QUADRATIC DIVISION ALGEBRAS(1)

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In this paper we shall investigate the structure of quadratic division algebras over an arbitrary field of characteristic not two. It is shown first that a quadratic algebra may be decomposed into a copy of the field and a skew-commutative algebra with a bilinear form. The standard theory of quadratic forms then rules out the existence of quadratic division algebras over some fields and imposes limitations on its structure over others. It is proved that no quadratic division algebra of order 3 exists over any field, and all quadratic division algebras of order 4 over an arbitrary field F are found in terms of the structure of the quadratic forms over F. If D is a finitely generated quadratic division algebra in which every two elements not in the same subalgebra of order 2 generate a subalgebra of order 4, it is shown that D has order 2^n, and the multiplication table of the skew-commutative algebra associated with such an algebra of order 8 is determined in terms of eight parameters. This gives a new class of division algebras of order 8 over any (formally) real field, and shows that any quadratic division algebra of order 4 over a real closed field may be embedded in a quadratic division algebra of order 8.

1. Let A be a (possibly infinite dimensional) algebra over a field F of characteristic not two, and let A have an identity element 1. Then A shall be called a a *quadratic* algebra if $1, a, a^2$ are linearly dependent over F for every a in A. We shall find it convenient to identify F with the subalgebra $F1$, and, hence, to replace the phrase "scalar multiple of the identity element of A" simply with the word "scalar." If an element x of A squares to a scalar but is not itself a scalar, we shall call it a *vector*. It follows immediately from these definitions that every element of a quadratic algebra A is uniquely expressible as the sum of a vector and a scalar. That the set of all vectors of A forms a subspace V complementary to F, follows from the following lemma due to L. E. Dickson [3].

LEMMA 1. *In a quadratic algebra, the sum of two vectors is also a vector. Equivalently, for any two vectors, x, y, the quantity $xy + yx$ is a scalar.*

Now, for any $x, y \in A$, let (x, y) denote the scalar component of the product xy. Since (x, y) is linear in both arguments, it is a bilinear form. In general, this form is not symmetric, and, in fact, does not satisfy the property that $(x, y) = 0$ implies

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$(y,x) = 0$. However, the modified bilinear form $[x,y] = ((x,y) + (y,x))/2 = (xy + yx)/2$ is symmetric, and when we speak of two elements or subspaces of A as being orthogonal, we shall have this latter form in mind. For example, we may characterize the subspace V of vectors as the orthogonal complement of F. Using these definitions one can easily prove

LEMMA 2. *The following three properties on A are equivalent*:

(i) *A contains no nilpotent elements.*

(ii) *For every nonzero vector $x \in A$, $x^2 \neq 0$.*

(iii) *For every nonzero vector $x \in A$, $(x,x) \neq 0$.*

Furthermore, these properties imply:

(iv) *The bilinear form $[x,y]$ is nonsingular on V (and hence on A).*

Restricting our attention to V, let us define the product "$\times$" by $x \times y = xy - (x,y)$ for any $x, y \in V$. Then V is closed under this product, and $x \times y + y \times x = xy + yx - (x,y) - (y,x) \in F \cap V = 0$, so that $y \times x = -x \times y$. If $\alpha + x$ and $\beta + y$ are any two elements of A ($\alpha, \beta \in F$; $x, y \in V$), then $(\alpha + x)(\beta + y) = \alpha\beta + \alpha y + \beta x + (x,y) + x \times y = [\alpha\beta + (x,y)] + [\alpha y + \beta x + x \times y]$, where the first bracket is in F and the second in V. From these remarks, it is trivial to prove

THEOREM 1. *Let V be a skew-commutative algebra (whose multiplication is denoted by "$\times$") over a field F of characteristic not two, let (x,y) be a bilinear form from V and V to F, and let A be the set of all formal sums $\alpha + x$ ($\alpha \in F, x \in V$) with addition defined by $(\alpha + x) + (\beta + y) = (\alpha + \beta) + (x + y)$, scalar multiplication defined by $\beta(\alpha + x) = \beta\alpha + \beta x$, and multiplication defined by $(\alpha + x)(\beta + y) = [\alpha\beta + (x,y)] + [\alpha y + \beta x + x \times y]$. Then A is a quadratic algebra over F. Conversely, every quadratic algebra over F arises in this manner.*

This theorem generalizes a well-known relation between the quaternions and the 3-dimensional space of real vectors under cross product and inner product (except that we have chosen to change the sign of the inner product in this more general context). Using this theorem, questions about quadratic algebras may be reduced to questions about bilinear forms and skew-symmetric algebras. For example, it is easy to show that the mapping $\alpha + x \to \alpha - x$ ($\alpha \in F$, $x \in V$) is an involution of A if and only if the bilinear form (x,y) is symmetric, and that A satisfies the flexible law if and only if the bilinear form (x,y) is symmetric and $(x, x \times y) = 0$ for all x, y in V.

2. We are now ready to study what properties characterize a quadratic division algebra. We begin with

THEOREM 2. *Let A be a quadratic algebra, V its subspace of vectors, let $u_1, u_2, \ldots,$ be an orthogonal basis of V under the bilinear form $[x,y]$, and let $u_i^2 = \alpha_i$ for $i = 1, 2, \ldots$. Then*

(i) *A has no nilpotent elements if and only if the quadratic form $\sum \alpha_i \lambda_i^2$ does not represent zero over F.*

(ii) *Every nonzero element of A generates a subalgebra which is a field if and only if the quadratic form $\sum \alpha_i \lambda_i^2 - \lambda^2$ does not represent zero over F.*

Part (i) of this theorem follows from the remark that $\sum \alpha_i \lambda_i^2$ is just the square of the element $x = \sum \lambda_i u_i$, since $u_i u_j + u_j u_i = 2[u_i, u_j] = 0$ for $i \neq j$. If, furthermore, A has the property that every nonzero element generates a subalgebra without zero divisors, then no vector x can have the property that x^2 is a square in F, since $x^2 = \beta^2$ implies that $(\beta + x)(\beta - x) = 0$. Conversely, if x^2 is not a square in F, then $F[x]$ will be a field. These remarks establish Part (ii).

In order for A to be a division algebra, the skew-commutative algebra V must satisfy another condition in addition to the condition on its bilinear form. Specifically, we shall prove

THEOREM 3. *Let A be a quadratic algebra with the property that the subalgebra generated by any nonzero element is a field. Then the following statements are equivalent:*

(i) *A has no zero divisors.*

(ii) *A contains no subalgebras of order* 3.

(iii) *For any two linearly independent vectors x, y of A, the vectors x, y, $x \times y$ are linearly independent.*

(iv) *There do not exist two linearly independent vectors x, y of A such that $x \times y = x$ or $x \times y = 0$.*

To prove this theorem, we shall establish the implications (i) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) by showing that the negatives of these statements imply each other in the reverse order. First of all, suppose that $0 = (\alpha + x)(\beta + y) = [\alpha\beta + (x, y)] + [\alpha y + \beta x + x \times y]$ for α, $\beta \in F$ and x, $y \in V$. Then $\alpha y + \beta x + x \times y = 0$, and $x, y, x \times y$ are linearly dependent. On the other hand, x and y are independent, since otherwise there would exist a field containing both $\alpha + x$ and $\beta + y$ by hypothesis. Secondly, if x, y are independent, $x, y, x \times y$ dependent, then xy is in the subspace B spanned by $1, x, y$, and B is a subalgebra of order 3.

Thirdly, if $1, x, y$ span a subalgebra B of order 3, then the cross product of any two vectors in B is a multiple of $x \times y$, since $(\alpha x + \beta y) \times (\gamma x + \delta y) = (\alpha\delta - \beta\gamma)$ $x \times y$ for any $\alpha, \beta, \gamma, \delta \in F$. Thus, either $x \times y = 0$, or we may set $x' = x \times y$, choose y' to be any vector of B independent from x', and have $x' \times y' = vx'$ for some nonzero scalar v. But then, setting $y'' = v^{-1} y'$ gives $x' \times y'' = x'$.

