# SYMMETRIC RANDOM WALK[1]

BY

L. A. SHEPP

Let $X_k$, $k=1, 2, 3, \cdots$, be a sequence of mutually independent random variables on an appropriate probability space which have a given common distribution function $F$. Let $S_n = X_1 + \cdots + X_n$, then the event $\liminf |S_n| = 0$ has probability either zero or one. If this event has zero chance, we say $F$ is *transient*; in the other case, $|S_n|$ tends to infinity almost surely, and $F$ is called *recurrent*. The proofs of these assertions are in [1].

If $F$ is *symmetric*, then transiency depends only on the tail of $F$. Theorem 1 gives a condition on the tail of the d.f. $F$ which is necessary and sufficient for transiency. Let $F$ and $G$ be symmetric and $F$ be less peaked than $G$, in the terminology introduced by Birnbaum. Theorem 4 shows that, if $F$ is unimodal, then the recurrence of $F$ implies the recurrence of $G$. The unimodality condition cannot be entirely removed as an illuminating example shows. Unimodal distribution functions play a central role and Theorem 4 shows the connection between this class and the uniform distributions via the representation of Khinchin. The condition in Theorem 1 can be very much simplified in case $F$ is unimodal, this is done in §5. Finally the results are shown to extend to the higher dimensional case.

1. **Preliminaries.** It is well known that a distribution function (d.f.) $F$ is transient if and only if (1.1) is finite.

$$\sum P\{ |S_n| < 1\}. \tag{1.1}$$

In terms of the characteristic function $\phi(z) = Ee^{iX_1 z}$ this is equivalent[2] to the finiteness of

$$\lim_{\rho \nearrow 1} \int_0^1 R(1 - \rho\phi(u))^{-1} du. \tag{1.2}$$

If $F$ is symmetric, i.e., at continuity points $x$ and $-x$ of $F$, we have

$$F(x) = 1 - F(-x) \tag{1.3}$$

then $\phi(z)$ is real and (1.2) becomes

LEMMA 1.1. *If $F$ is symmetric, then $F$ is transient ($F \in T$) if and only if*

Received by the editors August 30, 1961.

[1] The material presented in this paper constituted part of a doctoral dissertation at Princeton University. I wish to express my gratitude to Professor W. Feller for his inspiring teaching and guidance. Prepared with the partial support of the National Science Foundation, Grant G-14648.

[2] The equivalence is almost formal. For (1.1), see [1].

$$\int_0^1 (1 - \phi(u))^{-1}du \tag{1.4}$$

*is finite.*

We shall need the following simple lemma which expresses the fact that transiency depends only on the tail of the d.f. $F$. Here and in the following our distribution functions are intended to be symmetric, unless otherwise mentioned, for it is only in the symmetric case that transiency does depend only on the tail.

LEMMA 1.2. *If $F_1(x) = F_2(x)$ for $|x| \geqq T$ then $F_1$ and $F_2$ are either both transient or both recurrent.*

To see this, note that $1 - \phi_1(u) \sim 1 - \phi_2(u)$ as $u \to 0$ since

$$\phi_1(u) - \phi_2(u) = O(u^2) = o(1 - \phi_1(u)) \tag{1.5}$$

unless the variances of $F_1$ and $F_2$ are both finite, in which case $F_1$ and $F_2$ are both recurrent. Since transiency depends only on the finiteness of (1.4), the proof is complete.

2. **A tail condition.** In view of Lemma (1.2) our first task is to give a condition on the tail of $F$, necessary and sufficient for transiency. Let us define for $0 \leqq x \leqq t$,

$$Q(t, x) = \sum_{n=0}^{\infty} F(2(n + 1)t - x) - F(2nt + x) \tag{2.1}$$

which is the $F$-measure of a certain linear set.

