A CONJECTURE ON WEAK COMPACTNESS

BY

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There are several characterizations of weak compactness in terms of intersection properties of convex sets. The first of these was Šmulian's theorem [13] to the effect that if a closed convex set C in a normed linear space is not weakly compact, then C contains a decreasing sequence of nonempty closed convex sets whose intersection is empty. This result was sharpened in various ways by Dieudonné [4], Floyd and Klee [5], and Ptak [11]. (For a survey of these and related results, see Chapter III of Day [3].) However, in each of their constructions it may happen that all the sets in the sequence penetrate rather far into the set C; for example, if C is the unit cell $\{x \in E: \|x\| \leqq 1\}$ of a normed linear space E, it may happen for some $m \in [0, 1[$ that each set in the sequence intersects the smaller cell mC. On the other hand, R. C. James [7] has proved that if C is the unit cell of a separable Banach space E and C is not weakly compact (or, equivalently, E is nonreflexive), then there exists a linear functional $f \in E^*$ such that $\|f\| = 1$ but $fx < 1$ for all $x \in C$. With $K_n = \{x \in C: fx \geqq 1 - 1/n\}$, it is clear not only that $\bigcap_1^\infty K_i = \emptyset$ but also that for each $m \in [0, 1[$, the set mC meets only finitely many of the sets K_i.

The theorem of James lends support to the following

CONJECTURE. *A closed convex subset C of a separable Banach space E is weakly compact if and only if every continuous linear functional on E attains a maximum on C.*

The conjecture reduces at once to the case of a bounded set. In the present note we supply further support by proving the following

THEOREM. *Suppose C is a bounded closed convex subset of a separable Banach space, and C is not weakly compact. Then C contains a decreasing sequence K_α of nonempty closed convex sets such that for each $x \in C$ and each $m \in [0, 1[$, the set $x + m(C - x)$ meets only finitely many of the sets K_i.*

Note that from the above conjecture there would follow the same characterization of weak compactness for an arbitrary weakly closed subset of E. Further, James has announced (at a conference in Warsaw in 1960) the validity of his theorem for an arbitrary nonreflexive Banach space. Thus it may be true for any Banach space E that a weakly closed subset X of E is weakly compact if and only if every $f \in E^*$ attains a maximum on X.

Our proof of the Theorem stated above depends on the notion of *relative*

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extreme point introduced in [8] and considered also in [10]. When Z is a subset of a linear space and $X \subset Z$, a point z of Z is said to be extreme in Z relative to X provided there do not exist distinct points $x \in X$ and $z' \in Z$ such that $z \in]x, z'[$. The set of all such relative extreme points will be denoted by $ex_X Z$. Our basic tool is the existence of relative extreme points under certain circumstances, in conjunction with the following result:

1. PROPOSITION. *Suppose X and Z are subsets of a Hausdorff linear space, Z is closed and starshaped from X, and every open covering of X admits a subcovering of cardinality $\leqq \aleph$, where $\aleph$ is an infinite cardinal number. Suppose $z \in ex_X Z \sim X$. Then there is a set $\mathfrak{N}$ of neighborhoods of z such that* card $\mathfrak{N} \leqq \aleph$ *and $Z \cap (\bigcap_{N \in \mathfrak{N}} N) \subset ex_X Z \sim X$.*

Proof. Note that for every $x \in X$ and every positive integer n there exist disjoint open neighborhoods $U(x, n)$ of x and $V(x, n)$ of z such that whenever $u \in U(x, n) \cap X$ and $v \in V(x, n) \cap Z$, then the ray $u + [(n+1)/n, \infty[(v-u)$ lies entirely in the complement of Z. For otherwise there are nets x_α in X, z_α in Z, and r_α in $[(n+1)/n, \infty[$ (where α ranges over some directed set) such that $x_\alpha \to x$, $z_\alpha \to z$, and always $x_\alpha + r_\alpha(z_\alpha - x_\alpha) \in Z$. Since Z is starshaped from X, this implies that

$$x_\alpha + \frac{n+1}{n}(z_\alpha - x_\alpha) \in Z,$$

whence

$$x + \frac{n+1}{n}(z - x) \in Z$$

for Z is closed. But this contradicts the hypothesis that $z \in ex_X Z$.

