ITERATION METHODS FOR NONLINEAR PROBLEMS(1)

BY

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1. **Introduction.** The method of relaxation or the method of successive displacements has been used extensively to solve linear problems [1]. In particular this method has been applied to the solution of linear elliptic equations and has been especially successful with the use of automatic digital computers.

Analogous methods have been used in practice, with apparent success, on nonlinear problems as well. For the most part, these have not been justified mathematically and this work is an attempt to fill this gap. In particular it is shown that the relaxation methods yield solutions to problems arising from the minimization of certain convex functions. In practice, these functions are obtained by approximating multiple integrals in a calculus of variations problem.

It is shown that an approximate Plateau problem may be solved by a successive displacements method, or a method analogous to Liebmann's method.

We at the same time obtain an extension of a free steering theorem for positive definite symmetric matrices given as Theorem 4 of [3], and results of Ostrowski [2].

2. **Definitions.** Let E_n denote Euclidean n-space and let u be a column vector in E_n. Let $G(u)$ be a real valued, twice continuously differentiable function defined on a convex domain K in E_n. For each $u \in K$ we define the matrix $A = A(u) = (a_{ij})$ by

$$a_{ij} = a_{ij}(u) = \partial^2 G/\partial u_i \partial u_j = G_{u_i u_j}, \qquad i, j = 1, 2, \cdots, n.$$

We assume further that for all $u \in K$

(a) $A(u)$ is positive definite,

(b) $G(u) \geqq m$

for some constant m. If $u, v \in K$ then we write $w \in (u, v)$ if w lies on the line segment joining u and v, that is if there exists a number α, $0 < \alpha < 1$ such that $w = \alpha u + (1-\alpha)v$. Let $r(u) = \text{grad } G(u)$, then

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$$G(u) - G(v) = r(v)^T(u - v) + \frac{1}{2}(u - v)^T A(w)(u - v), \qquad w \in (u, v) \tag{2.1}$$

from the mean value theorem. (The superscript T denotes transpose.)

It then follows from condition (a), as is well known, that $G(u)$ is strictly convex. That is if $u, v \in K$ and $0 < \alpha < 1$ then

$$G(\alpha u + (1 - \alpha)v) \leqq \alpha G(u) + (1 - \alpha)G(v) \tag{2.2}$$

with equality if and only if $u = v$.

We seek solutions to the equation

$$r(u) = 0 \tag{2.3}$$

or stationary points for $G(u)$. From (2.1) and condition (a) it follows that the solution is unique and yields a global minimum for $G(u)$. For if u and v are distinct solutions then

$$G(u) - G(v) = \frac{1}{2}(u - v)^T A(w)(u - v) > 0$$

and by the same token $G(v) > G(u)$.

We proceed now to give a constructive existence theorem for solutions of (2.3).

Let D be a convex set such that $\overline{D} \subset K$, (the bar denoting closure); then we call D a *solvent set* if for every $u^0 \in D$ there exists, for each i, $i = 1, 2, \cdots, n$ a vector $u' \in D$ such that

$$\begin{aligned} r_i(\overset{0}{u}_1, \cdots, \overset{0}{u}_{i-1}, u_i', \overset{0}{u}_{i+1}, \cdots, \overset{0}{u}_n) &= 0 \\ u_k' = \overset{0}{u}_k, \qquad k &\neq i. \end{aligned} \tag{2.4}$$

For any given $u^0 \in D$ we now define a sequence $\{u^p\}$ in D as follows:

Let $Z = (1, 2, \cdots, n)$ and $\{i_p\}_{p=0}^{\infty}$ be an infinite sequence of integers chosen from Z such that every integer in Z occurs infinitely often in the sequence. If $u^p \in D$ then u^{p+1} is obtained from u^p by changing one component as follows:

Choose $u_k^{(p+1)}$ to be the solution to the equation in u_k

$$r_k(u_1^{(p)}, \cdots, u_{k-1}^{(p)}, u_k, u_{k+1}^{(p)}, \cdots, u_n^{(p)}) = 0, \qquad k = i_p \tag{2.5}$$

while

$$u_k^{(p+1)} = u_k^{(p)}, \qquad k \neq i_p.$$

The process defined here may be regarded as an extension of the method of successive displacements as used for linear problems. We continue to call the above process by this name or briefly, a relaxation process.

