ON EXTENSIONS OF H-SPACES

BY

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1. **Introduction.** The following is known:

THEOREM A [2]. *Let Y be a space with only two nonvanishing homotopy groups, $\pi_p(Y) = \pi$, $\pi_q(Y) = G$, $q > p$. Y admits a multiplication if and only if the k-invariant $k \in H^{q+1}(\pi,p;G)$ is primitive.*

By primitive we mean $m^*(k) = p_1^*(k) + p_2^*(k)$ where $m : K \times K \to K$ is the multiplication on $K = K(\pi,p)$ (there is only one up to homotopy) and $p_i : K_1 \times K_2 \to K_i$ are the projections. Since $k = f^*(\iota)$ for some map $f : K(\pi,p) \to K(G,q+1)$, we can rephrase primitivity in terms of the map f. Using obstruction theory, we find that k is primitive if and only if f is an H-map, in the sense of:

DEFINITION 1. Let (X,m) and (W,n) be H-spaces. A map $f : X \to W$ is an *H-map* if there exists a *sputnik homotopy* h_t: $X \times X \to W$ such that

$$h_0(x,y) = f(x)f(y), \quad h_1(x,y) = f(xy), \quad \text{i.e.,}$$

$$\begin{array}{ccc} X \times X & \xrightarrow{\;f \times f\;} & W \times W \\ {\scriptstyle m}\downarrow & \simeq & \downarrow{\scriptstyle n} \\ X & \xrightarrow[\;f\;]{} & W. \end{array}$$

In the present paper, we wish to examine a somewhat more general situation more fully with four goals in mind:

(1) to give multiplications on Y explicitly in terms of a sputnik homotopy for a map representing k,

(2) to examine more general conditions under which it is possible to define a multiplication in this way,

(3) to relate these multiplications to additive secondary cohomology operations,

(4) to extend the results to homotopy associative H-spaces.

We are indebted to J. C. Moore for suggesting these problems and for super-

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vising the thesis which grew out of them [8]. Our viewpoint is a direct descendent of one which he presented in a seminar on related problems [7]. A very similar approach, though disguised by different notation, is used by Hilton [3] to prove Theorem 2.

As for our own notation, most of it is standard but a few things may need explanation. If X and Y are spaces with basepoints e_x and e_y respectively, then $X \wedge Y$ is the space $X \times Y/X \times e_y \cup e_x \times Y$.

In a diagram, © indicates commutavity of the portion surrounding it, while $\simeq$ indicates homotopy commutativity.

Let X^I be the space of maps of the unit interval I into X and let R be the set of non-negative real numbers. The (Moore) *free path space* $\mathscr{P}(X)$ is the space of all pairs (f,r) where $r \in R$ and $f:[0,r] \to X$, topologized so that the map $h:\mathscr{P}(X) \to X^I \times R$, given by $h(f,r) = (f',r)$ with $f'(t) = f(rt)$, $0 \leqq t \leqq 1$, is a homeomorphism onto its image [5; 1]. Paths (f,r) and (g,s) such that $f(r) = g(0)$ are added by the rule $(f,r) + (q,s) = (h, r + s)$ where

$$\begin{aligned} h(t) &= f(t), && 0 \leqq t \leqq r, \\ h(t) &= g(t-r), && r \leqq t \leqq r+s. \end{aligned}$$

The (Moore) *based path space* $\mathscr{L}X$ is the subspace of $\mathscr{P}(X)$ consisting of paths (f,r) such that $f(0) = e_x$. The map $\pi: \mathscr{L}X \to X$ given by $\pi(f,r) = f(r)$ is a fibre map with fibre ΩX, the (Moore) *space of loops* on X. We will often write f for (f,r) since a function uniquely determines its own domain of definition and, when convenient, will write $\bar{1}$ for r. Thus we write $\pi(f) = (f(\bar{1}))$. With the rule of addition mentioned above, ΩX becomes an associative H-space.

An H-space (X, m) consists of a space X with basepoint e and a multiplication $m: X \times X \to X$ such that $m(x,e) = m(e,x) = x$.

