cycle $\bar{p}$ for p which avoids it: and clearly we may suppose that the only simplexes of $\bar{p}$ contained in $f_1(S^m \times D^{m+1})$ form $\bar{q}$, since the intersection number of p with x is unity. But now $\bar{q} - \bar{p}$ defines a chain in W'' whose boundary $\bar{z} = f_2(e \times S^m)$ determines z. Hence $z = 0$ in $H_m(W'')$ and a fortiori also in $H_m(W')$.

Now as in §6 of [M], W' is m-parallelisable if W is, so we can repeat the process to kill all elements of infinite order in $H_m(W)$. Hence we may assume $G = H_m(W)$ finite. Let its exponent (the l.c.m. of the orders of its elements) be θ. We shall take homology and cohomology with coefficient group Z_θ, but still represent the classes by integral chains. Now $H_m(W, Z_\theta) = G$ by the Universal Coefficient Theorem; we shall identify these groups by this isomorphism. Consider the map $\bar{y} \to \partial \bar{y}/\theta$ of chain groups: this induces a homomorphism γ: $H_{m+1}(W, Z_\theta) \to H_m(W, Z_\theta)$ dual to the Bockstein in cohomology. This is onto since each element of the latter group has a representative θ times which is a boundary, and (1-1) since if $\bar{y}$ represents a class y with $\gamma y = 0$, there exists a chain $\bar{w}$ with $\partial \bar{w} = \partial \bar{y}/\theta$, and since $H_{m+1}(W) = 0$ (by duality), $\bar{y} - \theta \bar{w}$, being a cycle, is a boundary, so $\bar{y}$ determines the zero element of $H_{m+1}(W, Z_\theta)$.

2. **The nonsingular bilinear form.** Combining γ with isomorphisms deduced from Poincaré duality and the Universal Coefficient Theorem we now have

$$G = H_m(W) \cong H_m(W, Z_\theta) \cong H_{m+1}(W, Z_\theta) \cong H^m(W, Z_\theta) = \text{Hom}(G, Z_\theta).$$

Hence we have a pairing of G with itself to Z_θ. Write b: $G \otimes G \to Z_\theta$.

LEMMA 2. *b is a nonsingular bilinear form on G, symmetric if m is odd and skew if m is even.*

Proof. We have already proved the first part. For the second it is more convenient to work in cohomology (isomorphic to homology by the above). Here, b is given by $b(x, y) = \beta x \cdot y$, evaluated on the fundamental class of $(W, \text{Bd } W)$, where β denotes the Bockstein. Now

$$\begin{aligned} b(x, y) + (-1)^m b(y, x) &= \beta x \cdot y + (-1)^m \beta y \cdot x \\ &= \beta x \cdot y + (-1)^m x \cdot \beta y \\ &= \beta(xy). \end{aligned}$$

But $xy \in H^{2m}(W, Z_\theta)$, so reverting to homology we get $H_1(W, \text{Bd } W; Z_\theta)$. But every element of this is the restriction of an integer class, so applying ∂/θ gives zero, as required.

Note. This result also follows by interpreting $b(x, y)$ as a linking number (mod θ).

We shall now show how the form b determines the effect of surgery on $H_m(W)$. Let x be the element chosen to operate on, and let y be of order r in $H_m(W)$. Since $ry = 0$, $\theta \mid rb(y, x)$, so $(r/\theta)b(y, x)$ is an integer defined modulo r. ($\mid$ denotes divisibility.) Represent y by an m-cycle $\bar{y}$ not meeting $f_1(S^m \times D^{m+1})$. In W'', $\bar{y}$ represents a homology class y', and ry' is a multiple of z.