3. **Convergence of the sequence.** We now prove the following:

THEOREM 3.1. *Let $G(u)$ be a C^2 function in K satisfying* (a) *and* (b), *and assume that a solvent set D exists which is bounded. Then* (2.3) *has a unique solution u^* and the sequence $\{u^p\}$ of the relaxation process converges to u^*.*

Proof. The uniqueness has already been shown above.

We assume now that x is a limit point of the sequence $\{u^p\}$ such that $r(x)\neq 0$. We note from (2.1) that

$$(3.1)\qquad \begin{aligned} G(u^p) - G(u^{p+1}) &= \frac{1}{2}(u^p - u^{p+1})^T A(v)(u^p - u^{p+1}), \qquad v \in (u^p, u^{p+1}) \\ &= \frac{1}{2}\left(u_k^{(p)} - u_k^{(p+1)}\right)^2 a_{kk}(v), \qquad k = i_p \end{aligned}$$

so that

$$(3.2)\qquad G(u^p) \geqq G(u^{p+1}).$$

It follows from condition (b) that $G(u^p)$ converges monotonically to some number $G_\infty = G(x)$.

Let $\rho = \min\{\,|r_i(x)|\,|\,r_i(x)\neq 0,\ i\in Z\}$ and since D is bounded there exist positive numbers λ, Λ such that

$$(3.3)\qquad \lambda w^T w \leqq w^T A(u) w \leqq \Lambda w^T w$$

for all $u \in \overline{D}$.

Let U be a neighborhood of x given by

$$(3.4)\qquad |u - x| < \delta < \frac{\rho}{2\Lambda}$$

where the distance is Euclidean. We choose an integer N sufficiently large that for all $p > N$

$$(3.5)\qquad G(u^p) - G(u^{p+1}) < \frac{\lambda\rho^2}{8\Lambda^2}\cdot$$

Then for all $p > N$ we get from (3.1), (3.3) and (3.5) that

$$(3.6)\qquad \begin{aligned} G(u^p) - G(u^{p+1}) &\geqq \frac{\lambda}{2}\,|u^p - u^{p+1}|^2, \\ |u^p - u^{p+1}| &< \frac{\rho}{2\Lambda}\cdot \end{aligned}$$

If now for some $p > N$, there is a $u^p \in U$ such that $r_{i_p}(x)\neq 0$ then we get that, for $w \in (u^p, x)$ and $k = i_p$,

$$|r_k(u^p) - r_k(x)| = \left|\sum_{j=1}^{n} a_{kj}(w)(u_j^{(p)} - x_j)\right| \leqq |A(w)(u^p - x)| < \rho/2$$

and thus

$$\rho - |r_k(u^p)| \leqq |r_k(x)| - |r_k(u^p)| < \rho/2$$

or

$$\rho/2 < |r_k(u^p)|.$$

This implies that $u^{p+1} \neq u^p$ and that

$$\rho/2 < |r_k(u^p)| = |r_k(u^p) - r_k(u^{p+1})| = a_{kk}(v)|u^p - u^{p+1}| \leqq \Lambda|u^p - u^{p+1}|$$

which contradicts (3.6).

We may therefore assume that for all $u^p \in U$, such that $p > N$, $r_{i_p}(x) = 0$ and $(u^{p+1} - u^p)^T r(x) = 0$.

Let K_∞ be the set of $u \in K$ such that $G(u) \leqq G(x)$ and let Γ_∞ be the bounding surface $G(u) = G_\infty = G(x)$ of K_∞. Let T be the tangent hyperplane $(u - x)^T r(x) = 0$ to Γ_∞ at x. Denote by $E^+(T)$ and $E^-(T)$ the open half spaces defined by T where $E^-(T)$ is the set $(u - x)^T r(x) < 0$, and $\overline{E}^-(T)$ contains K_∞.