If (X, m) is an H-space, a multiplication $\mathscr{P}m: \mathscr{P}X \times \mathscr{P}X \to \mathscr{P}X$ can be defined by

$$\begin{aligned} \mathscr{P}m\,[(f,r),(g,s)](t) &= m(f(t),g(t)) \text{ for } t \leqq \min\,[r,s], \\ &= m(f(t),g(s)) \text{ if } \min\,[r,s] = s \leqq t \leqq r, \\ &= m(f(r),g(t)) \text{ if } \min\,[r,s] = r \leqq t \leqq s. \end{aligned}$$

The unit is $\lambda_e: [0,0] \to e$.

2. **Explicit multiplications.** We turn to the achievement of our first goal.

THEOREM 2. *Let (X,m) and (W,n) be H-spaces. Let the fibring $\Omega W \to Y \xrightarrow{\mathfrak{p}} X$ be induced from $\Omega W \to \mathscr{L}W \to W$ by a map $f: X \to W$. If f is an H-map, then there exists a multiplication $s: Y \times Y \to Y$ such that*

(1)
$$\begin{array}{ccc} Y \times Y & \xrightarrow{\ s\ } & Y \\ {\scriptstyle \mathfrak{p} \times \mathfrak{p}} \downarrow & © & \downarrow {\scriptstyle \mathfrak{p}} \\ X \times X & \xrightarrow[\ m\ ]{} & X, \end{array}$$

(2)
$$\begin{array}{ccccc} \Omega W \times Y & \longrightarrow & Y \times Y & \xrightarrow{s} & Y \\ \downarrow & & \copyright & & \downarrow \\ \mathscr{L}W \times \mathscr{L}W & & \xrightarrow[\mathscr{L}n]{} & & \mathscr{L}W. \end{array}$$

Proof. Y can be represented as $\{(x,\lambda) \mid x \in X, \lambda \in \mathscr{L}W, \pi(\lambda) = f(x)\}$. If f is an H-map, we have a homotopy $h_t: X \times X \to W$ such that $h_0(x,y) = f(x)f(y)$ and $h_1(x,y) = f(xy)$. Consider $\mathscr{P}(W)$, the space of all maps λ: $[0,r] \to W$. Corresponding to h_t, there is a map $F^1: X \times X \to \mathscr{P}(W)$ such that $F^1(x,y)(t) = h_t(x,y)$. According to [4, Corollary (4.4)] we can assume without loss of generality that $h_t(x,e) = h_t(e,x) = f(x)$ for all t. If F^1 corresponds to such a homotopy, we can deform it (through maps F^s: $X \times X \to \mathscr{P}(W)$ such that $F^s(x,y)(0) = f(x)f(y)$ and $F^s(x,y)(\bar{1}) = f(xy)$) to a map $F: X \times X \to \mathscr{P}(W)$ such that $F(x,e) = F(e,x)$ is the path of length 0 and value $f(x)$. We call F a *homotopy multiplier* for f.

Now define s by
$$s[(x,\lambda),(x',\lambda')] = (xx', \lambda\lambda' + F(x,x')).$$
We see that s is well defined, i.e., $\pi(\lambda\lambda' + F(x,x')) = f(xx')$,and has (e,λ_e) as a unit. Diagrams (2.1) and (2.2) are clearly commutative. (F^1 would have done as good a job if we didn't require (2) and were satisfied with a homotopy unit.)

In certain situations, this is the only way that Y can have a multiplication.

THEOREM 3. *Let Y be as in Theorem* 2. *If there exists a multiplication satisfying* (2.1), *then f is an H-map provided that X is* $(p-1)$*-connected and for some $q \geqq p$, $\pi_i(W) = 0$ for $i \leqq q$ and $i > p+q$.*

Proof. The obstructions to f being an H-map would be classes in $H^i(X \wedge X; \pi_i(W))$. Since p is multiplicative, $(\mathrm{p} \wedge \mathrm{p})^*$ would take them into the corresponding obstructions for $f\mathrm{p}$. These must all vanish since $f\mathrm{p}$ is null-homotopic. However, $(\mathrm{p} \wedge \mathrm{p})^*$ is an isomorphism since $\pi_i(W) = 0$ for $i > p+q$ and $\mathrm{p}^*: H^i(X) \to H^i(Y)$ is an isomorphism for $i \leqq q$. Thus all the obstructions vanish; f is an H-map. (Note that, up to homotopy, W admits only one multiplication.)

