ON RING EXTENSIONS FOR COMPLETELY PRIMARY NONCOMMUTATIVE RINGS

BY
E. H. FELLER AND E. W. SWOKOWSKI

0. Introduction. It is the authors' purpose in this paper to initiate the study of ring extensions for completely $N$ primary noncommutative rings which satisfy the ascending chain condition for right ideals (A.C.C.). We begin here by showing that every completely $N$ primary ring $R$ with A.C.C. is properly contained in just such a ring. This is accomplished by first showing that $R[x]$, $x$ an indeterminate where $ax = xa$ for all $a \in R$, is $N$ primary and then constructing the right quotient ring $Q(R[x])$. The details of these results appear in §§1, 7 and 8. The corresponding results for the commutative case are given by E. Snapper in [7] and [8].

If $R \subset A$, where $A$ is completely $N$ primary with A.C.C. then, from the discussion in the preceding paragraph, it would seem natural to examine the structure of $R(\sigma)$ when $\sigma \in A$ and $a\sigma = \sigma a$ for all $a \in R$ in the cases where $\sigma$ is algebraic or transcendental over $R$. These structures are determined in §§ 6 and 8 of the present paper.

The definitions and notations given in [2] will be used throughout this paper. As in [2], for a ring $R$, $N$ or $N(R)$ denotes the union of nilpotent ideals(1) of $R$, $P$ or $P(R)$ denotes the set of nilpotent elements of $R$ and $J$ or $J(R)$ the Jacobson radical of $R$. The letter $H$ is used for the natural homomorphism from $R$ to $R/N = \bar{R}$. If $B$ is a subset of $R$ then $\bar{B}$ denotes the image of $B$ under $H$. If $N = P$ in $R$ and if $R'$ is a ring contained in $R$ then $N(R') = N \cap R'$ and $\bar{R'} = R'/N(R')$. Thus we consider the contraction of $H$ on $R'$ as the natural homomorphism from $R'$ onto $R'/N(R')$.

Unlike the commutative case, the results of this paper will at times depend on the three conditions (i), (ii) and (iii) of [2, §3]. Therefore, we make the following definition.

Definition 0.1. A ring $R$ with identity is called an extendable ring if it satisfies the three conditions:

(i) $P(q)$ is an ideal when $q$ is a right $P$ primary ideal(2).

(ii) $P(R) = N(R)$.

(iii) The nontrivial completely prime ideals of $R/N$ are maximal right ideals.

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(1) Ideal shall always mean two-sided ideal.

(2) See [2, §1] for the meaning of $P(q)$.
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1. Properties of $R[x]$. In [2] we defined a ring $R$ to be $N$ primary if $ab = 0$, $a \neq 0$ implies $b \in N$ and $b \neq 0$ implies $a \in N$. A ring $R$ with identity is completely $N$ primary if $R/N$ is a division ring. Note that if $R$ is $N$ primary or completely $N$ primary then $N$ is a completely prime ideal and $N = P$. For the rings considered in this paper $N$ and $P$ shall always be equal.

**Theorem 1.1.** If $R$ is $N$ primary then, in the polynomial ring $R[x]$, $N[x] = P[x] = N(R[x]) = P(R[x]) = J(R[x]).$

**Proof.** We may write $R[x]/N[x] = (R/N)[x]$ where $N[x] \subseteq J(R[x]).$ Since $R/N$ is an integral domain it contains no nonzero nil ideals and hence, from Theorem 4 of [6, p. 12], $R[x]/N[x]$ is semisimple. Thus $J(R[x]) = N[x].$

Certainly $N[x] \subseteq P[x] \subseteq P(R[x]).$ We now show that $P(R[x]) = P[x].$ If $f \in P(R[x])$ then $f^n = 0$ for some positive integer $n.$ Hence in $R[x]/P[x] \cong (R/P)[x]$ we have $(f^n) = 0.$ Since $(R/P)[x]$ has no divisors of zero it follows that $f \in P[x].$

Next we show that $N(P[x]) = N[x].$ Clearly $N(R[x]) = P(R[x]) = N[x].$ If $f \in N[x]$ then $f = a_n x^n + \ldots + a_1 x + a_0$ for $a_i \in q_i$ where $q_i$ is a unit of $R$ and $q_i' = 0$ for positive integers $t_i,$ $i = 0, 1, \ldots, n.$ Thus $f$ is contained in the ideal $k = q_0[x] + q_1[x] + \ldots + q_n[x]$ and $k^{t_0 + t_1 + \ldots + t_n + 1} = 0.$ Hence $f \in N(R[x]).$

This completes the proof.

