SUMMABILITY \( C \) OF SERIES OF SURFACE SPHERICAL HARMONICS

BY

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I. Introduction. 1.1. Surface spherical harmonics. Let \( \Omega \) denote the surface of the unit sphere in Euclidean 3-space, whose center is the origin \( O \) of a system of Cartesian coordinates \( x, y, z \). Let \( Q \) denote a point on \( \Omega \). The function \( Y_n(Q) \) is said to be a surface spherical harmonic of degree \( n \) if 
\[
H_n(x, y, z) \text{ is a homogeneous harmonic polynomial of degree } n \text{ and } H_n(x, y, z) = r^n Y_n(Q) \text{ where } (x, y, z) \text{ lies on the line through } O \text{ and } Q \text{ at a distance } r \text{ from } O.
\]

1.2. Laplace series. If \( f(Q) \) is a Lebesgue integrable function on \( \Omega \), the Laplace series of \( f(Q) \) is a series of surface spherical harmonics \( \sum_{n=0}^{\infty} Y_n(Q) \) where \( Y_n(Q) \) is defined by
\[
Y_n(Q) = \frac{(2n + 1)/4\pi}{\int_{\Omega} f(M) P_n([M, Q]) d\Omega_M}
\]
Here \([M, Q]\) denotes the inner product of the unit vectors \( OM \) and \( OQ \) and \( P_n([M, Q]) \) denotes the Legendre polynomial of order \( n \).

In this paper necessary and sufficient conditions for the Cesaro summability of series of surface spherical harmonics are obtained. The analogous results for trigonometric series were given by Plessner [11, p. 256]. In the field of Laplace series sufficient conditions for Cesaro summability was obtained by Gronwall [4, p. 213] and by Fejer [3, p. 267] and a necessary and sufficient condition for the convergence of a particular class of Laplace series were obtained by V. L. Shapiro [9, p. 514]. The latter also obtained sufficient conditions for the Cesaro summability of series of surface spherical harmonics [8, p. 212].

II. Generalized Laplacians. 2.1. Definition. For a point \( P \) on \( \Omega \) let \( C(P, h') \) denote the circle of intersection of \( \Omega \) and the sphere of radius \( 2 \sin (h'/2), 0 < h' < \pi \), whose center is at \( P \). Let \( f(Q) \) be a function defined in the neighborhood \( D(P, h') = \{Q \in \Omega : [Q, P] \geq \cos h'\} \) of \( P \), and integrable on the circumference of every circle \( C(P, h) \) contained in this neighborhood. If
\[
\frac{1}{2\pi \sin h} \int_{C(P, h)} f(Q) ds_Q = \alpha_0 + \alpha_1 \frac{(1 - \cos h)}{2} + \frac{\alpha_2}{(2l)^2} \frac{(1 - \cos h)^2}{2^2} + \cdots + \frac{\alpha_r}{(r!)^2} \frac{(1 - \cos h)^r}{2^r} + o(1 - \cos h)^r, \quad r = 0, 1, \ldots,
\]

(2.1.1)

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we say that \( f(Q) \) has an \( r \)th generalized Laplacian at the point \( P \) and denote it by \( \Delta^r f(P) \). This generalized Laplacian is defined from the expansion (2.1.1) by setting \( \Delta^0 f(P) = \alpha_0, \Delta^1 f(P) = \alpha_k, k = 1, 2, \ldots, r, \) where \( \Delta \cdot \Delta \cdot \cdots \cdot \Delta \) (\( k \) times) \( = \Delta_k \) and \( \Delta_k \cdot \Delta_j \) for \( j, k \geq 0 \) and \( j + k \leq r \).

Thus if \( \frac{1}{2\pi \sin h} \int_{C(P,A)} f(Q) dS_Q \) has the expansion (2.1.1) then:

\[
\frac{1}{2\pi \sin h} \int_{C(P,A)} f(Q) dS_Q = \Delta^0 f(P) + \Delta^1 f(P) \frac{1 - \cos h}{2} + \Delta^2 f(P) \frac{(1 - \cos h)^2}{2^2} + \cdots
\]

\[
= \Delta^0 f(P) + \Delta^1 f(P) \frac{(1 - \cos h)}{2} + \Delta^2 f(P) \frac{(1 - \cos h)^2}{2^2} + \cdots
\]

\[
= \Delta^0 f(P) + \Delta^1 f(P) \frac{(1 - \cos h)}{2} + \Delta^2 f(P) \frac{(1 - \cos h)^2}{2^2} + \cdots
\]

\[
+ \frac{(1 - \cos h)^r}{2^r}, \quad r = 0, 1, \ldots.
\]

It is clear that if \( \Delta^s f(P) \) exists then \( \Delta^s f(P), 0 \leq s \leq r \), exists. \( \Delta^{(k)} f(P) \) shall denote the \( k \)th Laplace-Beltrami operator on \( f(Q) \) at the point \( P, k = 0, 1, \ldots \) (see [8, p. 212]), and \( \Delta^{(k)} f \) its value when \( P \) is the north pole of a system of coordinates with origin at the center of \( \Omega \). It is well known that if \( Y_n(Q) \) is a surface spherical harmonic then \( \Delta^{(r)} Y_n(Q) = [-n(n+1)]^r Y_n(Q) \).

2.2. Theorem. If \( Y_n(Q), n = 0, 1, \ldots, \) are arbitrary surface spherical harmonics and \( P \) is an arbitrary point on \( \Omega \) then, for any non-negative integer \( r \), \( \Delta^r Y_n(P) \) exists and \( \Delta^r Y_n(P) = \Delta^{(r)} Y_n(P) \).

Proof. Since \( \Delta^r \) and \( \Delta^{(r)} \) are linear operators, it is sufficient to show that \( \Delta^r Y_n(P) \) exists and \( \Delta^r Y_n(P) = \Delta^{(r)} Y_n(P) \) where \( Y_n(Q) \) has been normalized so that \( Y_n(P) = 1 \). Then \( \Delta^{(r)} Y_n(P) = \Delta^{(r)} P_n([P, Q]) \) (since both are equal to \( [-n(n+1)]^r \)), and by [7, p. 298]:

\[
\Delta^r Y_n(P) = \Delta^r P_n([P, Q])|_{Q=P}. \quad \text{We therefore need only show that} \quad \Delta^r P_n([P, Q])|_{Q=P} = [-n(n+1)]^r. \quad \text{But by [6, p. 21, (18)]:}
\]

\[
P_n(\cos h) = \sum_{k=1}^{\infty} \left[ -n(n+1)(1 - \cos h)^k \right] (k!)^2
\]

and the theorem follows easily from (2.1.1) by induction. If \( Y_n(P) = 0 \), obviously \( \Delta^{(r)} Y_n(P) = 0 = \Delta^r Y_n(P) \).

III. Statement of main results. 3.1. Definitions and notation. For an arbitrary series \( \sum_{n=0}^{\infty} U_n(Q), U_n^{(a)}(Q) \) for \( \alpha \neq -1, -2, \ldots \), shall denote
the sum $\sum_{n=0}^{\infty} \gamma_{n}^{(a)} U_{l}(Q)$ where $\gamma_{n}^{(a)} = \frac{(\alpha+1) \cdots (\alpha+j)}{j!}$ are the ordinary Cesaro coefficients of order $\alpha$. $\Delta^{(k)} F_{n}$ shall denote the $k$th difference $k=0, 1, \cdots$, of the sequence $\{ F_{n} \}$ where $\Delta^{(1)} F_{n} = F_{n} - F_{n+1}$. $S[f(Q)]$ shall denote the Laplace series of the function $f(Q)$ defined on $\Omega$ and $\Delta^{(r)} S[f(P)]$ the value at the point $P$ of the series obtained by applying the Laplace-Beltrami operator term by term $r$ times, $r=0, 1, \cdots$, to $S[f(Q)]$, i.e., if $S[f(Q)] = \sum_{n=0}^{\infty} Y_{n}(Q)$ then $\Delta^{(r)} S[f(P)] = \sum_{n=0}^{\infty} \left[-n(n+1)^{r}\right] Y_{n}(P)$. 

**Definition.** If $Y_{n}(Q), n=0, 1, \cdots$, are surface spherical harmonics the anti-Laplace-Beltrami operator of order $r, r=0, 1, \cdots$, on $Y_{n}(Q)$, denoted $\Delta^{(-r)} Y_{n}(Q)$, is defined as:

$$\Delta^{(-r)} Y_{n}(Q) \equiv \left\{ Y_{n}(Q) \frac{P_{r}[P, Q]}{[-r(r+1)]^{r}} \right\},$$

$$\Delta^{(-r)} Y_{n}(Q) \equiv \frac{Y_{n}(Q)}{[-n(n+1)]^{r}}, \quad n = 1, 2, \cdots.$$ 

Obviously $\Delta^{(-r)} \{ \Delta^{(-r)} Y_{n}(Q) \} = \Delta^{(-r)} \{ \Delta^{(r)} Y_{n}(Q) \} = Y_{n}(Q)$ for $n$ and $r$ non-negative integers. Given $f(Q) \in L$ on $\Omega$, $\Delta^{(r)} S[f(Q)]$ shall denote the series obtained by applying the anti-Laplace-Beltrami operator term-by-term to $S[f(Q)]$, i.e., if $S[f(Q)] = \sum_{n=0}^{\infty} Y_{n}(Q)$:

$$\Delta^{(r)} S[f(Q)] = \sum_{n=0}^{\infty} Y_{n}(Q),$$

$$\Delta^{(-r)} S[f(Q)] = (-1)^{r} \left\{ Y_{0}(Q) \left[ \frac{P_{r}[P, Q]}{[r(r+1)]^{r}} \right] \right\}$$

$$+ \sum_{n=1}^{\infty} \frac{Y_{n}(Q)}{[n(n+1)]^{r}}, \quad r = 1, 2, \cdots.$$ 

(3.1.1)

3.2. **Main theorems.** The major results of this paper are embodied in four theorems. The second gives the sufficient conditions for Cesaro summability and follows as a direct consequence of the first. The third yields the necessary conditions and the last combines these results to give the necessary and sufficient conditions.