THEOREM 4. *Let Y be as in Theorem* 2. *If Y admits a multiplication s', then f is an H-map with respect to some multiplication on X provided X is* $(p-1)$*-connected and for some $q \geqq p$, $\pi_i(X) = 0$ for $i > 2q$ while $\pi_i(W) = 0$ for $i \leqq q$ and $i > p+q$.*

Proof. Since $\pi_i(W) = 0$ for i $\leqq q$, we can construct a cross-section $\chi: X^q \to Y$. Define a map m' by the following diagram:

$$\begin{array}{ccc} Y \times Y & \xrightarrow{s'} & Y \\ {\scriptstyle \chi \times \chi} \uparrow & & \downarrow {\scriptstyle \mathrm{p}} \\ X^q \times X^q & \xrightarrow{m'} & X. \end{array}$$

The obstructions to extending m' to all of $X \times X$ lie in $H^{i+1}(X \wedge X, X^q \wedge X^q; \pi_i(X))$ and hence all vanish since $\pi_i(X) = 0$ for $i + 1 \geqq 2q + 2$. Let $m: X \times X \to X$ be an extension of m'. The obstructions to $\mathfrak{p}$ being an H-map with respect to s' and m lie in $H^i(Y \wedge Y, Y^q \wedge Y^q;\ \pi_i(X))$ which again are trivial groups. Since $\mathfrak{p}$ is a fibring, if $\mathfrak{p}s'$ is homotopic to $m(\mathfrak{p} \times \mathfrak{p})$, then s' can be deformed to a multiplication s such that $\mathfrak{p}s = m(\mathfrak{p} \times \mathfrak{p})$. Thus Theorem 3 applies; f must be an H-map.

Combining Theorems 2 and 4 yields Theorem A. More generally, we can say:

THEOREM 5. *A space Y admits a multiplication if and only if each stage Y_q of the Postnikov system of Y admits a multiplication m_q: $Y_q \times Y_q \to Y_q$ such that*

(1) *the projection $Y_{q+1} \to Y_q$ is multiplicative, and*

(2) *the k-invariant $k_{q+1} \in H^{q+2}(Y_q; \pi_{q+1}(Y))$ is primitive with respect to m_q.*

3. **Additive cohomology operations and H-spaces.** Theorem 5 fails to be a completely satisfactory solution of our problem in the following sense:

In order to test k_{q+1} for primitivity, we must specify m_q. Now in light of Theorems 2 and 3, m_q corresponds to a chain $b \in C^q(Y_{q-1} \wedge Y_{q-1};\ \pi_q(Y))$ such that $\delta b = m^{\#}_{q-1}(a) - p^{\#}_1(a) - p^{\#}_2(a)$ where p_i, $i = 1,2$ are the projections and a represents k_q. Given such a cochain b, how we can tell if k_{q+1} is primitive with respect to the corresponding multiplication? We lack the technique for a general answer to this question, but in certain interesting cases the following approach is possible:

A cohomology class in $H^{q+1}(\pi, p; G)$ can also be regarded as a primary cohomology operation. Theorem A then says that Y admits a multiplication if and only if the k-invariant is an additive primary operation. It is this point of view which yields a neat generalization. Frank Adams [0] has defined secondary cohomology operations in terms of fibrings with fibre and base being abelian groups, while Spanier has suggested a somewhat different point of view. We have both these approaches in mind as we proceed.