THEOREM 1. *A symmetric d.f. $F$ is transient if and only if*

$$\int_1^{\infty} \left( \int_0^t xQ(t, x)dx \right)^{-1} dt \tag{2.2}$$

*is finite.*

To prove this result, integrate by parts to obtain,

$$1 - \phi(u) = 2u \int_0^{\infty} \sin xu(1 - F(x))dx \tag{2.3}$$

where the integral is taken in the Cauchy sense. If we integrate over the sets

$$\frac{2\pi n}{u} + \frac{r\pi}{2u} < x \leqq \frac{2\pi n}{u} + (r + 1)\frac{\pi}{2u} \tag{2.4}$$

$n = 0, 1, \cdots; r = 0, 1, 2, 3$; separately, the integral in (2.3) becomes

$$\int_0^{\pi/2u} \sin xu\, Q\left(\frac{\pi}{u}, x\right) dx + \int_0^{\pi/2u} \sin xu R\left(\frac{\pi}{u}, x\right) dx \tag{2.5}$$

where

$$R(t, x) = \sum_{n=0}^{\infty} F((2n+1)t + x) - F((2n+1)t - x). \tag{2.6}$$

It follows from the easily verifiable statements,

$$\bigcup_{n=0}^{\infty} ((2n+1)t - x, (2n+1)t + x] \subset \bigcup_{n=0}^{\infty} (2nt + x, 2(n+1)t - x] \tag{2.7}$$

and

$$\frac{2}{\pi} \leqq \frac{\sin r}{r} \leqq 1, \qquad 0 < r \leqq \frac{\pi}{2}, \tag{2.8}$$

that for $0 \leqq x \leqq t/2$,

$$Q(t, x) \leqq Q(t, x) + R(t, x) \leqq 2Q(t, x) \tag{2.9}$$

and

$$\frac{4}{\pi}\int_0^{\pi/2u} xQ\left(\frac{\pi}{u}, x\right) dx \leqq (1 - \phi(u))/u^2 \leqq 4\int_0^{\pi/2u} xQ\left(\frac{\pi}{u}, x\right) dx. \tag{2.10}$$

It is also easily checked that

$$\int_{t/2}^{t} xQ(t, x)dx \leqq 2\int_{t/4}^{t/2} xQ(t, x)dx. \tag{2.11}$$

Setting $t = \pi/u$ and using Lemma (1.1) and the observation that the lower limit in (2.2) has no essential importance to the statement, the theorem follows.

3. **Unimodal distributions.** A distribution function, not necessarily symmetric for the purposes of §3, is called *unimodal* (with vertex at $c$) if it is convex for $x < c$ and concave for $x > c$. Without loss of generality, we may take $c$ at zero. Khinchin [3] has shown that $G$ is a unimodal distribution function if and only if, at all $x$, continuity points of $F$,

$$F(x) = G(x) - xG'(x) \tag{3.1}$$

where $F$ is a distribution function.

We shall call the *unique* $G$ the unimodal correspondent of $F$, $G = UF$, satisfying (3.1). The characteristic functions, $\psi$ corresponding to $G$ and $\phi$ to $F$ stand in the following relationship:

$$\psi(z) = \frac{1}{z}\int_0^z \phi(u)du. \tag{3.2}$$

There is an equivalent form of the correspondence in terms of random variables.

THEOREM 2. *A d.f. $G$ is unimodal if and only if there is a random variable $Y$, having distribution function $G$, such that*

$$Y = X\cdot\theta \tag{3.3}$$

*where $X$ and $\theta$ are independent and $\theta$ is uniformly distributed on the unit interval. Moreover, this property characterizes the uniform distribution (up to a constant multiple).*

In view of (3.2), the first part of the theorem follows from

$$\psi(z) = Ee^{iYz} = E_\theta E_X e^{iX\cdot\theta z} = \frac{1}{z}\int_0^z \phi(t)dt. \tag{3.4}$$

To prove the second claim, suppose $\alpha$ has the property enjoyed by $\theta$, i.e., for each unimodal $Y$ on an appropriate sample space, $Y=X\cdot\alpha$ and also for each $X$, $X\cdot\alpha$ has a unimodal distribution if $X$ and $\alpha$ are independent. Taking $Y$ to be a uniform variable, $\theta_1$, we obtain $\theta_1=X_1\alpha_1$. Taking $X\equiv 1$ we obtain $\alpha_1 X=\alpha_1$ is unimodal and so $\alpha_1=X_2\theta_2$

$$\theta_1 = X_1\alpha_1, \qquad \alpha_1 = X_2\theta_2 \tag{3.5}$$

where $X_1$ and $X_2$ may be chosen independent of $\theta_2$. We have $\theta_1=(X_1X_2)\theta_2$ where $\theta_1\sim\theta_2$ (copies) and it follows using an elementary argument that $X_1$ and $X_2$ are constant, a.e., and so $\alpha=c\cdot\theta$. This completes the proof.