LEMMA 3. *If we write $ry' = \lambda z$, we have $\lambda \equiv (r/\theta)b(y, x)$ (mod r).*

Proof. Let $\bar{p}$ be an $(m+1)$-chain with $\partial \bar{p} = r\bar{y}$. As in the proof of Lemma 1, if the intersection number of $\bar{p}$ and x is λ, we may suppose that the only simplexes of $\bar{p}$ contained in $f_1(S^m \times D^{m+1})$ form $\lambda \bar{q}$. Now $\bar{p} - \lambda \bar{q}$ defines a chain in W'', of boundary $r\bar{y} - \lambda \bar{z}$, hence $ry' = \lambda z$. But as $(\partial/\theta)(\theta/r)\bar{p} = \bar{y}$, the class mod θ of $\theta \bar{p}/r$ corresponds under γ to y, so by definition of b,

$$b(y, x) \equiv (\theta \bar{p}/r) \cap x \equiv \theta\lambda/r \pmod{\theta}$$

i.e.

$$\lambda \equiv \frac{r}{\theta} b(y, x) \pmod{r}.$$

COROLLARY. *Let $b(y, x) = 0$. Then there exists a class y'' in $H_m(W'')$ inducing y in $H_m(W)$ and also of order r.*

Proof. $ry' = krz$ for some integer k. We may choose $y'' = y' - kz$.

Before we can prove our main theorem we need a number-theoretic lemma about bilinear forms b.

LEMMA 4. *Let $b: G \otimes G \to Z_\theta$ be a nonsingular bilinear form on the finite Abelian group G. Write $c(x)$ for $b(x, x)$.*

(i) *If b is symmetric and $c(x) = 0$ for all x, then $\theta = 2$ and we can find a basis $\{x_i, y_i: 1 \leq i \leq r\}$ for G such that*

$$b(x_i, y_j) = \delta_{ij}\, b(x_i, x_j) = b(y_i, y_j) = 0.$$

(ii) *If b is skew-symmetric, we can find elements x_i, y_i of order θ_i in G $(1 \leqq i \leqq r)$ such that*

$$b(x_i, x_j) = b(x_i, y_j) = b(y_i, y_j) = 0 \text{ for } i \neq j;$$
$$c(x_i) = 0, \qquad b(x_i, y_i) \text{ has order } \theta_i,$$

and G contains the direct sum of the cyclic subgroups generated by the x_i, y_i as a direct summand of index at most 2.

Corollary. *Under the conditions of* (ii), *if B is the subgroup generated by the x_i, then either*

$$G \cong B \oplus B \quad or \quad G \cong B \oplus B \oplus Z_2.$$

Proof. (i) Under these hypotheses, for all x, y in G,

$$2b(x, y) = b(x, y) + b(y, x) = c(x + y) - c(x) - c(y) = 0.$$

Hence the exponent of G is 2. We now pick x_i, y_i by induction. Choose any nonzero x_1, then since b is nonsingular there exists y_1 with $b(x_1, y_1) = 1$. Since $c(x_1) = 0$, $y_1 \neq x_1$. Now G is the direct sum of the subgroup $Gp\{x_1, y_1\}$ and H, the annihilator of $Gp\{x_1, y_1\}$, and b induces a nonsingular form on H, so we may continue the induction. (All this is of course well known.)

Note. If $x_1, x_2, \cdots$ belong to a group, $Gp\{x_1, x_2, \cdots\}$ denotes the subgroup which they generate.

(ii) Since b is skew, $c(x) = b(x, x) = -c(x)$, so has order 2. Moreover, $c(x+y) - c(x) - c(y) = b(x, y) + b(y, x) = 0$, so c is a homomorphism $G \rightarrow Z_2$. Now since G is a finite Abelian group it is the direct sum of its Sylow subgroups S_p, and these are clearly orthogonal under b, so we can take them separately.

First, suppose p odd. Let x_1 be an element of maximal order p^r in S_p. Then since b is nonsingular there exists y_1 such that $b(x_1, y_1)$ has order p^r. Then y_1 has order p^r (not greater, since this was maximal) and G contains the direct sum of the cyclic groups generated by x_1, y_1; for if $0 = \lambda x_1 + \mu y_1$, then

$$0 = b(\lambda x_1 + \mu y_1, y_1) = \lambda b(x_1, y_1) + \mu c(y_1) = \lambda(b x_1, y_1)$$

so λ is divisible by p^r; similarly, so is μ. Again we have $G = Gp\{x_1, y_1\} \oplus H$, where H is the annihilator of x_1, y_1, since any $z \in G$ can be written as

$$z = b(z, y_1)x_1 - b(z, x_1)y_1 + h$$

with $h \in H$. b induces a nonsingular form on H, so we may apply induction to obtain our theorem.