And finally, $x \times y = x$ for two vectors x and y leads to

$$x\left[1 + \frac{(x, y)}{x^2} x - y\right] = x + (x, y) - (x, y) - x \times y = 0,$$

and $x \times y = 0$ leads to

$$x[(x,y)x - x^2y] = (x,y)x^2 - x^2(x,y) = 0.$$

We restate part of Theorem 3 as the following

COROLLARY. *There do not exist quadratic division algebras of order* 3 *over any field of characteristic not two.*

Taking now a skew-commutative algebra V with a bilinear form on it, we see that the condition that the associated quadratic algebra A be a division algebra completely reduces to two unrelated conditions, one involving only the bilinear form, and the other involving only the skew-commutative algebra V. We may then show that no quadratic division algebra of order n exists over F either by showing that every quadratic form in n variables over F represents zero, or by showing that no skew-commutative algebra of order $n-1$ over F satisfies (iii) of Theorem 3. For example, it follows trivially from standard results in the theory of quadratic forms (cf. [5]) that a quadratic division algebra over an algebraic number field which is not real, or over a p-adic number field, must have order 1, 2, or 4. On the other hand, if we are given any quadratic form in n variables over F which does not represent zero (we may assume that it is in the form $\sum \alpha_i\lambda_i^2 - \lambda^2$), and any skew-commutative algebra V of order $n-1$ over F, we may easily define a bilinear form on V which induces the given quadratic form, and the quadratic algebra made from V using Theorem 1 will be a division algebra.

We may also make new quadratic division algebras out of well-known ones by changing the bilinear form. For example, if we change the usual bilinear form on the quaternions by adding to it any skew-symmetric form defined on the set of vectors, the corresponding quadratic form will be unchanged, so that the algebra will still be a division algebra. However, the modified algebra will not satisfy the flexible law nor will the mapping $\alpha + x \to \alpha - x$ be an involution.

Let us call a quadratic algebra A *homogeneous* if any two nonscalars of A generate isomorphic subalgebras. We prove next

THEOREM 4. *Let A be a homogeneous quadratic algebra without nilpotent elements over a field F of characteristic $p \neq 2$. Then A has order* 1, 2, *or* 3.

Suppose, to the contrary, that A has order $\geqq 4$ and hence contains three mutually orthogonal vectors u_1, u_2, u_3. Since A is homogeneous, we may replace u_2 and u_3 by appropriate scalar multiples of themselves so that $u_1^2 = u_2^2 = u_3^2$. Letting $x = \lambda_1u_1 + \lambda_2u_2 + \lambda_3u_3$, we have

$$x^2 = \lambda_1^2u^2 + \lambda_2^2u_2^2 + \lambda_3^2u_3^2 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)u_1^2,$$

and this must be nonzero for any choice of $\lambda_1, \lambda_2, \lambda_3 \in F$. But, the quadratic form $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$ represents zero over any field of characteristic p, giving the desired contradiction.

Since there are no quadratic division algebras of order 3, we also have the following

COROLLARY. *Let A be a homogeneous quadratic division algebra over a field of characteristic $p \neq 2$. Then A has order* 1 *or* 2, *and hence is a field.*

3. We proceed next to the problem of determining all quadratic division algebras of order 4 over an arbitrary field F of characteristic not two. We shall solve this problem completely modulo the theory of quadratic forms over F. That is to say, the solution of each part of the problem will be reduced to the solution of a standard problem in the theory of quadratic forms over F.

Using Theorems 1, 2, and 3, the problem breaks into the two distinct problems of finding all bilinear forms $\sum \alpha_{ij}\lambda_i\lambda_j$ in three variables such that $\sum \alpha_{ij}\lambda_i\lambda_j - \lambda^2$ does not represent zero over F, and of finding all skew-commutative algebras of order 3 over F that satisfy (iii) of Theorem 3. Concerning the first of these, we consider that the problem of finding all *symmetric* bilinear forms $\sum \alpha_{ij}\lambda_i\lambda_j$ such that $\sum \alpha_{ij}\lambda_i\lambda_j - \lambda^2$ does not represent zero, belongs to the theory of quadratic forms over F. The nonsymmetric bilinear forms with this property are then just the sum of a symmetric form with this property and an arbitrary skew-symmetric bilinear form.

There remains the problem of finding all skew-commutative algebras over F satisfying (iii) of Theorem 3. For convenience, we shall call a skew-commutative algebra *division-like* if it satisfies (iii) (or (iv)) of Theorem 3. Let V be a skew-commutative algebra of order 3, and let x, y, z be a basis of V. Then for some constants a_{ij} $(1 \leqq i, j \leqq 3)$, we have

$$\begin{aligned} y \times z &= a_{11}x + a_{12}y + a_{13}z, \\ z \times x &= a_{21}x + a_{22}y + a_{23}z, \\ x \times y &= a_{31}x + a_{32}y + a_{33}z, \end{aligned} \tag{1}$$

and these equations completely determine the multiplication in V. We shall call $\mathbf{A} = \| a_{ij} \|$ the matrix associated with the basis x, y, z of V. Defining the column matrices

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \mathbf{W} = \begin{bmatrix} y \times z \\ z \times x \\ x \times y \end{bmatrix},$$

we may express the equations (1) in matrix notation as $\mathbf{W} = \mathbf{AX}$. If $\mathbf{X}'$ is a second basis of V, there exists a nonsingular matrix $\mathbf{B} = \| b_{ij} \|$ such that $\mathbf{X} = \mathbf{BX}'$. Also, if $\mathbf{A}'$ is the matrix associated with $\mathbf{X}'$, and if

$$\mathbf{W}' = \begin{bmatrix} y' \times z' \\ z' \times x' \\ x' \times z' \end{bmatrix},$$

then $\mathbf{W}' = \mathbf{A}'\mathbf{X}'$. For the relationship between $\mathbf{W}$ and $\mathbf{W}'$, we have

$$\mathbf{W} = \begin{bmatrix} y \times z \\ z \times x \\ x \times y \end{bmatrix} = \begin{bmatrix} (b_{21}x' + b_{22}y' + b_{23}z') \times (b_{31}x' + b_{32}y' + b_{33}z') \\ (b_{31}x' + b_{32}y' + b_{33}z') \times (b_{11}x' + b_{12}y' + b_{13}z') \\ (b_{11}x' + b_{12}y' + b_{13}z') \times (b_{21}x' + b_{22}y' + b_{23}z') \end{bmatrix}$$

$$= \begin{bmatrix} (b_{22}b_{33} - b_{23}b_{32})y' \times z' + (b_{23}b_{31} - b_{21}b_{33})z' \times x' \\ (b_{32}b_{13} - b_{33}b_{12})y' \times z' + (b_{33}b_{11} - b_{31}b_{13})z' \times x' \\ (b_{12}b_{23} - b_{13}b_{22})y' \times z' + (b_{13}b_{21} - b_{11}b_{23})z' \times x' \end{bmatrix}$$

$$\begin{matrix} + (b_{21}b_{32} - b_{22}b_{31})x' \times y' \\ + (b_{31}b_{12} - b_{32}b_{11})x' \times y' \\ + (b_{11}b_{22} - b_{12}b_{21})x' \times y' \end{matrix} \Bigg] = (\text{adj. } \mathbf{B})^T\mathbf{W}'.$$

Substituting $\mathbf{W} = \mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{B}\mathbf{X}'$ and $\mathbf{W}' = \mathbf{A}'\mathbf{X}'$ in this equation gives $\mathbf{A}\mathbf{B}\mathbf{X}' = (\text{adj. } \mathbf{B})^T\mathbf{A}'\mathbf{X}'$, and multiplying on the left by $\mathbf{B}^T$ gives $\mathbf{B}^T\mathbf{A}\mathbf{B}\mathbf{X}' = \mathbf{B}^T(\text{adj. } \mathbf{B})^T\mathbf{A}'\mathbf{X}' = |\mathbf{B}|\mathbf{A}'\mathbf{X}'$. We have proved

LEMMA 3. *Let* $\mathbf{X}$ *and* $\mathbf{X}'$ *be two bases of a skew-commutative algebra* V *of order* 3, *let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be the matrices associated with these bases, and let the matrix* $\mathbf{B}$ *be defined by* $\mathbf{X} = \mathbf{B}\mathbf{X}'$. *Then* $\mathbf{A}$ *and* $\mathbf{A}'$ *are related by* $\mathbf{A}' = |\mathbf{B}|^{-1}\mathbf{B}^T\mathbf{A}\mathbf{B}$.

If c is any element of F and $\mathbf{X}$ any basis of V, let $\mathbf{X}' = c\mathbf{X}$, and we may conclude from Lemma 3 that $\mathbf{A}' = |c^{-1}\mathbf{I}|^{-1}c^{-1}\mathbf{I}\cdot\mathbf{A}\cdot c^{-1}\mathbf{I} = c\mathbf{A}$. Thus a change of basis exists which just multiplies $\mathbf{A}$ by a given scalar. From this remark and from Lemma 3, it is clear that we can also change $\mathbf{A}$ into any matrix congruent to it. If $\mathbf{A}$ is symmetric but not zero, it is clear that we may use these two operations to put $\mathbf{A}$ in the form

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{vmatrix}$$

for some scalars α and γ. If we restrict ourselves to division-like algebras, α and γ will both be nonzero.

