SEMI-GAUSSIAN SUBSPACES[1]

BY

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1. **Introduction.** Let M be a subspace (=closed linear manifold) of an L_2 space. Then M may or may not have the following property: If $\{x_k\}$ is any orthogonal sequence in M such that $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$, then the series $\sum_{k=1}^{\infty} x_k$ converges almost everywhere. That is, for orthogonal expansions in M, convergence in the mean implies convergence almost everywhere. If M does have this property, we say that M is *semi-Gaussian.*

Clearly, Gaussian subspaces are semi-Gaussian. For if $\{x_k\}$ is an orthogonal sequence in a Gaussian subspace, then $\{x_k\}$ is a mutually independent random variable sequence—this property and the property that if x is in the subspace, then $Ex=0$, $Ex^2 \geqq 0$, and the distribution of x is Gaussian, characterize Gaussian subspaces in both the real and the complex cases—and the mean convergence of the series $\sum_{k=1}^{\infty} x_k$ implies its convergence almost everywhere. Here, and throughout the paper, the same symbol may denote, depending on the context, either an equivalence class (of measurable functions equal almost everywhere) or one of its members. Any finite dimensional subspace is semi-Gaussian. So is any subspace of a semi-Gaussian subspace. Other examples may be obtained by using the methods of §4 or the result of §5.

In this paper, we study semi-Gaussian subspaces and obtain some results that are new even for Gaussian subspaces. In §2, we give necessary and sufficient conditions for a subspace to be semi-Gaussian. In §3, these results are used to answer a question about the almost everywhere convergence of sequences of the form $\{T^n x\}$ where $T = T_1 T_2 T_1$ and T_j is an orthogonal projection in L_2, $j=1, 2$. In §4, it is shown that if M_1 and M_2 are orthogonal semi-Gaussian subspaces then $M_1 \vee M_2$ is semi-Gaussian, and, furthermore, that this is not necessarily true if the assumption of orthogonality is dropped. In §5, we give a necessary and sufficient condition for L_2 to be semi-Gaussian.

Throughout, L_2 is the (real or complex) L_2 of a positive measure space (W, F, μ). Any subspace entering the discussion is implicitly assumed to be a subspace of L_2. If M is a subspace, P_M denotes the orthogonal projection in L_2 with range M. We write $\sum_{k=1}^{\infty} x_k$ to denote both a series and its limit in the mean if this limit exists.

Received by the editors August 18, 1961.

[1] This research was supported by the National Science Foundation under Grant No. G11382.

2. Necessary and sufficient conditions for a subspace to be semi-Gaussian.

THEOREM 1. *Suppose that M is a subspace of L_2. Then the following statements are equivalent:*

(i) *M is semi-Gaussian.*

(ii) *If $\{x_k\}$ is any orthogonal sequence in L_2 satisfying $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$, then the series $\sum_{k=1}^{\infty} P_M x_k$ converges almost everywhere.*

(iii) *If T is any positive definite self-adjoint operator in L_2 with norm not greater than one, then the sequence $\{P_M T^n x\}$ converges almost everywhere for each x in L_2.*

Note that each term of the series $\sum_{k=1}^{\infty} P_M x_k$ is in M but that different terms are not necessarily orthogonal. The content of (ii) and (iii) is that certain kinds of sequences $\{y_n\}$ in L_2, converging in the mean but not necessarily converging almost everywhere, are mapped into sequences converging almost everywhere, as well as in the mean, by the transformation $\{y_n\} \to \{P_M y_n\}$. Of course, if $\{y_n\}$ is already in M, then this transformation leaves $\{y_n\}$ fixed. This happens in (iii), for example, if T maps into M.

Proof. (i)$\Rightarrow$(ii): Let M be semi-Gaussian and suppose that $\{x_k\}$ satisfies the condition of (ii). If M is finite dimensional, the desired result clearly follows from the convergence in the mean of the partial sums $\sum_{k=1}^{n} P_M x_k$. Suppose that M is infinite dinensional. Then there exist orthogonal infinite dimensional subspaces M_1 and M_2 such that $M = M_1 + M_2$. Since

$$\sum_{k=1}^{n} P_M x_k = \sum_{k=1}^{n} P_{M_1} x_k + \sum_{k=1}^{n} P_{M_2} x_k, \qquad n \geqq 1,$$

it is enough, by symmetry, to show that the series $\sum_{k=1}^{\infty} P_{M_1} x_k$ converges almost everywhere.