As a consequence of this theorem we have, for an $N$ primary ring $R,$ that $R[x]/N(R[x]) = R[x]/N[x] \cong (R/N)[x].$ Thus we can consider $(R[x])$ as $R[x].$ In this case, the natural homomorphism $H$ from $R[x]$ onto $R[x]$ maps the polynomial $\sum a_i x^i$ on $\sum a_i x^i.$

**Theorem 1.2.** Let $R$ be an $N$ primary ring. Then $f = a_n x^n + \ldots + a_0$ is a unit of $R[x]$ if and only if $a_0$ is a unit of $R$ and $a_1, \ldots, a_n \in N.$

**Proof.** If $f$ is of this form then $f$ is a unit of $R[x].$ Hence by 2a of [2] $f$ is a unit of $R[x].$

Conversely, if $f = a_n x^n + \ldots + a_0$ is a unit of $R[x]$ then $f = \bar{a}_n x^n + \ldots + \bar{a}_0$ is a unit of $R[x].$ But since $R$ is an integral domain it follows that $f = \bar{a}_0$ is a unit of $R$ and $\bar{a}_1 = \ldots = \bar{a}_n = 0.$ Thus $a_0$ is a unit of $R$ and $a_1, \ldots, a_n \in N(R).$

**Theorem 1.3.** If $R$ is completely $N$ primary and $N^n = 0$ for some positive integer $n,$ then $R[x]$ is $N[x]$ primary.

**Proof.** This proof will be by induction on the smallest integer $n$ such that $N^n = 0.$ If $n = 1$ obviously $R[x]$ is $N(R[x])$ primary. By Theorem 1.1 $N(R[x]) = N[x]$ and thus $R[x]$ is $N[x]$ primary.

(1) Here the symbol $\sim$ denotes the coset modulo $P[x].$
Suppose the theorem is true for all rings \( R \) where \( N^n = 0, N^{n-1} \neq 0 \). Let \( R \) satisfy the conditions of the theorem and \( N^{n+1} = 0, N^n \neq 0 \). Consider the ring \( R/N^n \). The radical of this ring is \( N/N^n \) and \( (N/N^n)^n \) is the zero coset. In addition since \( (R/N^n)/(N/N^n) \cong R/N \) we have that \( R/N^n \) is completely \( N/N^n \) primary. Hence, by the induction hypothesis \( (R/N^n)[x] \) is \( (N/N^n)[x] \) primary.

Suppose in \( R[x] \) we have \( fg = 0, g \neq 0 \) and \( f \notin N[x] \). Since \( (R/N)[x] \) is an integral domain \( g \in N[x] \). If \( g \notin N^n[x] \), then \( f_g = 0 \) in \( (R/N^n)[x] \) and from above \( f \in (N/N^n)[x] \). Thus \( f \in N[x] \), a contradiction. Suppose \( g \in N^n[x] \). Let \( a_i \) be the coefficient of the highest power of \( x \) in \( f \) which is not in \( N \). Since \( N^{n+1} = 0 \) and \( fg = 0 \), we have \( a_i b_s = 0 \) where \( g = b_0 x^s + \ldots + b_0, b_s \neq 0 \). Since \( R \) is completely \( N \) primary \( a_i \in N \). Thus in any case the assumption that \( fg = 0, g \neq 0, f \notin N[x] \) leads to a contradiction. Hence if \( g \neq 0 \) we can only conclude that \( f \in N[x] \). Similarly if \( gf = 0 \) and \( f \neq 0 \) then \( g \in N[x] \). Therefore, \( R[x] \) is \( N[x] \) primary and the proof by induction is complete.

2. Extensions of rings. A ring \( A \) is an extension of a ring \( R \) if \( R \subseteq A \). In the remainder of this paper, unless otherwise stated, we assume that if \( A \) is an extension of \( R \) then \( R \) and \( A \) have the same identity element. If \( R \subseteq A \) then \( A \) can be considered as an \( R \) module(4) with submodule \( R \). Thus, in order to develop a theory for extensions of a ring, it is convenient to discuss some notions concerning modules.

Let \( R \) be a ring with identity and \( M \) a unital \( R \) module. If \( R_1 \) and \( M_1 \) are nonempty subsets of \( R \) and \( M \) respectively, then \( R_1 M_1 \) denotes the set of all finite sums \( \sum r_i m_i \) where \( r_i \in R_1 \) and \( m_i \in M_1 \). A subset \( S \) of \( M \) is called a generating system of a submodule \( Q \) of \( M \) if \( Q = RS \). If \( S \) contains a finite number of elements then \( Q \) is said to be finitely generated. A finite generating system \( S \) of \( Q \) is called a basis of \( Q \) if \( Q \) does not have a generating system containing fewer elements than \( S \). If \( Q \) is finitely generated then the rank \( \rho(Q) \) of \( Q \) is the number of elements in a basis of \( Q \).

The following theorem was proved in [7, p. 685] for commutative rings. The proof carries over immediately to the noncommutative case.

**Theorem 2.1.** Let \( M \) be an \( R \) module and \( Q \) a submodule of \( M \) such that \( M - Q \) is finitely generated. If \( q \) is any \( J \) ideal of \( R \) and \( Q' \) any submodule of \( M \) let \( \bar{S} \) denote the image of the subset \( S \) of \( M \) under the natural homomorphism from \( M \) onto \( M - qQ' \). Then \( M = Q + RS \) if and only if \( \bar{M} = \bar{Q} + R\bar{S} \). Consequently \( \rho(M - Q) = \rho(\bar{M} - \bar{Q}) \) and if \( \bar{M} = \bar{Q} \) then \( M = Q \).