If $\mathbf{A}$ is nonsymmetric, it may be expressed as the sum of a symmetric matrix $\mathbf{A}_1$ and a skew-symmetric matrix $\mathbf{A}_2$ of rank 2 (since the rank of a skew-symmetric matrix is always even). Letting $\mathbf{B}$ be any nonsingular matrix such that

$$\mathbf{B}^T\mathbf{A}_2\mathbf{B} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix},$$

we see that $\mathbf{B}^T\mathbf{A}\mathbf{B} = \mathbf{B}^T\mathbf{A}_1\mathbf{B} + \mathbf{B}^T\mathbf{A}_2\mathbf{B}$ has the property that its first row is the same as its first column. Hence, we may assume that $a_{12} = a_{21}$ and $a_{13} = a_{31}$.

If V is division-like, we also know that the first row of $\mathbf{A}$ is nonzero. Then, using elementary operations on the first row and column of $\mathbf{A}$, followed by an appropriate scalar multiplication, we may assume that $a_{11}=1, a_{12}=a_{13}=a_{21}=a_{31}=0$. Since $a_{22}=0$ implies that $z \times x = a_{23}z$, which cannot happen in a division-like algebra, we may subtract an appropriate multiple of the second row from the third (and the same for the columns) to make $a_{32}=0$. The matrix $\mathbf{A}$—whether symmetric or nonsymmetric—may thus be put in the form

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \gamma \end{vmatrix}. \tag{2}$$

When $\mathbf{A}$ is nonsymmetric, $\beta \neq 0$, and an appropriate multiplication of the second row and column makes $\beta = 1$. We have seen that $\alpha \neq 0$, and a similar argument also shows that $\gamma \neq 0$.

Although the reduction of $\mathbf{A}$ effected above is the best that can be done over an arbitrary field, we can do better over a real closed field. Starting with $\mathbf{A}$ in the form (2) we may multiply the second and third rows and columns by appropriate elements to make α and γ equal to ± 1. If $\alpha = -1$, then $z \times x = -y + \beta z$ and $y \times z = x$, leading to $(x+y) \times z = (x+y) - \beta z$, which cannot happen if V is division-like. Hence $\alpha = 1$, and similarly $\gamma = 1$. And finally, if β is negative, we may multiply the second row and column of $\mathbf{A}$ by -1 to make it positive. We have proved the first statement from each of the following two theorems:

THEOREM 5. *In any skew-commutative division-like algebra V of order* 3 *over a field F of characteristic not two, there exists a basis x, y, z such that*

$$y \times z = x, \qquad z \times x = \alpha y + \beta z, \qquad x \times y = \gamma z \tag{3}$$

where $\beta = 0$ or 1, *and where $\alpha, \gamma \in F$. Conversely, if the basis elements of a skew-commutative algebra V multiply as in* (3), *then V is division-like if and only if the quadratic form $\lambda_1^2 + \beta\lambda_1\lambda_2 + \alpha\gamma\lambda_2^2 + \alpha\lambda_3^2 + \beta\lambda_3\lambda_4 + \gamma\lambda_4^2$ does not represent zero over F. A skew-commutative algebra whose basis elements multiply as in* (3) *with $\beta = 1$ is never isomorphic to an algebra whose basis elements multiply as in* (3) *with $\beta = 0$, and it is isomorphic to an algebra whose basis elements multiply as in* (3) *with $\beta = 1$ if and only if the new α and γ may be expressed in terms of the old as $\alpha\eta$ and $\gamma\eta^{-1}$ respectively where η is a nonzero element of F represented by the form $\lambda_1^2 + \lambda_1\lambda_2 + \alpha\gamma\lambda_2^2$ over F.*

THEOREM 6. *In any skew-commutative division-like algebra V of order* 3 *over a real closed field F, there exists a basis x, y, z such that*

$$y \times z = x, \qquad z \times x = y + \beta z, \qquad x \times y = z \tag{4}$$

where $\beta \geqq 0$. Two skew-commutative algebras whose basis elements multiply

as in (4) *are isomorphic if and only if they have the same* β*'s. A skew-commutative algebra whose basis elements multiply as in* (4) *is division-like if and only if* $|\beta| < 2$.

To prove the rest of these two theorems, we consider first under what conditions the algebra whose multiplication is given by (3) is division-like. Suppose that $0 = (ax + by + cz) \times (dx + ey + fz) = (bf - ce)y \times z + (cd - af)z \times x + (ae - bd)x \times y = (bf - ce)x + (cd - af)\alpha y + [(cd - af)\beta + (ae - bd)\gamma]z$ for some choice of $a, b, c, d, e, f \in F$. But this implies the relations $bf - ce = 0$, $cd - af = 0$, and $ae - bd = 0$, which easily imply that $ax + by + cz$ and $dx + ey + fz$ are linearly dependent. Hence, using (iv) of Theorem 3, V will be division-like if and only if the relation $(ax + by + cz) \times (dx + ey + fz) = ax + by + cz \neq 0$ holds for some $a, b, c, d, e, f \in F$. Multiplying out the left side of this relation and equating the coefficients of x, y, z gives

$$(5) \qquad bf - ce = a, \qquad (cd - af)\alpha = b, \qquad (cd - af)\beta + (ae - bd)\gamma = c.$$

Substituting the first equation of (5) into the second gives $(cd - bf^2 + cef)\alpha = b$, or $b = c(d + ef)(\alpha^{-1} + f^2)^{-1}$ and substituting the second into the first gives $(cdf - af^2)\alpha - ce = a$, or $a = c(df - \alpha^{-1}e)(\alpha^{-1} + f^2)^{-1}$. Then, substituting these expressions for a and b into the third equation of (5) and multiplying by $\alpha^{-1} + f^2$ yields

$$[cd(\alpha^{-1} + f^2) - cf(df - \alpha^{-1}e)]\beta + [ce(df - \alpha^{-1}e) - cd(d + ef)]\gamma = c(\alpha^{-1} + f^2),$$

which simplifies to

$$0 = c[(1 + \alpha f^2) + (e^2 + \alpha d^2)\gamma - (d + ef)\beta].$$

Since $c = 0$ clearly implies that a and b are zero, this last equation may be reduced to

$$(6) \qquad 1 - \beta d + \alpha\gamma d^2 + \alpha f^2 - \beta ef + \gamma e^2 = 0.$$

Conversely, if d, e, f exist satisfying (6), then we may easily find a, b, c not all zero satisfying (5), so that the existence of a solution of (5) is equivalent to the existence of a solution of (6).

But if d, e, f exist satisfying (6), then letting $\lambda_1 = 1$, $\lambda_2 = -d$, $\lambda_3 = f$, $\lambda_4 = -e$ gives a representation of zero by the quadratic form $\lambda_1^2 + \beta\lambda_1\lambda_2 + \alpha\gamma\lambda_2^2 + \alpha\lambda_3^2 + \beta\lambda_3\lambda_4 + \gamma\lambda_4^2$. Conversely, if this quadratic form represents zero with $\lambda_1 \neq 0$, we can clearly get a solution of (6). If the form represents zero with $\lambda_1 = 0$ and $\lambda_2 \neq 0$, we may obtain a solution of (6) by setting $d = 0$, $e = -\gamma\lambda_3\lambda_2^{-1}$, and $f = \alpha\lambda_4\lambda_2^{-1}$. And finally, if the form represents zero with $\lambda_1 = \lambda_2 = 0$, we may set $e = f = 0$ and $d = -\gamma\lambda_3\lambda_4^{-1}$ to get a solution of (6). This proves the second sentence of Theorem 5, and the last sentence in Theorem 6 follows easily by setting $\alpha = \gamma = 1$ and using the fact that $\lambda_1^2 + \beta\lambda_1\lambda_2 + \lambda_2^2$ represents only posi-

tive numbers over a real closed field if $|\beta| < 2$, and that it represents zero if $|\beta| \geqq 2$.

To discuss the question of when two skew-commutative algebras of the type given by (3) are isomorphic, we shall return to a consideration of the matrices (2) associated with these two algebras. If $\beta = 0$ for one of the algebras but not for the other, then one matrix is symmetric and the other nonsymmetric, and they cannot be related by congruence and scalar multiplication, and hence the two algebras could not be isomorphic. If $\beta = 0$ for both algebras, the isomorphism problem reduces to the classical problem of when two diagonal matrices are congruent over F, which we shall not treat. We may thus assume that $\beta = 1$ for both algebras.