**Theorem.** Let $f(Q)$ be a bounded Borel measurable function on $\Omega$. If $\Delta f(P)$ exists for some non-negative integer $r$ then $\Delta^{(r)} S[f(P)]$ is $(C-\alpha)$ summable, $\alpha > 2r+1$, to $\Delta f(P)$.

**Sufficiency theorem.** Let $\sum_{n=0}^{\infty} Y_{n}(Q)$ be a series of surface spherical harmonics on $\Omega$. Let $r$ be a non-negative integer great enough so that $\Delta^{(-r)} \{ \sum_{n=0}^{\infty} Y_{n}(Q) \}$ converges uniformly to $F_{r}(Q)$ on $\Omega$. If $\Delta F_{r}(P)$ exists then $\sum_{n=0}^{\infty} Y_{n}(Q)$ is $(C-\alpha)$ summable at the point $P, \alpha > 2r+1$, to $\Delta F_{r}(P)$. 
Necessity theorem. Let \( \sum_{n=0}^{\infty} Y_n(Q) \) be a series of surface spherical harmonics on \( \Omega \). Let \( r \) be a non-negative integer great enough so that \( \Delta^{(r)} \{ \sum_{n=0}^{\infty} Y_n(Q) \} \) converges uniformly to \( F_r(Q) \) on \( \Omega \). If \( \sum_{n=0}^{\infty} Y_n(Q) \) is \((C-\alpha)\) summable, \( \alpha \) a non-negative integer, to \( s \) at the point \( P \) on \( \Omega \) then for \( r \) an integer greater than \((\alpha+2)/2\), \( \Delta_r F_r(P) \) exists and equals \( s \).

Necessity and sufficiency theorem. Let \( \sum_{n=0}^{\infty} Y_n(Q) \) be a series of surface spherical harmonics with \( Y_n(Q) = O(n^k) \) uniformly on \( \Omega \) for some \( k \). A necessary and sufficient condition that \( \sum_{n=0}^{\infty} Y_n(Q) \) be summable \( C \) to \( s \) at an arbitrary point \( P \) on \( \Omega \) is that there exist a non-negative integer \( r > (k+1)/2 \) such that \( \Delta_r F_r(P) \) exists and equals \( s \) where \( F_r(Q) = \Delta^{(r)} \{ \sum_{n=0}^{\infty} Y_n(Q) \} \).

IV. The sufficiency theorem. In order to facilitate the proof of this theorem we prove the following sublemmas and lemmas:

4.1. Lemma 1. Let \( f(Q) \) be a bounded Borel measurable function and \( r \) a non-negative integer. If \( \Delta^k f(P) = 0, k = 0, \ldots, r \), implies \( \Delta^k S[f(P)] \) is \((C-\alpha)\) summable to zero then \( \Delta_r f(P) = s \) implies \( \Delta_r S[f(P)] \) is \((C-\alpha)\) summable to \( s \).

Proof. Suppose \( \Delta_r f(P) = s \). We observe first that there exists a finite sum of surface spherical harmonics \( T(Q) = \sum_{j=0}^{r} a_j P_j([P, Q]) \) such that \( \Delta^k T(P) = \Delta^k f(P), k = 0, \ldots, r \). For, by the theorem of §2.2 and the linearity of the operators \( \Delta_k \) and \( \Delta^k \), \( \Delta^k T(Q) = \sum_{j=0}^{r} a_j \Delta^k P_j([P, Q]) = \Delta^k T(Q) \), and it is possible to choose the \( a_j \) so that:

\[
\sum_{j=0}^{r} a_j = \Delta_0 f(P),
\]

\[
\sum_{j=1}^{r} [-j(j+1)]^k a_j = \Delta_k f(P), \quad k = 1, 2, \ldots, r.
\]

This can clearly be done since the determinant of the above system of \((r+1)\) linear equations is a nonvanishing Vandermonde determinant. Let \( F(Q) = f(Q) - T(Q) \). Then \( \Delta_k F(P) = 0, k = 0, 1, \ldots, r \). Therefore, by hypothesis, \( \Delta^k S[f(P)] \) is \((C-\alpha)\) summable to zero. But since the Laplace series of a surface spherical harmonic is the surface spherical harmonic itself, \( \Delta^k S[F(P)] = \Delta^k S[f(P)] - \Delta^k T(P) = \Delta^k S[f(P)] - s \). The lemma follows immediately.

For the following lemmas and sublemmas we shall assume, unless otherwise stated, that \( \alpha \) is non-negative.

4.2. Sublemma 1.

\[
\sum_{j=0}^{n} \frac{\gamma_{n-j}^{(a)}}{\gamma_n^{(a)}} \left[ -j(j+1) \right]^{(2j+1)} \sin(j+1/2) t \quad (4.2.1)
\]

\[
\equiv \sum_{k=0}^{r} a_k \sum_{j=0}^{n} \frac{\gamma_{n-j}^{(a)}}{\gamma_n^{(a)}} d^{(2k+1)} \frac{d^{(2j+1)}}{d [j+1/2] d^j} \cos(j+1/2) t.
\]
Proof. The above identity is a simple application of the following two facts:

\[(4.2.2)\] \((-1)^{k+1} \frac{d^{(2k+1)}}{dt^{(2k+1)}} \cos(j + 1/2)t = (j + 1/2)^{2k+1} \sin(j + 1/2)t, \quad k = 0, \ldots, r.\]

\[(4.2.3)\] \([-j(j+1)]^r(2j+1)\] is an odd polynomial of degree \((2r + 1)\) in \((j + 1/2)\).

Statement \((4.2.2)\) is obvious. Statement \((4.2.3)\) follows easily since

\[[-j(j+1)]^r = [(-1)^r/4^r][(2j + 1)^2 - 1]^r = (-1)^r[(j + 1/2)^2 - 1/4]^r.\]

4.3. Sublemma 2. For \(0 < t < 2\pi, k = 0, \ldots, r; r \geq 0, \text{ and } s > \alpha + 2r + 1:\n
\[
\sum_{j=0}^{n} \frac{\gamma_{n-j}}{\gamma_n} \frac{d^{(2k+1)}}{dt^{(2k+1)}} \cos(j + 1/2)t
\]

\[\equiv \frac{1}{\gamma_n^{(\alpha)}} \cdot \left\{ - \sum_{j=1}^{n} \frac{\gamma_{n-j}^{(\alpha-j)}}{\gamma_n^{(\alpha)}} \frac{d^{(2k+1)}}{dt^{(2k+1)}} \left[ \frac{1}{(2 \sin(t/2))(1 - e^{-it})^\alpha} \right] 

\quad + \frac{d^{(2k+1)}}{dt^{(2k+1)}} \left[ \frac{e^{i(n+1)t}}{(2 \sin(t/2))(1 - e^{-it})^\alpha} \right] 

\quad - \frac{d^{(2k+1)}}{dt^{(2k+1)}} \left[ \sum_{j=n+1}^{\infty} \frac{\gamma_j^{(\alpha-s-1)} - i(j-n-1)t}{(2 \sin(t/2))(1 - e^{-it})^\alpha} \right] \right\}.
\]

Proof. Since \(\sum_{j=0}^{n} \cos(j + 1/2)t = [\sin(j + 1)t]/2 \sin(t/2)\) for \(0 < t < 2\pi,\)

\[
\sum_{j=0}^{n} \frac{d^{(2k+1)}}{dt^{(2k+1)}} \left[ \frac{\gamma_{n-j}}{\gamma_n} \right] [\cos(j + 1/2)t] = \sum_{j=0}^{n} \frac{d^{(2k+1)}}{dt^{(2k+1)}} \left[ \frac{\gamma_{n-j}^{(\alpha-1)}}{\gamma_n^{(\alpha)}} \right] [\sin(j + 1)t]/2 \sin(t/2) \quad \text{for } 0 < t < 2\pi.
\]

Also,

\[
\sum_{j=0}^{n} \left[ \frac{\gamma_{n-j}}{\gamma_n} \right] [\sin(j + 1)t]/2 \sin(t/2) = \frac{1}{2} \gamma_n^{(\alpha)} \sin(t/2)] \cdot \left\{ \sum_{j=0}^{n} \frac{\gamma_{n-j}^{(\alpha-1)}}{\gamma_n} \exp[i(j + 1)t] \right\}

= [1/2 \gamma_n^{(\alpha)} \sin(t/2)] \cdot \left\{ \exp[i(n + 1)t] \sum_{j=0}^{n} \gamma_j^{(\alpha-1)} \exp(-ijt) \right\},
\]
and applying Abel's partial summation formula $s$ times to the expression on the right we obtain (see [11, p. 258]):

\[
\sum_{j=0}^{n} \frac{\gamma_{n-j}^{(a)}}{\gamma_{n}^{(a)}} \sin(j + 1/2)t = \frac{1}{2\gamma_{n}^{(a)} \sin(t/2)} \left\{ - \sum_{j=1}^{s} \frac{\gamma_{n-j}^{(a-j)}}{(1 - e^{-it})j} + \frac{e^{i(n+1)t}}{(1 - e^{-it})^n} \left( \sum_{j=n+1}^{\infty} \gamma_{j}^{(a-s-1)} e^{-(j-n-1)t} \right) \right\}.
\]

But since $\alpha - s < -1$ implies $\sum_{j=0}^{\infty} |\gamma_{j}^{(a-1)}| < \infty$, by Abel's limit theorem we have $1/(1-\text{exp}(it))^{n-s} = \sum_{j=0}^{\infty} \gamma_{j}^{(a-s-1)} \exp(-ijt)$. Thus:

\[
\sum_{j=0}^{n} \frac{\gamma_{n-j}^{(a)}}{\gamma_{n}^{(a)}} \cos(j + 1/2)t = \frac{1}{2\gamma_{n}^{(a)} \sin(t/2)} \left\{ - \sum_{j=1}^{s} \frac{\gamma_{n-j}^{(a-j)}}{(1 - e^{-it})j} + \frac{e^{i(n+1)t}}{(1 - e^{-it})^n} \left( \sum_{j=n+1}^{\infty} \gamma_{j}^{(a-s-1)} e^{-(j-n-1)t} \right) \right\}.
\]