The latter follows from the convexity of K_∞ and (2.1). This also implies that T has only the point x in common with Γ_∞.

Let T_τ be a hyperplane parallel to T such that $T_\tau \subset E^-(T)$ and is a distance $\tau > 0$ from T. Let

$$C_\tau = \overline{E}^+(T_\tau) \cap K_\infty$$

then it follows from the strict convexity of $G(u)$ that, for sufficiently small τ, C_τ is bounded.

Let τ_0 be so chosen and let μ be a positive constant such that $w^T A(u) w \geqq \mu w^T w$ for all $u \in C_{\tau_0}$ and all $w \in E_n$. If $0 < \tau < \tau_0$ then for $u \in C_\tau$, $0 \geqq G(u) - G(x) = (u - x)^T r(x) + (u - x)^T A(v)(u - x)/2$ for some $v \in (u, x)$. Thus

$$|r(x)|\tau \geqq -r(x)^T(u - x) \geqq \frac{1}{2}\mu|u - x|^2.$$

This implies that for a sufficiently small neighborhood U of x, satisfying (3.4), $C_{\tau_1} \subset U$, for some $\tau_1 > 0$. Let $T_1 = T_{\tau_1}$ and let $S = E^-(T_1) \cup U$.

Since $S \supset K_\infty$, we may assume that all but a finite number of the u^p are in S. Let $u^p \in E^+(T_1) \cap U$ then $u^{p+1} \in S$ and $(u^{p+1} - u^p)^T r(x) = 0$. Thus $u^{p+1} \in E^+(T_1)$ and therefore $u^{p+1} \in U$. Thus all but a finite number of u^p are in U. If, say, $r_1(x) \neq 0$ then $i_p = 1$ for at most a finite number of values of p.

This contradicts the hypothesis and thus $r(x) = 0$. Since the solution is unique the sequence converges, and the proof is complete.

4. **Converse.** A partial converse to Theorem 3.1 can be obtained in the following sense:

THEOREM 4.1. *Let $G(u)\in C^2$, assume that $A(u)$ is nonsingular, and $a_{ii}(u)>0$ for $u\in K$, $i\in Z$. Let K be an open solvent set such that for every $u^0\in K$ the resulting relaxation process converges to the same solution u^* of* (2.3) *and $u^*\in K$. Then $A(u^*)$ is positive definite.*

Proof. Let $w^TA(u^*)w<0$ for some $w\neq 0$, then this is still true in some neighborhood N: $|u-u^*|<\delta$. We may assume without loss that $|w|<\delta$ and let $u^0 = w + u^*$. Since $u^0 \in N$, for $v \in (u^0, u^*)$, $0 > G(u^0) - G(u^*) = (u^0-u^*)^TA(v)(u^0-u^*)/2$. But if a relaxation process is begun with u^0, $G(u^0)\geqq G(u^*)$ from (3.1). Thus $w^TA(u^*)w\geqq 0$ for all $w\in E_n$.

Since $A(u^*)$ is nonsingular $w^TA(u^*)w=0$ implies that $w=0$.

Theorems 3.1 and 4.1 are well known for the special case where $A(u)$ is a constant matrix and where $\{i_p\}$ is periodic in p (see [1, Theorem 21.1]). The existence of solvent sets in this case is covered by the remarks of the next section.

5. **Remarks.** 1. Let $K=E_n$ and assume that K is a solvent set. Assume furthermore that the set $K_0=\{u\mid G(u)\leqq G(v^0)\}$, for some $v^0\in K$, is bounded. If the existence of the set D in Theorem 3.1 is replaced by the above hypothesis this theorem remains valid.

This follows readily from (3.1) which implies that K_0 may be chosen for D.