For each n, the class $\{U(x, n) : x \in X\}$ is an open covering of X, and hence there is a subset S_n of X such that card $S_n \leqq \aleph$ and $X \subset \bigcup_{x \in S_n} U(x, n)$. Let $\mathfrak{N} = \bigcup_{n=1}^{\infty} \{V(x, n) : x \in S_n\}$. Then card $\mathfrak{N} \leqq \aleph_0 \cdot \aleph = \aleph$. If $u \in X$ and $v \in Z \cap (\bigcap_{N \in \mathfrak{N}} N)$, then

$$u +]0, \infty[(v - u) = \bigcup_{n=1}^{\infty} \left(u + \left[\frac{n+1}{n}, \infty\right[(v - u)\right),$$

and by the choice of $\mathfrak{N}$ the latter set cannot intersect Z. Consequently $v \in ex_X Z \sim X$ and the proof is complete.

2. THEOREM. *Suppose C is a sequentially complete bounded convex subset of a locally convex space, E is the linear extension of C, and C is not weakly compact but does enjoy the Lindelöf property with respect to the weak topology $\sigma(E, E^*)$. Let ξ denote the natural embedding of E in the space $E^{*\prime}$ (the algebraic dual of E^*), C_σ the closure of ξC in the topology $\sigma(E^{*\prime}, E^*)$, E_2 the linear extension of*

C_σ, and σ_2 the restriction to E_2 of $\sigma(E^{*\prime}, E^*)$. Then there exist a point $Y \in C_\sigma$ and a sequence f_α in E^* such that with $\phi X = \sum_1^\infty 2^{-n}|Xf_n - Yf_n|$ for all $X \in E_2$, the function ϕ is finite-valued, σ_2-continuous, and

$$\{X \in C_\sigma: \phi X = 0\} = \{X \in C_\sigma: Xf_\alpha = Yf_\alpha\} \subset ex_{\xi C} C_\sigma \sim \xi C.$$

If $K_n = \{x \in C: \phi\xi x \leqq 1/n\}$, then K_α is a decreasing sequence of nonempty closed convex subsets of C such that for each $x \in C$ and each $m \in [0, 1[$, the set $x + m(C - x)$ meets only finitely many of the sets K_i.

Proof. Let τ denote the given topology for E and σ_0 the weak topology $\sigma(E, E^*)$.

Since C is σ_0-bounded, the set ξC is $\sigma(E^{*\prime}, E^*)$-bounded and hence C_σ is $\sigma(E^{*\prime}, E^*)$-compact. Consequently, both C_σ and the set $U = C_\sigma - C_\sigma$ must be σ_2-compact. Since U is convex and symmetric, the gauge functional μ of U is a norm for E_2 relative to which the unit cell is U itself. And since $E = R(C - C)$, the function $\mu\xi$ is a norm for E. Let μ_2 and μ_0 denote the metric topologies for E_2 and E which are generated by the respective norms μ and $\mu\xi$. Since U is σ_2-bounded, μ_2 must be finer than σ_2, and μ_2-completeness of the sets U and C_σ is a consequence of the following fact (Grothendieck [6, p. 163]): If τ_1 and τ_2 are two locally convex Hausdorff linear topologies for a linear space E, τ_1 is finer than τ_2, and there exists a τ_1-fundamental system of τ_1-neighborhoods of 0 which are τ_2-closed, then every τ_2-complete subset of E is τ_1-complete.

Note also that σ_2-boundedness of U implies σ_0-boundedness of the set $\xi^{-1}(U \cap \xi E)$ which is the unit cell of E relative to the norm $\mu\xi$; since this set is σ_0-bounded it must be τ-bounded (according to a theorem of Mackey [9]) and consequently μ_0 is finer than τ. This fact can be used to show that the set ξC is μ_2-closed. For consider a sequence X_α in ξC which is μ_2-convergent to a point $X \in C_\sigma$. For each n there exists $x_n \in C$ such that $X_n = \xi x_n$, and of course x_α is a μ_0-Cauchy sequence in C. But μ_0 is finer than τ, so x_α is also τ-Cauchy, and since C is τ-sequentially-complete the sequence x_α must be τ-convergent to a point $x \in C$. For each $f \in E$ we have

$$(\xi x)f = fx = \lim fx_\alpha = \lim X_\alpha f = Xf,$$

and from this it follows that $X = \xi x \in \xi C$.