If a ring \( A \) has finite rank as a module over a subring \( R \) we call the rank the degree \( [A : R] \) of the ring extension and say that \( A \) is a finite extension of \( R \).

(4) \( R \) module shall always mean left \( R \) module. For a discussion of left \( R \) modules read Chapter I of [6].
Theorem 2.2. Let \( R \) and \( A \) be rings where \( R \subseteq A \) and suppose every right unit of \( A \) is a left unit. Then \( R = A \) if and only if \([A:R] = 1\).

Proof. If \( R = A \) then the identity element of \( A \) is a basis of \( A \) and hence \([A:R] = 1\). Conversely if \([A:R] = 1\), let \( z \) be a basis of \( A \). Then since \( 1 \in A \) we have \( az = 1 \) for some \( a \in R \). Thus \( z \) is a right and left unit of \( A \) and certainly regular. Then for \( z^2 \in A \) we have \( bz = z^2 \) for some \( b \in R \) which implies that \( b = z \in R \). Thus \( R = A \).

By statements 1.1 and 1.2 of [2] it follows that the conditions of Theorem 2.2 are satisfied when \( A \) is a \( J \)-primary ring. In addition, if \( R \) and \( A \) have the same identity element and \( A/N \) is a principal ideal domain, then by the discussion in §5 of [2] we know that Theorem 2.2 is valid.

If \( R \) and \( A \) are rings where \( R \subseteq A \), the contraction \( q_+ \) of an ideal \( q \) of \( A \) is defined as the largest ideal of \( R \) which is contained in \( q \), i.e., \( q_+ = q \cap R \). The extension \( q* \) of an ideal \( q \) of \( R \) in \( A \) is defined as the smallest ideal of \( A \) which contains \( q \), i.e., \( q = AqA \), the set of all sums \( \sum a_1 q_1 b_i \) where \( q_1 \in q \) and \( a_i, b_i \in A \). Thus, by definition, \( N(R)^* = AN(R)A \). Of particular importance to us is the case when \( N(R)^* = N(R)A \) and in addition \( N(R)^* = N(A) \). We make the following

**Definition 2.1.** A ring \( A \) is called a principal extension of a subring \( R \) if \( N(A) = N(R)^* = N(R)A \).

If \( R \) is \( N \)-primary then, by Theorem 1.1, the ring \( R[x] \) is a principal extension of \( R \).

**Theorem 2.3.** Let \( A \) be a finite principal extension of \( R \). If \( S \) is a subset of \( A \) then \( A = RS \) if and only if \( \bar{A} = \bar{R} \). Hence \([A:R] = [\bar{A}:\bar{R}]\) and if \( R = \bar{A} \) then \( R = A \).

Proof. If \( A \) is a principal extension of \( R \) then \( N(A) = N(R)A \). Consider \( A \) as an \( R \) module with \( N(R) \subseteq J(R) \). The first part of the theorem then follows from Theorem 2.1 by setting \( A = Q' = M \), \( Q = 0 \) and \( q = N(R) \). If in Theorem 2.1 we let \( A = Q' = M \), \( Q = R \) and \( q = N(R) \) it follows that \( R = \bar{A} \) implies \( R = A \).

**Theorem 2.4.** Let \( R = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n \), where \( A_i \) is a finite principal extension of \( A_{i-1} \) of degree \([A_i : A_{i-1}] = r_i \) for \( i = 1, 2, \ldots, n \). Then, if all the rings \( R, A_1, \ldots, A_{n-1} \) are completely \( N \)-primary, \([A_n : R] = r_1 r_2 \ldots r_n \).

Proof. Since \( N(A_n) = N(R)A_1 A_2 \ldots A_n = N(R)A_n \) it follows that \( A_n \) is a finite principal extension of \( R \). Hence \([A_n : R] = [A_n : \bar{R}] \). Since \( A_i \) is a finite principal extension of \( A_{i-1} \), \( r_i = [A_i : A_{i-1}] \). Consequently, since \( R, \bar{A_1}, \ldots, \bar{A}_{n-1} \) are division rings we have \([A_n : R] = r_1 r_2 \ldots r_n \).

3. Degrees of ideals.

**Definition 3.1.** Let \( R \) and \( A \) be rings where \( R \subseteq A \). An ideal \( p \) of \( A \) has finite degree \( \deg(p) \) if the ring \( A/p \) is a finite extension of \( R/p \). If \( p \) has finite degree then \( \deg(p) = [A/p : R/p] \).
Definition 3.1 is equivalent to saying that if the rank of the \( R \) module \( A - p \) is finite then \( \deg(p) = \rho(A - p) \).

**Theorem 3.1.** Let \( A \) be a principal extension of \( R \) and \( p \) an ideal of \( A \) of finite degree. If \( S \) is a subset of \( A \) then \( A = p + RS \) if and only if \( \bar{A} = \bar{p} + R\bar{S} \). Hence \( \deg(p) = \deg(\bar{p}) \).