Following [7], we generalize our notion of cohomology by replacing maps into $K(\pi, n)$ by maps into abelian topological groups (which have the homotopy type of products of $K(\pi, n)$'s). Given an abelian topological group A, for $n \geqq 0$ denote by $\Omega^n A$ an abelian group whose underlying space has the indicated homotopy type. For $n > 0$, denote $\Omega^{-n} A$ an abelian group B such that $\Omega^n B \simeq A$. We define $H^n(X; A)$ as $\pi(X, \Omega^n A)$, the group of homotopy classes of X into $\Omega^n A$, with the obvious addition. We abbreviate $H^0(X; A)$ as $H(X; A)$. Using obstruction theory, it is easy to show that a universally defined cohomology operation $\mathcal{O}: H^n(\ ; A) \to H^m(\ ; B)$ corresponds to an element of $H^m(\Omega^n A; B)$, i.e. to a homotopy class of maps of $\Omega^n A$ into $\Omega^m B$. We can also show that $\mathcal{O}$ is additive if and only if it is represented by an H-map.

Now let A,B,C be abelian groups and $\mathcal{O}: H(\ ; A) \to H(\ ; B)$, $\mathcal{N}: H(\ ; B) \to H(\ ; C)$ be operations such that $\mathcal{N}\mathcal{O} = 0$.

DEFINITION 6. The *rough* secondary operation

$$(\mathcal{O},\mathcal{N}):\mathrm{Ker}\ \mathcal{O}\cap H^0(X;A)\to H^{-1}(X;C)/\Omega\mathcal{N}\circ H^{-1}(X;B)+H^{-1}(A;C)\circ[u]$$

is defined as follows: Let $f:A\to B$ and $g:B\to C$ represent $\mathcal{O}$ and $\mathcal{N}$ respectively, so that $gf\simeq 0$. Let $\Omega B\to Y\to A$ be the fibre space over A induced by f from $\Omega B\to \mathscr{L}B\to B$. Let $h:Y\to\Omega C$ be any map such that $h\,|\,\Omega B\simeq\Omega g$. Now let $u:X\to A$ be any map representing a class in the kernel of $\mathcal{O}$. Since $fu\simeq 0$, fu can be lifted to $\bar{u}:X\to Y$. The coset $(\mathcal{O},\mathcal{N})[u]$ is represented by $h\bar{u}$.

DEFINITION 7. Given f and g representing $\mathcal{O}$ and $\mathcal{N}$ and a particular lifting $j:A\to\mathscr{L}C$ of gf, the *precise* secondary operation $(\mathcal{O},\mathcal{N},j):\mathrm{Ker}\ \mathcal{O}\to\mathrm{Coker}\,\Omega\mathcal{N}$ is defined as follows: Let $\Omega C\to D\to B$ be the fibre space induced by g. Define $h:Y\to\Omega C$ by $h(x,\lambda)=\mathscr{L}g(\lambda)-j(x)$ for $x\in A$, $\lambda\in\mathscr{L}B$, $\lambda(\bar{1})=f(x)$, observing that $h\,|\,\Omega B=\Omega g$. Given $\bar{u}:X\to Y$ lifting $u:X\to A$ (which represents a class in the kernel of $\mathcal{O}$), the coset $(\mathcal{O},\mathcal{N},j)[u]$ is represented by $h\bar{u}$. (Notice that we obtain the same coset if we let j vary through homotopic liftings, i.e., there is a homotopy $j_t:A\to\mathscr{L}C$ such that $j_t(x)(\bar{1})=gf(x)$ for all t.)

DEFINITION 8. A precise secondary operation $(\mathcal{O},\mathcal{N},j)$ is *additive* if $\mathcal{O}$ is additive and $(\mathcal{O},\mathcal{N},j)\,[u+v]=(\mathcal{O},\mathcal{N},j)[u]+(\mathcal{O},\mathcal{N},j)[v]$ for any two classes $[u]$, $[v]$ in Ker $\mathcal{O}$. A similar definition applies to rough operations.