2. Let $K=E_n$ and assume that there exists a positive constant λ such that

$$w^TA(u)w \geqq \lambda w^Tw \tag{5.1}$$

for all u, $w\in K$. Then K is a solvent set and K_0 is bounded for any $v^0\in K$. Thus by Remark 1 the conclusion of Theorem 3.1 is valid with the above assumptions.

To show that K is a solvent set we let $u^0\in K$ and let, for any i, $u'\in K$ such that $u_k' = u_k^0$, $k\neq i$. We seek to show that there exists a solution to $r_i(u')=0$ as an equation in u_i.

Since $a_{ii}(u)\geqq\lambda$ for all u, it follows from

$$r_i(u') - r_i(\overset{0}{u}) = a_{ii}(v)(u_i - \overset{0}{u_i}), \qquad v \in (u^0, u')$$

that for $u_i>u_i^0+|r_i(u^0)|/\lambda$, $r_i(u')>0$ while $r_i(u')<0$ for $u_i<u_i^0-|r_i(u^0)|/\lambda$. Thus $r_i(u')$ has a zero in K.

To show that K_0 is bounded we note from (2.1) that for $w\in(u, v^0)$, $0 \geqq G(u) - G(v^0) = (u - v^0)^Tr(v^0) + (u - v^0)^TA(w)(u - v^0)/2$ or that $|u - v^0| \leqq 2|r(v^0)|/\lambda$.

6. **Approximate relaxation.** In the previously defined relaxation process it was assumed only that the equation (2.4) had a solution. In practice one would like to have a process which is more constructive. The natural approach would be to take several steps in Newton's method to find an approximation to the root. Since in the linear case only one such step is taken we consider this possibility first.

Assume that $K=E_n$ and that, for $G(u)$ satisfying the (a), (b) conditions of §2, the set $K_0=\{u|G(u)\leqq G(u^0)\}$ is bounded, for some $u^0\in K$. Let

$$\alpha_i = \min_{u\in K_0} a_{ii}(u), \qquad i \in Z,$$

and assume that there exist positive constants β_i such that $a_{ii}(u)\leqq\beta_i$ for all $u\in K$.

Given a sequence of indices $\{i_p\}$ which exhaust Z infinitely often, and a sequence of numbers $\{\omega_p\}$, $p=0, 1, 2, \cdots$, then for a given u^0 we define an *approximate relaxation* process by

$$\begin{aligned} u_i^{(p+1)} &= u_k^{(p)} + \omega_p h_p, & k &= i_p \\ u_k^{(p+1)} &= u_k^{(p)}, & k &\neq i_p \end{aligned} \tag{6.1}$$

where $h_p=-r_k(u^p)/a_{kk}(u^p)$. For this process we prove

THEOREM 6.1. *For a suitable choice of* $\{\omega_p\}$ *the approximate relaxation process* (6.1) *converges to a unique solution of* (2.3).

Proof. The proof is similar to that for Theorem 3.1. Let $\gamma_0=\min_i(\alpha_i/\beta_i)$ and let γ be chosen such that $0<\gamma<\gamma_0\leqq 1$. If the ω_p are such that

$$0 < \gamma \leqq \omega_p \leqq 2\gamma_0 - \gamma, \qquad p = 0, 1, 2, \cdots$$

then from (2.1), for $v\in(u^p, u^{p+1})$, $k=i_p$,

$$\begin{aligned} G(u^{p+1}) - G(u^p) &= r_k(u^p)(u_k^{(p+1)} - u_k^{(p)}) + \frac{1}{2}(u_k^{(p+1)} - u_k^{(p)})^2 a_{kk}(v) \\ &= \frac{1}{2}\omega_p h_p^2(\omega_p a_{kk}(v) - 2a_{kk}(u^p)) \\ &\leqq -\gamma\omega_p h_p^2 a_{kk}(u^p)/2\gamma_0. \end{aligned}$$

Thus it follows that the $G(u^p)$ are again monotone non-increasing, all the u^p are in K_0 and that $r_{i_p}(u^p)\to 0$. The remainder of the proof follows in the same manner as for Theorem 3.1.