We know now that C_σ is μ_2-bounded and μ_2-complete, and ξC is a μ_2-closed convex subset of C_σ. And of course $\xi C \neq C_\sigma$, for we are assuming that C is not σ_0-compact. It follows from a theorem in [8] that there exists a point

$$Y \in ex_{\xi C} C_\sigma \sim \xi C.$$

Since C enjoys the Lindelöf property relative to σ_0, the same must be true of ξC relative to σ_2 and thus by Proposition 1 there is a sequence W_α of σ_2-neighborhoods of Y in E_2 such that

$$C_\sigma \cap \left(\bigcap_1^\infty W_i \right) \subset ex_{\xi C} C_\sigma \sim \xi C.$$

Let $S^* = \{f \in E^*: \sup_{X \in U} Xf = 1\}$. Recalling the definition of the topology σ_2, we see that for each n there is a finite subset G_n of S^* such that W_n contains the set of all $X \in E_2$ for which $Xg = Yg$ for all $g \in G_n$. Let f_α be an enumeration of the set $\bigcup_{n=1}^\infty G_n$ and for each $X \in E_2$ let

$$\phi X = \sum_{n=1}^\infty \mid Xf_n - Yf_n \mid /2^n.$$

Of course for each n the function $Xf_n | X \in E_2$ is σ_2-continuous, and since $|Xf_n - Yf_n| \leqq \mu(X-Y) < \infty$ we see that ϕ is a σ_2-continuous real-valued convex function on E_2. From the choice of sequence f_α it is evident that

$$\{X \in C_\sigma: \phi X = 0\} = \{X \in C_\sigma: Xf_\alpha = Yf_\alpha\} \subset ex_{\xi C} C_\sigma \sim \xi C.$$

Since ϕ is convex and σ_2-continuous, $\inf \phi C_\sigma = 0$, and ξC is σ_2-dense in C_σ, it is evident that each set K_i is convex, closed, and nonempty. Now with $x \in C$ and $m \in [0, 1[$, suppose the set $x + m(C - x)$ meets infinitely many of the sets K_i. Then $\inf \phi(\xi x + m(\xi C - \xi x)) = 0$, and since C_σ is σ_2-compact there exists $Z \in C_\sigma$ such that $\phi(x + m(Z - \xi x)) = 0$. Then $\xi x + m(Z - \xi x) \in ex_{\xi C} C_\sigma$, whence $\xi x = Z$ and $\phi \xi x = 0$, a contradiction completing the proof.

To establish a slight generalization of the THEOREM stated earlier, it suffices to apply Theorem 2 in conjunction with the following observation:

3. PROPOSITION. *Suppose E is a separable metrizable locally convex space. Then in the weak topology $\sigma(E, E^*)$, every subset of E enjoys the Lindelöf property.*

Proof. We note first that if Y is a separable metric space and X is a topological space which is a union of countably many separable metrizable subsets, then every set of continuous maps of X into Y enjoys the Lindelöf property with respect to the topology of pointwise convergence. Under the assumption that X and Y are both separable metric, this was proved by M. E. Rudin and the author [12], but the proof required of X only that the product space $]0, \infty[\times X^n \times Y^n$ should be separable for each n.

Now let U_α be a fundamental sequence of neighborhoods of 0 in E, and for each i and j let $V_{ij} = \{f \in E^*: \sup fU_i \leqq j\}$. Then $E^* = \bigcup_{i,j} V_{ij}$, and since E is separable every set V_{ij} must be compact and metrizable under the topology $\sigma(E^*, E)$ (p. 66 of Bourbaki [1]). Let ξ denote the natural map of E into $E^{*\prime}$. Then ξ is a linear homeomorphism of the space $(E, \sigma(E, E^*))$ into the space $(E^*, \sigma(E^*, E))^*$ under its weak topology, and every subset of the latter space is Lindelöfian by the result of the preceding paragraph. This completes the proof.

We note in closing that the THEOREM surely is valid for every Banach space which is Lindelöfian in its weak topology, and that there exist non-separable spaces of this sort (see Corson [2]).

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