We shall need the following definition. A distribution function $F$ is *convex at infinity* if there exists $a$, $b$ such that $F$ is convex (from below) for $X<a$ and convex (from above) for $X>b$.

4. **Peakedness.** Following Z. W. Birnbaum, a distribution function $F$ is *less peaked* than a distribution function $G$ (about zero) if for all $a>0$,

$$1 - F(a) + F(-a) \geqq 1 - G(a) + G(-a). \tag{4.1}$$

Here $F$ and $G$ are not necessarily symmetric, but in the symmetric case (4.1) becomes

$$F(x) \leqq G(x), \qquad x > 0. \tag{4.2}$$

If $F$ and $G$ are as above, write $F<G$. We can readily construct random variables $X$ and $Y$ corresponding to $F$ and to $G$ respectively and such that

$$|Y| \leqq |X|, \qquad \text{sign } Y = \text{sign } X. \tag{4.3}$$

THEOREM 3. *Let $F_i$, $G_i$ be symmetric and unimodal with $F_i<G_i$, $i=1, 2$.*

*The convolutions* $F_1 * F_2$, $G_1 * G_2$ *are symmetric and unimodal and* $F_1 * F_2 < G_1 * G_2$.

The first claim is usually ascribed to Wintner [6] and the second is due to Birnbaum [2](3).

THEOREM 4. *Let* $F$, $G$ *be symmetric and* $F$ *be unimodal. If* $F < G$, *then the recurrence of* $F$ *implies the recurrence of* $G$.

REMARK. The condition that $F$ is unimodal can be weakened to convexity at infinity (§3), but an example will show it cannot be entirely removed.

We shall need some lemmas for the proof. As above, let $UF$ denote the unimodal correspondent of $F$.

LEMMA 4.1. *For any d.f.* $F$, $F < UF$.

This is immediate from the definitions and (3.1). Let $F$ be symmetric.

LEMMA 4.2. *If* $UF$ *is recurrent, then* $F$ *is recurrent.*

By Lemma (1.1), $(1-\psi)^{-1}$ is not in $L'(0, 1)$. For some $c > 0$,

$$1 - \cos u \leqq c\left(1 - \frac{\sin u}{u}\right), \qquad -\infty < u < \infty. \tag{4.4}$$

Hence,

$$1 - \phi(z) \leqq c\int\left(1 - \frac{\sin xz}{xz}\right)dF(x) = c(1 - \psi(z)). \tag{4.5}$$

It follows that $(1-\phi)^{-1}$ is not in $L'(0, 1)$ and again using Lemma (1.1), Lemma (4.2) is proved.

LEMMA 4.3. *Let* $F_1$, $F_2$ *be symmetric and unimodal. If* $F_1 < F_2$ *then the recurrence of* $F_1$ *implies the recurrence of* $F_2$.

To prove this special case of the theorem, note that by Theorem 3,

$$F_1^{n^*} < F_2^{n^*}, \qquad n = 1, 2, 3, \cdots, \tag{4.6}$$

where the exponent, $n^*$, means $n$-fold convolution. Hence, for $x > 0$,

$$F_1^{n^*}(x) - F_1^{n^*}(-x) \leqq F_2^{n^*}(x) - F_2^{n^*}(-x). \tag{4.7}$$

Summing,

$$\sum_{n=0}^{\infty} F_1^{n^*}(x) - F_1^{n^*}(-x) \leqq \sum_{n=0}^{\infty} F_2^{n^*}(x) - F_2^{n^*}(-x). \tag{4.8}$$

(3) Birnbaum assumes, unnecessarily, that $F_i$ and $G_i$ are absolutely continuous. This follows from unimodality (except for a possible jump at the vertex) see [4, p. 92].