Let $N = \bigvee_{k=1}^{\infty} \{x_k - P_{M_1} x_k\}$, the smallest subspace containing the sequence $\{x_k - P_{M_1} x_k\}$. Then, since M_2 is infinite dimensional, there is an operator U from N into M_2 such that

$$(Ux, Uy) = (x, y), \qquad x \in N, y \in N.$$

For each positive integer k, let

$$y_k = P_{M_1} x_k + U(x_k - P_{M_1} x_k), \; z_k = P_{M_1} x_k - U(x_k - P_{M_1} x_k).$$

Then $(y_j, y_k) = (z_j, z_k) = (x_j, x_k)$, and $\sum_{k=1}^{\infty} y_k$ and $\sum_{k=1}^{\infty} z_k$ are mean convergent orthogonal expansions in the semi-Gaussian subspace M. Consequently, they converge almost everywhere. Since

$$2 \sum_{k=1}^{n} P_{M_1} x_k = \sum_{k=1}^{n} y_k + \sum_{k=1}^{n} z_k, \qquad n \geqq 1,$$

the desired result follows.

(ii)$\Rightarrow$(iii): Let M satisfy (ii) and suppose that T satisfies the condition of (iii). Let x be in L_2. Then, by a result in [1, Lemma 2 and a related remark],

$$\sum_{n=1}^{\infty} n\|T^n x - T^{n+1}x\|^2 < \infty,$$

implying that

$$\sum_{n=1}^{\infty} n\|P_M T^n x - P_M T^{n+1}x\|^2 < \infty.$$

Interchange of summation and integration shows that

$$\sum_{n=1}^{\infty} n\,|P_M T^n x - P_M T^{n+1}x|^2 < \infty$$

almost everywhere which is easily seen to imply that

$$\lim_{n\to\infty} \max_{2^n \leqq j \leqq k \leqq 2^{n+1}.} |P_M T^j x - P_M T^k x| = 0 \tag{1}$$

almost everywhere. Thus, it is enough to show that $\{P_M T^{2^n} x\}$ converges almost everywhere.

Let $\{A_k\}$ be a strictly increasing real number sequence converging to 1 and satisfying $A_0 = 0$, $A_1^4 < 1/2$, $A_k^2 \leqq A_{k-1}$, $k \geqq 2$. (An example of such a sequence is $A_0 = 0$, $A_k = \exp(-1/2^k)$, $k \geqq 1$.) Then

$$\sum_{n=1}^{\infty} \left(A_k^{2^n} - A_{k-1}^{2^n}\right)^2 < 1, \qquad k \geqq 1. \tag{2}$$

This is obvious for $k = 1$. For $k > 1$, the inequality follows from the fact that the left hand side of (2) is less than

$$\sum_{n=1}^{\infty} \left(A_k^{2^n} - A_{k-1}^{2^n}\right) \leqq \sum_{n=1}^{\infty} \left(A_{k-1}^{2^{n-1}} - A_{k-1}^{2^n}\right) = A_{k-1} < 1.$$

There is a spectral measure E defined on the Borel subsets of $I = [0, 1]$ such that

$$T = \int_I \lambda E(d\lambda).$$

Let $I_1 = [0, A_1]$, $I_k = (A_{k-1}, A_k]$ for $k > 1$, $J = \{1\}$, and define the function f on I by $f(\lambda) = A_k$ if $\lambda \in I_k$, $= 1$ if $\lambda \in J$. Let $\hat{T}$ be the operator defined by

$$\hat{T} = \int_I f(\lambda) E(d\lambda).$$

Let $x_k = E(I_k)x$ for $k \geqq 1$, $y = E(J)x$. Then, clearly, $x_1, x_2, \cdots, y$ are pairwise orthogonal, $\sum_{k=1}^{\infty} \|x_k\|^2 \leqq \|x\|^2 < \infty$, and

$$\hat{T}^n x = \sum_{k=1}^{\infty} A_k^n x_k + y, \qquad n \geqq 1. \tag{3}$$

For each positive integer n,

$$\begin{aligned} \|P_M T^{2^n} x - P_M \hat{T}^{2^n} x\|^2 &\leqq \|T^{2^n} x - \hat{T}^{2^n} x\|^2 \\ &= \int_I [\lambda^{2^n} - f^{2^n}(\lambda)]^2 \|E(d\lambda)x\|^2 \\ &\leqq \sum_{k=1}^{\infty} \int_{I_k} (A_k^{2^n} - A_{k-1}^{2^n})^2 \|E(d\lambda)x\|^2 \\ &= \sum_{k=1}^{\infty} (A_k^{2^n} - A_{k-1}^{2^n})^2 \|x_k\|^2. \end{aligned}$$

Thus, using (2),

$$\sum_{n=1}^{\infty} \|P_M T^{2^n} x - P_M \hat{T}^{2^n} x\|^2 \leqq \sum_{k=1}^{\infty} \|x_k\|^2 < \infty.$$

Therefore,

$$P_M T^{2^n} x - P_M \hat{T}^{2^n} x \to 0 \tag{4}$$

almost everywhere as $n \to \infty$.