THEOREM 9. *A precise secondary operation $(\mathcal{O},\mathcal{N},j)$ is additive if and only if there exists a multiplication s on Y such that $\mathrm{p}:Y\to A$ is multiplicative and h is an H-map.*

Proof. Suppose there exists such a multiplication on Y. Let $u,v:X\to A$ be in the kernel of $\mathcal{O}$ so that $fu\simeq 0\simeq fv$. Let $\bar{u},\bar{v}$: $X\to Y$ be liftings of u and v respectively, then $q(\bar{u}\times\bar{v})$ is a lifting of $u+v$. Now $(\mathcal{O},\mathcal{N},j)\,[u+v]$ is represented by $hq(\bar{u}\times\bar{v})$ which is homotopic to $h(\bar{u})+h(\bar{v})$, a representative of $(\mathcal{O},\mathcal{N},j)[u]+(\mathcal{O},\mathcal{N},j)[v]$, i.e., $(\mathcal{O},\mathcal{N},j)$ is additive.

Conversely, suppose $(\mathcal{O},\mathcal{N},j)$ is additive and let $u=v=\mathrm{p}:Y\to A$. Since $\mathcal{O}$ is additive, f is an H-map and by Theorem 2, Y admits some multiplication $s':Y\times Y\to Y$ such that p is multiplicative. Since h itself represents $(\mathcal{O},\mathcal{N},j)[u]$ and hs' represents $(\mathcal{O},\mathcal{N},j)\,[u+u]$, the homotopy class of $h+h$ must differ from that of hs' by a class in the indeterminacy. Thus we can write $h+h\simeq hs'+(\Omega\mathcal{N})\phi$ for some map $\phi:Y\times Y\to\Omega B$. Define $s:Y\times Y\to Y$ by $s=s'+\phi$ where $+$ denotes the usual action of ΩB on T, corresponding to that of ΩB on $\mathscr{L}B$. Thus we have $h+h\simeq hs$ which proves the theorem.

COROLLARY 10. *Let $(\mathcal{O},\mathcal{N},j)$ be a precise secondary operation represented by the map $h:Y\to\Omega C$. Let $\Omega^2C\to Z\to Y$ be induced by h. For the space Z to admit a multiplication, it is sufficient for $(\mathcal{O},\mathcal{N},j)$ to be additive. The condition is necessary if for some $p\leqq q\leqq r$ we have*

$$\pi_i(A)=0 \qquad \text{for } i<p \text{ and } i>2q,$$
$$\pi_i(B)=0 \qquad \text{for } i<q+1 \text{ and } i>p+q,$$
$$\pi_i(C)=0 \qquad \text{for } i<r+2 \text{ and } i>2q.$$

To apply Theorem 9, can we determine the additivity of $(\mathcal{O},\mathcal{N},j)$ without constructing Y or its multiplication s?

DEFINITION 11. Given H-maps $f:U\to V$, $g:V\to W$ with homotopy multipliers $F:U\times U\to\mathcal{P}(V)$ and $G:V\times V\to\mathcal{P}(W)$, the *composite homotopy multiplier* (G,F) for gf is defined by

$$(G,F)(u_1,u_2)=G(f(u_1),f(u_2))+(\mathcal{P}g)F(u_1,u_2).$$

THEOREM 12. *Let a precise secondary operation* $(\mathcal{O},\mathcal{N},j)$ *be determined by* f,g *and* j *as above.* $(\mathcal{O},\mathcal{N},j)$ *is additive if*

(1) *f and g are H-maps (i.e., $\mathcal{O}$ and $\mathcal{N}$ are additive) with homotopy multipliers F and G, and*

(2) *j is an H-map with a homotopy multiplier J which covers the composite homotopy multiplier (G,F).*

The converse is true if for some $p\leqq q\leqq r$

$$\begin{aligned}\pi_i(A)&=0\quad\text{for}\quad i<p\quad\text{and}\quad i>2q,\\ \pi_i(B)&=0\quad\text{for}\quad i<q+1\quad\text{and}\quad i>p+q,\\ \pi_i(C)&=0\quad\text{for}\quad i<r+2\quad\text{and}\quad i>p+q+1.\end{aligned}$$