REMARK 3. A converse similar to Theorem 4.1 can be formulated for the approximate relaxation process providing α_i, β_i exist and $0\leqq\omega_p\leqq 2\gamma_0$. In fact Theorem 4.1 can be formulated for all processes $\{u^p\}$ which begin in an open set and yield non-increasing sequences $\{G(u^p)\}$.

7. **Application to elliptic equations.** We now consider an application of the preceding theorems to the problem of solving certain nonlinear difference equations. In particular we consider those equations which arise from minimizing problems which are discretizations of variational problems. (Cf. Courant, Friedrichs and Lewy [4]; and L. Bers [5].)

Let Ω be a given bounded domain in the (x, y)-plane with boundary $\dot{\Omega}$ and closure $\bar{\Omega}$. Let $F(x, y, p, q)$ be a given function defined for $(x, y)\in\bar{\Omega}$ and all (p, q) such that F is of class C^2 in p, q and

$$(*) \qquad \Phi \equiv \begin{pmatrix} F_{pp} & F_{pq} \\ F_{pq} & F_{qq} \end{pmatrix} \text{ is positive definite}$$

for $(x, y) \in \bar{\Omega}$ and all p, q. (Subscripts in p, q indicate differentiation in those variables.)

Let a square lattice of mesh width $h>0$ be defined by the nodes $x=\mu h$, $y=\nu h$; μ, $\nu=0, \pm 1, \pm 2, \cdots$. To each node $P_i=(x, y)$ we associate four neighbors $(x+h, y)$, $(x, y+h)$, $(x-h, y)$, $(x, y-h)$ and denote these by $P_{i1}, P_{i2}, P_{i3}, P_{i4}$ respectively. Denote by Ω_h the set of nodes $P_1, P_2, \cdots, P_n$ in Ω all of whose neighbors lie in $\bar{\Omega}$. We choose h sufficiently small that Ω_h is not null.

Denote by $\dot{\Omega}_h$ the set of neighbors of nodes of Ω_h which are themselves not in Ω_h. We list these as $P_{n+1}, \cdots, P_{n+m}$ and call them boundary nodes. The union of Ω_h and $\dot{\Omega}_h$ is denoted by $\bar{\Omega}_h$.

Let $u(P)$ be a real valued function on $\bar{\Omega}_h$ then we write $u_i=u(P_i)$, $i=1, \cdots, n+m$, and for those neighbors $P_{i\nu}$ of P_i which are in $\bar{\Omega}_h$ let $u_{i\nu}=u(P_{i\nu})$. Let $P_1, P_2, \cdots, P_N$ be the nodes of $\bar{\Omega}_h$ such that P_{i1} and P_{i2} are in $\bar{\Omega}_h$, so that $N \geqq n$. For each such node P_i we associate the pair of numbers $p_i=(u_{i1}-u_i)/h$, $q_i=(u_{i2}-u_i)/h$.

Let $b(P)$ be a given function defined on the boundary nodes and let $u(P)=b(P)$ for $P \in \dot{\Omega}_h$. Let $u=(u_1, u_2, \cdots, u_n)^T$ and

$$G(u) = h^2 \sum_{k=1}^{N} F(P_k, p_k, q_k). \tag{7.1}$$

It follows readily from (*) that $G(u)$ is strictly convex. We however will prove the following stronger result for the matrix $A(u)$ as defined in §2:

THEOREM 7.1. *The matrix $A(u)$ is positive definite for all u.*

Proof. Let w be any nonzero vector in E_n; then we show that $w^T A(u) w > 0$. Let $F^{(k)}=h^2 F(P_k, p_k, q_k)$, let

$$A^{(k)} = (a_{ij}^{(k)}) = \left(\frac{\partial^2 F^{(k)}}{\partial u_i \partial u_j}\right) \text{ and set } V_k = w^T A^{(k)} w \text{ then } w^T A w = \sum_{k=1}^{N} V_k.$$