We complete the proof by showing that $\{P_M \hat{T}^n x\}$ converges almost everywhere, which will imply that $\{P_M T^n x\}$ converges almost everywhere by (4) and (1). By (3), it is enough to show that $\sum_{k=1}^{\infty} A_k^n P_M x_k \to 0$ almost everywhere as $n \to \infty$.

So far we have not used the assumption that M satisfies (ii). We do so now to obtain the fact that almost everywhere

$$\lim_{K \to \infty} \sum_{k=1}^{K} A_k^n P_M x_k$$

exists, $n = 0, 1, \cdots$. Summation by parts gives

$$\sum_{k=1}^{K} A_k^n P_M x_k = A_{K+1}^n \sum_{k=1}^{K} P_M x_k - \sum_{k=1}^{K} \left(\sum_{j=1}^{k} P_M x_j \right) (A_{k+1}^n - A_k^n), \qquad K \geqq 1, n \geqq 1.$$

Therefore, using the fact that $\{A_k\}$ converges to 1, we get that

$$\sum_{k=1}^{\infty} A_k^n P_M x_k = \sum_{k=1}^{\infty} P_M x_k - \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} P_M x_j \right) (A_{k+1}^n - A_k^n), \qquad n \geqq 1, \tag{5}$$

each of the three series converging almost everywhere since the first two do. Clearly, $\{A_{k+1}^n - A_k^n\}$ is a regular matrix of summation. Therefore, letting $n \to \infty$ in (5) we get that $\sum_{k=1}^{\infty} A_k^n P_M x_k \to 0$ almost everywhere, the desired result.

(iii)$\Rightarrow$(i): Let M satisfy (iii) and suppose that $\{x_k\}$ is an orthogonal sequence in M such that $\sum_{k=1}^{\infty}\|x_k\|^2<\infty$. Let A_k be in $(0, 1)$, $y_k=0$ if $x_k=0$, $=x_k/\|x_k\|$ if $x_k\neq 0$, $k\geqq 1$. Let T be the linear operator defined by

$$Tx = \sum_{k=1}^{\infty} A_k(x, y_k)y_k, \qquad x \in L_2.$$

Clearly, T is positive definite, self-adjoint, and has norm not greater than one. Since M satisfies (iii), $\{T^n x\}=\{P_M T^n x\}$ converges almost everywhere for x in L_2. Let $x=\sum_{k=1}^{\infty} x_k$. Then

$$T^n x = \sum_{k=1}^{\infty} A_k^n x_k, \qquad n \geqq 1,$$

and since $\|T^n x\|^2=\sum_{k=1}^{\infty} A_k^{2n}\|x_k\|^2\rightarrow 0$ as $n\rightarrow\infty$, we have that

$$\lim_{n\rightarrow\infty} \sum_{k=1}^{\infty} A_k^n x_k = 0$$

almost everywhere.

Now consider $y_n=\sum_{k=1}^{\infty} A_k^n x_k$, $z_n=\sum_{k=n}^{\infty} x_k$ for the particular sequence $A_k=\exp(-1/(2k)!)$. In this case we show that $y_{r_n}-z_n\rightarrow 0$ almost everywhere as $n\rightarrow\infty$, where $r_n=(2n-1)!$, and, by the above paragraph, this implies the desired result.

If $n>1$, then $A_{n-1}^{r_n}=e^{-(2n-1)}\leqq e^{-n}$, $1-A_n^{r_n}=1-\exp(-1/2n)\leqq 1/2n$, and, since $\{A_k\}$ is increasing,

$$\begin{aligned}\|y_{r_n}-z_n\|^2 &= \sum_{k=1}^{n-1} A_k^{2r_n}\|x_k\|^2 + \sum_{k=n}^{\infty}(1-A_k^{r_n})^2\|x_k\|^2\\ &\leqq (e^{-2n}+1/4n^2)\sum_{k=1}^{\infty}\|x_k\|^2.\end{aligned}$$

Therefore, $\sum_{n=2}^{\infty}\|y_{r_n}-z_n\|^2<\infty$, implying that $y_{r_n}-z_n\rightarrow 0$ almost everywhere as $n\rightarrow\infty$. This completes the proof.