For $p = 2$ we apply the same argument, if $1 < r$. The proof of independence of x_1, y_1 must be modified as follows. By the equation above, $b(x_1, y_1)$ has order at most 2, so λ is divisible by 2^{r-1}, so by 2. Similarly, so is μ. Hence $\mu c(y_1) = 0$, and we may proceed as before. (The modification of the direct sum argument is left to the reader.) We may suppose that $c(x_1)=0$, for if not, and $c(y_1)=0$, we interchange x_1, y_1; whereas if $c(x_1)=c(y_1)\neq 0$, we may replace x_1 by x_1+y_1.

Finally, suppose G has exponent 2. If the order of G is two, G has the required form. If it is greater, let x_1 be any nonzero element of Ker c, and y_1 such that $b(x_1, y_1)\neq 0$; then we can split off the direct summand $Gp\{x_1, y_1\}$ as before. This concludes the proof.

Note. (i) We can be somewhat more precise in our reduction of (G, b), but this is of no advantage for the applications we shall make of the lemma.

(ii) The above proof is complicated by the possibility $c\neq 0$ in (ii). We shall show in §5 that for m-parallelisable W, c must in fact vanish.

3. **Proof of theorem.**

THEOREM. *Let W be m-parallelisable, of dimension $2m+1$. If the boundary of W has no homology in dimensions m, $m+1$, W is χ-equivalent to an m-connected manifold.*

Proof, m even. By Theorem 3 of [M], we may suppose W $(m-1)$-connected, and by Lemma 1, $H_m(W)$ finite. By Lemma 2 it admits a nonsingular skew form b, so by Lemma 4, we may express G in the special form there given. First suppose B is not zero. Take the class x_1, represent by a sphere, and perform surgery. Then $H_m(W'')$ is generated by elements x_i', y_i', z; where x_i',y_i' are classes mapping to x_i, y_i in $H_m(W)$, for uniformity of notation we have denoted the generator of the "extra" Z_2 in G (if there is one) by x_0, and x_1',z are the classes of $f_2(S^m\times e), f_2(e\times S^m)$. By the corollary to Lemma 3, we may suppose that for $i\neq 1$, x_i',y_i' have the same orders as x_i, y_i. Also by Lemma 3, we may choose y_1' such that $\theta_1 y_1'=-z$, and since $c(x_1)=0$, $\theta_1 x_1'=\lambda\theta_1 z$, for some integer λ.

Suppose if possible $\lambda\neq 0$. Then in W',x_1' becomes zero, so we have (using primes to denote corresponding elements)

$$\theta_1 y_1'' = -z', \qquad \lambda\theta_1 z' = 0$$

so y_1'' has order $\lambda\theta_1^2$. The orders of other basic elements are unchanged from G, and there are no new ones. We see that the resulting group fails to have the form required by the corollary to Lemma 4. Hence $\lambda=0$. Then in W' we have $\theta_1 y_1''=-z'$, and y_1'' has infinite order. By Lemma 1, we may now kill y_1'', and we have then simplified the finite group G. Hence by induction we may simplify till G is 0 or Z_2. In the latter case perform surgery starting with the nonzero element x of G. Then $2x'=\lambda z$ for some odd λ. Hence $H_m(W')$ is cyclic of some odd order, which by Lemma 4 must be unity, so in this case also we can make W m-connected.