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \gamma \end{bmatrix}, \qquad \mathbf{A}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha' & 1 \\ 0 & 0 & \gamma' \end{bmatrix}.$$

Then our problem is to determine when there exists a nonsingular matrix $\mathbf{B} = \| b_{ij} \|$ such that $\mathbf{B}^T\mathbf{A}\mathbf{B} = |\mathbf{B}|\mathbf{A}'$. By computation,

$$\mathbf{B}^T\mathbf{A}\mathbf{B} = \left| \begin{matrix} b_{11}^2 + \alpha b_{21}^2 + \gamma b_{31}^2 + b_{21}b_{31} & b_{11}b_{12} + \alpha b_{21}b_{22} + \gamma b_{31}b_{32} + b_{21}b_{32} & b_{11}b_{13} + \alpha b_{21}b_{23} + \gamma b_{31}b_{33} + b_{21}b_{33} \\ b_{11}b_{12} + \alpha b_{21}b_{22} + \gamma b_{31}b_{32} + b_{22}b_{31} & b_{12}^2 + \alpha b_{22}^2 + \gamma b_{32}^2 + b_{22}b_{32} & b_{12}b_{13} + \alpha b_{22}b_{23} + \gamma b_{32}b_{33} + b_{22}b_{33} \\ b_{11}b_{13} + \alpha b_{21}b_{23} + \gamma b_{31}b_{33} + b_{23}b_{31} & b_{12}b_{13} + \alpha b_{22}b_{23} + \gamma b_{32}b_{33} + b_{23}b_{32} & b_{13}^2 + \alpha b_{23}^2 + \gamma b_{33}^2 + b_{23}b_{33} \end{matrix} \right|,$$

and taking the difference between corresponding off-diagonal elements of this matrix and of $|\mathbf{B}|\mathbf{A}'$, and equating, gives

$$\begin{gathered} b_{21}b_{32} - b_{22}b_{31} = 0, \qquad b_{21}b_{33} - b_{23}b_{31} = 0, \\ b_{22}b_{33} - b_{23}b_{32} = |\mathbf{B}|. \end{gathered} \tag{7}$$

If $b_{21} \neq 0$, the first two of these equations give $b_{32} = b_{22}b_{31}b_{21}^{-1}$ and $b_{33} = b_{23}\, b_{31}b_{21}^{-1}$, and substituting these into the third equation of (7) gives

$$b_{22}b_{23}b_{31}b_{21}^{-1} - b_{23}b_{22}b_{31}b_{21}^{-1} = |\mathbf{B}| = 0.$$

But $\mathbf{B}$ was assumed nonsingular, implying that $b_{21} = 0$ and $b_{31} = 0$ (by symmetry). A direct computation of $|\mathbf{B}|$ now yields $|\mathbf{B}| = b_{11}(b_{22}b_{33} - b_{23}b_{32})$, and comparing with the last equation of (7) gives $b_{11} = 1$. Equating correspon-

ding elements in the first column of $\mathbf{B}^T\mathbf{AB}$ and $|\mathbf{B}|\mathbf{A}'$ now gives $1 = |\mathbf{B}|$, $b_{12} = 0$, and $b_{13} = 0$. And finally, equating the remaining elements of $\mathbf{B}^T\mathbf{AB}$ and $|\mathbf{B}|\mathbf{A}'$ gives the relations

$$\begin{aligned} &\alpha b_{22}b_{23} + \gamma b_{32}b_{33} + b_{23}b_{32} = 0, && b_{22}b_{33} - b_{23}b_{32} = 1, \\ &\alpha b_{22}^2 + \gamma b_{32}^2 + b_{22}b_{32} = \alpha', && \alpha b_{23}^2 + \gamma b_{33}^2 + b_{23}b_{33} = \gamma'. \end{aligned} \tag{8}$$

From $|\mathbf{B}| = 1$, we have $|\mathbf{A}| = |\mathbf{A}'|$, or $\alpha\gamma = \alpha'\gamma'$, which says that there exists $\eta \in F$ such that $\alpha' = \alpha\eta$ and $\gamma' = \gamma\eta^{-1}$. We would like to know which η may arise in this manner from the effect of some matrix $\mathbf{B}$. First of all, if $b_{22} = 0$, then $|\mathbf{B}| \neq 0$ implies that $b_{32} \neq 0$, and $b_{23} \neq 0$, so that the first two equations of (8) reduce to $b_{33} = -\gamma^{-1}b_{23}$ and $b_{32} = -b_{23}^{-1}$, which leads to $\eta = \gamma\alpha^{-1}b_{23}^{-2} = \alpha\gamma(\alpha^{-1}b_{23}^{-1})^2$. Similarly, $b_{33} = 0$ gives $\eta = \alpha\gamma(\alpha^{-1}b_{32})^2$. Also, using $|\mathbf{B}| \neq 0$, we may readily deduce from the first equation of (8) that $b_{23} = 0$ if and only if $b_{32} = 0$. But, if both of these are zero, (8) yields $b_{22}b_{33} = 1, \alpha b_{22}^2 = \alpha', \gamma b_{33}^2 = \gamma'$, which gives $\eta = b_{22}^2$. In each of these cases, η is trivially of the form $\lambda_1^2 + \lambda_1\lambda_2 + \alpha\gamma\lambda_2^2$.

If b_{22}, b_{23}, b_{32}, b_{33} are all nonzero, we multiply the first equation of (8) by b_{22}, the third by b_{23}, and subtract, to get $\alpha' b_{23} = \gamma b_{32}(b_{23}b_{32} - b_{22}b_{33}) = -\gamma b_{32}$. This gives $b_{23} = -\gamma b_{32}(\alpha')^{-1} = -\gamma b_{32}(\alpha\eta)^{-1} = -\gamma\alpha^{-1}\eta^{-1}b_{32}$, and allows the second equation of (8) to be written in the form $b_{22} = (1 + b_{23}b_{32})b_{33}^{-1} = (1 - \gamma\alpha^{-1}\eta^{-1}b_{32}^2)b_{33}^{-1}$. Substituting this into the first equation of (8) gives

$$\alpha(1 - \gamma\alpha^{-1}\eta^{-1}b_{32}^2)b_{33}^{-1}b_{23} + \gamma b_{32}b_{33} + b_{23}b_{32} = 0,$$

and multiplying by $\eta\alpha^{-1}b_{33}b_{23}^{-1}$ gives

$$\eta - \gamma\alpha^{-1}b_{32}^2 + \gamma\eta\alpha^{-1}b_{32}b_{23}^{-1}b_{33}^2 + \eta\alpha^{-1}b_{32}b_{33} = 0.$$

Thus,

$$\begin{aligned} \eta &= \gamma\alpha^{-1}b_{32}^2 - \eta\alpha^{-1}b_{32}b_{33} - \gamma\eta\alpha^{-1}b_{32}(-\gamma^{-1}\alpha\eta b_{32})b_{33}^2 \\ &= \gamma\alpha(\alpha^{-1}b_{32})^2 + (\alpha^{-1}b_{32})(-\eta b_{33}) + (-\eta b_{33})^2, \end{aligned}$$

and η is of the form $\lambda_1^2 + \lambda_1\lambda_2 + \alpha\gamma\lambda_2^2$ in this case also.

Conversely, if $\eta = \lambda_1^2 + \lambda_1\lambda_2 + \alpha\gamma\lambda_2^2$, it is easy to check that η is induced by the matrix B whose components are $b_{22} = -(\lambda_1 + \lambda_2)$, $b_{23} = -\gamma\eta^{-1}\lambda_2$, $b_{32} = \alpha\lambda_2$, and $b_{33} = -\eta^{-1}\lambda_1$. This completes the proof of Theorem 5.

To finish the proof of Theorem 6, suppose that we are given two matrices of the form (2) with $\alpha = \gamma = 1$ for both, and with different positive β's. Then we may multiply the second row and column of these matrices by β^{-1} and $(\beta')^{-1}$ respectively to get

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta^{-2} & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{A}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\beta')^{-2} & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

But we have deduced above that the two algebras with these associated matrices may only be isomorphic if $|\mathbf{A}| = |\mathbf{A}'|$, which implies $\beta^{-2} = (\beta')^{-2}$, or $\beta = \beta'$.

4. The problem of determining all quadratic division algebras of order n seems to be much harder to deal with for $n \geqq 5$ than for $n = 4$. In fact, for $n \geqq 5$, $n \neq 8$, we do not even know if there are any fields over which quadratic division algebras of order n exist. Since Lemma 3 has no analogue for any other order, the approach of the last section does not work for $n \geqq 5$.