Consequently, for $0 < t < 2\pi$, $k = 0, \cdots, r$; $r \geq 0$, and $s > \alpha + 2r + 1$:

\[
\sum_{j=0}^{n} \frac{\gamma_{n-j}^{(a)}}{\gamma_{n}^{(a)}} \frac{d^{(2k+1)}}{dt^{(2k+1)}} \cos(j + 1/2)t = \frac{1}{\gamma_{n}^{(a)}} \left\{ - \sum_{j=1}^{s} \gamma_{n-j}^{(a-j)} \left\{ \frac{d^{(2k+1)}}{dt^{(2k+1)}} \left[ \frac{1}{(2 \sin(t/2))(1 - e^{-it})j} \right] \right\} + \frac{d^{(2k+1)}}{dt^{(2k+1)}} \left[ \frac{e^{i(n+1)t}}{(2 \sin(t/2))(1 - e^{-it})^n} \right] - \frac{d^{(2k+1)}}{dt^{(2k+1)}} \left[ \frac{\sum_{j=n+1}^{\infty} \gamma_{j}^{(a-s-1)} e^{-(j-n-1)t}}{(2 \sin(t/2))(1 - e^{-it})^s} \right] \right\}
\]

where the last expression on the right exists since for $s > \alpha + 2r + 1$ the absolute value of $\sum_{j=n+1}^{\infty} d^{(2k+1)}/dt^{(2k+1)} \{ \gamma_{j}^{(a-s-1)} \exp[-i(j-n-1)t] \}$ is bounded by a constant multiple of the convergent series $\sum_{j=n+1}^{\infty} j^{(a+2k+1)-(s+1)}$, and consequently converges uniformly for $0 < t < 2\pi$.

**4.4. Sublemma 3.** For $0 < t < 2\pi$, $h \geq 0$, and $n \geq 1$ there exist positive constants $K_1$, $K_2$, $K_3$ such that:

\[
(4.4.1) \quad \left| - \sum_{j=1}^{s} \gamma_{n-j}^{(a-j)} \frac{d^{h}}{dt^{h}} \left[ \frac{1}{(2 \sin(t/2))(1 - e^{-it})j} \right] \right| < K_1 \sum_{j=1}^{s} \frac{n^{(a-j)}}{t^{(j+h+1)}},
\]

\[
(4.4.2) \quad \left| \frac{d^{(h)}}{dt^{(h)}} \frac{e^{i(n+1)t}}{(2 \sin(t/2))(1 - e^{-it})^n} \right| < K_2 \sum_{n=0}^{h} \frac{n^{h}}{t^{(a+1+h-n)}},
\]
Proof. We observe that the $h$th derivative of a product of two functions $A(t)$ and $B(t)$ is given by:

\[
\frac{d^h}{dt^h} A(t) \cdot B(t) = \sum_{\mu=0}^{h} \binom{h}{\mu} A^{(\mu)}(t) B^{(h-\mu)}(t).
\]

Since $\left| \frac{d^{(\mu)}}{dt^{(\mu)}} \left[ \frac{1}{\sin(t/2)} \right] \right| < C_1/t^{n+1}$ and $\left| \frac{d^{(\mu)}}{dt^{(\mu)}} \left[ \frac{1}{1-e^{-it}} \right] \right| < C_2/t^{n+s}$ where $C_1, C_2 > 0$, there exists a positive constant $C$ such that:

\[
\left| \frac{d^{(h)}}{dt^{(h)}} \left( 2 \sin(t/2) \right) \left( 1 - e^{-it} \right)^s \right| < C \frac{1}{t^{n+s}}.
\]

Thus from (4.4.4) and (4.4.5) we obtain (4.4.1) and (4.4.2) noticing that for the former inequality $\gamma_n^{(h)} = O(n^k)$. With

\[
A(t) = - \sum_{j=n+1}^{\infty} \gamma_j^{(\alpha-s-1)} \exp[-i(j-n-1)t]
\]

and $B(t) = 1/\left[ 2 \sin(t/2) \right] \left[ 1 - \exp(-it) \right]$, we again employ (4.4.4) and (4.4.5) to obtain (4.4.3), noticing that:

\[
\left| - \frac{d^{(\mu)}}{dt^{(\mu)}} \sum_{j=n+1}^{\infty} \gamma_j^{(\alpha-s-1)} e^{-i(j-n-1)t} \right| < K \sum_{j=n+1}^{\infty} j^{(\alpha-s-1)} (j-n-1)^\mu < K n^{(\alpha+s-1)}
\]

for $\mu = 0, \ldots, h$ and $s > \alpha + h$ where $K > 0$.

4.5. Sublemma 4. For $\beta > 1$ and $0 < \theta < \pi/2$ there exists a positive constant $K$ such that:

\[
\int_0^{\pi/2} \frac{dt}{\beta^3(\cos \theta - \cos t)^{1/2}} < K \frac{1}{\theta^2}.
\]

Proof. Using the identity $\cos \theta - \cos t = 2 \sin \left( (t+\theta)/2 \right) \sin \left( (t-\theta)/2 \right)$ we see that:

\[
\int_0^{\pi/2} \frac{dt}{\beta^3(\cos \theta - \cos t)^{1/2}} = \frac{1}{2^{1/2}} \int_0^{\pi/2} \frac{dt}{\beta^3 \sin((t+\theta)/2) \sin((t-\theta)/2))^{1/2}}.
\]
But for \( \theta \leq t \leq \pi \), \( \sin [(t+\theta)/2] = \sin(t/2)\cos(\theta/2) + \cos(t/2)\sin(\theta/2) \geq \sin(\theta/2)\cos(\theta/2) + \cos(t/2)\sin(\theta/2) \geq \sin(\theta/2)\cos(\theta/2) = \sin(\theta/2) \) and hence:

\[
\frac{1}{2^{1/2}} \int_{t}^{\pi} \frac{dt}{\theta^{1/2} \left[ \sin((t+\theta)/2) \sin((t-\theta)/2) \right]^{1/2}} \leq \frac{1}{(\sin \theta)^{1/2}} \int_{t}^{\pi} \frac{dt}{\theta^{1/2} \left[ \sin((t-\theta)/2) \right]^{1/2}}
\]

for \( 0 < \theta < \pi \).

Also, for \( 0 < \theta \leq t \leq \pi \), \( \sin [(t-\theta)/2] \geq \left[ 2/\pi \right] \left[ (t-\theta)/2 \right] \) since \( (t-\theta)/2 < \pi/2 \). Thus:

\[
\frac{1}{(\sin \theta)^{1/2}} \int_{t}^{\pi} \frac{dt}{\theta^{1/2} \left[ \sin((t-\theta)/2) \right]^{1/2}} \leq \frac{\pi^{1/2}}{(\sin \theta)^{1/2}} \int_{t}^{\pi} \frac{dt}{\theta^{1/2} \left[ (t-\theta) \right]^{1/2}}
\]

for \( 0 < \theta < \pi \).

Obviously \( \int_{t}^{\pi} 1/\theta^{(t-\theta)^{1/2}} dt < \int_{0}^{\pi} 1/\theta^{(t-\theta)^{1/2}} dt \) and dividing the integral on the right into two parts,

\[
\int_{t}^{\pi} 1/\theta^{(t-\theta)^{1/2}} dt = \int_{t}^{2\theta} 1/\theta^{(t-\theta)^{1/2}} dt + \int_{2\theta}^{\pi} 1/\theta^{(t-\theta)^{1/2}} dt.
\]

Noticing that

\[
\int_{t}^{2\theta} 1/\theta^{(t-\theta)^{1/2}} dt \leq \left[ 1/\theta^{\theta} \right] \int_{t}^{2\theta} 1/(t-\theta)^{1/2} dt = 2/\theta^{(\theta-1/2)}
\]

and

\[
\int_{2\theta}^{\pi} 1/\theta^{(t-\theta)^{1/2}} dt \leq \left[ 1/\theta^{1/2} \right] \int_{2\theta}^{\pi} 1/\theta^{(t-\theta)^{1/2}} dt = \left[ 1/2 \gamma(\gamma-1) \right] \left[ 1/\theta^{(\theta-1/2)} \right]
\]

for \( \beta > 1 \),

we see that there exists a constant \( C > 0 \) such that:

\[
\int_{t}^{\pi} \frac{dt}{\theta^{(\cos \theta - \cos t)^{1/2}}} < \frac{C}{\theta^{(\theta-1/2)}(\sin \theta)^{1/2}} \quad \text{for} \quad \beta > 1, \quad 0 < \theta < \pi.
\]

But for \( 0 < \theta < \pi/2 \), \( \sin \theta \geq (2/\pi)\theta \) and consequently \( (4.5.1) \) follows.

4.6. Sublemma 5. For \( 0 < \theta < \pi/2 \) there exist positive constants \( K_{1}, K_{2}, K_{3} \) such that:

\[
(4.6.1) \quad \frac{2^{1/2}}{\pi} \int_{t}^{\pi} \frac{G(n, t, \alpha)}{(\cos \theta - \cos t)^{1/2}} dt < K_{1} \sum_{j=1}^{n} \frac{n^{n-j}}{\theta^{(j+k+1)}}
\]

where
\[ G(n, t, \alpha) = \frac{1}{\gamma_n^{(\alpha)}} \sum_{j=1}^{h} \frac{n^{(\alpha-j)}}{t(j+\alpha+1)}, \quad h = 1, 3, \ldots, 2r + 1, \]

\[ \frac{2^{1/2}}{\pi} \int_{\theta}^{\pi} \frac{H(n, t, \alpha)}{(\cos \theta - \cos t)^{1/2}} \, dt < K_2 \frac{n^{(h-\alpha)}}{\theta^{(\alpha+1)}} \]

where

\[ H(n, t, \alpha) = \frac{1}{\gamma_n^{(\alpha)}} \cdot \frac{n^h}{t^{(\alpha+1)}}, \quad h = 1, 3, \ldots, 2r + 1, \]

\[ \frac{2^{1/2}}{\pi} \int_{\theta}^{\pi} \frac{L(n, t, \alpha)}{(\cos \theta - \cos t)^{1/2}} \, dt < K_3 \frac{n^{(h-\alpha)}}{\theta^{(\alpha+1)}} \]

where

\[ L(n, t, \alpha) = \frac{1}{\gamma_n^{(\alpha)}} \cdot \frac{n^{(h-\alpha)}}{t^{(\alpha+1)}}, \quad h = 1, 3, \ldots, 2r + 1. \]

Proof. These results follow immediately from 4.5 Sublemma 4 and the fact that \( \gamma_n^{(\alpha)} \sim n^\alpha / \Gamma(\alpha+1) \) for \( \alpha \neq -1, -2, \ldots \).