3. **Iterates of orthogonal projections.** The theory of semi-Gaussian subspaces helps to provide an answer to the following question: If $T=T_1T_2T_1$ where T_j is an orthogonal projection in L_2, $j=1, 2$, and x is in L_2, does $\{T^n x\}$ or, less demandingly, $\{\sum_{k=1}^{n} T^k x/n\}$ converge almost everywhere? In [1] it was proved that if T_1 and T_2 are members of a special class of orthogonal projections, the class of conditional expectation operators (restricted to L_2), then, for each x in L_2, the sequence $\{T^n x\}$ does converge almost everywhere. However, the following theorem shows that if T_1 and T_2 are assumed merely to belong to the class of orthogonal projections and are not otherwise restricted, then even the sequence $\{\sum_{k=1}^{n} T^k x/n\}$ need not converge almost everywhere. For otherwise, in view of the following theorem, every L_2 space would be a semi-Gaussian subspace of itself, contradicting the fact that

there do exist orthogonal expansions in some L_2 spaces that converge in the mean but not almost everywhere.

THEOREM 2. *Suppose that M is a subspace of L_2 such that if T_j is an orthogonal projection from L_2 into M, $j=1, 2$, $T=T_1T_2T_1$, and x is in L_2, then the sequence $\{\sum_{k=1}^n T^kx/n\}$ converges almost everywhere. Then M is semi-Gaussian.*

Of course, the converse is also true since it is easy to check that T satisfies the condition of (iii) in Theorem 1, and, therefore, if M is semi-Gaussian and x is in L_2, then $\{T^nx\}=\{P_MT^nx\}$ converges almost everywhere. Note that for the converse the range of T_2 need not be restricted.

Proof. Let $\{x_k\}$ be an orthogonal sequence in M such that $\sum_{k=1}^\infty \|x_k\|^2 < \infty$. We shall prove that if $\{A_k\}$ is a sequence in (0, 1), then $\sum_{k=1}^\infty A_k^n x_k \to 0$ almost everywhere as $n\to\infty$. An examination of the last part of the proof of Theorem 1, indicates that this implies that the series $\sum_{k=1}^\infty x_k$ converges almost everywhere, the desired result.

The result is clear if only a finite number of the x_k's are different from 0. In the other case, suppose that $x_k\neq 0$ for all k. Let $y_k=x_k/\|x_k\|$, $a_k^2=A_{2k-1}$, $b_k^2=1-a_k^2$, and $z_k=a_ky_{2k-1}+b_ky_{2k}$. Then $\{y_k\}$ and $\{z_k\}$ are orthonormal sequences and T_1 and T_2 defined by

$$T_1x = \sum_{k=1}^{\infty} (x, y_{2k-1})y_{2k-1},$$

$$T_2x = \sum_{k=1}^{\infty} (x, z_k)z_k$$

are orthogonal projections from L_2 into M. Let $T=T_1T_2T_1$ and $x=\sum_{k=1}^\infty x_{2k-1}$. Then

$$T^n x = \sum_{k=1}^{\infty} T^n x_{2k-1} = \sum_{k=1}^{\infty} A_{2k-1}^n x_{2k-1}, \qquad n \geqq 1.$$

By the assumptions of the theorem, $\{\sum_{k=1}^n T^kx/n\}$ converges almost everywhere. But this implies [1, Theorem 1] that $\{T^nx\}$ converges almost everywhere, since here T is a positive definite self-adjoint operator with norm not greater than one. Thus, as $n\to\infty$, $\sum_{k=1}^\infty A_{2k-1}^n x_{2k-1}\to 0$ almost everywhere using the fact that the limit in the mean is 0. Similarly, $\sum_{k=1}^\infty A_{2k}^n x_{2k}\to 0$ almost everywhere and the desired result is implied.

4. On the supremum of two semi-Gaussian subspaces.

THEOREM 3. *If M_1 and M_2 are orthogonal semi-Gaussian subspaces, then M_1+M_2 is semi-Gaussian.*

Proof. Let $M=M_1+M_2$. Suppose that T satisfies the condition of (iii) in Theorem 1. Then, for x in L_2,

$$P_MT^nx = P_{M_1}T^nx + P_{M_2}T^nx.$$

By Theorem 1, since M_1 and M_2 are semi-Gaussian, the right hand side approaches a limit almost everywhere as $n \to \infty$. Therefore, M satisfies (iii) of Theorem 1, and, consequently, is semi-Gaussian.