We must now consider the case when m is odd. The main difference from the earlier case is that there (using Lemma 4) the effect of surgery was already determined by the choice of the class x. But for m odd there is the additional question of product structure for $S^m\times S^m$. Now $H^m(S^m\times S^m)$ is the free Abelian group on two generators induced from the projections on the factors. Any autohomeomorphism of $S^m \times S^m$ induces an automorphism of this group and so a linear transformation of determinant ± 1. We represent this by the appropriate matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

over Z. If $m \neq 1, 3, 7$, there is no element of Hopf invariant odd in $\pi_{2m+1}(S^{m+1})$ and so no map $S^m \times S^m \to S^m$ with both degrees odd (by [2; 5]). Hence ab, cd are even, i.e., a, d have the opposite parity to b, c. However,

LEMMA 5. *$S^m \times S^m$ admits diffeomorphisms corresponding to any matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which is unimodular, and with a, d of opposite parity to b, c.

Proof. This falls naturally into two parts. First we produce a diffeomorphism for the matrix

$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

and then prove that this, together with the trivially representable matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

generates the group of all matrices satisfying the conditions above. We define the diffeomorphism using a map of Hopf [5]. Let $(p, q) \in S^m \times S^m$. Then draw the great circle through the points p, q of S^m, and let q' be the other point of it at the same distance from p as q is. Thus if q is p or its antipode, $q' = q$ is unique. Then consider the map $S^m \times S^m \to S^m \times S^m$ defined by $(p, q) \to (p, q')$. It is clearly (1-1) and infinitely differentiable (and its own inverse), and since m is odd it corresponds to the matrix

$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix},$$

as promised.

Alternatively we may use a map $f: S^m \to SO_{m+1}$ of index 2 (it is well known that such exist), and define a diffeomorphism by $F(x, y) = (x, f(x) \cdot y)$: this corresponds to the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The proof about generators for the group parallels Kuroš [6, Appendix B]. The only change is where he sets $a=qc+a'$, $0\leqq a'<c$, we must put $a=2q'c+a''$, $-c<a''\leqq c$. But $a''=c$ is impossible, as this would imply that a had the same parity as c. The remainder of the proof is unaltered (working with

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

which is easily expressed by the matrices above). In fact the corresponding projective group, a subgroup of index 3 in the modular group, is $Z_2 * Z$.

Proof of theorem, *m* odd. First suppose $c(x)$ not identically zero. Choose x such that $c(x)\neq 0$. We represent x by an S^m and perform surgery. We shall adhere to our earlier notation, denoting corresponding classes with primes. We consider the elements x', z of $H_m(W'')$. By Lemma 3, $rx'=sz$, say, where r is the order of x, and $r\nmid s$ since $c(x)\neq 0$. Hence the h.c.f. $(r, s)=h<r$. Set $r=r'h$, $s=s'h$. Choose λ, μ such that $\lambda r'+\mu s'=1$: we may suppose λ, μ of opposite parity since if they are both odd, r', s' must be of opposite parity and we may take $\lambda+s'$, $\mu-r'$.

Write $y=\lambda z+\mu x'$. Since λ, μ have opposite parity, by Lemma 5 we may choose the product structure in $S^m\times S^m$ so that y corresponds to one of the factors. Then glue in $D^{m+1}\times S^m$ to kill y and give W'(1). Now in W',

$$rx'' = sz' \qquad \text{i.e., } h(r'x'' - s'z') = 0$$

and $0=y'=\lambda z'+\mu x''$, so

$$x'' = (\lambda r' + \mu s')x'' = \lambda(r'x'' - s'z'),$$
$$z' = (\lambda r' + \mu s')z' = -\mu(r'x'' - s'z'),$$

hence the group generated by x'', z' has order a factor of h (in fact equal to it) which is less than r. Since the index of this group in $H_m(W')$ equals that of $Gp\{x\}$ in $H_m(W)$, (for $Gp\{x', z\}$ contains the kernels of both $H_m(W'')\to H_m(W)$ and $H_m(W'')\to H_m(W')$) we have succeeded in decreasing the order of $H_m(W)$, or more precisely, in replacing it by a divisor of itself.

We may repeat the above process as long as c is not zero. Hence by induction (G being finite) we may suppose $c=0$, and (G, b) as in (i) of Lemma 4. Perform surgery on the class x_1. If A denotes the subgroup of G generated by x_i, y_i for $1<i\leqq r$, then by Lemma 3, and corollary, $H_m(W'')=A\oplus Gp\{x_1', y_1', z\}$, $2x_1'$ is an even multiple of z, and we may suppose $2y_1'=z$.