Now an application of the result in (1.1) completes the proof. To prove the theorem, observe that

$$F < G < UG. \tag{4.9}$$

Hence $F < UG$ and since $F$ is recurrent, $UG$ is recurrent by Lemma (4.3). Thus $G$ is recurrent, as was to be proved. If $F_1$ is merely convex at infinity then there is a unimodal distribution function $F_2$ such that for $|x| \geqq T$, $F_1(x) = F_2(x)$. Using Lemma (1.2), it may be shown that the theorem holds for $F_1$. The details are omitted.

If the convexity is altogether removed from the hypothesis, then the theorem becomes false. In terms of random variables, this means that there is a sequence $X_1, X_2, \cdots, X_n, \cdots$, of mutually independent random variables with the common d.f. $F$, and a second sequence $Y_1, Y_2, \cdots, Y_n, \cdots$, of mutually independent random variables with the common d.f. $G$, defined on the same probability space, such that except for a set of probability zero the following conditions hold:

$$\liminf |X_1 + \cdots + X_n| = 0, \tag{4.10}$$

$$\lim |Y_1 + \cdots + Y_n| = \infty, \tag{4.11}$$

$$|Y_n| \leqq |X_n|, \tag{4.12}$$

$$\operatorname{sign} Y_n = \operatorname{sign} X_n. \tag{4.13}$$

In fact, the example of §6, shows that (4.10) holds in the stronger form

$$X_1 + \cdots + X_n = 0 \text{ infinitely often.} \tag{4.14}$$

5. **Tail conditions in the unimodal case.** The problem of giving a condition on the tail of a symmetric d.f. equivalent to transiency of the generated random walk was solved with Theorem 1. However, the condition of (2.2) involves the $F$-measures of certain complicated linear sets in (2.1) and is difficult to deal with. One would expect to improve the convergence of (2.2) in the case of smooth $F$ and this is the content of this section. The following theorem gives a condition similar to that of (2.2) except that the integrand is now monotonic, which greatly simplifies the task of determining convergence.

THEOREM 5. *If $F$ is unimodal (or merely convex at infinity) then $F$ is transient if and only if*

$$\int_1^\infty \left( \int_0^t x(1 - F(x))dx \right)^{-1} dt \tag{5.1}$$

*is finite.*

If one compares (5.1) with (2.2) it follows from

$$Q(t, x) \leqq 1 - F(x) \tag{5.2}$$

that if (5.1) diverges then $F$ is recurrent, without the assumption of unimodality. From Lemmas (4.1), (4.2), and (4.3) it follows that $F\in T$ (transient) if and only if $UF\in T$. By Lemma (1.1), $UF\in T$ if and only if $(1-\psi(u))^{-1} \in L'(0, 1)$. It remains to show only that if (5.1) is finite then $F$ is transient, or that

$$\int_0^1 (1 - \psi(u))^{-1} du \tag{5.3}$$

is finite. However, $1-\psi(u)=2\int_0^\infty(1-(\sin xu)/xu)dF(x)$ and using the fact that $1-(\sin r/r)\geqq c\min(r^2, 1)$ for some $c>0$, it follows that

$$1 - \psi(u) \geqq 2c\left(u^2\int_0^{1/u} x^2 dF(x) + \int_{1/u}^\infty dF\right) = 4cu^2\int_0^{1/u} x(1 - F(x))dx. \tag{5.4}$$

If (5.1) converges then by (5.4) setting $u=1/t$, (5.3) also converges and this proves the theorem. It is possible to give a direct proof of Theorem 5 but it is rather detailed and will be omitted. Again, if the convexity assumption is removed from the hypothesis, the theorem becomes false. The example previously referred to in §6, also serves to demonstrate this, i.e., (5.1) is finite and (2.2) is infinite for the common distribution of the variables $X_n$ of (6.1) and (6.2).