We note first that $a_{ij}^{(k)}=0$ except for a principal submatrix of order at most, three arising from the three variables u_k, u_{k1}, u_{k2}. For notational convenience we drop the k and write these variables as $\bar{u}_0, \bar{u}_1, \bar{u}_2$. If $P_k, P_{k1}, P_{k2} \in \Omega_h$ then this submatrix has the form

$$\phi^{(k)} = \begin{bmatrix} F_{pp} + 2F_{pq} + F_{qq} & -F_{pp} - F_{pq} & -F_{qq} - F_{pq} \\ -F_{pp} - F_{pq} & F_{pp} & F_{pq} \\ -F_{qq} - F_{pq} & F_{pq} & F_{qq} \end{bmatrix} \tag{7.2}$$

where $\phi^{(k)}=(\partial^2F^{(k)}/\partial\bar{u}_i\partial\bar{u}_j)_{i,j=0,1,2}$ and the elements of $\phi^{(k)}$ are evaluated at (P_k, p_k, q_k).

We note first that every principal subdeterminant of $\phi^{(k)}$, of order not greater than two, is positive. That the $\phi_{ii}^{(k)}>0$ follows readily from (*) and the fact that the sum of the elements of a positive definite matrix is positive. Every principal subdeterminant of order two has the value $\Delta=\det\Phi$ and is therefore also positive. Since the row sums of $\phi^{(k)}$ are zero, $\phi^{(k)}$ has rank two and is therefore positive semidefinite.

If at least one, but not all, of the P_k, P_{k1}, P_{k2} is in $\dot{\Omega}_h$ then $\phi^{(k)}$ is, by the above argument, positive definite. Thus in all cases $V_k\geqq 0$, and if for some $w\neq 0$, $w^TA(u)w=0$ then $V_k=0$ for $k=1, 2, \cdots, N$.

Let P_k be any point in Ω_h and suppose that P_{k1} or P_{k2} is in $\dot{\Omega}_h$. Then, since $\phi^{(k)}$ is positive definite, $w_k=0$. Now let P_k, P_{k1}, P_{k2} all belong to Ω_h. Then $\phi^{(k)}$ is given by (7.2) and may be factored uniquely, by the LDU theorem, into the form

$$\phi^{(k)}=\begin{bmatrix}1&0&0\\c_{10}&1&0\\c_{20}&c_{21}&1\end{bmatrix}\begin{bmatrix}\phi_{00}^{(k)}&0&0\\0&\Delta&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&c_{10}&c_{20}\\0&1&c_{21}\\0&0&1\end{bmatrix}=U^TDU.$$

If we use the same local indices with the w_k as with the u_k, associate $\bar{w}_0$, $\bar{w}_1$, $\bar{w}_2$ with the points P_k, P_{k1}, P_{k2} and set $\bar{w}=(\bar{w}_0, \bar{w}_1, \bar{w}_2)^T$, $U\bar{w}=\bar{v}$, $\bar{v}=(\bar{v}_0, \bar{v}_1, \bar{v}_2)^T$ then $\bar{v}^TD\bar{v}=0$. This yields $\bar{v}_0=\bar{v}_1=0$ and therefore $\bar{w}_2=\bar{v}_2$, $\bar{w}_1=-c_{21}\bar{w}_2$, $\bar{w}_0=-c_{10}\bar{w}_1-c_{20}\bar{w}_2$. That is $\bar{w}_0=0$ if $\bar{w}_2=0$. We may then proceed to the triple of points with vertex at P_{k2}. If we set $P_j=P_{k2}$ and if P_{j1} or $P_{j2}\in\dot{\Omega}_h$ then $w_j=\bar{w}_2=0$. If not we may repeat the above argument until we arrive at such a point and the theorem is proved.