Proof. Suppose f,g and $j,F,G,$ and J are as above. Define a multiplication on Y as in Theorem 2, i.e., for $x_1,x_2\in A$, λ_1, $\lambda_2\in\mathcal{L}B$ with $\lambda_i(\bar{1})=f(x_i)$, we have

$$(x_1,\lambda_1)(x_2,\lambda_2)=(x_1x_2,\lambda_1\lambda_2+F(x_1,x_2)).$$

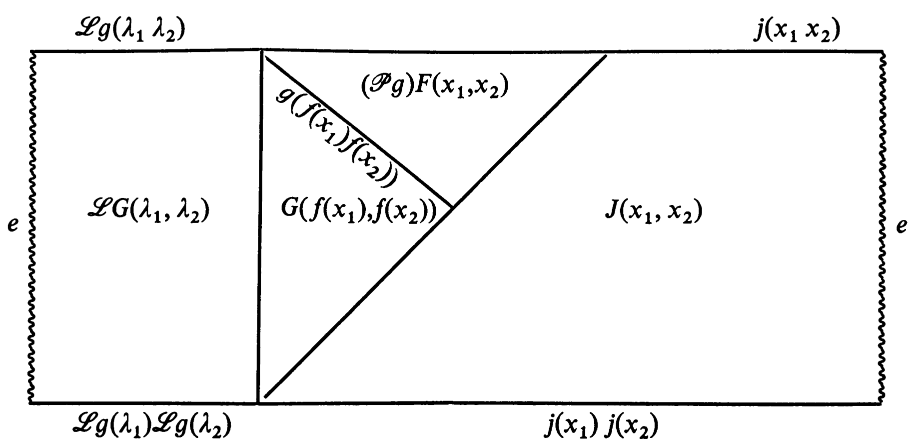
Thus

$$h((x_1,\lambda_1)(x_2,\lambda_2))=\mathcal{L}g(\lambda_1\lambda_2)+(\mathcal{P}g)F(x_1,x_2)-j(x_1x_2)$$

while

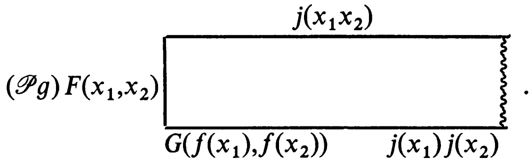
$$h(x_1,\lambda_1)h(x_2,\lambda_2)=\mathcal{L}g(\lambda_1)\mathcal{L}g(\lambda_2)-j(x_1)j(x_2).$$

A sputnik homotopy for h is described by the following diagram, the homotopy parameter running vertically and the loop parameter horizontally:



where the right hand trapezoid can be filled in by J.

As for the converse, consider the space Z induced over Y by h. It is the same as the space induced over A by $\psi : A \to D$ defined by $\psi(x) = (f(x), j(x))$. Since $\pi_i(C) = 0$ for $i > p + q + 1 < 2q + 1$, $[g]$ is a loop class (i.e., in the image of "suspension" $\Omega : H^{+1}(\Omega^{-1}B, C) \to H(B, C)$); thus g is an H-map, and D admits a multiplication. In fact, up to homotopy D admits only one multiplication and g only one homotopy multiplier G. Now, if $(\mathcal{O}, \mathcal{N}, j)$ is additive, h is an H-map and Z admits a multiplication. Since $\pi_i(D) = 0$ for $i < q + 1$ and $i > p + q$, ψ must also be an H-map by Theorem 4. Let Ψ be the homotopy multiplier for ψ. Projecting into $\mathcal{P}(B)$ we obtain a homotopy multiplier F for f. Projecting into $\mathcal{P}(\mathcal{L}C)$ we obtain a map which looks like:



From this, it is easy to derive the required homotopy multiplier J. This completes Theorem 12.