Of course, the equivalence of (i) and (ii) in Theorem 1 could also have been used to prove Theorem 3.

The following is a simple and amusing application of the above result.

THEOREM 4. *Let $\{x_k\}$ be an orthogonal sequence of real random variables such that the distribution of $x_1, \cdots, x_n$ is Gaussian for all n. Then, if the series $\sum_{k=1}^{\infty} x_k$ converges in the mean, it converges almost everywhere.*

We note that $x_1, x_2, \cdots$ need not be mutually independent and this is where the novelty of the result is centered.

Proof. Here, of course, the underlying measure space is a probability space. We consider the real L_2 space associated with it. Let M_1 be the smallest subspace containing the equivalence class of measurable functions equal to 1 almost everywhere, and let $M_2 = \bigvee_{k=1}^{\infty} \{x_k - Ex_k\}$. Since M_1 is finite dimensional and M_2 is Gaussian, both are semi-Gaussian subspaces. They are orthogonal. Consequently, by Theorem 3, $M_1 + M_2$ is semi-Gaussian. Since $M_1 + M_2$ contains x_k, $k \geqq 1$, the desired result follows.

The supremum of two semi-Gaussian subspaces need not be semi-Gaussian. This follows from Theorem 5, which indicates that the ordinary L_2 space over the unit interval, certainly not a semi-Gaussian subspace, is the supremum of two semi-Gaussian subspaces.

THEOREM 5. *If N is a separable subspace, then there are semi-Gaussian subspaces M_1 and M_2 such that $N \subset M_1 \vee M_2$. In particular, if L_2 is separable, then L_2 is the supremum of two semi-Gaussian subspaces. Moreover, both M_j can be chosen to have the stronger property that every mean convergent sequence $\{x_n\}$ in M_j converges almost everywhere.*

Proof. If L_2 has the property that every sequence $\{x_n\}$ converging in the mean converges almost everywhere, let $M_1 = M_2 = L_2$ and the desired result holds. Suppose that L_2 does not have this property. Then there must be a set B in F of positive measure such that B does not contain an atom[2]. For it is easily seen that if every set of positive measure contains an atom, then, for sequences, convergence in the mean implies convergence almost everywhere. We may and do suppose that $\mu(B)$ is finite. The restriction of μ to $\{A \cap B \mid A \in F\}$ is nonatomic. Therefore (for example, by [2]), there is a sequence $\{B_k\}$ of measurable subsets of B such that for every finite nonempty set K of positive integers

$$\mu\left(\bigcap_{k \in K} B_k\right) = \mu(B) \prod_{k \in K} \frac{1}{2} \cdot$$

(2) A set A is an atom if A is measurable with positive measure and each measurable subset of A with positive measure has the same measure as A.

If k is a positive integer, let

$$\begin{aligned} y_k(w) &= 2^k\mu^{-1/2}(B) \qquad \text{if } w \in B_{k^2} \cap B_{k^2+1} \cap \cdots \cap B_{k^2+2k}, \\ &= -2^k\mu^{-1/2}(B) \qquad \text{if } w \in (B - B_{k^2}) \cap B_{k^2+1} \cap \cdots \cap B_{k^2+2k}, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then $\{y_k\}$ is an orthonormal sequence. Let $M_1 = \bigvee_{k=1}^{\infty}\{y_k\}$. Let $\{x_n\}$ be a sequence in M_1 such that $x_n \to x$ in the mean as $n \to \infty$. We now show that $\{x_n\}$ converges to x almost everywhere. We may and do suppose in the following that $x = 0$. Since

$$\sum_{k=1}^{\infty} \mu(\{w \mid y_k(w) \neq 0\}) < \infty,$$

we have, by Borel-Cantelli, that for almost all w, $y_k(w) = 0$ for all but a finite number of integers k. Consequently, for almost all w,

$$|x_n(w)| = \left| \sum_{k=1}^{\infty} (x_n, y_k) y_k(w) \right| \leqq \|x_n\| \sum_{k=1}^{\infty} |y_k(w)| \to 0$$

as $n \to \infty$. Thus, M_1 has the desired property.