**Proof.** Apply Theorem 2.1 with \( M = Q' = A \), \( Q = p \) and \( q = N(R) \).

The following two results are proved in [7] for commutative rings. Using the definitions and results listed above the proofs now carry over, without essential modification, to the noncommutative case.

**Statement 3.1.** An ideal \( p \) of \( R[x] \) has finite degree if and only if \( p \) contains a monic polynomial.(5)

**Theorem 3.2.** If \( R \) is \( N \) primary, then an ideal \( p \) of \( R[x] \) has finite degree if and only if \( \bar{p} \) has finite degree in \( R[x] \). In this case if \( B \) is a subset of \( R[x] \) then \( R[x] = RB + p \) if and only if \( R[x] = RB + \bar{p} \). Consequently \( \deg(p) = \deg(\bar{p}) \).

Notice that if \( R \) is a division ring then \( R[x] \) is a principal ideal domain and if \( \bar{p} = (\bar{f}) \) then \( \deg \bar{p} = D(\bar{f}) \). (The symbol \( D(f) \) denotes the degree of the polynomial \( f \).)

**Definition 3.2.** The order of a regular polynomial \( f \) of \( R[x] \) is the minimal degree of the nonzero polynomials of \( fR[x] \), The order of \( f \) is denoted by \( O(f) \).

Note that \( O(f) = D(f) \) when \( R \) is an integral domain.

**Lemma 3.1.** If \( R \) is completely \( N \) primary and \( N^n = 0 \) for a positive integer \( n \), then \( O(f) \) is equal to the exponent of the highest power of \( x \) in \( f \) whose coefficient is a unit of \( R \).

**Proof.** Let \( f = a_{m}x^m + \ldots + a_{0} \) where \( a_{m+1}, \ldots, a_{m+1} \in N \) and \( a_{m} \notin N \). Certainly \( O(f) \leq m \). If \( N = 0 \), the theorem is obviously true. Assume inductively that the theorem is true for rings \( R \) where \( N^r = 0 \), \( r \leq n \). We shall show that the theorem is true for rings \( R \) where \( N^n \neq 0 \) and \( N^{n+1} = 0 \). Suppose \( D(fg) < m \) where \( g = b_{m}x^m + \ldots + b_{0} \), \( b_{m} \neq 0 \). Since \( (R/N)[x] \) is an integral domain, we have \( g \in N[x] \). Since \( R \) is completely \( N \) primary, \( R/N^n \) is completely \( N/N^n \) primary. Also, by Theorem 1.3, the coset of \( f \) is regular in \( R/N^n[x] \). If \( g \in N[x] \) but \( g \notin N^n[x] \) then, by the induction hypothesis, \( D(fg) \) is not less than \( m \). If, on the other hand, \( g \in N^n[x] \) and \( D(fg) < m \) then \( a_{m}b_{m} = 0 \) which is impossible because \( a_{m} \) is a unit of \( R \). Thus, in any case, the assumption \( D(fg) < m \) leads to a contradiction. Consequently \( O(f) = m \).

**Theorem 3.3.** Let \( R \) be a completely \( N \) primary ring with \( N^n = 0 \) for some positive integer \( n \). If, for \( f \in R[x] \), the principal ideal \( (f) = fR[x] = R[x]f \) then \( (f) \) is generated by a monic polynomial of degree \( m \) if an only if \( f = a_{m}x^m + \ldots + a_{0} \) where \( a_{m+1}, \ldots, a_{m+1} \in N \) and \( a_{m} \) is a unit of \( R \).

(5) A nonzero polynomial of \( R[x] \) is called monic if its leading coefficient is a unit element of \( R \).
Proof. If \( f \) is of this form let \( B \) denote the set of elements \( x^{m-1}, \ldots, x, 1 \) of \( R[x] \). Then we may write \( R[x] = RB + (f) \). From Theorem 3.2 it follows that \( R[x] = RB + (f) \). Hence \( g = x^m + b_{m-1}x^{m-1} + \ldots + b_0 \) where \( f_1 \in (f) \) and \( b_1 \in R, i = 0, 1, \ldots, m - 1 \). Hence \( g = x^m + b_{m-1}x^{m-1} + \ldots + b_0 \). Now there exist polynomials \( h \) and \( k \) such that \( f = hg + k \) where \( k = 0 \) or \( D(k) < m \). By Lemma 3.1, \( O(f) = m \) and hence \( k = 0 \). Thus \( R[x]g = (f) \). Similarly \( gR[x] = (f) \). Thus \( (g) = (f) \) where \( g \) is monic.

Conversely, let \( (f) = (g) \) where \( g \) is monic of degree \( m \). Then \( O(f) = O(g) \) (by the definition of \( O( ) \)), while \( O(g) = m \) by Lemma 3.1 (because \( g \) is monic). Thus \( O(f) = m \), which means again by Lemma 3.1 that \( f \) is of the required form.