However, there is a generalization of quadratic division algebras of order 4 which leads in what is probably a more interesting direction—namely, quadratic algebras with the property that any two elements not in the same subalgebra of order two, generate a subalgebra of order four. The skew-commutative vector algebra V associated with such a quadratic algebra A may be characterized by the property that every two independent elements generate a subalgebra of order three. Then V is division-like, showing that our deletion of the requirement that A have no divisors of zero does not enlarge the class of algebras being considered in any essential way (in this connection, it might be remarked that it is an open question whether every skew-commutative division-like algebra arises as the vector algebra of some quadratic division algebra).

THEOREM 7. *Let A be a quadratic algebra with the property that any two elements not in the same subalgebra of order* 2 *generate a subalgebra of order* 4, *and let B be a subalgebra of A generated by n elements $x_1, \ldots, x_n$ but generated by no proper subset of these elements. Then the order of B is 2^n.*

We shall prove first, by induction on n, that the order of B is at least 2^n. The case $n = 1$ is trivial, and the inductive step is contained in

LEMMA 4. *Let B be a quadratic algebra whose associated vector algebra V is division-like, let C be a subalgebra of order m with basis $u_1, \ldots, u_m$, and let v be any element of B not in C. Then $u_1, \ldots, u_m, u_1 v, \ldots, u_m v$ are linearly independent, and hence, the order of B is at least $2m$.*

Suppose that $\alpha_1 u_1 + \ldots + \alpha_m u_m + \beta_1 u_1 v + \ldots + \beta_m u_m v = 0$ for some elements $\alpha_i, \beta_i \in F$. Then, letting $w = -(\alpha_1 u_1 + \ldots + \alpha_m u_m)$ and $u = \beta_1 u_1 + \ldots + \beta_m u_m$, we have $uv = w$ where u, w are in C and v is not in C. If $u = 0$, it follows easily that the α_i's and β_i's are zero; and if u is a nonzero scalar, we cannot have $w \in C$ and $v \notin C$. Hence, we may assume that $u = \beta + x$ where $\beta \in F$, $x \in V$, and $x \neq 0$. Letting L_u denote the linear transformation on B induced by left multiplication by u, and letting $\gamma + y$ be any element of B, we observe that the L_u sends $\gamma + y$ into the subalgebra D spanned by 1 and x only when

$$(\beta + x)(\gamma + y) = [\beta\gamma + (x, y)] + [\gamma x + \beta y + x \times y] = \delta + \eta x$$

for some $\delta, \eta \in F$, which implies that x, y and $x \times y$ are linearly dependent. Since

V is division-like, this means that x and y are dependent, or that $\gamma + y$ is in D. In particular, this shows that the kernel of L_u, or of any power of L_u, lies in D. Because D has order 2, we also see that L_u^2 and L_u^3 have the same kernel. Then, letting H be the image of the subalgebra C under L_u^2, it follows that HL_u has the same dimension as H and is contained in H, so that L_u is nonsingular on H.

Returning to the equation $uv = w$, we set $w = w_1 + w_2$ where $w_2 \in H$, and where w_1 is in the kernel of L_u^2, and hence in D. Denoting by v_2 the inverse image of w_2 under L_u restricted to H, we have $u(v - v_2) = w_1 \in D$. Hence, $(v - v_2) \in D$ by the argument in the last paragraph, and $v \in C$, to give the desired contradiction.

Continuing with the proof of Theorem 7, we shall show next that the order of B is no more than 8 when $n = 3$. If C is the subalgebra of B generated by the first two of the three generators of B, then C has a basis $1, u_1, u_2, u_3$ where we may take the u_i's to be vectors. Denoting the third generator of B by v (we may assume that it is a vector also), and defining $w_i = u_i \times v$ for $1 \leqq i \leqq 3$, we may conclude from Lemma 4 that $u_1, u_2, u_3, v, w_1, w_2, w_3$ are linearly independent vectors. It is then sufficient to show that the space V' spanned by these seven vectors is closed under vector multiplication. But for each $i = 1, 2, 3$, all products between the elements u_i, v, w_i are in V', since they are in the subalgebra of B generated by u_i and v which has order 4 by hypothesis, and hence is spanned by $1, u_i, v$, and w_i.

It remains to show that all products of the form $u_i \times w_j$ and $w_i \times w_j$ lie in V' for $1 \leqq i, j \leqq 3$, $i \neq j$. For the first of these, consider the equation

$$\begin{aligned}(u_i + v) \times [(u_i + v) \times u_j] &= (u_i + v) \times [u_i \times u_j - w_j] \\ &= u_i \times (u_i \times u_j) - u_i \times w_j + v \times (u_i \times u_j) - v \times w_j.\end{aligned}$$

The left side is in the subalgebra generated by $u_i + v$ and u_j, which is spanned by $1, u_i + v, u_j$, and $(u_i + v) \times u_j = u_i \times u_j - w_j$, and therefore it is in V'. On the other hand, all the terms on the right side except $u_i \times w_j$ are easily seen to be in V', and hence $u_i \times w_j$ is also. Similarly, the expression

$$\begin{aligned}(w_i + tv) \times [(w_i + tv) \times u_j)] &= (w_i + tv) \times [w_i \times u_j - tw_j] \\ &= w_i \times (w_i \times u_j) - tw_i \times w_j + tv \times (w_i \times u_j) - t^2 v \times w_j\end{aligned}$$

is in V' for any t in F, and the last two terms on the right side are also in V'. Thus $w_i \times (w_i \times u_j) - tw_i \times w_j$ is in V' for any value of t, implying that $w_i \times w_j$ is in V'.

The proof of Theorem 7 may now be finished by any easy induction. If B is generated by $x_1, \ldots, x_n$, but not by any proper subset of these elements, let C be the subalgebra generated by $x_1, \ldots, x_{n-1}$. Then C has a basis $1, u_1, \ldots, u_m$ of $m + 1 = 2^{n-1}$ elements by the inductive hypothesis, and $1, u_1, \ldots, u_m, x_n, u_1 x_n, \ldots, u_m x_n$ are linearly independent. If B' is the subspace spanned by these 2^n elements,

it is sufficient to show that any product of two of these elements lies in B'. But for fixed $i,j = 1,\ldots,m, i \neq j$, the elements $1, u_i, u_j, u_iu_j, x_n, u_ix_n, (u_iu_j)x_n$ form a basis of the subalgebra D_{ij} generated by u_i, u_j and x_n, and hence, the product of any two of them will lie in B'. Since any two of the basis elements of B' are in at least one of the subalgebras D_{ij}, B' is a subalgebra, and B has order 2^n.

The structure of the class of algebras of Theorem 7 seems sufficiently interesting to merit a closer look at the algebras of order 8 belonging to this class. Since the quadratic form associated with these algebras plays no part in their structure, we shall deal just with their associated vector algebras. We shall devote the remainder of this section to proving a theorem about the form of the multiplication table of such an algebra, and, in the following section, we shall use this information to exhibit a new class of division algebras over any real field.

THEOREM 8. *Let V be a skew-commutative algebra of order 7 with the property that any two independent elements of V generate a subalgebra of order 3, let U be any such subalgebra of V, and let u_1, u_2, u_3 be a basis of U such that*

$$u_1 \times u_2 = u_3, \qquad u_2 \times u_3 = \alpha u_1 + \beta u_2, \qquad u_3 \times u_1 = \gamma u_2. \tag{9}$$

Then there exists independent elements v, w_1, w_2, w_3 of V not in U, and five elements $\delta, \varepsilon_1, \varepsilon_2, \varepsilon_3, \eta$ of F, such that the rest of the multiplication table of V in terms of the basis $u_1, u_2, u_3, v, w_1, w_2, w_3$ is given by

$$\begin{aligned}
u_i \times v &= w_i, \quad w_i \times v = \delta u_i + \varepsilon_i v + \eta w_i \quad \textit{for } i = 1,2,3,\\
u_1 \times w_1 &= -\gamma v, \quad u_2 \times w_2 = -\alpha v, \quad u_3 \times w_3 = -\alpha\gamma v - \beta w_3,\\
u_1 \times w_2 &= -\varepsilon_2 u_1 + \varepsilon_1 u_2 + \eta u_3 - w_3,\\
u_2 \times w_1 &= \varepsilon_2 u_1 - \varepsilon_1 u_2 - \eta u_3 + \beta v + w_3,\\
u_2 \times w_3 &= \alpha\eta u_1 + (\beta\eta - \varepsilon_3)u_2 + \varepsilon_2 u_3 - \alpha w_1 - \beta w_2,\\
u_3 \times w_2 &= -\alpha\eta u_1 + (\varepsilon_3 - \beta\eta)u_2 - \varepsilon_2 u_3 + \alpha w_1,\\
u_3 \times w_1 &= \varepsilon_3 u_1 + \gamma\eta u_2 - \varepsilon_1 u_3 - \beta w_1 - \gamma w_2,\\
u_1 \times w_3 &= -\varepsilon_3 u_1 - \gamma\eta u_2 + \varepsilon_1 u_3 + \gamma w_2\\
w_2 \times w_1 &= \eta\varepsilon_2 u_1 - \eta\varepsilon_1 u_2 - (\eta^2 + \delta)u_3 + (\beta\eta - \varepsilon_3)v + \eta w_3,\\
w_3 \times w_2 &= -\alpha(\eta^2 + \delta)u_1 + \eta(\varepsilon_3 - \beta\eta)u_2 - \eta\varepsilon_2 u_3 - \alpha\varepsilon_1 v + \alpha\eta w_1\\
&\quad + \beta\eta w_2,\\
w_1 \times w_3 &= -(\eta\varepsilon_3 + \beta\delta)u_1 - \gamma(\eta^2 + \delta)u_2 + \eta\varepsilon_1 u_3 - (\gamma\varepsilon_2 + \beta\varepsilon_1)v\\
&\quad + \gamma\eta w_2 .
\end{aligned} \tag{10}$$