Interpreting our results in cohomology we have:

COROLLARY 13. *A space Z is determined by cochains* $u \in C^{q+1}(G, p; H)$, $v \in C^{r+2}(H, q+1; K)$ *and* $w \in C^{r+1}(G, p; K)$ *such that* $\delta w = v \circ u$. *For Z to admit a multiplication, it is sufficient that there exist cochains*

$$x \in C^q((G,p) \wedge (G,p), H),\ y \in C^{r+1}((H, q+1) \wedge (H, q+1), K)$$

and

$$z \in C^r((G,p) \wedge (G,p), K)$$

such that

(1) $\delta x = \{m^{\#}(u)\}$,
(2) $\delta y = \{n^{\#}(v)\}$,
(3) $\delta z = \{m^{\#}(w)\} - v \circ x - y \circ (u \wedge u)$,

(*where* $\{m^{\#}(u)\}$ *is the component of* $m^{\#}(u)$ *in* $C^{q+1}((G,p) \wedge (G,p), H)$, *etc.*). *The condition is necessary if* $r < p + q$.

The complicated details of this corollary can be summarized by defining, in the category of H-spaces and H-maps, a secondary operation

$$\mathcal{M} : \operatorname{Prim} H(A,B) \cap \operatorname{Ker} \mathcal{N} \to$$
$$H^{-1}(A \wedge A; C)/(\Omega\mathcal{N}) \circ H^{-1}(A \wedge A; B) + \{m^{\#}\} H^{-1}(A; C)$$
$$+ H^{-1}(B \wedge B; C) \circ [u \wedge u].$$

This may perhaps be convenient for theoretical purposes, but it gives us no new information.

4. **Homotopy associative extensions.** Now let us return to the multiplication constructed in Theorem 2. Even if m and n are associative, the multiplication s need not be homotopy associative. To study this problem more thoroughly, we need some terminology.

DEFINITION 14. *A homotopy associative H-space* (X,m,Q) consists of an H-space (X,m) and *associating homotopy* $Q: I \times X^3 \to X$ such that

(0) $Q(0,x,y,z) = x(yz)$,

(1) $Q(1,x,y,z) = (xy)z$,

i.e., the diagram

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{1 \times m} & X \times X \\ {\scriptstyle m \times 1}\downarrow & \simeq & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow[m]{} & X \end{array}$$

is homotopy commutative.

Maps of homotopy associative H-spaces are somewhat complicated to describe. Suppose that (W,n,R) and (Y,m,Q) are homotopy associative H-spaces and that $f:(X,n) \to (W,n)$ is an H-map. Given a sputnik homotopy between $n(f \times f)$ and fm, we can define a homotopy between $R \circ (1 \times f \times f \times f) \mid \dot{I} \times X^3$ and $fQ \mid \dot{I} \times X^3$. We can then ask if this homotopy can be extended to one between $R \circ (1 \times f \times f \times f)$ and fQ.

DEFINITION 15. A map $f: Y \to X$ is a *homotopy associative map* of a homotopy associative H-space (X,m,Q) into a homotopy associative H-space (W,n,R) if

(1) there exists a sputnik homotopy $h_t: X \times X \to W$ such that $h_0(x,x') = f(x)f(x')$ and $h_1(x,x') = f(xx')$ [i.e., f is an H-map], and

(2) there exists a homotopy $d_s: I \times X^3 \to W$, $0 \leqq s \leqq 2$, such that

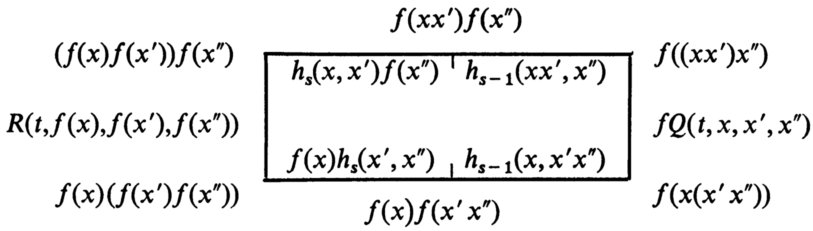
(a) $$d_0(t,x,x',x'') = R(t,f(x),f(x'),f(x'')),$$

(b) $$d_2(t,x,x',x'') = fQ(t,x,x',x''),$$

(c) $$\begin{aligned} d_s(0,x,x',x'') &= f(x)h_s(x',x''), && 0 \leqq s \leqq 1, \\ &= h_{s-1}(x,x',x''), && 1 \leqq s \leqq 2, \end{aligned}$$