4. Primary ideals in \( R \), where \( R/N \) is a principal ideal domain. If \( R \) is a principal ideal domain then the A.C.C. holds for right ideals and hence \( P(q) \) is an ideal when \( q \) is a right \( P \) primary ideal of \( R \). Moreover, in this case \( P = N = 0 \) and the completely prime ideals of \( R/N \) are maximal right ideals. Hence, by Definition 0.1, a principal ideal domain is an extendable ring.

If \( q \) is a \( P \) primary ideal then \( P(q) = N(q) \) since \( P(q) \) is a completely prime ideal. Thus \( P(q) \) is also a maximal left ideal.

Theorem 4.1. If \( R \) is a principal ideal domain and \( q = (a) \) is a \( P \) primary ideal in \( R \) with \( P(q) = (b) \) then \( b \) is irreducible and \( a = vb^n = b^n \) where \( u \) and \( v \) are units of \( R \).

Proof. If \( b = cd \) then \( c \) or \( d \) must be in \( (b) \). Suppose \( d \in (b) \), say \( d = eb(a) \). Then \( b = ceb \) and \( c \) is a unit of \( R \). If \( c \in (b) \), say \( c = be \), then \( b = bed \) and \( d \) is a unit of \( R \). Hence \( b \) is irreducible.

For the second part of the proof we have \( (a) \subseteq (b) \) where \( b \) is irreducible. Let \( a = bc \) where \( c \in R \). If \( b \notin (a) \) then \( c^a \in (a) \) since \( (a) \) is \( P \) primary. Hence \( c \in P((a)) = (b) \). Thus \( c = bd \) and \( a = b^2d \). This process continues until \( a = b^n d' \) where \( b^n \in (a) \) and \( b^n - 1 \notin (a) \). Consequently, \( b^n = ae \) and \( a = aed' \). Thus \( d' \) is a unit and \( a = b^nv = ub^n \) where \( u \) and \( v \) are units of \( R \).

Lemma 4.1. If \( R/N \) is a principal ideal domain and \( q \) is an ideal of \( R \), then there is an element \( a \) in \( q \) such that \( q = aR + N' = Ra + N' \) where \( N' = q \cap N \).

Proof. Let \( N' = q \cap N \). Since \( R \) is a principal ideal domain \( \tilde{q} = (\tilde{a}) = \tilde{a}R = \tilde{R} \tilde{a} \) for some \( \tilde{a} \in \tilde{R} \). Let \( a \) be any element of \( q \) such that \( aH = a \). Obviously \( aR + N' \subseteq q \). Moreover, if \( b \in q \) then \( bH = \tilde{a} \tilde{r} \) where \( \tilde{r} \in \tilde{R} \). Thus \( b = ar + n \) where \( r \in R \) and \( n \in N \). Since \( n = b - ar, n \) is also in \( q \) and therefore \( n \in N' \). Hence \( q = aR + N' \).

Similarly \( q = Ra + N' \).

Theorem 4.2. Let \( R \) be an extendable ring such that \( R/N \) is a principal ideal domain. If \( q \) is a \( P \) primary, not nil, nontrivial ideal of \( R \) then
\[
q = (vn^2 + n)R + N' = R(vn^2 + n) + N'
\]

(9) Note that if \( b \) is any element of a principal ideal domain \( R \) such that \( bR \) is a left, whence two-sided ideal, then from [5, p. 37] \( bR = Rb \), which one may denote by \( (b) \).
where \( v \) is a unit of \( R \), \( n \in N \), \( \pi \) is an irreducible element of \( \overline{R} \) and \( N' = N \cap q \). Moreover, \( P(q) = \pi R + N \) and \( R/P(q) \) is a division ring.

**Proof.** By Lemma 4.1, \( q = aR + N' = Ra + N' \) where \( a \in q \). From 3b of [2] the not nil ideal \( aR + N' \) is \( P \) primary in \( R \) if and only if the ideal \( (a) \) is \( P \) primary in \( \overline{R} \). From Theorem 4.1 we may write \( a = \overline{\pi}^n = \overline{\pi}^n \overline{u} \) where \( \overline{\pi} \) is irreducible in \( \overline{R} \) and \( \overline{u}, \overline{v} \) are units of \( \overline{R} \). Hence \( a = \pi^nu + n_1 = v\pi^k + n_2 \) where \( u \) and \( v \) are units of \( R \) and \( n_1, n_2 \in N \).

To find \( P(q) \), write \( q = (\overline{v}\overline{\pi}^k) = (\overline{\pi}^k) = \overline{\pi}^k R \). Clearly \( \pi R \subseteq P(q) \). Moreover, since \( \overline{\pi} \) is irreducible, the ideal \( \pi R \) is maximal in \( \overline{R} \) and hence \( P(q) = \pi R \). From 2g of [2] we have \( (P(q))^- = (\pi R)^- \) and hence \( P(q) = \pi R + N \). The fact that \( R/P(q) \) is a division ring is a consequence of statement 3.1 of [2].

**Theorem 4.3.** If \( R \) is a completely \( N \) primary ring which satisfies the A.C.C. for right ideals then \( R[x] \) is an extendable ring, \( \overline{R}[x] \) is a principal ideal domain and \( \overline{R}[x] \) is \( N[x] \) primary.