Let $M_3 = (M_1 \vee N) \cap M_1^{\perp}$. Then $N \subset M_1 \vee M_3$. If M_3 is finite dimensional, let $M_2 = M_3$ and the desired result is implied. Suppose that M_3 is infinite dimensional. Then M_3 is spanned by an orthonormal sequence $\{z_k\}$, using the fact that M_3 is separable since M_1 and N are. Let $\{b_k\}$ be a sequence in (0, 1) such that

$$\sum_{k=1}^{\infty} b_k(|y_k| + |z_k|) < \infty \tag{6}$$

almost everywhere. For example, by Borel-Cantelli, b_k may be chosen to be $1/k^2 m_k$ where $1 < m_k < \infty$ and

$$\mu(\{w \mid |y_k(w)| + |z_k(w)| > m_k\}) < 1/2^k.$$

Then, let $M_2 = \bigvee_{k=1}^{\infty}\{a_k y_k + b_k z_k\}$ where a_k satisfies $a_k^2 + b_k^2 = 1$. Suppose that $\{x_n\}$ is a sequence in M_2 such that the limit in the mean of the sequence is 0. Then it converges to 0 almost everywhere as we now show. Clearly,

$$\begin{aligned} |P_{M_1} x_n - x_n| &= \left| \sum_{k=1}^{\infty} (x_n, y_k) y_k - \sum_{k=1}^{\infty} (x_n, a_k y_k + b_k z_k)(a_k y_k + b_k z_k) \right| \\ &\leqq 2\|x_n\| \sum_{k=1}^{\infty} b_k(|y_k| + |z_k|), \end{aligned}$$

which converges to 0 almost everywhere as $n \to \infty$, using (6). But $\{P_{M_1} x_n\}$ converges in the mean to 0 and since it is a sequence in M_1 it must converge almost everywhere to 0; accordingly, $\{x_n\}$ converges almost everywhere to 0.

Therefore, M_2 has the property described in the last sentence of Theorem 5.

Clearly, z_k is in $M_1 \vee M_2$ since y_k and $a_k y_k + b_k z_k$ are, $k \geqq 1$. Thus, $M_3 \subset M_1 \vee M_2$. Consequently, $N \subset M_1 \vee M_3 \subset M_1 \vee M_2$, completing the proof.

5. **A necessary and sufficient condition for L_2 to be semi-Gaussian.**

THEOREM 6. *The space L_2 is semi-Gaussian if and only if each measurable set of positive measure contains an atom.*

Proof. The "if" part of the proof rests on the easily proven fact that if each measurable set of positive measure contains an atom, then every sequence converging in the mean converges almost everywhere.

To prove the "only if" part let L_2 be semi-Gaussian and suppose that there is a set B in F, of positive measure, and containing no atoms. We may and do suppose that $\mu(B)$ is finite. Since the restriction of μ to $\{A \cap B \mid A \in F\}$ is nonatomic, there is (by [2], for example) a Borel measurable function z from B into $I = [0, 1]$ such that

$$\mu(\{w \mid w \in B, z(w) < t\}) = t\mu(B), \qquad t \in I. \tag{7}$$

Let $\{f_k\}$ be a sequence of real Borel measurable functions on I such that $\sum_{k=1}^{\infty} \int_I f_k^2 dv < \infty$, $\int_I f_j f_k dv = 0$ for all j, k with $j \neq k$, where V is Lebesgue measure in I, and such that the series $\sum_{k=1}^{\infty} f_k(t)$ diverges for every t in I. Such a sequence exists [3]. If k is a positive integer, let $x_k(w) = f_k(z(w))$ if w is in B, $=0$ if w is not in B. Then, for all j, k,

$$\int_W x_j x_k d\mu = \int_B f_j(z) f_k(z) d\mu = \mu(B) \int_I f_j f_k dv,$$

using (7). Thus, $\{x_k\}$ is an orthogonal sequence in L_2 satisfying $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$. However, $\sum_{k=1}^{\infty} x_k$ diverges almost everywhere on B, a set of positive measure. This is a contradiction of the assumption that L_2 is semi-Gaussian, and the desired result is implied.

6. **An unsolved problem.** The following problem suggests itself. Let $\{x_k\}$ be a mutually independent random variable sequence such that $E|x_k|^2 < \infty$, $k \geqq 1$, and let $M = \bigvee_{k=1}^{\infty} \{x_k\}$. Is M necessarily semi-Gaussian? At the present time, we do not know. A positive answer is easily seen to imply a positive answer to the analogous question for uncountable mutually independent families of random variables.

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