6. **An example.** Let the distributions of the $X_n$ process be given by the assignments,

$$P\{X_n = a_k\} = P\{X_n = - a_k\} = \frac{1}{2} p_k, \qquad k \geqq 1, n \geqq 1, \tag{6.1}$$

with

$$a_k = 2^{k^2}, \qquad p_k = c\, 2^{k-k^2}k^2, \tag{6.2}$$

where $c$ is so chosen that $\sum p_k=1$. The variables $Y_n$ are then defined in terms of $X_n$ by the unimodal correspondence, i.e.,

$$Y_n = X_n\cdot\theta_n, \qquad n \geqq 1, \tag{6.3}$$

where $\theta_n$ is uniformly distributed on the interval (0, 1) and the family $\{X_n, \theta_m\}$ is a family of mutually independent random variables. We shall show that (4.10) and (4.11) hold for these assignments(4). This will also prove the assertion in the last paragraph of §5. It is clear that (4.12) and (4.13) hold. The calculations involved are very unpleasant and so they will only be outlined. Let $F$ and $G$ be the common distribution function of $X_n$ and $Y_n$ respectively. To prove that (4.10) holds we must show that (2.2) is infinite. Let the quantity (2.2) be denoted by $A$. For $1\leqq n\leqq 2^k$; $k\geqq 1$, let

(4) Another example satisfying these conditions and having still larger steps may be given by choosing $a_k=2^{2^k}$, $p_k=c\,2^{k-(2/3)2^k}$. The proof is similar.

$$(6.4)\qquad I_{n,k} = \left\{t\colon a_k < t < a_{k+1};\ \frac{a_{k+1}}{2n+1} < t \leqq \frac{a_{k+1}}{2n}\right\}.$$

It is clear that

$$(6.5)\qquad A \geqq \sum_{k\geqq 1}\sum_{n=1}^{2^k}\int_{t\in I_{n,k}}\left(\int_0^t xQ(t,x)dx\right)^{-1}dt.$$

If $t\in I_{n,k}$ and $a_l\in\bigcup_{n\geqq 1}(2nt+x,\ 2(n+1)t-x)$ then for $l\leqq k$, $x\leqq a_l\leqq t$. It follows that

$$(6.6)\qquad \int_0^t xQ(t,x)dx \leqq \sum_{l=1}^{k}\int_0^{a_l} xp_l dx + \int_0^{a_{k+1}-2tn} xp_{k+1}dx + \frac{t^2}{2}\sum_{l\geqq k+2} p_l.$$

The right-hand side is less than $c_1(a_k^2p_k+u^2p_{k+1})$ for some $c_1$ an absolute positive constant, with $u=a_{k+1}-2tn$. This is so because the last term in (6.6) is, for all $t\in I_{n,k}$ smaller than the last term of the sum. Thus there is $c_2>0$, such that

$$(6.7)\qquad A \geqq c_2\sum_{k\geqq 1}\sum_{n=1}^{2^k}\frac{1}{n}\int_{0\leqq u\leqq a_{k+1}/2n+1}(a_k^2p_k + u^2p_{k+1})^{-1}du.$$

This last quantity, by direct integration, is larger than $c_3\sum_{k\geqq 1}\sum_{n=1}^{2^k} 1/nk^2$ which diverges. This proves the first part. To prove that (4.11) holds we must show that (5.1) is finite, because of Theorem 5, since $G$ is unimodal. The integrand in (5.1) is monotonic and this is comparatively simple. If $a_k<t<a_{k+1}$, there exists $c_1>0$ independent of $k$, such that

$$(6.8)\qquad \int_0^t x(1-F(x))dx \geqq c_1(a_k^2p_k + t^2p_{k+1}).$$

By direct integration, it follows that

$$(6.9)\qquad \sum_{k\geqq 1}\int_{a_k}^{a_{k+1}}\left(\int_0^t x(1-F(x))dx\right)^{-1}dt \leqq c_2\sum_{k\geqq 1}\frac{1}{k^2} < \infty.$$

This completes the proof. It is possible to construct such an example where $F$ is in the class $C^\infty$, or even infinitely divisible. We omit the details which are contained in [5]. It would seem that for some distribution function $G$, $F<G$ implies $F\in T$. In a forthcoming paper we shall prove that there is no such $G$; for every random walk there is a recurrent random walk having larger steps. Thus the example above gives a true picture of the situation for the case of non-unimodal walks.

7. **Two dimensional case.** It is proved in [1] that the generalization of the notion of recurrence to dimensions higher than two is vacuous. Thus, the only remaining case of interest is the case of two dimensions, where our results carry over in an almost formal way.