Write $2x_1'=(4k+d)z$, where $d=2$ or 4, and kill $x_1'-2kz$. Then $dz'=0$, and so $H_m(W')=A\oplus Gp\{y_1''\}$, and y_1'' has order $2d$.

(1) This form of surgery is rather more general than that used in [M], but it follows from our proof of Lemma 5 that it is equivalent to a series of the spherical modifications of [M].

Write U for W', u for y_1''. Then since b is nonsingular, and u is the only basis element of $H_m(U)$ of order greater than 2, the order of $c(u)$ equals the order of u. Now perform surgery starting with the class u. Then in $H_m(U'')$ we have $2du$ equal to an odd multiple of the new class w. Then we kill u (we have no need to worry about the product structure this time), and

$$H_m(U') = A \oplus Z_k \qquad \text{if } k \text{ is the odd order of } w.$$

(The sum is direct as A is a 2-group.) But now, by the first part of the proof, we can replace the order of the group by a divisor of itself such that the new group has the form of Lemma 4 (i), and so has order not exceeding that of A. Hence in the second case also we have succeeded in decreasing the order of $H_m(W)$, so our induction is complete, and we may reduce the group to zero.

COROLLARY. *Let T^{2m} be a homotopy sphere which bounds a π-manifold. Then it bounds a contractible manifold.*

For the result is trivial if $m=1$, and otherwise we may apply the theorem to find an m-connected manifold with boundary T. But by relative Poincaré duality, such a manifold must be contractible.

COMPLEMENT. Let T^{4m} be a homotopy sphere, and W a π-manifold with boundary T. Then there is a contractible manifold C with boundary T, such that if W' is formed by glueing W to C along T, there is a parallelisable manifold M, with boundary W'.

Proof. Our construction of C from W by surgery was by choosing at each stage a class on which to perform the construction. By Lemma 5 of [M], if we choose the correct trivialisation of the normal bundle at each stage, the manifolds $\omega(W, f)$ are parallelisable: this goes also for the proof of Theorem 2 of [M]. Since the trivialisations given for the tangent bundles of these manifolds fit together on the boundary, we may form M by glueing these manifolds together, and it will then be parallelisable.

These results are of use for computing the groups Θ_m of J-equivalence classes of homotopy spheres. Our reference is [7]. In the notation of those notes, the above corollary states $\Theta_{2m}(\partial\pi)=0$. Since Milnor proves that $\Theta_{2m}/\Theta_{2m}(\partial\pi)$ is finite, it follows that for each m, Θ_{2m} is a finite group. Also, using other results of Milnor, Θ_4 and Θ_{12} vanish. We may also show $\Theta_6=0$, and will sketch the proof (we omit details since a simpler proof is known). By Thom [8], the spinor cobordism group in dimension 6 is isomorphic to the stable homotopy group $\pi_{n+6}(M(\text{Spin } n))$. Results of Adams [1] relate these to a spectral sequence which starts with

$$\text{Ext}_{A_2}^{**}(H^*(M(\text{Spin } n), Z_2), Z_2),$$

where A_2 denotes the Steenrod algebra mod 2. A straightforward computation of this in low dimensions now shows that the group in question vanishes.

Hence a homotopy 6-sphere, being a spin manifold, bounds another, W say. But W is a spin manifold, and so 3-parallelisable, and the result now follows by the theorem above.

All these results have been obtained independently by M. Kervaire (including a stronger form of the above complement), and will appear in a joint paper by M. Kervaire and J. Milnor entitled *Groups of homotopy spheres*, which will also contain the substance of [7]. Recent results of Smale and Munkres have emphasised the importance of the groups Θ_m.

4. **Simplifying certain $(m-1)$-parallelisable $(2m+1)$-manifolds.** Suppose that U is an $(m-1)$-parallelisable $(2m+1)$-manifold, and in addition that $H_{m-1}(U)$ is torsion free, hence free Abelian. By Theorem 3 of [M], U is χ-equivalent to an $(m-1)$-connected manifold. We wish to obtain a slight refinement of this result. Now since $H_{m-1}(U)$ is free, $H^m(U, A) = \text{Hom}(H_m(U), A)$. The obstruction p to m-parallelisability of U lies in $H^m(U, \pi_{m-1}(O))$, where O denotes the stable orthogonal group. We make the convention of regarding p as a function on $H_m(U)$.

We may now state the reduction lemma.

LEMMA 6. *If U is a compact $(m-1)$-parallelisable $(2m+1)$-manifold, with $H_{m-1}(U)$ torsion free, then there is a sequence of surgeries taking U to an $(m-1)$-connected manifold U^*, and such that*

(i) *If $m>2$, there are induced isomorphisms of $H_m(W)$, $H_{m+1}(W, Z_\theta)$ at each stage, which commute with the Bockstein operator, with intersection numbers* mod θ, *and with p.*

(ii) *If $m=2$, there are forwards maps of $H_2(W)$ at each stage, inducing isomorphisms of its torsion subgroup, and backwards maps of $H_3(W, Z_\theta)$, commuting with the same three invariants, and inducing isomorphisms*

$$H_2(U) = H_2(U^*), \qquad H_3(U^*, Z_\theta) = H_3(U, Z_\theta).$$

Proof. If $m<2$, we can take $U^*=U$ (supposed connected).

(i) If $m>2$, we may first use the procedure of [M] to kill successively the $\pi_i(U): 0<i<m-1$. We note that this induces natural isomorphisms of $H_m(W)$, $H_{m+1}(W, Z_\theta)$ at each stage, and if the resulting manifold is U_1, $H_{m-1}(U_1)$ is naturally imbedded in $H_{m-1}(U)$, hence it also is torsion free. Since $m-1>1$, by the Hurewicz isomorphism, $\pi_{m-1}(U)=H_{m-1}(U)$, so is free Abelian. Now since U_1 is $(m-1)$-parallelisable, by construction, we may kill the generators in turn: it is easy to see that $H_m(W)$ and $H^{m+1}(W, Z_\theta)$ remain unaltered. The required commutativities now follow from the naturality of the several invariants for the successive inclusion maps $W''\rightarrow W$ and $W''\rightarrow W'$.

(ii) If $m=2$, we may first choose elements of $\pi_1(U)$ inducing generators of $H_1(U)$, and kill these as before. Hence we may assume $H_1(U)=0$. We now select a set of generators of $\pi_1(U)$ and kill them in order. At each stage, we have exact sequences

$$0 \to H_2(W'') \to H_2(W') \to Z \to 0$$
$$H_2(W'') \cong H_2(W)$$
$$0 \to Z_\theta \to H_3(W'', Z_\theta) \to H_3(W, Z_\theta) \to 0.$$
$$H_3(W'', Z_\theta) \cong H_3(W', Z_\theta)$$

Let the resulting manifold be U_1. $H_2(U)$ is contained in $H_2(U_1)$ with free Abelian quotient group. We lift a set of generators of this quotient group to $H_2(U_1)$: we may suppose that p vanishes on each. For if U is 2-parallelisable, by Theorem 3 of [M], we may suppose that U_1 is also, so p vanishes identically; yet if not, p is a nonzero homomorphism $H_2(U) \to Z_2$, and to each lifted generator on which p does not vanish we may add an element of $H_2(U)$ with the same property.

Since p vanishes on these generators, they are representable by imbeddings of $S^2 \times D^3$, and we may perform χ-constructions to kill them. At each stage of this process we have exact sequences (by Lemma 1)

$$0 \to Z \to H_2(W'') \to H_2(W') \to 0,$$
$$H_2(W'') \cong H_2(W)$$
$$0 \to H_3(W'', Z_\theta) \to H_3(W, Z_\theta) \to Z_\theta \to 0.$$
$$H_3(W'', Z_\theta) \cong H_3(W', Z_\theta)$$

The resulting manifold is the required U^*. We have exhibited maps of the homology groups as stated, which induce isomorphisms as stated (this is clear for H_2 and will follow by duality for H_3). From the diagrams above, and from the naturality of the invariants for the inclusion maps, follow again the various commutation relations.

COROLLARY. *Suppose in addition that the boundary of U has no homology in dimensions m, m+1, so that a bilinear form can be set up as in* §2. *Then the transition from U to U* preserves the bilinear form.*

This is clear, since the form is defined by Bocksteins and intersection numbers.

5. Topology of certain $(m-1)$-parallelisable $(2m+1)$-manifolds, (m even). We may now apply the above lemma to make our manifolds $(m-1)$-connected, and the methods of the rest of this paper will then apply. We shall study the homomorphism c of Lemma 4 (ii), and show in particular that if W is m-parallelisable, then $c=0$.

We shall suppose in the following that W satisfies the condition:

(A) W is a compact $(m-1)$-parallelisable $(2m+1)$-manifold, such that $H_{m-1}(W)$ is torsion free and $H_m(W)$ finite, and the boundary of W has no homology in dimensions m, $m+1$; where m is even.

The obstruction p to m-parallelisability has coefficient group $\pi_{m-1}(O)$, which was evaluated by Bott [3] as Z if $m=0 \pmod 4$; as 0 if $m=6 \pmod 8$; and as Z_2 if $m=2 \pmod 8$. But under (A), $H^m(W)$ vanishes, so $p=0$ unless $m=2 \pmod 8$, when the coefficient group is Z_2.

LEMMA 7. *If W satisfies* (A), $x \in H_m(W)$ *and* $p(x)=0$, *then* $c(x)=0$.

Proof. By Lemma 6, we may suppose W $(m-1)$-connected. Note that $p(x)=0$ is the condition that x be representable (by an imbedding of $S^m \times D^{m+1}$). We take a base of $H_m(W)$ as in Lemma 4. Let y_1 be an element of this base of order greater than 2 with $p(y_1)=0$, $c(y_1)\neq 0$. We shall deduce a contradiction.

Let $2n$ be the order of y_1 (it is even since $c(y_1)\neq 0$). Let A be the subgroup of $H_m(W)$ generated by x_j, y_j for $j\neq 1$. Since y_1 is representable, we can perform surgery. As in the proof of the theorem, using Lemma 3, we have $H_m(W'')=A\oplus Gp\{x_1', y_1', z\}$ where $2nx_1'=z$, $2ny_1'=(2\lambda n+n)z$. Hence $H_m(W')=A\oplus Gp\{x_1'', z'\}$ and this last group is cyclic of order $\geqq 2n>2$, which contradicts the corollary to Lemma 4 (A being of the type admitted by that corollary).

Now suppose that $H_m(W)$ contains an element x for which $p(x)=0$, $c(x)\neq 0$. Let $M_{2\theta}$ be obtained from $S^m\times S^{m+1}$ by performing surgery on 2θ times a generator of $H_m(S^m\times S^{m+1})$. Clearly, M satisfies (A). It is easy to see that $H_m(M)=2Z_{2\theta}$, and since $p=0$ for $S^m\times S^{m+1}$, by [M] we may suppose that it is 0 for M, hence $c=0$, since by what we have already proved c vanishes on each generator. Let x_0 be a generator of $H_m(M)$ (of order 2θ).

Form $W \# M$. Now $H_m(W\# M)=H_m(W)\oplus H_m(M)$, and it is clear that b admits the direct sum decomposition and c and p are additive. Consider the element $x+x_0$ of order 2θ. We have

$$p(x+x_0)=p(x)+p(x_0)=0, \qquad c(x+x_0)=c(x)+c(x_0)=c(x)\neq 0.$$

By the proof of Lemma 4, an odd multiple y of $x+x_0$ can be chosen as a basis element of $H_m(W\# M)$; this will have order greater than 2, and $p(y)=0$, $c(y)\neq 0$, which contradicts what we proved above. This proves the lemma.