8. **Example of a solvent set.** Let $F(p, q)=f(\omega)$ where $\omega=p^2+q^2$ and assume that $f'(\omega)>0$ for all $\omega\geqq 0$. If we denote $\omega_k=\omega(P_k)$ then

$$\begin{aligned}r_i(u) &= h^2\sum_{k=1}^{N}\frac{\partial f(\omega_k)}{\partial u_i}\\ &= 2h[f'(\omega_{i4})q_{i4}+f'(\omega_{i3})p_{i3}-f'(\omega_i)(p_i+q_i)]\\ &= 2\sigma\left[u_i-\frac{1}{\sigma}\sum_{\alpha=1}^{4}\sigma_{i\alpha}u_{i\alpha}\right]\end{aligned}$$

where $\sigma_{i\alpha}=f'(\omega_i)$, $\alpha=1, 2$; $\sigma_{i\alpha}=f'(\omega_{i\alpha})$, $\alpha=3, 4$ and

$$\sigma=\sum_{\alpha=1}^{4}\sigma_{i\alpha}.$$

Let $b_m=\min b(P)$, $b_M=\max b(P)$ for P taken over the boundary $\dot{\Omega}_h$, and let D be the hypercube $b_m\leqq u_i\leqq b_M$, $i=1, \cdots, n$. Then D is a solvent set.

For if $u^0 \in D$ and for any $i \in Z$ consider

$$r_i(u_{i'}^{(0)}, u_i) = 2\sigma\left[u_i - \frac{1}{\sigma}\sum_{\alpha=1}^{4} \sigma_{i\alpha} u_{i\alpha}^{(0)}\right]$$

where we write $u_{i'}^{(0)}$ for the variables other than $u_i^{(0)}$. This function of the one variable u_i has the property $r_i(u_{i'}^{(0)}, u_i) \geqq 0$ for $u_i = b_M$ while $r_i(u_{i'}^{(0)}, u_i) \leqq 0$ for $u_i = b_m$. Thus there is a zero on the interval $[b_m, b_M]$ and $u' \in D$. Thus we obtain, in particular, that the solution u^* satisfies the inequalities $b_m \leqq u_i^* \leqq b_M$, $i \in Z$. If Ω_h is connected then the equality will hold if and only if $u^*(P)$ is constant on $\bar{\Omega}_h$.

In the case of minimal surfaces $f(\omega) = (1+\omega)^{1/2}$ and the above results will be valid. Since here

$$G(u)^2 \geqq h^4 \sum_{k=1}^{N} \omega_k \tag{8.1}$$

we get that for any $u^0 \in E_n$, K_0 is bounded.

Since, by an argument similar to that given above, E_n is solvent, Remark 1 applies, and any initial guess u^0 may be used to start the iteration.

Since, for any $i \in Z$, u_i enters in at most three terms of the sum (7.1), we get that

$$\begin{aligned} r_i(u) &= \frac{\partial}{\partial u_i}\left(F^{(i)} + F^{(i3)} + F^{(i4)}\right) \\ &= \frac{1}{h}\left(-F_p^{(i)} - F_q^{(i)} + F_p^{(i3)} + F_q^{(i4)}\right), \end{aligned} \tag{8.2}$$

$$a_{ii}(u) = \frac{\partial r_i(u)}{\partial u_i} = \frac{1}{h^2}\left[F_{pp}^{(i)} + 2F_{pq}^{(i)} + F_{qq}^{(i)} + F_{pp}^{(i3)} + F_{qq}^{(i4)}\right]. \tag{8.3}$$

For the case $f(\omega) = (1+\omega)^{1/2}$,

$$a_{ii}(u) = \frac{1 + f(\omega_i)^2 - 2p_i q_i}{f(\omega_i)^3} + \frac{1 + q_{i3}^2}{f(\omega_{i3})^3} + \frac{1 + p_{i4}^2}{f(\omega_{i4})^3} \tag{8.4}$$

we get that $a_{ii}(u) \leqq 4$ for all $u \in E_n$. Thus all the conditions of Theorem 6.1 are satisfied with $\beta_i \equiv 4$.