We begin the proof by taking v to be any vector in V but not in U, and by setting $w_i = u_i \times v$ for $i = 1,2,3$. Then, for each i, $w_i \times v = (u_i \times v) \times v$ is in the

subalgebra of V generated by u_i and v, and hence spanned by u_i, v, and w_i. Thus, there exist scalars $\delta_i, \varepsilon_i, n_i$ such that $w_i \times v = \delta_i u_i + \varepsilon_i v + \eta_i w_i$ for $1 \leqq i \leqq 3$. Considering the equation

$$[(u_1 + u_2 + u_3) \times v] \times v = (w_1 + w_2 + w_3) \times v = \delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3 + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)v + \eta_1 w_1 + \eta_2 w_2 + \eta_3 w_3,$$

we observe that the left side is in the subspace spanned by $(u_1 + u_2 + u_3), v$, and $(w_1 + w_2 + w_3)$. Since the right side is also in this subspace, we have $\delta_1 = \delta_2 = \delta_3$ and $\eta_1 = \eta_2 = \eta_3$, enabling us to drop the subscripts on the δ's and η's.

For each $i = 1, 2, 3$, we also have that $u_i \times w_i$ is in the subspace spanned by u_i, v, and w_i, so that there exist scalars $\lambda_i, \theta_i, \nu_i$ such that $u_i \times w_i = \lambda_i u_i + \theta_i v + \nu_i w_i$. Then

$$u_1 \times [u_1 \times (v + u_2)] = u_1 \times (w_1 + u_3) = \lambda_1 u_1 + \theta_1 v + \nu_1 w_1 - \gamma u_2$$

is in the subspace spanned by $u_1, v + u_2$, and $w_1 + u_3$, which implies that $u_1 \times [u_1 \times (v + u_2)] = f u_1 + g(v + u_2) + h(w_1 + u_3)$ for some scalars f, g, h. Equating the coefficients of u_2, u_3, v, and w_1 in these two expressions gives $-\gamma = g$, $0 = h$, $\theta_1 = g$, and $\nu_1 = h$ respectively, implying that $\nu_1 = 0$ and $\theta_1 = -\gamma$. Similarly, comparing coefficients in

$$\begin{aligned} u_2 \times [u_2 \times (v + u_1)] &= u_2 \times (w_2 - u_3) \\ &= \lambda_2 u_2 + \theta_2 v + \nu_2 w_2 - \alpha u_1 - \beta u_2 \\ &= f' u_2 + g'(v + u_1) + h'(w_2 - u_3) \end{aligned}$$

and

$$\begin{aligned} u_3 \times [u_3 \times (v + u_1)] &= u_3 \times (w_3 + \gamma u_2) \\ &= \lambda_3 u_3 + \theta_3 v + \nu_3 w_3 - \alpha\gamma u_1 - \beta\gamma u_2 \\ &= f'' u_3 + g''(v + u_1) + h''(w_3 + \gamma u_2), \end{aligned}$$

we get the other two equations of

$$(11) \quad u_1 \times w_1 = \lambda_1 u_1 - \gamma v, \; u_2 \times w_2 = \lambda_2 u_2 - \alpha v, \; u_3 \times w_3 = \lambda_3 u_3 - \alpha\gamma v - \beta w_3.$$

We show next that we could have picked v so that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Setting $v' = v + r u_1 + s u_2 + t u_3$, where r, s, t are scalars yet to be determined, we define w_1', w_2', w_3' by

$$\begin{aligned} w_1' &= u_1 \times v' = w_1 + s u_3 - t\gamma u_2, \\ w_2' &= u_2 \times v' = w_2 - r u_3 + t\alpha u_1 + t\gamma u_2, \\ w_3' &= u_2 \times v' = w_3 + r\gamma u_2 - s\alpha u_1 - s\beta u_2, \end{aligned}$$

where we have used (9) for the second equality in each case. Then,

$$\begin{aligned}
u_1 \times w_1' &= \lambda_1 u_1 - \gamma v - s\gamma u_2 - t\gamma u_3 = (\lambda_1 + r\gamma)u_1 - \gamma v', \\
u_2 \times w_2' &= \lambda_2 u_2 - \alpha v - r\alpha u_1 - r\beta u_2 - t\alpha u_3 = (\lambda_2 + s\alpha - r\beta)u_2 - \alpha v', \\
u_3 \times w_3' &= \lambda_3 u_3 - \alpha\gamma v - \beta w_3 - r\alpha\gamma u_1 - r\beta\gamma u_2 - s\alpha\gamma u_2 + s\alpha\beta u_1 s\beta^2 u_2 \\
&= (\lambda_3 + t\alpha\gamma)u_3 - \alpha\gamma v' - \beta w_3'.
\end{aligned}$$

But, since α and γ are never zero, there exist values of r, s, t so that $\lambda_1 + r\gamma = \lambda_2 + s\alpha - r\beta = \lambda_3 + t\alpha\gamma = 0$, showing that we could have achieved $\lambda_1 = \lambda_2 = \lambda_3 = 0$ by picking an appropriate v' instead of v. Thus, with the correct choice of v, (11) reduces to the second line of (10).

It remains to compute the products $u_i \times w_j$ and $w_i \times w_j$ for $1 \leqq i, j \leqq 3$, $i \neq j$, which will be uniquely determined in terms of the constants that we have already introduced. To evaluate the products of the form $u_i \times w_j$, we let r, s, t be three parameters, and compute

$$\begin{aligned}
&[(ru_i + su_j + tv) \times (u_k + v)] \times (u_k + v) \\
&= [ru_i \times u_k + rw_i + su_j \times u_k + sw_j - tw_k] \times (u_k + v) \\
&= r(u_i \times u_k) \times u_k + r(u_i \times u_k) \times v + rw_i \times u_k + rw_i \times v + s(u_j \times u_k) \times u_k \\
&\quad + s(u_j \times u_k) \times v + sw_j \times u_k + sw_j \times v - tw_k \times u_k - tw_k \times v \\
&= -f(ru_i + su_j + v) - g(u_k + v) \\
&\quad - h[ru_i \times u_k + rw_i + su_j \times u_k + sw_j - tw_k]
\end{aligned}$$

for some constants f, g, h depending on the parameters r, s, t. Solving the last equation for $rw_i \times u_k + sw_j \times u_k$, expanding the terms of the form $w_i \times v$ and $u_i \times w_i$, and collecting similar terms, we get

$$\begin{aligned}
&ru_k \times w_i + su_k \times w_j \\
&\quad = r(f + \delta)u_i + s(f + \delta)u_j + (g - t\delta)u_k \\
(12) &\quad + [rhu_i \times u_k + shu_j \times u_k + r(u_i \times u_k) \times u_k + s(u_j \times u_k) \times u_k] \\
&\quad + (r\varepsilon_i + s\varepsilon_j + g + tf + t\theta_k - t\varepsilon_k)v + r(h + \eta)w_i + s(h + \eta)w_j \\
&\quad - t(h - \eta - v_k)w_k + [r(u_i \times u_k) \times v + s(u_j \times u_k) \times v].
\end{aligned}$$