**Proof.** If \( R \) satisfies the A.C.C. then \( N \) is nilpotent and, from Theorem 1.3, \( R[x] \) is \( N[x] \) primary. Since \( \overline{R} \) is a division ring, \( \overline{R}[x] \) is a principal ideal domain(7). It remains to show that \( R[x] \) is an extendable ring. If \( q \) is a right \( P \) primary ideal in \( R[x] \) then, since \( R[x] \) satisfies the A.C.C., \( P(q) \) is an ideal of \( R[x] \) and, from Theorem 1.1, \( P(R[x]) = N[R[x]] \). Finally, condition (iii) of Definition 0.1 holds since \( R[x]/N(R[x]) = R[x]/N[x] = \overline{R}[x] \) is a principal ideal domain.

5. \( R[x] \), where \( R \) is a completely \( N \) primary ring. In this section let \( R \) denote a completely \( N \) primary ring which satisfies the A.C.C. for right ideals. By Theorem 4.3, \( R[x] \) is an extendable ring, \( N(R[x]) = N[x] \) and \( (R/N)[x] \) is a principal ideal domain. In addition, by [6, p. 199], \( (N[x])^t = 0 \) for some integer \( t \).

Certainly the not nil ideals of \( R[x] \) are the regular ideals of \( R[x] \)(8). If \( q \) is a regular ideal in \( R[x] \) then, by \$4 \), \( \overline{q} = (\overline{f}) \) in \( \overline{R}[x] \) where \( \overline{f} \) is regular in \( \overline{R}[x] \). Hence all the regular ideals of \( R[x] \) are of the form \( q = fR[x] + N' \) where \( f \) is regular in \( R[x] \) and \( N' = q \cap N[x] \). By \$3 \) we have \( \Omega(f) = D(f) \).

**Theorem 5.1.** An ideal \( q \) of \( R[x] \) has finite degree if and only if \( q \) is a regular ideal. In this case, \( \deg(q) = \Omega(f) = D(f) \).

**Proof.** By Theorem 3.2, \( q \) has finite degree if and only if \( \overline{q} \) has finite degree, i.e., if and only if \( q \notin N[x] \). Thus a necessary and sufficient condition that \( q \) have finite degree is that \( \overline{q} \) be regular. Again, by Theorem 3.2, the degree of \( q \) is the same as the degree of \( \overline{q} = (\overline{f}) \). Since \( \overline{R} \) is a division ring \( D(f) = \Omega(f) \).

(7) See [5, Chapter 3].

(8) An ideal \( q \) of a ring \( R \) is called regular if it contains at least one regular element. Thus if \( R \) is \( N \) primary then it contains only regular ideals and nil ideals, for if an ideal \( q \) is not regular then every element of \( q \) is a divisor of zero and hence \( q \subseteq N \).
If \( q \) is a regular, nontrivial \( N \) primary ideal of \( R[x] \), then by Theorem 4.2, \( q = (v(x)p(x)^k + n(x))R[x] + N' \) where \( v(x) \) is a unit of \( R[x] \), \( \bar{p}(x) \) is an irreducible polynomial of \( \bar{R}[x] \), \( n(x) \in N[x] \) and \( N' = N[x] \cap q \). The radical \( N(q) = p(x)R[x] + N[x] \). Then \( \deg(q) = k \deg(\bar{p}) \) and \( \bar{R}[x]/(\bar{p}(x)) \) is a division ring. Thus \( R[x]/q \) is a completely \( N \) primary ring.

6. Simple algebraic extensions. In this section let \( R \subseteq A \) where \( R \) and \( A \) are completely \( N \) primary rings which satisfy the A.C.C. If \( \sigma \in A \), where \( a\sigma = \sigma a \) for all \( a \in R \), the symbol \( R[\sigma] \) shall denote the smallest subring of \( A \) containing \( R \) and \( \sigma \). The symbol \( R(\sigma) \) shall denote the smallest completely \( N \) primary ring containing \( R \) and \( \sigma \). In the latter case, \( R(\sigma) \) is called a simple extension of \( R \). Certainly \( R[\sigma] \subseteq R(\sigma) \) and, if \( x \) is an indeterminate, \( R[\sigma] \) is the homomorphic image of the polynomial ring \( R[x] \) under the homomorphism \( f(x) \to f(\sigma) \). Since \( R[\sigma] \) is a subring of a completely \( N \) primary ring we know that \( R[\sigma] \) is \( P \) primary. The kernel of the homomorphism must then be a \( P \) primary ideal \( q \) of \( R[x] \) and \( R[x]/q \cong R[\sigma] \). In addition \( q_\sigma = q \cap R \) is the zero ideal since \( q \) is the set of polynomials which have \( \sigma \) as a root. As in §5, \( q \) is either a regular ideal or a nil ideal.