The lemma may be rephrased: $c=p$ or $c=0$. For if the kernel of c properly contains that of p, which has index at most 2, the kernel of c is the whole group, so $c=0$. If $m\neq 2 \pmod 8$, this simply states $c=0$. If $m=2 \pmod 8$, we shall now show that whether c is p or 0 depends only on m. In fact we shall produce a closed manifold V satisfying (A), and with $p(V)\neq 0$. Form $W\# V$. p and c are additive. There are now two cases.

If $c(V)=p(V)$, $c(W\# V)=c(W)+c(V)\neq 0$ since $c(V)\neq 0$. Hence it equals $p(W\# V)=p(W)+p(V)$, and we deduce $c(W)=p(W)$.

If $c(V)=0$, $c(W)+c(V)\neq p(W)+p(V)$ since $c(V)\neq p(V)$. Hence $c(W)=c(W\# V)=0$.

The manifold V may be constructed as follows. Take the nontrivial S^{m+1}

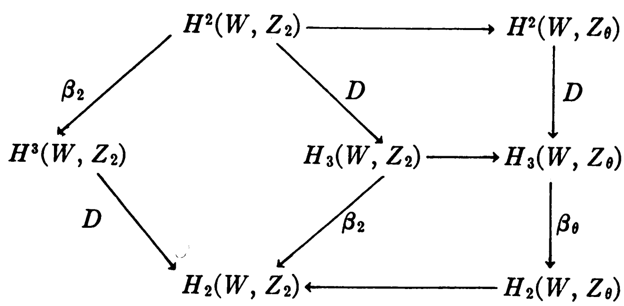
bundle U over S^m (defined since $m=2 \pmod 8$). Let x generate $H_m(U)$. $p(2x)=2p(x)=0$, so we may perform surgery and kill $2x$. This yields a manifold V which satisfies (A), and x determines a class x' in V with $p(x')\neq 0$.

In the case $m=2$, we can show that $c=p$. (We have not yet succeeded in deciding the question in any other cases.) For the Wu manifold $P(1, 2)$ (see [4]) satisfies (A) and has $H_m(P)=Z_2$. Since b is nonsingular, $c\neq 0$. We may sum up these results as

PROPOSITION 1. *Let W satisfy* (A). *If* $m\neq 2 \pmod 8$, $c(W)=p(W)=0$. *If* $m=2 \pmod 8$, *there is an integer* r_m mod 2 *such that* $c(W)=r_m p(W)$ *for all W. Moreover,* $r_2=1$.

Now for $m=2$, p is the second Stiefel class w^2. For any closed 5-manifold W satisfying (A), we know c by elementary homology theory, and may now use Wu's formulae to deduce from w^2 the operation of the Steenrod squares in W.

We finally turn to the problem of deciding when in Lemma 4 (ii) there is an extra Z_2. Since c is a homomorphism and b nonsingular, G has an element y_0 with $c(x)=b(x, y_0)$ for all x. It is easy to show that the extra Z_2 appears if and only if $c(y_0)\neq 0$. For 5-manifolds, this fact admits an interesting interpretation. We know that $c=p=w^2$. Now we have the commutative diagram



where D denotes duality isomorphisms, β_2 is the Bockstein, and the horizontal maps are induced by the obvious homomorphisms of coefficient groups. But $w^2\in H^2(W, Z_2)$ maps under β_2 to $w^3\in H^3(W, Z_2)$, and $c\in H^2(W, Z_2)$ maps to $y_0\in H_2(W, Z_\theta)$, so each of w^2w^3, $c(y_0)$ is equal to the Kronecker product of c with its image in $H_2(W, Z_2)$. Now since a closed oriented 5-manifold W is cobordant to zero if and only if the Stiefel number $w^2w^3[W]$ vanishes by [8], we have proved

PROPOSITION 2. *Let W be a closed oriented* 5-*manifold such that $H_1(W)$ is torsion free, $H_2(W)$ finite. Then there exists a finite Abelian group B such that either*

(i) $H_2(W) = B \oplus B$ *or* (ii) $H_2(W) = B \oplus B \oplus Z_2$.

W is cobordant to zero if and only if (i) *holds.*

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Trinity College,
Cambridge, England