Setting $i = 1, j = 2, k = 3$, the two expressions in brackets in (12) reduce to

$$-rh\gamma u_2 + sh\alpha u_1 + sh\beta u_2 - r\alpha\gamma u_1 - r\beta\gamma u_2 - s\alpha\gamma u_2 + s\alpha\beta u_1 + s\beta^2 u_2$$

and

$$-r\gamma w_2 + s\alpha w_1 + s\beta w_2,$$

respectively, and (12) becomes

$$
\begin{aligned}
&ru_3 \times w_1 + su_3 \times w_2 \\
&\quad = [r(f+\delta-\alpha\gamma)+s\alpha(h+\beta)]u_1 \\
&\qquad + [-r\gamma(h+\beta)+s(f+\delta-\alpha\gamma+h\beta+\beta^2)]u_2 + [g-t\delta]u_3 \\
&\qquad + [r\varepsilon_1+s\varepsilon_1+g+t(f-\alpha\gamma-\varepsilon_3)]v + [r(h+\eta)+s\alpha]w_1 \\
&\qquad + [-r\gamma+s(h+\eta+\beta)]w_2 - t(h+\eta+\beta)w_3 .
\end{aligned} \tag{13}
$$

Since the left side of (13) is independent of t, each component on the right side is also, from which it is easy to see that h and f are independent of t. But, for any fixed values of r and s, the coefficient of w_3 can only be independent of t if $h+\eta+\beta=0$. Using this relation and setting $r=0$, we see that the u_1 coefficient of $u_3 \times w_2$ is $-\alpha\eta$. Hence, for any values of $r \neq 0$ and s, the u_1 coefficient of $u_3 \times w_1$ is $(f+\delta-\alpha\gamma)$, which shows that f is independent of s. Making the same argument with respect to the coefficient of u_2 with the roles of r and s reversed, we see that f is independent of r. Then, replacing g by $g(r,s,t)$ and setting $r=s=0$ in (13) yields $g(0,0,t)-t\delta=0=g(0,0,t)+t(f-\alpha\gamma-\varepsilon_3)$ for all t, from which we get $f-\alpha\gamma-\varepsilon_3=-\delta$, or $f+\delta-\alpha\gamma=\varepsilon_3$. We may now substitute for f and h in (13) and set first $r=1$ and $s=0$, and then $r=0$ and $s=1$ to obtain respectively

$$
\begin{aligned}
u_3 \times w_1 &= \varepsilon_3 u_1 + \gamma\eta u_2 + k_1 u_3 + (\varepsilon_1+k_1)v - \beta w_1 - \gamma w_2, \\
u_3 \times w_2 &= -\alpha\eta u_1 + (\varepsilon_3-\beta\eta)u_2 + k_2 u_3 + \varepsilon_3 + k_2)v + \alpha w_1 ,
\end{aligned} \tag{14}
$$

where $k_1 = g(1,0,t)-t\delta$ and $k_2 = g(0,1,t)-t\delta$.

Secondly, if we make the substitution $i=1$, $j=3$, $k=2$ in (12), we get

$$
\begin{aligned}
&ru_2 \times w_1 + su_2 \times w_3 \\
&\quad = [r(f+\delta-\alpha)-sh\alpha]u_1 + [rh+s(f+\delta-\alpha)]u_3 \\
&\qquad + [-r\beta-sh\beta+g-t\delta]u_2 + [r\varepsilon_1+s\varepsilon_2+g+t(f-\alpha-\varepsilon_2)]v \\
&\qquad + [r(h+\eta)-s\alpha]w_1 + [r+s(h+\eta)]w_3 - [s\beta+t(h+\eta)]w_2 .
\end{aligned}
$$

This time we may deduce that $h+\eta=0$ and $f+\delta-\alpha=\varepsilon_2$, leading to

$$
\begin{aligned}
u_2 \times w_1 &= \varepsilon_2 u_1 + (k_3-\beta)u_2 - \eta u_3 + (\varepsilon_1+k_3)v + w_3, \\
u_2 \times w_3 &= \alpha\eta u_1 + (\beta\eta+k_4)u_2 + \varepsilon_3 u_3 + (\varepsilon_3+k_4)v - \alpha w_1 - \beta w_2,
\end{aligned} \tag{15}
$$

where $k_3 = g(1,0,t)-t\delta$ and $k_4 = g(0,1,t)-t\delta$. And finally, the substitution $i=2$, $j=3$, $k=1$ in (12) yields

$$
\begin{aligned}
ru_1 \times w_2 + su_1 \times w_3 = {} & [r(f+\delta-\gamma)+sh\gamma]u_2 + [-rh+s(f+\delta-\gamma)]u_3 \\
& + (g-t\delta)u_1 + [r\varepsilon_2+s\varepsilon_3+g+t(f-\gamma-\varepsilon_1)]v \\
& + [r(h+\eta)+s\gamma]w_2 + [-r+s(h+\eta)]w_3 \\
& - t(h+\eta)w_1 ,
\end{aligned}
$$

from which we deduce that $h+\eta=0$, $f+\delta-\gamma=\varepsilon_1$, and

$$
\begin{aligned}
u_1 \times w_2 &= k_5 u_1 + \varepsilon_1 u_2 + \eta u_3 + (\varepsilon_2 + k_5)v - w_3, \\
u_1 \times w_3 &= k_6 u_1 - \gamma\eta u_2 + \varepsilon_1 u_3 + (\varepsilon_3 + k_6)v + \gamma w_2,
\end{aligned} \tag{16}
$$

where $k_5 = g(1,0,t) - t\delta$ and $k_6 = g(0,1,t) - t\delta$.

To determine $k_1, \ldots, k_6$, we compute

$$
\begin{aligned}
(u_1 + qu_2) \times [(u_1 + qu_2) \times v] &= (u_1 + qu_2) \times (w_1 + qw_2) \\
&= -\gamma v + qu_1 \times w_2 + qu_2 \times w_1 - q^2\alpha v \\
&= l(u_1 + qu_2) + mv + n(w_1 + qw_2),
\end{aligned}
$$

where l, m, n depend on the parameter q. The last part of this equation may be rewritten as

$$
q(u_1 \times w_2 + u_2 \times w_1) = lu_1 + qlu_2 + (m + \gamma + q^2\alpha)v + n(w_1 + qw_2). \tag{17}
$$

But, adding the first equation of (15) to the first equation of (16) and multiplying by q gives

$$
\begin{aligned}
q(u_1 \times w_2 + u_2 \times w_1) = q(k_5 + \varepsilon_2)u_1 + q(\varepsilon_1 + k_3 - \beta)u_2 \\
+ (\varepsilon_2 + k_5 + \varepsilon_1 + k_3)v,
\end{aligned} \tag{18}
$$

and comparing the coefficients of u_1 and u_2 in (17) and (18) yields $l = q(k_5 + \varepsilon_2) = \varepsilon_1 + k_3 - \beta$. Since this holds for all q, we have $k_5 = -\varepsilon_2$ and $k_3 = \beta - \varepsilon_1$. Similarly, starting with $(u_1 + qu_3) \times [(u_1 + qu_3) \times v]$ leads to $k_1 = -\varepsilon_1$ and $k_6 = -\varepsilon_3$, and starting with $(u_2 + qu_3) \times [(u_2 + qu_3) \times v]$ leads to $k_2 = -\varepsilon_2$ and $k_4 = -\varepsilon_3$. Substituting these values into (14), (15), and (16) reduces these equations to the form given in (10).

We now have only the products $w_i \times w_j$ to determine. For $w_2 \times w_1$, we compute

$$
\begin{aligned}
&[(su_3 + tv) \times w_1] \times w_1 \\
&\quad = [s\varepsilon_3 u_1 + s\gamma\eta u_2 - s\varepsilon_1 u_3 - s\beta w_1 - s\gamma w_2 - t\delta u_1 - t\varepsilon_1 v - t\eta w_1] \times w_1 \\
&\quad = (t\delta - s\varepsilon_3)\gamma v + s\gamma\eta\varepsilon_2 u_1 - s\gamma\eta\varepsilon_1 u_2 + s\beta\gamma\eta v + s\gamma\eta w_3 - s\varepsilon_1\varepsilon_3 u_1 - s\gamma\eta\varepsilon_1 u_2 \\
&\qquad + s\varepsilon_1^2 u_3 + s\beta\varepsilon_1 w_1 + s\gamma\varepsilon_1 w_2 + t\delta\varepsilon_1 u_1 + t\varepsilon_1^2 v + t\varepsilon_1\eta w_1 - s\gamma w_2 \times w_1 \\
&\quad = -f(su_3 + tv) - gw_1 \\
&\qquad - h[(s\varepsilon_3 - t\delta)u_1 + s\gamma\eta u_2 - s\varepsilon_1 u_3 - t\varepsilon_1 v - (s\beta + t\eta)w_1 - s\gamma w_2],
\end{aligned}
$$

where f, g, h depend on the parameters s and t. Solving for $s\gamma w_2 \times w_1$ gives

$$
\begin{aligned}
s\gamma w_2 \times w_1 = {} & [s(h\varepsilon_3 - \varepsilon_1\varepsilon_3 + \gamma\eta\varepsilon_2) + t\delta(\varepsilon_1 - h)]u_1 \\
& + s\gamma\eta(h - 2\varepsilon_1)u_2 + s(-\gamma\eta^2 + \varepsilon_1^2 + f - h\varepsilon_1)u_3 \\
& + [s\gamma(\beta\eta - \varepsilon_3) + t(\gamma\delta + \varepsilon_1^2 + f - h\varepsilon_1)]v \\
& + [s\beta(\varepsilon_1 - h) + g + t\eta(\varepsilon_1 - h)]w_1 + s\gamma(\varepsilon_1 - h)w_2 \\
& + s\gamma\eta w_3.
\end{aligned} \tag{19}
$$