4.7. Sublemma 6. For \( s > \alpha + h, h = 1, 3, \ldots, 2r + 1, \alpha > 2r + 1, \) and \( n \) a positive integer, there exists a positive constant \( K \) such that:

\[ \int_{1/n}^{\pi} \sum_{j=1}^{s} \frac{n^{-j}}{\theta^{(j+\alpha+1)}} \sin(2\pi+1) \, d\theta < K. \]

\[ \int_{1/n}^{\pi} \frac{n^{(h-\alpha)}}{\theta^{(\alpha+1)}} \sin(2\pi+1) \, d\theta < K. \]

\[ \int_{1/n}^{\pi} \frac{n^{(h-\alpha)}}{\theta^{(\alpha+1)}} \sin(2\pi+1) \, d\theta < K. \]

Proof. For \( h = 1, 3, \ldots, 2r + 1: \)

\[ \int_{1/n}^{\pi} \sum_{j=1}^{s} \frac{n^{-j}}{\theta^{(j+\alpha+1)}} \sin(2\pi+1) \, d\theta \leq \int_{1/n}^{\pi} \sum_{j=1}^{s} \frac{n^{-j}}{\theta^{(j+2r+2)}} \sin(2\pi+1) \, d\theta, \]

and since \( \sin \theta \leq 1 \) for \( 0 < \theta \leq \pi: \)

\[ \int_{1/n}^{\pi} \sum_{j=1}^{s} \frac{n^{-j}}{\theta^{(j+2r+2)}} \sin(2\pi+1) \, d\theta \leq \int_{1/n}^{\pi} \sum_{j=1}^{s} \frac{n^{-j}}{\theta^{(j+1)}} \, d\theta. \]

Integrating the expression on the right we obtain:

\[ \int_{1/n}^{\pi} \sum_{j=1}^{s} \frac{n^{-j}}{\theta^{(j+1)}} \, d\theta = \sum_{j=1}^{s} \frac{1}{j} \left[ 1 - \frac{1}{(\pi n)^j} \right] < K. \]
where $K$ is a positive constant, and (4.7.1) follows. Similarly we find that for $h=1, 3, \cdots, 2r+1$ and $\alpha > 2r+1$:

\[
\int_{1/n}^{\pi} \frac{n^{(h-\alpha)}}{\theta^{(a+1)}} \sin^{(2r+1)} \omega d\theta \leq n^{(2r+1-\alpha)} \int_{1/n}^{\pi} \frac{\sin^{(2r+1)}}{\theta^{(a+1)}} d\theta \leq n^{(2r+1-\alpha)} \int_{1/n}^{\pi} \frac{d\theta}{\theta^{(a-2r)}}
\]

\[
= \frac{1}{\alpha - (2r + 1)} \left[ 1 - \frac{1}{(\pi n)^{(a-2r+1)}} \right].
\]

Thus for $\alpha > 2r+1$ we see that with $K=1/[\alpha - (2r+1)] > 0$ (4.7.2) holds.

Finally, for $h=1, 3, \cdots, 2r+1$:

\[
\int_{1/n}^{\pi} \frac{n^{(\alpha+h)}}{\theta^{(a+1)}} \sin^{(2r+1)} \omega d\theta \leq n^{(\alpha+2r+1)} \int_{1/n}^{\pi} \frac{d\theta}{\theta^{(s-2r)}}
\]

\[
\leq \frac{1}{s - (2r + 1)} \left[ 1 - \frac{1}{(\pi n)^{(s-2r+1)}} \right],
\]

and therefore, for $s > \alpha + h$, we have (4.7.3) with $K=1/[s - (2r+1)] > 0$.

The following notation will be used in the remaining lemmas and theorems of this section: We let $K_n^{(\alpha, r)}(\cos \theta)$ denote the $n$th $(C - \alpha)$ partial sum of

\[
\sum_{j=0}^{\infty} \left[ -j(j+1) \right](2j+1)P_j(\cos \theta),
\]

i.e.,

\[
K_n^{(\alpha, r)}(\cos \theta) = \sum_{j=0}^{n} \left[ \gamma_n^{(\alpha)} \right] \left[ -j(j+1) \right](2j+1)P_j(\cos \theta).
\]

4.8. Lemma 2. Let $1/n \leq \eta \leq \pi/2$ where $n$ is a positive integer. For $\alpha > 2r+1$ there exists a positive constant $K$ independent of $\eta$ such that:

\[
\int_{1/n}^\eta \left| K_n^{(\alpha, r)}(\cos \theta) \right| \sin^{(2r+1)} \omega d\theta < K.
\]

Proof. From Mehler's integral representation of the Legendre polynomials [6, p. 27, (27)]:

\[
P_j(\cos \theta) = \frac{2^{1/2}}{\pi} \int_{0}^{\pi} \frac{\sin(j + 1/2)t}{(\cos \theta - \cos t)^{1/2}} dt \quad \text{for } 0 < \theta < \pi; \ j = 0, 1, \cdots,
\]

we see that for $0 \leq \theta < \pi$:

\[
K_n^{(\alpha, r)}(\cos \theta) = \frac{2^{1/2}}{\pi} \int_{0}^{\pi} \frac{\sum_{j=0}^{n} \left[ \gamma_n^{(\alpha)} \right] \left[ -j(j+1) \right](2j+1) \sin(j + 1/2)t}{(\cos \theta - \cos t)^{1/2}} dt
\]
and thus by 4.2 Sublemma 1, for \(0 < \theta < \pi\):

\[
K_n^{(a,r)}(\cos \theta) = \frac{2^{1/2}}{\pi} \int_{\theta}^{\pi} \frac{\sum_{k=0}^{r} a_k \sum_{j=0}^{n} \frac{\gamma_{n-j}^{(a)}}{\gamma_{n}^{(a)}} \frac{d}{dt(2k+1)} \cos(j + 1/2)t}{(\cos \theta - \cos t)^{1/2}} dt.
\]

Utilizing the fact that \(\left| f(z) \right| \leq |f(z)|\) we have, by 4.3 Sublemma 2 and 4.4 Sublemma 3, for \(0 < t < \pi, k = 0, \cdots, r, \) and \(r \geq 0,\) the existence of a constant \(K > 0\) such that:

\[
\left| \sum_{j=0}^{n} \gamma_{n-j}^{(a)} \frac{d}{dt(2k+1)} \cos(j + 1/2)t \right| < \frac{K}{\gamma_{n}^{(a)}} \left\{ \sum_{j=1}^{s} \frac{n_j^{(a-j)}}{t(j+2k+2)} + \sum_{\mu=0}^{2k+1} \frac{n_{\mu}}{t(\mu+2k+2-\mu)} + \sum_{\mu=0}^{2k+1} \frac{n_{\mu}}{t(\mu+2k+1-\mu)} \right\}
\]

where \(s\) is fixed so that \(s > a + 2r + 1.\) For \(1/n \leq t,\) noticing that there exists a constant \(K > 0\) such that

\[
\sum_{\mu=0}^{2k+1} n_{\mu}/t(a+2k+2-\mu) < Kn_{2k+1}/t^{a+1}
\]

and

\[
\sum_{\mu=0}^{2k+1} n_{\mu}/t(a+2k+1-\mu) < K \frac{n_{2k+1}}{t^{a+1}}
\]

for \(k = 0, \cdots, r,\) we see that with \(1/n \leq t < \pi\) there exists a \(K > 0\) such that:

\[
\left| \sum_{j=0}^{n} \gamma_{n-j}^{(a)} \frac{d}{dt(2k+1)} \cos(j + 1/2)t \right| < \frac{K}{\gamma_{n}^{(a)}} \left\{ \sum_{j=1}^{s} \frac{n_j^{(a-j)}}{t(j+2k+2)} + \frac{n_{2k+1}}{t^{a+1}} \frac{n_{(a+2k+1)}}{t+1} \right\}, \quad k = 0, \cdots, r.
\]

Thus by 4.6 Sublemma 5, for \(0 < 1/n \leq \theta \leq \pi/2\) there exists a constant \(K > 0\) such that:

\[
\frac{2^{1/2}}{\pi} \int_{\theta}^{\pi} \frac{\sum_{j=0}^{n} \gamma_{n-j}^{(a)} \frac{d}{dt(2k+1)} \cos(j + 1/2)t}{(\cos \theta - \cos t)^{1/2}} dt
\]

\[
< K \left\{ \sum_{j=1}^{s} \frac{n_j^{(a-j)}}{\theta(j+2k+2)} + \frac{n_{2k+1}}{\theta(a+1)} + \frac{n_{(a+2k+1)}}{\theta+1} \right\}, \quad k = 0, \cdots, r.
\]
Therefore from (4.8.3) and (4.8.4) we see that, for $0 < 1/n \leq \theta \leq \pi/2$, there exists a constant $K > 0$ such that:

$$ | K_n^{(a, r)}(\cos \theta) | < K \left\{ \sum_{k=0}^{r} \sum_{j=1}^{\infty} \frac{\eta_j^{-1}}{\theta^{j+2k+2}} + \sum_{k=0}^{r} \frac{\eta_k^{2k+1}}{\theta^{(a+1)}} + \sum_{k=0}^{r} \frac{\eta_k^{-(r+2k+1)}}{\theta^{(r+1)}} \right\}. $$

Employing 4.7 Sublemma 6 and noticing that the integrands in (4.7.1), (4.7.2), and (4.7.3) are positive for the domain of integration $[1/n, \eta]$, $\eta \leq 2$, the result follows.

4.9. Lemma 3. Let $f(\theta, \phi)$ be a bounded measurable function on $\Omega$, $r$ a non-negative integer, and $\eta$ any positive number less than $\pi$. If $\alpha > 2r + 1$ then:

$$(4.9.1) \quad \left[ 1/4\pi \right] \int_\Omega \int_{Q-D(P, \eta)} f(M) K_n^{(a, r)}([P, M])d\Omega_M \to 0 \quad \text{as} \quad n \to \infty. $$

Proof. From Lemma 5 of [8, p. 217] we have: If $0 < h_1 < h_2 < \pi$ and $F_1(x)$ is a bounded measurable function on $\Omega$ which is equal to zero in $D(x_0, h_2)$ then for $x$ in $D(x_0, h_1)$ $\Delta^{(r)}S[F_1(x)]$ is uniformly summable $(C-\alpha)$ to zero, $\alpha > 2r + 1$.

Define $F_1(Q)$ by:

$$(4.9.2) \quad F_1(Q) = 0 \quad \text{in} \quad D(P, \eta),$$

$$F_1(Q) = f(Q) \quad \text{in} \quad \Omega - D(P, \eta).$$

Then by the above mentioned lemma we have $\Delta^{(r)}S[F_1(P)]$ summable $(C-\alpha)$ to zero $\alpha > 2r + 1$. But $\Delta^{(r)}S[F_1(P)] = [1/4\pi] \int_\Omega f_1(M) K_n^{(a, r)}([P, M])d\Omega_M$ and by (4.9.2),

$$\Delta^{(r)}S[F_1(P)] = 1/4\pi \int_\Omega \int_{Q-D(P, \eta)} f(M) K_n^{(a, r)}([P, M])d\Omega_M.$$ 

4.10. Lemma 4. For $r$ a non-negative integer and $n$ a positive integer, there exists a constant $K > 0$ such that:

$$(4.10.1) \quad \int_0^{1/n} | K_n^{(a, r)}(\cos \theta) | \sin^{(2r+1)} \theta d\theta < K.$$ 

Proof. Since $|P_j(\cos \theta)| \leq 1$, $j = 0, 1, \ldots$,

$$| K_n^{(a, r)}(\cos \theta) | \leq \sum_{j=0}^{n} \left[ \gamma_j^{(a)} / \gamma_j^{(a)} \right] [j(j + 1)]^r [2j + 1].$$

Consequently there exists a constant $K > 0$ such that
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\[ |K_n^{(a,r)}(\cos \theta)| < Kn^{2r+1} \sum_{j=0}^{n} \frac{\gamma_{n-j}/\gamma_n}{\gamma_n} \leq Kn^{2r+1} \frac{\gamma_n^{(a+1)}}{\gamma_n^{(a)}}. \]

But since \( \lim_{n \to \infty} \frac{\gamma_n^{(a)}}{\gamma_n} = 1/\Gamma(\alpha+1) \) for \( \alpha \neq -1, -2, \cdots \), \( |K_n^{(a,r)}(\cos \theta)| < Kn^{2r+1} \) and (4.10.1) follows.

4.11. Theorem. Let \( f(Q) \) be a bounded Borel measurable function on \( \Omega \). If \( \Delta_r f(P) \) exists for some non-negative integer \( r \) then \( \Delta^{(r)}S[f(P)] \) is \( (C-\alpha) \) summable, \( \alpha > 2r+1 \), to \( \Delta_r f(P) \).

Proof. Without loss of generality it may be assumed that \( P \) is the north pole of a system of spherical coordinates whose origin is at the center of \( \Omega \). Furthermore by 4.1 Lemma 1 it may be assumed that \( \Delta_k f(P) = 0, k = 0, \cdots, r \), and it remains to show that \( \Delta^{(r)}S[f(P)] \) is \( (C-\alpha) \) summable, \( \alpha > 2r+1 \), to zero. We have

\[ S[f(P)] = \left[ \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{2n+1}{\Gamma(2n+1)} \frac{f(M)P_n(M)}{R^2} \right] d\Omega_M \]

and

\[ \Delta^{(r)}S[f(P)] = \left[ \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{f(M)P_n(M)}{R^2} \right] \frac{(2n+1)^r}{(2n+1)!} \frac{f(M)P_n(M)}{R^2} d\Omega_M. \]

Denoting the \( (C-\alpha) \) partial sums of this latter series by \( C_n^{(a,r)}(P) \), clearly

\[ C_n^{(a,r)}(P) = \left[ \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} f(M)K_n^{(a,r)}(M, P)d\Omega_M \right]. \]

Since \( \Delta_k f(P) = 0, k = 0, \cdots, r \), given an arbitrary \( \epsilon > 0 \) there exists a \( \delta(\epsilon/2K) > 0 \) such that:

\[ \left| \frac{(1/2\pi \sin h)}{\sin^{2r} \theta} \int_{\Omega(\theta)} f(Q) d\theta Q \right| < \frac{\epsilon}{2K} \]

whenever \( \theta < \delta(\epsilon/2K) \) where \( K \) is chosen such that \( K = \max(K_1, K_2) \) and \( K_1, K_2 \) are the constants of 4.8 Lemma 2 and 4.10 Lemma 4 respectively. Let \( D(P, \eta) \) be the spherical cap of radius \( \eta \) about \( P \) with \( \eta \) chosen such that \( \eta = \min(\delta(\epsilon/2K), \pi/2) \) and let \( n \) be a positive integer chosen so that \( 1/n \leq \eta \). Then since:

\[ |C_n^{(a,r)}(P)| = \left[ \frac{1}{4\pi} \int_0^\pi \int_0^{2\eta} f(\theta, \phi)K_n^{(a,r)}(\cos \theta) \sin \theta d\phi d\theta \right. \]

\[ + \int_0^{\Omega-D(P, \eta)} f(M)K_n^{(a,r)}(M, P) d\Omega_M \left. \right|, \]

\[ |C_n^{(a,r)}(P)| < \left[ \frac{\epsilon}{2K} \right] \int_0^\pi |K_n^{(a,r)}(\cos \theta)| \sin^{2r+1} \theta d\theta \]

\[ + \left[ \frac{1}{4\pi} \int_0^{\Omega-D(P, \eta)} f(M)K_n^{(a,r)}(M, P) d\Omega_M \right| \]

by (4.11.1). Dividing the first integral on the right into two parts it follows that:
Thus by 4.8 Lemma 2 and 4.10 Lemma 4, for \( \alpha > 2r + \gamma \):

\[
| C_n^{(\alpha, \gamma)}(P) | < \epsilon + \frac{1}{4\pi} \left| \int \int_{\Omega \setminus D(P, \gamma)} f(M) K_n^{(\alpha, \gamma)}([P, M]) d\Omega_M \right|
\]

and therefore by 4.9 Lemma 3, \( \lim \sup_{n \to \infty} C_n^{(\alpha, \gamma)}(P) \leq \epsilon \). Since \( \epsilon > 0 \) was chosen arbitrarily, the theorem follows.

4.12. Sufficiency theorem. Let \( \sum_{n=0}^\infty Y_n(Q) \) be a series of surface spherical harmonics on \( \Omega \). Let \( r \) be a non-negative integer great enough so that \( \Delta^{(\gamma)} \{ \sum_{n=0}^\infty Y_n(Q) \} \) converges uniformly to \( F_r(Q) \) on \( \Omega \). If \( \Delta_r F_r(P) \) exists then \( \sum_{n=0}^\infty Y_n(Q) \) is \( (C-\alpha) \) summable at the point \( P, \alpha > 2r + 1 \), to \( \Delta_r F_r(P) \).

Proof. The theorem follows immediately from the preceding one after noticing that \( \Delta^{(\gamma)} \{ \sum_{n=0}^\infty Y_n(Q) \} \) converges uniformly on \( \Omega \) to a continuous function \( F_r(Q) \), and that \( \Delta^{(\gamma)} S[F_r(P)] = \sum_{n=0}^\infty Y_n(P) \).

V. The necessary conditions for \( C \) summability. The following three lemmas facilitate the proof of the main theorem of this section:

5.1. Lemma. Let \( Y_n(Q), n = 1, 2, \ldots, \) be surface spherical harmonics such that \( \sum_{n=1}^\infty Y_n(P) \) is \( (C-\alpha) \) summable, \( \alpha \) a non-negative integer, for \( P \) an arbitrary point on \( \Omega \). Then for \( r > (\alpha + 1)/2 \):

\[
(5.1.1) \quad (-1)^r \sum_{n=1}^\infty Y_n(P) \Delta^{(n+1)} \left\{ \frac{P_n(\cos k)}{[n(n+1)]^r} \right\} = \sum_{k=0}^r a_k \frac{(1 - \cos k)^k}{2^k} + L^r(\cos k)
\]

where

\[
(5.1.2) \quad a_0 = \text{the (C - \alpha) sum of the series} \quad (-1)^r \sum_{n=1}^\infty Y_n(P) \cdot \frac{1}{[n(n+1)]^r},
\]

\[
(5.1.2) \quad a_k = \text{the (C - \alpha) sum of the series} \quad (-1)^r \sum_{n=1}^\infty Y_n(P) \frac{\Delta_0(\Delta_0 + 1 \cdot 2 \cdots) [\Delta_0 + (k - 1)k] P_n}{[n(n+1)]^r},
\]

\( k = 1, 2, \ldots, r \), and \( L^r(\cos k) \) is a convergent series.

Proof. From [1, p. 21, (8)] we know that if \( \{f_n\} \) and \( \{g_n\} \) are two sequences then:
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(5.1.3) \[ \Delta^{(j)}[f_n \cdot g_n] = \sum_{k=0}^{j} \binom{j}{k} \Delta^{(k)}f_n \cdot \Delta^{(j-k)}g_{n+k}. \]

Letting \( f_n = P_n(\cos k) \), \( g_n = 1/[n(n+1)] \), and noting that

\[ \Delta^{(k)}\left[1/n^k\right] = O(1/n^{k+1}) \quad \text{for } k \geq 0, \quad \left| P_n(\cos k) \right| \leq 1, \]

and \( Y_n^{(a)}(P) = O(n^a) \), we see that for \( r > (\alpha+1)/2 \):

(5.1.4) \[ \sum_{n=1}^{\infty} \left| Y_n^{(a)}(P) \right| \Delta^{(a+1)} \left\{ \frac{P_n(\cos h)}{(n(n+1))^r} \right\} < K \sum_{n=1}^{\infty} \frac{1}{n^{(2r-a)}} < \infty. \]

Now from [1, p. 19], if \( \{f_n\} \) is an arbitrary sequence then:

(5.1.5) \[ \Delta^{(a)}f_n = \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} f_{n+j}. \]

Consequently:

(5.1.6) \[ (-1)^r \sum_{n=1}^{\infty} Y_n^{(a)}(P) \Delta^{(a+1)} \left\{ \frac{P_n(\cos h)}{(n(n+1))^r} \right\} = (-1)^r \sum_{n=1}^{\infty} Y_n^{(a)}(P) \sum_{j=0}^{\alpha+1} (-1)^j \binom{\alpha+1}{j} \frac{P_{n+j}(\cos h)}{((n+j)(n+j+1))^r}, \]

and therefore by (2.2.1):

(5.1.7) \[ (-1)^r \sum_{n=1}^{\infty} Y_n^{(a)}(P) \Delta^{(a+1)} \left\{ \frac{P_n(\cos h)}{(n(n+1))^r} \right\} = (-1)^r \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} Y_n^{(a)}(P) \frac{(-1)^j \binom{\alpha+1}{j}}{((n+j)(n+j+1))^r} C_{n+j,k} \frac{(1 - \cos k)^k}{2^k}, \]

where \( C_{n+j,k} \) is defined by:

(5.1.8) \[ C_{n+j,0} = 1, \quad C_{n+j,k} = \frac{\triangle_0(\Delta_0 + 1 \cdot 2) \cdots [\Delta_0 + (k - 1) \Delta_0]}{(k!)^2} P_{n+j}, \quad \text{for } k \leq n + j, \]

(5.1.9) \[ C_{n+j,k} = 0, \quad \text{for } k \geq n + j + 1. \]

Thus with \( a_{nk} \) defined by:

(5.1.10) \[ a_{nk} = Y_n^{(a)}(P) \sum_{j=0}^{\alpha+1} \frac{(-1)^j \binom{\alpha+1}{j}}{((n+j)(n+j+1))^r} C_{n+j,k} \frac{(1 - \cos k)^k}{2^k}, \]

we have:
(5.1.10) \((-1)^{r} \sum_{n=1}^{\infty} Y_{n}^{(\alpha)}(P) \Delta \left(\alpha+1\right) \left\{ \frac{P_{n}(\cos h)}{[n(n+1)]^r} \right\} = (-1)^{r} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} a_{nk}.\)

It will now be shown that:

(5.1.11) \[\sum_{n=1}^{\infty} \sum_{k=0}^{r} a_{nk} = \sum_{k=0}^{r} \sum_{n=1}^{\infty} a_{nk} + \sum_{n=1}^{\infty} \sum_{k=r+1}^{\infty} a_{nk}\]

where \(L^{r}(\cos h) = \sum_{n=1}^{\infty} \sum_{k=r+1}^{\infty} a_{nk}\) is a convergent series. To do this it need only be shown that \(\sum_{n=1}^{\infty} \sum_{k=0}^{r} a_{nk}\) is absolutely convergent for, from the convergence of \(\sum_{n=1}^{\infty} \sum_{k=0}^{r} a_{nk}\) and \(\sum_{n=1}^{\infty} \sum_{k=r+1}^{\infty} a_{nk}\) (see (5.1.4)), the result is immediate. From (5.1.5) and (5.1.9),

\[\sum_{n=1}^{\infty} \left| a_{nk} \right| = \sum_{n=1}^{\infty} \left| Y_{n}^{(\alpha)}(P) \right| \left| \Delta^{(\alpha+1)} \left\{ C_{nk}/[n(n+1)]^{r} \right\} \right| (1 - \cos h)^{k}/2^{k},\]

and thus from (5.1.8) we have:

\[\sum_{n=1}^{\infty} \left| a_{nk} \right| = \sum_{n=1}^{\infty} \left| Y_{n}^{(\alpha)}(P) \right| \left| \Delta^{(\alpha+1)} \left\{ \frac{\Delta_{0}^2 \cdot \cdots \cdot \Delta_{0} + (k - 1)k}{[n(n+1)]^{r}} P_{n} \right\} \right| (1 - \cos h)^{k}/2^{k},\]

or.

(5.1.12) \[\sum_{n=1}^{\infty} \left| a_{nk} \right| = \frac{1}{(k)!^{2}} \sum_{n=1}^{\infty} \left| Y_{n}^{(\alpha)}(P) \right| \left| \Delta^{(\alpha+1)} \left\{ \frac{\Delta_{0}^2 \cdot \cdots \cdot \Delta_{0} + (k - 1)k}{[n(n+1)]^{r}} P_{n} \right\} \right| \frac{(1 - \cos h)^{k} k!}{2^{k}}, \quad k = 1, \ldots, r.\]

The highest degree of the polynomial \(\Delta_{0}^2 \cdot \cdots \cdot \Delta_{0} + (k - 1)k P_{n}\)

\[\equiv \left[ -n(n+1) \right] \left[ -n(n+1) + 1 \right] \cdots \left[ -n(n+1) + (k - 1)k \right] \text{for} \quad n = 1, \ldots, r,\]

is obviously \(2r\). Thus since \(\Delta^{(\alpha+1)}(-1)^{r} = 0\) for \(\alpha \geq 0\),

\[\left| \Delta^{(\alpha+1)} \left\{ \frac{\Delta_{0}^2 \cdot \cdots \cdot \Delta_{0} + (k - 1)k}{[n(n+1)]^{r}} F_{n} \right\} \right| < K \Delta^{(\alpha+1)} \left\{ \frac{1}{[n(n+1)]^{r}} \right\}\]

\(< \frac{K}{n^{(\alpha+1)}} \text{ for} \quad k = 1, \ldots, r.\]

Also, \(\Delta^{(\alpha+1)} \left\{ 1/[n(n+1)]^{r} \right\} = O(1/n^{2(\alpha+1)})\) for \(r > (\alpha+1)/2.\) Thus since \(Y_{n}^{(\alpha)}(P) = O(n^{\alpha}),\) from (5.1.12) we see that \(\sum_{n=1}^{\infty} a_{nk}, k = 0, \ldots, r,\) converges absolutely. Therefore from (5.1.7), (5.1.8), (5.1.9), and (5.1.10) we see that:
(5.1.13) \((-1)^r \sum_{n=1}^{\infty} Y_n^{(a)}(P) \Delta^{(a+1)} \left( \frac{P_n(\cos h)}{[n(n+1)]^r} \right) = \sum_{k=0}^{r} a_k (1 - \cos h)^k + L^r(\cos h)

where

\begin{align*}
a_0 &= (-1)^r \sum_{n=1}^{\infty} Y_n^{(a)}(P) \Delta^{(a+1)} \left( \frac{1}{[n(n+1)]^r} \right), \\
a_k &= (-1)^r \sum_{n=1}^{\infty} Y_n^{(a)}(P) \Delta^{(a+1)} \left( \frac{\Delta_0(\Delta_0 + 1; 2) \cdots \Delta_0 + (k - 1)k P_n}{[n(n+1)]^r} \right),
\end{align*}

\(k = 1, \ldots, r\).

But from [5, p. 128, Theorem 1] we know that if (i) \(\sum b_n\) is summable or bounded \((C - k), k \text{ an integer, (ii) } F_n \to 0, \text{ and (iii) } \sum |n+1|^k |\Delta^{(k+1)} F_n| < \infty\), then \(\sum b_n F_n\) is summable \((C - k)\) to \(\sum B_n^{(a)} \Delta^{(k+1)} F_n\) the last series being absolutely convergent. It is to be noted that the theorem is valid if (ii) is replaced by (ii') \(P_n = \sum_{i=0}^{\infty} (1 + i^2)\), where \(K\) is a positive constant. With \(F_n = 1/[n(n+1)]^r\) for \(k = 0, F_n = \Delta_0(\Delta_0 + 1; 2) \cdots \Delta_0 + (k - 1)k [P_n/[n(n+1)]^r\) for \(k = 1, \ldots, r\), and \(b_n = Y_n(P)\) it is obvious that the hypotheses (i), (ii'), and (iii) of the above theorem are satisfied. Consequently we have:

\begin{align*}
a_0 &= \text{the } (C - \alpha) \text{ sum of the series } (-1)^r \sum_{n=1}^{\infty} Y_n(P) \cdot \frac{1}{[n(n+1)]^r}, \\
a_k &= \text{the } (C - \alpha) \text{ sum of the series } (-1)^r \sum_{n=1}^{\infty} Y_n(P) \frac{\Delta_0(\Delta_0 + 1; 2) \cdots \Delta_0 + (k - 1)k P_n}{[n(n+1)]^r}, \quad k = 1, \ldots, r.
\end{align*}

and the proof of the lemma is complete.

5.2. Lemma 2. Let \(R_n^{(a)}(\cos h)\) be defined by:

\[ R_n^{(a)}(\cos h) = P_n(\cos h) - \frac{\Delta_0(\Delta_0 + 1; 2) \cdots \Delta_0 + (r - 1) P_n}{(r!)^2} \frac{1 - \cos h}{2^r} + \cdots \]

where \(0 < h < \pi\) and \(r\) is a positive integer. If \(j\) is an integer such that \((j+1)/2 < r\) and \(n \leq \left(1/(1 - \cos h)\right)^{1/2}\), then there exists a positive constant \(K\) (independent of \(r, n, \text{ and } h\)) such that:

(5.2.1) \[ \left| \frac{\Delta^{(j)} R_n^{(a)}(\cos h)}{(1 - \cos h)^r} \right| < Kn^{(r+1)-j}(1 - \cos h). \]
Proof. We may assume that \( n \geq r + 1 \) for if \( n \leq r \), \( \Delta \alpha (\Delta \alpha + 1 \cdot 2) \cdots [\Delta \alpha + (k - 1)k] P_n = 0 \) for \( n \leq k \leq r \). Thus \( R_n^\alpha (\cos h) = 0 \) and \( \Delta (j) \left\{ \frac{R_n^\alpha (\cos h)}{[1 - \cos h]^r} \right\} = 0 \). Employing the expansion for \( P_n (\cos h) \) we see that:

\[
\frac{R_n^\alpha (\cos h)}{[1 - \cos h]^{r+1}} = (-1)^{r+1} \frac{(n - r) \cdots n(n + 1) \cdots (n + r + 1)}{[(r + 1)!]^2 2^{r+1}} + \cdots
\]

\[
+ (-1)^{k+r+1} \frac{(n - (k + r)) \cdots n(n + 1) \cdots [n + (k + r) + 1]}{[(k + r + 1)!]^2} \cdot \frac{(1 - \cos h)^k}{2^{k+r+1}} + \cdots
\]

(5.2.2)

But since (see [1, p. 6,(3)]) \( \Delta (j)x^{(m)} = m(m - 1) \cdots (m - j + 1)x^{(m-j)} \) where \( x^{(m)} \equiv x(x - 1) \cdots (x - m + 1) \) we see, letting \( x = n + k + r + 1 \) and \( m = 2(k + r + 1) \), that:

\[
\Delta (j) \left\{ \frac{(n - (k + r)) \cdots n(n + 1) \cdots [n + (k + r) + 1]}{[(k + r + 1)!]^2} \cdot \frac{(1 - \cos h)^k}{2^{k+r+1}} \right\}
\]

(5.2.3)

\[
\frac{2^j[k + r + 1] \left[ k + r + \frac{1}{2} \right] \cdots \left[ k + r + 1 - \left( \frac{j - 1}{2} \right) \right]}{[(k + r + 1)!]^2} \cdot \frac{(n + k + r + 1) \cdots (n + 1)n \cdots [n - (k + r) + j]}{2^{k+r+1}}
\]

for \( k = 0, 1, \cdots, n - (r + 1) \).

Since \( (n + k + r + 1) \cdots (n + 1) \) is, for \( k = 0, 1, \cdots, n - (r + 1) \), clearly majorized by \( (2n)^{k+r+1} \) and \( n \cdots [n - (k + r) + j] \) by \( n^{k+r+1-j} \), for \( n^2(1 - \cos h) \leq 1 \) we have the expression on the right of (5.2.3) majorized by \( \left\{ 2^j/[1 \cdot 2 \cdots ((k+r)-(j+1)/2)]^2 \right\} n^{2(r+1)-j} \). But for \( r > (j+1)/2 \), this expression is in turn majorized by \( \left\{ 2^j/[k!]^2 \right\} n^{2(r+1)-j} \). Hence from (5.2.2) and (5.2.3) there exists a positive constant \( K \) such that:

\[
\left| \Delta (j) \left\{ \frac{R_n^\alpha (\cos h)}{[1 - \cos h]^{r+1}} \right\} \right| = \sum_{k=0}^{n-(r+1)} \Delta (j) \left\{ (-1)^k \frac{(n - (k + r)) \cdots n(n + 1) \cdots [n + (k + r) + 1]}{[(k + r + 1)!]^2} \cdot \frac{(1 - \cos h)^k}{2^{k+r+1}} \right\} < Kn^{2(r+1)-j}
\]
The lemma then follows immediately.

5.3. Lemma 3. Let $P_n(\cos h)$ be the Legendre polynomials of order $n$, $n = 1, 2, \cdots$. If $n > \left[1/(1 - \cos h)\right]^{1/2}$ where $0 < h < \pi$ then for any non-negative integer $r$:

(5.3.1) \[ \Delta^{(r)} P_n(\cos h) = O[n(1 - \cos h)^{r+1/2}] \]

Proof. The proof is by induction. The case $r = 0$ is trivial since $|P_n(\cos h)| \leq 1$ and $n(1 - \cos h)^{1/2} > 1$. Assuming that

(5.3.2) \[ \Delta^{(m)} P_n(\cos h) = O[n(1 - \cos h)^{(m+1)/2}] \quad \text{for } m = 0, 1, \cdots, k, \]

it will be shown that $\Delta^{(k+1)} P_n(\cos h) = O[n(1 - \cos h)^{(k+2)/2}]$. From the Christoffel-Darboux formula [2, p. 159] we know that:

(5.3.3) \[ \frac{\Delta P_n(\cos h)}{(1 - \cos h)} = K \frac{1}{n+1} \sum_{j=0}^{n} (j + 1/2) P_j(\cos h) \]

where $K$ is a positive constant. Thus it is easily seen from (5.3.2) that:

(5.3.4) \[ \Delta^{(m-1)} \left\{ \frac{1}{n+1} \sum_{j=0}^{n} (j + 1/2) P_j(\cos h) \right\} = O[n(1 - \cos h)^{(m-1)/2}] \]

for $m = 1, \cdots, k$.

Also, utilizing (5.3.3),

\[ \Delta^{(k+1)} \left\{ \frac{P_n(\cos h)}{(1 - \cos h)} \right\} = K \Delta^{(k-1)} \left\{ \Delta \left[ \frac{1}{n+1} \sum_{j=0}^{n} (j + 1/2) P_j(\cos h) \right] \right\} \]

\[ = K \left[ \Delta^{(k-1)} \left\{ \frac{n + 3/2}{n+1} P_{n+1}(\cos h) \right\} \right] \]

\[ + \Delta^{(k-1)} \left\{ \frac{1}{n+2} \cdot \frac{n}{n+1} \sum_{j=0}^{n} (j + 1/2) P_j(\cos h) \right\} \].

Consequently, from (5.1.3), the induction hypothesis (5.3.2), and (5.3.4), we have:

\[ \Delta^{(k+1)} \left\{ \frac{P_n(\cos h)}{(1 - \cos h)} \right\} = O \left[ n \left\{ \sum_{j=0}^{k-1} \left( \begin{array}{c} k-1 \\ j \end{array} \right) \frac{(1 - \cos h)^{(j+1)/2}}{n^{k-1-j}} + \frac{(1 - \cos h)^{j/2}}{n^{k-j}} \right\} \right] \]

Therefore, since $1/n < (1 - \cos h)^{1/2}$,

\[ \Delta^{(k+1)} \left\{ \frac{P_n(\cos h)}{(1 - \cos h)} \right\} = O[n(1 - \cos h)^{k/2}], \]

and the proof is complete.
We now proceed to state and prove the necessary conditions for $C$ summability.

5.4. Theorem. Let $\sum_{n=0}^{\infty} Y_n(Q)$ be a series of surface spherical harmonics on $\Omega$. Let $r$ be a non-negative integer great enough so that $\Delta^{(r)} \{ \sum_{n=0}^{\infty} Y_n(Q) \}$ converges uniformly to $F_r(Q)$ on $\Omega$. If $\sum_{n=0}^{\infty} Y_n(Q)$ is $(C-\alpha)$ summable, $\alpha$ a non-negative integer, to $s$ at the point $P$ on $\Omega$ then for $r$ an integer greater than $(\alpha+2)/2$, $\Delta_r F_r(P)$ exists and equals $s$.

Proof. Without loss of generality we may assume that $Y_0=0$ and $\sum_{n=1}^{\infty} Y_n(Q)$ is $(C-\alpha)$ summable to 0 at $P$. Since $\Delta^{(r)} \{ \sum_{n=0}^{\infty} Y_n(Q) \}$ converges uniformly to $F_r(Q)$, by [7, p. 298] it is to be seen that:

$$\int_{C(P,h)} F_r(Q) ds_Q = (-1)^r \sum_{n=1}^{\infty} \frac{Y_n(P) P_n(\cos h)}{[n(n+1)]^r}.$$

Applying Abel's partial summation formula $(\alpha+1)$ times to the right side of this equation we obtain:

$$(5.4.1) \quad \frac{1}{2\pi \sin h} \int_{C(P,h)} F_r(Q) ds_Q = (-1)^r \sum_{n=1}^{\infty} Y_n^{(\alpha)}(P) \Delta^{(\alpha+1)} \left\{ \frac{P_n(\cos h)}{[n(n+1)]^r} \right\}.$$

Note that the hypothesis for the application of this formula is satisfied since $Y_n^{(k)}(P) = o(n^k)$ for $k=0, \ldots, \alpha$, and as shown in §(5.1.4), for $r>(\alpha+1)/2$, $\Delta^{(k+1)} \left\{ \frac{P_n(\cos h)}{[n(n+1)]^r} \right\} = O(1/n^{2r})$ for $k=0, \ldots, \alpha$. Thus $\sum_{n=1}^{\infty} Y_n^{(k)}(P) \Delta^{(k+1)} \left\{ \frac{P_n(\cos h)}{[n(n+1)]^r} \right\}$ converges absolutely and $Y_n^{(k)}(P) \Delta^{(k+1)} \left\{ \frac{P_n(\cos h)}{[n(n+1)]^r} \right\} = o(1)$ for $k=0, \ldots, \alpha$. Applying $(5.1.1)$ to the right member of $(5.4.1)$ and comparing the result with $(2.12)$ and the definition $(2.1.1)$, it is seen that if $L_r(\cos h) = o(1-\cos h)^r$ then:

$$(5.4.2) \quad \frac{1}{2\pi \sin h} \int_{C(P,h)} F_r(Q) ds_Q = \Delta_0 F_r(P) + \Delta_1 F_r(P) \frac{(1-\cos h)}{2} + \Delta(\Delta + 1 \cdot 2) \frac{F_r(P)}{(2!)^2} \frac{(1-\cos h)^2}{2^2} + \cdots$$

$$+ \Delta(\Delta + 1 \cdot 2) \cdots \Delta + (r-1)r \frac{F_r(P)}{(r!)^2} \frac{(1-\cos h)^r}{2^r} + o(1-\cos h)^r$$

where $\Delta_k F_r(P)$, $k=0, \ldots, r$ are defined uniquely by the following set of $(r+1)$ equations:

$$(5.4.3) \quad \Delta_k F_r(P) = \text{The (} C - \alpha \text{) sum of the series} (-1)^r \sum_{n=1}^{\infty} Y_n(P) \frac{1}{[n(n+1)]^r},$$

and

$$\Delta(\Delta + 1 \cdot 2) \cdots \Delta + (k-1)k \frac{F_r(P)}{(k!)^2} = \text{The (} C - \alpha \text{) sum of the series}$$

$$(-1)^r \sum_{n=1}^{\infty} Y_n(P) \frac{\Delta(\Delta + 1 \cdot 2) \cdots \Delta + (k-1)k P_n}{[n(n+1)]^r}, \quad k = 1, \ldots, r.$$
Thus $\Delta_r F_r(P)$ will be equal to the $(C-\alpha)$ sum of

$$(-1)^r \sum_{n=1}^{\infty} Y_n(P) \Delta_0^{(r)} P_n(\cos h)/[n(n+1)]^r = \sum_{n=1}^{\infty} Y_n(P).$$

We therefore need only show that

$$L^*(\cos h) = \left[1/2 \pi \sin h\right] \int_{C(P,h)} F_r(Q) d\mathcal{Q} - \left\{ \Delta_0 F_r(P) + \sum_{k=0}^{r} \left[ \Delta(\Delta + 1.2) \cdots \sum_{j=k}^{r} \left[ \Delta + (k-1)k \right] F_r(P)/(k!)^2 \right] (1 - \cos h)^k/2^k \right\} = o(1 - \cos h)^r$$

where $\Delta_k F_r(P), k=0, \cdots, r$, are given by (5.4.3). From (5.4.1), (5.1.14), and (5.1.15) we have:

$$L^*(\cos h)$$

$$= (-1)^r (1 - \cos h)^r \sum_{n=1}^{\infty} Y_n^{(a)}(P) \Delta_0^{(a+1)} \left\{ \frac{R_n^r(\cos h)}{[n(n+1)]^r(1 - \cos h)^r} \right\}$$

where

$$R_n^r(\cos h) = P_n(\cos h)$$

$$= \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} Y_n^{(a)}(P) \Delta_0^{(a+1)} \left\{ \frac{R_n^r(\cos h)}{[n(n+1)]^r(1 - \cos h)^r} \right\}$$

It thus only remains to show that

$$\sum_{n=1}^{\infty} Y_n^{(a)}(P) \Delta_0^{(a+1)} \left\{ \frac{R_n^r(\cos h)}{[n(n+1)]^r(1 - \cos h)^r} \right\}$$

is $o(1)$. Let

$$\sum_{n=1}^{\infty} Y_n^{(a)}(P) \Delta_0^{(a+1)} \left\{ \frac{R_n^r(\cos h)}{[n(n+1)]^r(1 - \cos h)^r} \right\}$$

$$= \sum_{n=1}^{N} Y_n^{(a)}(P) \Delta_0^{(a+1)} \left\{ \frac{R_n^r(\cos h)}{[n(n+1)]^r(1 - \cos h)^r} \right\}$$

$$+ \sum_{n=N+1}^{\infty} Y_n^{(a)}(P) \Delta_0^{(a+1)} \left\{ \frac{R_n^r(\cos h)}{[n(n+1)]^r(1 - \cos h)^r} \right\}$$

where

$$N = \left[ \left( \frac{1}{1 - \cos h} \right)^{1/2} \right].$$
Applying (5.1.3) to \( R_n'(\cos h)/[n(n+1)]^r(1-\cos h)^r \) with
\[
f_n = R_n'(\cos h)/(1 - \cos h)^r \quad \text{and} \quad g_n = \sqrt{n(n-1)}'\]
we see from (5.2.1) Lemma 2 that for \( n \leq N \) there exists a \( K > 0 \) such that
\[
| \Delta^{(\alpha)} \{ R_n'(\cos h)/[n(n+1)]^r(1-\cos h)^r \} | < Kn^{(1-\alpha)}(1-\cos h). \]
Consequently,
\[
\sum_{n=1}^{N} V_n^{(\alpha)}(P) \Delta^{(\alpha+1)} \left\{ \frac{R_n'(\cos h)}{[n(n+1)]^r(1-\cos h)^r} \right\} = o\left[N^2(1-\cos h)\right] = o(1). \tag{5.4.7}
\]
Again applying (5.1.3) to \( P_n'(\cos h)/[n(n+1)]^r(1-\cos h)^r \) with \( f_n = P_n'(\cos h)/(1 - \cos h)^r \) and \( g_n = 1/[n(n+1)]^r \) we see from (5.3.1) Lemma 3 that for \( n > N \),
\[
\Delta^{(\alpha+1)} \{ P_n'(\cos h)/[n(n+1)]^r(1-\cos h)^r \} = O \left[ 1/n^{(2\alpha-1)} (1 - \cos h)^{r-(\alpha+2)/2} \right]. \tag{5.4.8}
\]
Consequently,
\[
\sum_{n=N+1}^{\infty} V_n^{(\alpha)}(P) \Delta^{(\alpha+1)} \left\{ \frac{P_n'(\cos h)}{[n(n+1)]^r(1-\cos h)^r} \right\} = o\left[N^2(1-\cos h)^{r-\alpha+2}/2 \right] = o(1) \quad \text{for} \quad r > (\alpha+2)/2. \tag{5.4.8}
\]
Now for \( k = 1, \cdots, r, \)
\[
\Delta^{(\alpha+1)} \left\{ \frac{\Delta_0(\Delta_0 + 1:2) \cdots [\Delta_0 + (k-1)k]P_n (1 - \cos h)^k}{(k!)^2 2^k [n(n+1)]^r(1-\cos h)^r} \right\} = O \left[ 1/n^{2(r-k)+\alpha+1}(1 - \cos h)^{r-k} \right]
\]
since for \( j = 1, \cdots, k; k = 1, \cdots, r, \) and \( j \neq r, \)
\[
\Delta^{(\alpha+1)} \left\{ \frac{\Delta_0 \Delta_0^{(\alpha)}P_n (1 - \cos h)^k}{(k!)^2 2^k [n(n+1)]^r(1-\cos h)^r} \right\} = O \left[ 1/n^{2(r-k)+\alpha+1}(1 - \cos h)^{r-k} \right]
\]
and for \( j = k = r, \)
Thus,

$$
\sum_{n=1}^{\infty} Y_n^{(\alpha)}(P) \Delta^{(a+1)} \left[ \frac{\Delta^r P_n}{(r!)^2} \left(1 - \cos h\right)^r \right] = \Delta^{(a+1)} \left[ \frac{(-1)^r}{2^r(r!)^2} \right] = 0.
$$

and from (5.4.6), (5.4.7), (5.4.8), and (5.4.9) it follows that for \( r > \frac{\alpha + 2}{2} \),

$$
\sum_{n=1}^{\infty} Y_n^{(\alpha)}(P) \Delta^{(a+1)} \left[ \frac{\Delta^r P_n}{(r!)^2} \left(1 - \cos h\right)^r \right] = o(1).
$$
follows immediately from that of $\Delta f(P)$ for $k=2, \cdots , r$. If $r=1$, that $f \in C^2$ in $I(P, h)$ implies the existence of $\Delta f(P)$ is shown as follows: By definition, $f(Q) \in C^2$ in $I(P, h)$ implies $g(x, y) \in C^2$ on \{(x, y) \mid x^2+y^2<\sin^2 h\}$ where $g(x, y)=f(x, y, (1-x^2-y^2)^{1/2})$ and $f(x, y, z)=f(Q)$ for all points $Q$ in $I(P, h)$. Then by Taylor’s Theorem, for $0 < t < h$, $(1/2\pi \sin t) \int_{C(P, t)} f(Q) ds Q = (1/2\pi) \int_0^t g(\sin t \cos \theta, \sin t \sin \theta) d\theta = g(0, 0) + (1/4) [g_x(0, 0) + g_y(0, 0)] \sin^2 t + o(\sin^2 t)$. Thus:

$$\frac{(1/2\pi \sin t) \int_{C(P, t)} f(Q) ds Q - f(P)}{1 - \cos t} = (1/2) [g_x(0, 0) + g_y(0, 0)],$$

and consequently $\Delta f(P)$ exists and in fact equals $[g_x(0, 0) + g_y(0, 0)]$.

VI. The necessary and sufficient conditions for $C$ summability.

**Theorem.** Let $\sum_{n=0}^{\infty} Y_n(Q)$ be a series of surface spherical harmonics with $Y_n(Q) = O(n^k)$ uniformly on $\Omega$ for some $k$. A necessary and sufficient condition that $\sum_{n=0}^{\infty} Y_n(Q)$ be summable $C$ to $s$ at an arbitrary point $P$ on $\Omega$ is that there exist a non-negative integer $r>(k+1)/2$ such that $\Delta_r F_s(P)$ exists and equals $s$ where $F_s(P) = \Delta_s \{ \sum_{n=0}^{\infty} Y_n(Q) \}$.

**Proof.** The sufficiency follows immediately from the theorem of §4.12 with the order of summability $\alpha > 2r+1$. Choosing $r$ an integer greater than $\max\{ (k+1)/2, ([\alpha]+3)/2 \}$ where $\alpha$ is the order (not necessarily integral) of summability of $\sum_{n=0}^{\infty} Y_n(Q)$ at $P$, the necessity follows immediately from the theorem of §5.4.

**Bibliography**


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