It follows from (8.4) that if u^0 is any guess then $a_{ii}(u) \geqq 4h^6/G(u^0)^3 = \alpha_i$ and γ^0 may be chosen to be $h^6/G(u^0)^3$. Thus we find that approximate relaxation can be used to solve an approximate Plateau problem.

9. **Uniformly elliptic problems.** Assume now that in addition to the previous conditions, given in §7, on $F(P, p, q)$ we assume that there exists a positive constant μ such that for all $(x, y) \in \bar{\Omega}$, and all (p, q)

$$\lambda(\Phi) \geqq \mu > 0 \tag{9.1}$$

where $\lambda(\Phi)$ denotes the smallest eigenvalue of Φ.

From (8.3) it follows immediately that $a_{ii}(u) \geqq 4\mu$ and, by the argument of §5, E_n is a solvent set. Condition (9.1) also implies that for every $v^0 \in E_n$, K_0 is bounded. We will prove the following stronger result:

THEOREM 9.1. *If* (9.1) *holds then there exists a constant* α *such that*

$$\lambda(A(u)) \geqq \alpha > 0 \tag{9.2}$$

for all $u \in E_n$.

Proof. Let $v \in E_{2N}$ be defined by $v = (p_1, q_1, p_2, q_2, \cdots, p_N, q_N)^T$ and let $H(v)$ be the right hand side of (7.1). If B is the matrix of order $2N$ given by $B = (\partial^2 H/\partial v_i \partial v_j)_{i,j=1}^{2N}$ then it is easily seen that B is the quasidiagonal matrix $B = h^2 \operatorname{diag}[\Phi_1, \Phi_2, \cdots, \Phi_N]$, where Φ_k is the second order matrix corresponding to (p_k, q_k). Thus $\lambda(B) \geqq \mu h^2$ and, for any pair $v, v^0 \in E_{2N}$, we get

$$H(v) - H(v^0) \geqq (\operatorname{grad} H(v^0))^T(v - v^0) + \frac{\mu h^2}{2} \left| v - v^0 \right|^2 \tag{9.3}$$

from the mean value Theorem (2.1).

Given a function $u(P)$ on $\overline{\Omega}_h$, which takes the boundary values $b(P)$ on $\dot{\Omega}$, we may write $v = Du + c$ where D is a constant matrix and c is a constant vector. Since $|Du|$ is a discrete Dirichlet norm it is well known that there exists a positive constant ρ such that

$$|Du|^2 \geqq \rho |u|^2. \tag{9.4}$$

Now let $u(P)$, $u^0(P)$ be any two functions on $\overline{\Omega}_h$ which take the same boundary values and v, v^0 the corresponding difference vectors. Since $G(u) = H(Du + c)$, the linear terms in the respective expansions about u^0 and v^0 are equal. Thus it follows from (9.3), (2.1) and (9.4) that for some $z \in (u, u^0)$

$$(u - u^0)^T A(z)(u - u^0) \geqq h^2\mu |v - v^0|^2 \geqq h^2\mu\rho |u - u^0|^2. \tag{9.5}$$

If w is any vector in E_n such that $|w| = 1$ then choose $u = u^0 + \epsilon w$. Letting $\epsilon \to 0$ we get that $z \to u^0$ and

$$w^T A(u^0) w \geqq h^2 \mu\rho. \tag{9.6}$$

Since u^0 was arbitrary this completes the proof.

We note from the proof that, since the positive definiteness of A was not used, this yields an independent proof of this fact under condition (9.1).

That K_0 is bounded for every v^0 then follows from Remark 2. If we assume that the largest eigenvalue $\Lambda(\Phi)$ satisfies the condition $\Lambda(\Phi) \leqq \nu$ for all $(x, y) \in \overline{\Omega}$ and all (p, q) then $a_{ii}(u) \leqq 4\nu$. In this case Theorem 6.1 is valid where we may choose $\gamma_0 = \mu/\nu$.

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