DEFINITION. A distribution function $G=G(x, y)$ is called *unimodal* if a density $g(x, y)$ exists (except perhaps at the vertex which we shall take at the origin) and if, for $R$ a rectangle with lower-left corner $(a, b)$ and upper-right corner $(c, d)$ and sides parallel to the axes, the expression

$$g(a, b) - g(a, d) - g(c, b) + g(c, d) \tag{7.1}$$

is positive if $R$ is contained in quadrants I or IV and negative if $R$ is contained in quadrants II or III. If $g \in D'$, then this definition simplifies to $g_{xy} \geqq 0$ if $x \cdot y > 0$ and $g_{xy} \leqq 0$ if $xy < 0$, subscripts denoting partial derivatives. Khinchin's theorem generalizes in the following way:

THEOREM 6. *A distribution function $G$ is unimodal if and only if for all $(x, y)$ points of continuity of $F$,*

$$F(x, y) = G(x, y) - xG_x(x, y) - yG_y(x, y) + xyG_{xy}(x, y) \tag{7.2}$$

*where $F$ is a distribution function.*

The proof is quite similar to the linear case. The analogous result to Theorem 2 is equivalent to the above and is given as follows.

THEOREM 7. *A. d.f. $G$ is unimodal if and only if there is a random vector $(Y_1, Y_2)$ with distribution function $G$, such that*

$$(Y_1, Y_2) = (X_1\theta_1, X_2\theta_2) \tag{7.3}$$

*where $(X_1, X_2)$ and $(\theta_1, \theta_2)$ are independent vectors and $(\theta_1, \theta_2)$ is uniformly distributed on the unit square.*

The proofs will be omitted. The relation (7.2) can be inverted to yield

$$G(x, y) = xy \int_x^\infty \int_y^\infty \frac{F(u, v)}{u^2v^2} dudv \tag{7.4}$$

for $x>0$ and $y>0$. It then follows that for $x>0$ and $y>0$,

$$F(x, y) \leqq G(x, y). \tag{7.5}$$

This suggests the following definitions. A d.f. $F$ is symmetric if

$$F\{A\} = F\{-A\} = F\{\overline{A}\} \tag{7.6}$$

when viewed as a measure on Borel sets $A$, the operations denoting, as usual, reflection about the $y$ and $x$ axis respectively. A distribution function $F$ is less peaked than a distribution function $G$ (for symmetric $F$ and $G$) provided (7.5) holds for $x>0$ and $y>0$. The theorems of Birnbaum and Wintner (Theorem 3) have the same statement and a similar proof and so need not be restated. We can now repeat the proofs of Theorems 4 and 5 practically word for word to obtain the following theorems.

THEOREM 8. *Let $F$, $G$ be symmetric, and $F$ be unimodal. If $F$ is less peaked than $G$, then the recurrence of $F$ implies the recurrence of $G$.*

THEOREM 9. *If $F$ is symmetric and unimodal then $F$ is transient if and only if*

$$\int_1^\infty \int_1^\infty (D(u, v))^{-1} du dv \tag{7.7}$$

*is finite, where*

$$D(u, v) = \int_0^u \int_0^v (x^2v^2 + y^2u^2) dF(x, y) + u^2v^2 \iint_{\{x>u\}\cup\{y>v\}} dF(x, y).$$

We may weaken the unimodality assumption to convexity at infinity (suitably taken) but a modification of the given example shows it cannot be entirely removed.

## REFERENCES

1. K. L. Chung and J. Fuchs, *On the distribution of sums of random variables*, Mem. Amer. Math. Soc. No. 6 (1956).

2. Z. W. Birnbaum, *On random variables with comparable peakedness*, Ann. Math. Statist. 19 (1948), 76–81.

3. B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions of sums of independent random variables*, Addison-Wesley, Cambridge, Mass., 1954.

4. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge, 1952.

5. L. A. Shepp, *Recurrent sums of random variables*, Thesis, Princeton University, 1961.

6. A. Wintner, *Asymptotic distributions and infinite convolutions*, Edwards Brothers, Ann Arbor, Mich., 1938.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIFORNIA