THE PYTHAGOREAN THEOREM IN CERTAIN
SYMMETRY CLASSES OF TENSORS

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The Hadamard determinant theorem [1] states that if \( A = (a_{ij}) \) is a positive semi-definite \( k \)-square Hermitian matrix then

\[
\det A \leq \prod_{i=1}^{k} a_{ii}
\]

with equality if and only if \( A \) has a zero row or \( A \) is diagonal.

In a recent paper [3] it was conjectured that an analogous result to (1) holds for the permanent of \( A \). We recall that the permanent is defined by

\[
\text{per}(A) = \sum_{\sigma \in S_k} \prod_{i=1}^{k} a_{\sigma(i)i}
\]

where the summation is over the whole symmetric group of degree \( k \). Recent interest in the permanent function stems from its application to a variety of combinatorial problems [4] and a partly unresolved conjecture of B. L. van der Waerden [2]. In [3] it was suggested that if \( A \) is once again a positive semi-definite \( k \)-square Hermitian matrix then

\[
\text{per}(A) \geq \prod_{i=1}^{k} a_{ii}
\]

with equality if and only if \( A \) has a zero row or \( A \) is diagonal. We are as yet unable to prove this but the subsequent inequality (3) is a step in this direction.

The first purpose of this note is to exhibit (1) as a case of the Pythagorean Theorem in a suitable symmetry class of tensors. Of course, many proofs of (1) are extant and our purpose in reproving it here is to exhibit a technique that is proving itself useful for examining a wide variety of matrix functions. We then show by a similar approach that

\[
\text{per}(A) \geq \left( \prod_{i=1}^{k} a_{ii} \right)^{1/k} \frac{k^k}{k^{2k}}
\]

where the inequality is strict unless \( A \) has a zero row.

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The technique here is to regard the determinant and permanent functions as analytic expressions for inner products in suitable symmetry classes of tensors. To be explicit, let $U$ be an $n$-dimensional unitary vector space with an inner product $(x, y)$. Let $U^{(k)}$ be the space of $k$-tensors over $U$; i.e., the $n^k$-dimensional dual space of the vector space $M^{(k)}$ of multilinear functionals $\phi$ of $k$-tuples of vectors from $U$. Certain distinguished “pure” vectors in $U^{(k)}$ are denoted by $f = x_1 \otimes \cdots \otimes x_k$ where $x_i \in U$ and $f$ is defined by $f(\phi) = \phi(x_1, \cdots, x_k)$ for each $\phi \in M^{(k)}$. The pure vectors span $U^{(k)}$ and the conjugate bilinear functional defined on pure vectors by

$$\langle x_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k \rangle = \prod_{i=1}^{k} (x_i, y_i)$$

is extendable to a unitary inner product on $U^{(k)}$. Let $T$ and $S$ be the symmetry operators of $U^{(k)}$ into itself defined by

$$T(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

$$S(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

where $\epsilon(\sigma) = \pm 1$ according as $\sigma$ is even or odd.

It is well known that $T$ and $S$ are Hermitian (with respect to the inner product in (4)) and idempotent. Moreover, one computes easily that

$$\langle T(x_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k \rangle = \frac{1}{k!} \det((x_i, y_j))$$

and

$$\langle S(x_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k \rangle = \frac{1}{k!} \per((x_i, y_j)).$$

To prove (1) we first remark that if $A$ is singular, $\det A = 0$, then the inequality obviously holds and equality requires that some $a_{ij} = 0$. But then $0 \leq a_i a_j - |a_{ij}|^2 = - |a_{ij}|^2$ and hence $a_{ij} = 0$, $j = 1, \cdots, k$, and row $i$ of $A$ is zero. Hence assume $A$ is nonsingular and let $x_1, \cdots, x_k$ be a set of linearly independent vectors such that $(x_i, x_j) = a_{ij}$, $i, j = 1, \cdots, k$. Let $u_1, \cdots, u_k$ be the E. Schmidt orthonormalizing sequence for $x_1, \cdots, x_k$. That is, the space $\langle x_1, \cdots, x_p \rangle$ spanned by $x_1, \cdots, x_p$ is the same as the space $\langle u_1, \cdots, u_p \rangle$ spanned by $u_1, \cdots, u_p$, $p = 1, \cdots, k$. Then since $u_i \otimes \cdots \otimes u_{i_\alpha}$, $1 \leq i_\alpha \leq k$, $\alpha = 1, \cdots, k$, is an orthonormal basis in $U^{(k)}$ we have from the Pythagorean theorem that
\[ \frac{1}{k!} \det A = \frac{1}{k!} \det((x_i, x_j)) = (T x_1 \otimes \cdots \otimes x_k, T x_1 \otimes \cdots \otimes x_k) \]

\[ = \sum |(T x_1 \otimes \cdots \otimes x_k, u_i \otimes \cdots \otimes u_{i_k})|^2 \]

\[ = \left( \frac{1}{k!} \right)^2 \sum |\det((x_s, u_t))|^2, \quad s, t = 1, \ldots, k, \]

where the summation extends over all \( k^k \) ordered selections \((i_1, \ldots, i_k)\) from 1, \( \ldots, k \). Since the determinant vanishes when two columns are the same, the last summation may be taken over sets of distinct ordered choices \((i_1, \ldots, i_k)\), i.e., over all \( k! \) permutations of 1, \( \ldots, k \). Hence

\[ \frac{1}{k!} \det A = \left( \frac{1}{k!} \right)^2 \sum_{\sigma \in S_k} |\det((x_s, u_{\sigma(t)}))|^2 \]

\[ = \frac{1}{k!} |\det((x_s, u_t))|^2 \]

\[ = \frac{1}{k!} \left| \begin{array}{ccc}
(x_1, u_1) & 0 & \cdots & 0 \\
(x_2, u_1) & (x_2, u_2) & 0 & \cdots \\
& & & \ddots \\
& & & \ddots \\
& & & & (x_k, u_1) & (x_k, u_2) & \cdots & (x_k, u_k)
\end{array} \right|^2 \]

\[ = \frac{1}{k!} \prod_{i=1}^{k} |(x_i, u_i)|^2 \]

\[ \leq \frac{1}{k!} \prod_{i=1}^{k} (x_i, x_i) = \frac{1}{k!} \prod_{i=1}^{k} a_{ii}. \]

Now the equality holds by Schwarz's inequality if and only if \( x_a \) is a multiple of \( u_a \). But since \( u_1, \ldots, u_k \) is an orthonormal set, it follows that \( A = ((x_i, x_j)) \) is diagonal.

From (6) we compute that

\[ \operatorname{per} A = k!(S x_1 \otimes \cdots \otimes x_k, x_1 \otimes \cdots \otimes x_k) \]

\[ = k! ||S x_1 \otimes \cdots \otimes x_k||^2 \]

\[ \geq k! |(S x_1 \otimes \cdots \otimes x_k, u \otimes \cdots \otimes u)|^2 \]

\[ = k! |(x_1 \otimes \cdots \otimes x_k, S u \otimes \cdots \otimes u)|^2 \]

\[ = k! |(x_1 \otimes \cdots \otimes x_k, u \otimes \cdots \otimes u)|^2 \]

\[ = k! \prod_{i=1}^{k} |(x_i, u)|^2, \]
where \( u \) is any unit vector in \( U \).

The problem then is to find for fixed \( x_i, i = 1, \ldots, k \), a significant lower bound for the expression \( \prod_{i=1}^{k} | (x_i, u) |^2 \) as a function of the unit vector \( u \). We have the

**Lemma.** If \( x_1, \ldots, x_k \) are vectors then there exists a unit vector \( u \) such that

\[ | (x_s, u) | \geq \| x_s \| / k, \quad s = 1, \ldots, k. \]

**Proof.** Let \( y_i = x_i / \| x_i \| \) or \( -x_i / \| x_i \| \) so that \( \sum_{i=1}^{k} y_i \) is of maximal length. That is, \( \| \sum_{i=1}^{k} y_i \| \geq \| \sum_{i=1}^{k} \pm x_i / \| x_i \| \| \) for all choices of signs on the \( x_i \). We assert that

\[ \text{Re} \left( y_s, \sum_{i=1, i \neq s}^{k} y_i \right) \geq 0, \quad s = 1, \ldots, k. \]  

For let \( z_s = \sum_{i=1, i \neq s}^{k} y_i \) and \( z = \sum_{i=1}^{k} y_i \) and we have

\[ \| z \|^2 = \| y_s + z_s \|^2 \geq \| -y_s + z_s \|^2, \]

\[ \| y_s \|^2 + 2 \text{Re}(y_s, z_s) + \| z_s \|^2 \geq \| y_s \|^2 - 2 \text{Re}(y_s, z_s) + \| z_s \|^2 \]

and (8) follows.

It is clear that

\[ \| z \| \leq \sum_{i=1}^{k} \| y_i \| = k \]

and thus

\[ \left| \left( y_s, \frac{z}{\| z \|} \right) \right| = \left| \left( y_s, \frac{y_s + z_s}{\| z \|} \right) \right| \]

\[ = \frac{1}{\| z \|} \left| \| y_s \|^2 + (y_s, z_s) \right| \]

\[ \geq \frac{1}{\| z \|} \text{Re}(1 + (y_s, z_s)) \geq \frac{1}{\| z \|} \frac{1}{k}. \]

In (7) we take \( u = z / \| z \| \) to obtain

\[ \text{per} \ A \geq k! \prod_{i=1}^{k} | (y_s, u) |^2 \| x_i \|^2 \]

\[ \geq (k! / k^k) \prod_{i=1}^{k} \| x_i \|^2 \]

\[ = (k! / k^k) \prod_{i=1}^{k} a_{ii}. \]
Clearly if any \( x_j = 0 \) then \( \per A = 0 \), \( a_{jj} = 0 \) and (3) is equality. If no \( x_j = 0 \) then (9) can be equality only if \( z = \theta ky_n \), where \( |\theta| = 1, s = 1, \ldots, k \), in which case

\[
|\langle y_s, u \rangle| = \left| \left( y_s, \frac{z}{\|z\|} \right) \right| = \frac{k}{k - 1} > 1 \frac{1}{k}
\]

and hence the inequality in the lemma is strict for \( k > 1 \).

Actually a refinement of the above argument shows that

\[
\prod_{i=1}^{k} |\langle y_s, u \rangle| \geq \frac{k^{1/2}}{k - 1} \cdots \frac{k^{1/2} - r}{k - r} \frac{1}{k^{k-r-1}},
\]

where \( r = \lfloor k/(k^{1/2}+1) \rfloor \), and therefore,

\[
\per(A) \geq \prod_{i=1}^{k} a_{ii} k^1 / \left( \frac{k^{1/2}}{k - 1} \cdots \frac{k^{1/2} - r}{k - r} \frac{1}{k^{k-r-1}} \right)^2.
\]

For

\[
\|z\|^2 = \langle z, z \rangle = \sum_{s=1}^{k} \langle y_s, z \rangle = k + \sum_{s=1}^{k} \langle y_s, z_s \rangle = k + \sum_{s=1}^{k} \Re(y_s, z_s) \geq k,
\]

i.e., \( \|z\| \geq k^{1/2} \). We can assume without loss of generality that \( \Re(y_1, z) \geq \cdots \geq \Re(y_k, z) \). Now from (11) we have

\[
\|z\| = \sum_{s=1}^{k} \Re(y_s, \frac{z}{\|z\|}) \geq k^{1/2}
\]

and therefore \( \Re(y_1, z/\|z\|) \geq k^{1/2}/k \), and a fortiori, \( |\langle y_1, z/\|z\| \rangle| \geq k^{1/2}/k \).

Further

\[
\left| \sum_{s=2}^{k} \left( y_s, \frac{z}{\|z\|} \right) \right| \geq \left| \sum_{s=1}^{k} \left( y_s, \frac{z}{\|z\|} \right) \right| - \left| \left( y_1, \frac{z}{\|z\|} \right) \right| \geq k^{1/2} - 1
\]

and thus \( |\langle y_2, z/\|z\| \rangle| \geq (k^{1/2} - 1)/(k - 1) \).

In general

\[
\left( y_{s+1}, \frac{z}{\|z\|} \right) \geq k^{1/2} - s \frac{1}{k - s}.
\]

Comparing (12) with (10) we find, by an elementary computation, that

\[
\max \left\{ \frac{k^{1/2} - s}{k - s}, \frac{1}{k} \right\} = \begin{cases} \frac{k^{1/2} - s}{k - s} & \text{for } s \leq r, \\ \frac{1}{k} & \text{for } s > r, \end{cases}
\]

where \( r \) is the greatest integer in \( k/(k^{1/2}+1) \).
Thus (12) gives a better lower bound for \( s = 1, \ldots, r \) while (10) gives a better lower bound for \( s = r + 1, \ldots, k \). The result follows.

References