(d) $$\begin{aligned} d_s(1,x,x',x'') &= h_s(x,x')f(x''), && 0 \leqq s \leqq 1, \\ &= h_{s-1}(xx',x''), && 1 \leqq s \leqq 2. \end{aligned}$$

Perhaps a picture would be helpful. Corresponding to d_s, we have a map of $[0,2] \times I$ into W^{X^3} which for each point (x,x',x'') can be depicted as follows:



THEOREM 16. *Let (X,m,Q) and (W,n,R) be homotopy associative H-spaces. Let Y be as in Theorem 2. If $f: X \to W$ is a homotopy associative map, then there exists a homotopy associative multiplication on Y.*

Proof. Let f be a homotopy associative map with h_t and d_s as in Definition 15. Let $s: Y \times Y \to Y$ be constructed using h_t as in Theorem 2. It is easy to see that $s(1 \times s)$ can be deformed to $\phi_0: Y^3 \to Y$ given by

$$\phi_0(y,y',y'') = (x(x'x''), \lambda(\lambda'\lambda'') + f(x) \cdot F(x',x'') + F(x,x'x''))$$

for $y = (x,\lambda)$, $y' = (x',\lambda')$, $y'' = (x'',\lambda'')$, while $s(s \times 1)$ can be deformed to

$$\phi_1(y,y',y'') = ((xx')x'', (\lambda\lambda')\lambda'' + F(x,x') \cdot f(x'') + F(xx',x'')).$$

Now corresponding to d_s we construct a map $D: I \times X^3 \to \mathscr{P}(W)$ in the usual way. An associating homotopy for Y is given by means of

$$\phi_t(y,y'y'') = (Q(t,x,x',x''), \mathscr{L}R(t,\lambda,\lambda',\lambda'') + D(t,x,x',x'')).$$

As for H-maps, a converse theorem can be obtained in restricted situations. Since the same methods are used as for Theorem 3, we state only the results.

THEOREM 17. *Let Y be as in Theorem 2. Suppose that f is an H-map with sputnik homotopy h_t and that $s: Y \times Y \to Y$ is obtained from h_t as in Theorem 2. If there is an associating homotopy $S: I \times Y^3 \to Y$ for s such that* $\mathrm{p}S(t,y_1,y_2,y_3) = Q(t,\mathrm{p}(y_1)\mathrm{p}(y_2),\mathrm{p}(y_3))$, *then f is a homotopy associative map (with this h_t as sputnik homotopy) under the following conditions: X is $(p-1)$-connected and for some $q > p$, $\pi_i(W) = 0$ for $i < q$ and $i > 2p+q$.*

THEOREM 18. *Let Y be a space with only two nonvanishing homotopy groups $\pi_p(Y) = \pi$ and $\pi_q(Y) = G$, $q > p$. Y admits a homotopy associative multiplication if and only if the k-invariant $k \in H^{q+1}(\pi,p;G)$ is represented by some homotopy associative map $f: K(\pi,p) \to K(G,q+1)$.*

APPLICATION 19. *Let α be a generator of $H^{n+1}(Z_p,n;Z)$, n odd, p an odd prime. The space with k-invariant α^p admits a multiplication but not one which is homotopy associative.*

That α^p is primitive is easy since $m^*(\alpha^p) = (m^*(\alpha))^p = (p_1^*(\alpha) + p_2^*(\alpha))^p = \sum(i_1 p - i)\ p_1^*(\alpha^i) \otimes p_2^*(\alpha^{p-i}) = p_1^*(\alpha^p) + p_2^*(\alpha^p)$ because α is of order p and p divides $(i, p - i)$ for $1 < i < p$. In [8] we perform the calculations necessary to show that α^p is not represented by a homotopy associative map. This requires some work, though basically it is a straightforward problem in obstruction theory. For $p = 3$, it is a consequence of Moore's statement [6] that α^p is not in the image of "suspension"

$$\sigma: H^{p(n+1)+1}(Z_p, n + 1; Z) \to H^{p(n+1)}(Z_p, n; Z).$$

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