From the coefficients of u_2 and u_3 in (19), we see that h and f are independent of t, and from the coefficient of u_1, we see that $t\delta(\varepsilon_1 - h)$ is independent of t, or that $\delta(\varepsilon_1 - h) = 0$. Since $\delta = 0$ would imply that $w_i \times v, v$, and w_i are linearly dependent, we must have $h = \varepsilon_1$. Similarly, the expression $\gamma\delta + \varepsilon_1^2 + f - h\varepsilon_1$ from the coefficient of v is zero, leading to $f = -\gamma\delta$. Substituting for h and f in (19) and setting $s = \gamma^{-1}$ now gives the equation for $w_2 \times w_1$ in (10) except for the coefficient of w_1, which cannot be determined from (19) because we do not have any way of finding the value of g. To evaluate this coefficient, we start with

$$
\begin{aligned}
&[(su_3 + tv) \times w_2] \times w_2 \\
&= [-s\alpha\eta u_1 + s(\varepsilon_3 - \beta\eta)u_2 - s\varepsilon_2 u_3 + s\alpha w_1 - t\delta u_2 - t\varepsilon_2 v - t\eta w_2] \times w_2 \\
&= -f(su_3 + tv) - gw_2 \\
&\quad - h[-s\alpha\eta u_1 + s(\varepsilon_3 - \beta\eta)u_2 - t\delta u_2 - s\varepsilon_3 u_3 - t\varepsilon_2 v + s\alpha w_1 - t\eta w_2].
\end{aligned} \tag{20}
$$

If we worked through all the details of this one as in the one above, we would get all of the coefficients of $w_2 \times w_1$ except for the coefficient of w_2. However, since we are only interested in the coefficient of w_1, we need only retain those terms that will have a bearing on this. Solving (20) for $s\alpha w_1 \times w_2$, the relevant terms are

$$s\alpha w_1 \times w_2 = \ldots + [\ldots + t\delta(\varepsilon_2 - h)]u_2 + \ldots + s\alpha(h - \varepsilon_2)w_1 + \ldots,$$

and arguing as above, $h = \varepsilon_2$, and the coefficient of w_1 is zero.

By an identical argument, $w_3 \times w_2$ may be computed using $[(su_1 + tv) \times w_2] \times w_2$, except for the coefficient of w_2, for which we use $[(su_1 + tv) \times w_3] \times w_3$; and $w_1 \times w_3$ may be computed using $[(su_2 + tv) \times w_1] \times w_1$, except for the coefficient of w_1, for which we use $[(su_2 + tv) \times w_3] \times w_3$.

5. If the field F is real closed, we already know that we may take α and γ to be 1 in Theorem 8, and that $|\beta| < 2$. We can also show that we may take $\delta = -1$, and that ε_1, ε_2, ε_3, and η then each satisfy the same condition as β. For, by replacing v, w_1, w_2, w_3 in (10) by cv, cw_1, cw_2, cw_3 where $c^{-2} = |\delta|$, we may make $\delta = \pm 1$. And considering the equation

$$
\begin{aligned}
(w_i + su_i) \times v &= \delta u_i + \varepsilon_i v + (\eta + s)w_i \\
&= [\delta - s(\eta + s)]u_i + \varepsilon_i v + (\eta + s)(w_i + su_i),
\end{aligned}
$$

we see that $(w_i + su_i) \times v$, $(w_i + su_i)$, and v will be linearly dependent whenever $s(s + \eta) - \delta = 0$, and that an s satisfying this condition exists unless δ is negative. Thus, we may assume that $\delta = -1$, and the condition that $s(s + \eta) + 1 = 0$ has no solution gives $|\eta| < 2$. Also, from the equation

$$
\begin{aligned}
w_i \times (u_i + tv) &= -tu_i + (1 + \varepsilon_i t)v + ?w_i \\
&= -t(u_i + tv) + (1 + \varepsilon_i t + t^2)v + ?w_i,
\end{aligned}
$$

we see that $1 + \varepsilon_i t + t^2$ cannot be zero for any $t \in F$, which gives the condition $|\varepsilon_i| < 2$.

These necessary conditions for V to be division-like are, unfortunately, not sufficient. The question of when V is division-like, as well as the question of when two V's are isomorphic, is too difficult for us to handle, even when F is assumed to be the real field. However, we can prove that the algebras of this type with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \eta = 0$ and $|\beta| < 2$ are division-like.

THEOREM 9. *Let F be a (formally) real field, and let V be the algebra of Theorem 8 with $\alpha = \gamma = -\delta = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \eta = 0$ (and β arbitrary). Then V is division-like if $|\beta| < 2$. This condition is also necessary if F is real closed*(2).

Combining this theorem with Theorem 6, we immediately have the following

COROLLARY. *Over a real closed field, every quadratic division algebra of order 4 may be embedded in a quadratic division algebra of order 8.*

For the proof of Theorem 9, it is convenient to define a bilinear form on V, and to deal with the corresponding quadratic algebra A instead of V. We select the symmetric bilinear form defined by $(u_i, u_i) = (v, v) = (w_i, w_i) = -1$ for $1 \leqq i \leqq 3$. Letting $x = a + bu_1 + cu_2 + du_3 + fv + gw_1 + hw_2 + kw_3$ and $y = a' + b'u_1 + c'u_2 + d'u_3 + f'v + g'w_1 + h'w_2 + k'w_3$ $(a, \ldots, k, a', \ldots, k' \in F)$ be any two elements of A, we may use (9) and (10) with $\alpha = \gamma = -\delta = 1$ αnd $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \eta = 0$ to express the relation $xy = 0$ in terms of its components, which gives eight homogeneous bilinear equations in the real numbers $a, \ldots, k$ and $a', \ldots, k'$. Thinking of the primed letters as the variables, the coefficient matrix of this set of equations will be

$$\mathbf{M} = \begin{bmatrix} a & -b & -c & -d & -f & -g & -h & -k \\ b & a & -d & c & -g & f-\beta k & k & -h+\beta g \\ c & d & a-\beta d & -b+\beta c & -h & -k & f & g \\ d & -c & b & a & -k & h & -g & f \\ f & g & h-\beta g & k & a & -b+\beta c & -c & -d \\ g & -f & k & -h+\beta g & b & a-\beta d & d & -c \\ h & -k & -f+\beta k & g & c & -d & a & b-\beta c \\ k & h & -g & -f+\beta k & d & c & -b & a-\beta d \end{bmatrix}. \tag{21}$$

Then, A will contain zero divisors if and only if $\mathbf{M}$ is singular for some choice of $a, \ldots, k$, not all zero. Defining

$$\Gamma = a^2 + b^2 + \ldots + k^2, \quad \Delta = \Gamma - \beta(ad + bc + k + gh),$$

and

(2) The author is indebted to the referee for suggesting a simplification of the original proof of this theorem.

$$M' = \begin{bmatrix} a & b-\beta c & c & d-\beta a & f-\beta k & g & h-\beta g & k \\ -b & a-\beta d & d & -c+\beta b & g & -f+\beta k & -k & h-\beta g \\ -c & -d & a & b & h & k & -f & -g \\ -d & c & -b & a & k & -h & g & -f \\ -f & -g & -h+\beta g & -k+\beta f & a-\beta d & b-\beta c & c & d \\ -g & f & -k & h & -b & a & -d & c \\ -h & k & f-\beta k & -g+\beta h & -c & d & a-\beta d & -b+\beta c \\ -k & -h & g & f & -d & -c & b & a \end{bmatrix},$$

we may compute that the product MM' has Γ in the upper left hand corner, Δ elsewhere on the diagonal, and zeros everywhere else except for the first row. But M is singular only if MM' is singular, and the determinant of MM' is $\Gamma\Delta^7$, which cannot vanish for F real and $|\beta| < 2$ unless $a = b = \ldots = k = 0$.

If F is real closed, the necessity of the condition $|\beta| < 2$ follows by applying Theorem 8 to the subalgebra of V with basis $1, u_1, u_2, u$.

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