DEFINITION 6.1. Let \( R \subseteq A \) and let \( \sigma \in A \) where \( a\sigma = \sigma a \) for all \( a \in R \). If \( \sigma \) satisfies at least one regular polynomial of \( R[x] \) then \( \sigma \) is called central algebraic with respect to \( R \). If \( \sigma \) satisfies only nilpotent polynomials of \( R[x] \) then \( \sigma \) is called central transcendental with respect to \( R(\sigma) \). The ideal \( q \) consisting of the polynomials of \( R[x] \) which have \( \sigma \) as a root is called the defining ideal of \( \sigma \).

We call \( R(\sigma) \) a simple algebraic extension of \( R \) if \( \sigma \) is algebraic with respect to \( R \) and a simple transcendental extension of \( R \) if \( \sigma \) is transcendental with respect to \( R \).

Let \( S = R(\sigma) \) be a simple algebraic extension of \( R \) and let \( q \) be the defining ideal of \( \sigma \). Then \( q \) is a not nil, nontrivial \( P \) primary ideal of \( R[x] \). By §§4 and 5 we may write \( q = (v(x)p(x)^k + n(x))R[x] + N' \) where the symbols have the same meanings as before. Then \( R[x]/q \) is a completely \( N \) primary ring whose residue class ring is isomorphic to \( \bar{R}[x]/(\bar{p}(x)) \) and \( R[x]/q \) is an extension of degree \( kD(\bar{p}(x)) \) of \( R/q_\sigma = R[\sigma] \). Since \( R[x]/q \cong R[\sigma] \) and \( R[\sigma] \) satisfies the A.C.C. we have

THEOREM 6.1. If \( S = R(\sigma) \) is a simple algebraic extension of \( R \), the defining ideal \( q \) of \( \sigma \) has the form \( q = (v(x)p(x)^k + n(x))R[x] + N' \) where \( v(x) \) is a unit of \( R[x] \), \( \bar{p}(x) \) is irreducible in \( \bar{R}[x] \), \( n(x) \in N[x] \) and \( N' = q \cap N[x] \). Then \( S = R(\sigma) = R[\sigma] \) which satisfies the A.C.C. Moreover, \( S \) is a finite extension of \( R \) where \( [S:R] = kD(\bar{p}(x)) \). The division ring \( S = R(\bar{\sigma}) \) is obtained from \( R \) by the adjunction of the zero \( \bar{\sigma} \) of the irreducible polynomial \( \bar{p}(x) \in \bar{R}[x] \) and hence \( [S:R] = k[S:R] \).

(9) Hereafter we shall refer to central algebraic (central transcendental) elements as algebraic (transcendental) elements.
Next we shall prove

**Theorem 6.2.** Let $S = R(\sigma)$ be a simple algebraic extension of $R$ where $[S : R] = k[S : R]$. Then, $S$ is a principal extension of $R$ if and only if $k = 1$. For any $k$, $N(S) = p(\sigma)R[\sigma] + N[\sigma]$ and hence there exists a positive integer $h$ such that $N(S)^h = 0$.

**Proof.** Writing $q$ in the form stated in Theorem 6.1 we have, as in §4, $N(q) = p(x)R[x] + N(R[x])$. Since $R[x]$ is a principal extension of $R$ this can be written $N(q) = p(x)R[x] + N \cdot R[x] = p(x)R[x] + N[x]$. It follows from 2h of [2] that $N(R[x]/q) = N(q)/q = (p(x)R[x] + N[x])/q$. The isomorphism from $R[x]/q$ onto $S = R(\sigma)$ maps $(p(x)R[x] + N[x])/q$ onto $p(\sigma)R[\sigma] + N[\sigma]$ and hence $N(S) = p(\sigma)R[\sigma] + N[\sigma]$. Now $N(q)/q$ is a nil ideal in $R[x]/q$ and, since the A.C.C. holds, the ideal $N(q)$ of $R$ is nilpotent modulo $q$. Thus there is a positive integer $h$ such that $N(S)^h = 0$. Finally, $S$ is a principal extension of $R$ if and only if $p(\sigma) \in N[\sigma]$; i.e., if and only if $p(x) \in q'$, where $q' = p(x)R[x] + N[x]$. If $p(x) \in q'$ then $q'$ contains a regular polynomial of degree $D(p(x))$. However, from §3, the minimal degree of the regular polynomials in $q'$ is $D(p(x)) = kD(p(x))$. Hence $kD(p(x)) \leq D(p(x))$ and therefore $k = 1$. Conversely if $k = 1$, the extension is clearly principal.

An element $\sigma$ of a ring $A$ is called principal with respect to a subring $R$ if $R(\sigma)$ is a principal extension of $R$. It follows from Theorem 6.2 that an algebraic element $\sigma$ is principal if and only if it is a root of a nontrivial fundamental irreducible(10) of $R[x]$.

**Example 6.1.** Let $R$ be a completely $N$ primary ring satisfying the A.C.C. If $x$ is an indeterminate, the ring $R[x]$ is $N$ primary. Let $q$ be a regular, nontrivial $N$ primary ideal of $R[x]$ such that $q_\ast = 0$. As above, we have that $q = (a(x)p(x)^k + n(x))R[x] + N'$ and $R[x]/q$ is a completely $N$ primary ring which contains $R$. Setting $\sigma = \tilde{x}$, where $\tilde{x}$ is the coset of $x$ in $R[x]/q$, then $\sigma$ is algebraic over $R$ with defining ideal equal to $q$.

**Example 6.2.** Let $D$ be the division ring of quaternions with coefficients in the rational numbers and $D^*$ the division ring of quaternions with coefficients in the real numbers. For an indeterminate $x$, $R = D[x]/(x^*)$ is completely $N$ primary and is contained in the completely $N$ primary ring $R^* = D^*[x]/(x^*)$. If $\sigma = \sqrt{2}$, then $R_1 = R(\sqrt{2}) = D_1[x]/(x^*)$ where $D_1$ is the ring of quaternions with coefficients in the set of all real numbers of the form $a + b\sqrt{2}$ where $a$ and $b$ are rational numbers. Thus $R_1$ is a simple algebraic extension of $R$ of degree 2.

For an indeterminate $y$, the ring $R[y]$ is $N$ primary. Then $y^2 - 2$ is a minimal degree polynomial satisfied by $\sqrt{2}$. One can use the division algorithm to show that the defining ideal of $\sqrt{2}$ is $q = (y^2 - 2)R[y]$. Then $q$ is $N$ primary and $R[y]/q \cong R_1$ where $R_1$ is an extension of $R$ of degree 2. Similarly, we could adjoin

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(10) See [2,2e] for the meaning of this term.
to \( R_1 \) the element \( \sqrt{3} \). Thus \( R_2 = R_1(\sqrt{3}) = D_2[x]/(x^n) \) where \( D_2 \) is the ring of quaternions with coefficients in the set of all real numbers of the form \( a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \) where \( a, b, c \) and \( d \) are rational numbers. Then \( R_2 \) is an extension of \( R_1 \) of degree 2 and an extension of \( R \) of degree 4.

7. Quotient rings. If a ring \( R \) has a right quotient ring as described in [5, p. 118] we shall denote this ring by \( Q(R) \). Then \( Q(R) \) is a ring containing \( R \) such that every regular element of \( R \) has an inverse in \( Q(R) \) and any element of \( Q(R) \) may be written in the form \( ab^{-1} = a/b \) where \( a, b \in R \) and \( b \) is regular. A necessary and sufficient condition for the existence of \( Q(R) \) is that for any pair of elements \( a, b \) in \( R \), \( b \) regular, there exists a common right multiple \( m = ab_1 = ba_1 \) such that \( b_1 \) is regular. We shall use this criterion to establish the following theorem, which generalizes a result of A. W. Goldie [4, p. 592].

**Theorem 7.1.** Let \( R \) be a ring with identity which satisfies the A.C.C. for right ideals and suppose the elements not in \( N(R/N^k) \) are regular in \( R/N^k \) for all positive integers \( k \). Then \( Q(R) \) exists.

**Proof.** Since the elements not in \( N \) are regular in \( R, R/N \) is an integral domain. Hence, if \( N = 0 \), then by Theorem 1 of [4], \( Q(R) \) exists. We proceed by induction on the smallest integer \( n \) such that \( N^n = 0 \). Assume that the theorem is true when \( N^k = 0 \) and \( N^{k-1} \neq 0 \). In \( R \) suppose \( N^{k+1} = 0 \) and \( N^k \neq 0 \). Then the ring \( \bar{R} = R/N^k \) satisfies the hypothesis of the theorem and, by the induction hypothesis, \( Q(\bar{R}) \) exists.

Let \( a \neq 0 \) and \( b \notin N \). If \( a \in N^k \) we consider the right ideals \( I_n = aR + baR + \ldots + b^naR, n = 0, 1, 2, \ldots \). By the A.C.C., \( I_i = I_{i+1} \) for some integer \( t \) and we may write \( b^{t+1}a = ar_0 + bar_1 + \ldots + bar_n, r_i \in R \). Since \( b^{t+1}a \neq 0 \) and \( N^{k+1} = 0 \), not all the \( r_i \) are in \( N \). Let \( i = h \) be the first subscript for which \( r_i \notin N \). It follows that \( b^{t+1-k}a = ar_h + \ldots + b^{t-k}ar_t \) and hence \( a(b^{t-k}a - ar_{h+1} - \ldots - b^{t-k}ar_t) = ar_h \) where \( r_h \) is regular. Thus \( a \) and \( b \) have a common right multiple. If, on the other hand, \( a \notin N^k \), then, since \( Q(\bar{R}) \) exists, we have \( a\bar{c} = \bar{b}\bar{d} \) in \( \overline{R} = R/N^k \) where \( c \notin N \). Thus \( a\bar{c} = \bar{b}\bar{d} + e \) with \( e \in N^k \). If \( e = 0 \), we stop. Otherwise, as above, write \( b\bar{f} = \bar{e}g \) where \( g \notin N \), whence \( acg = b(dg + f) \) with \( cg \) regular. Thus \( Q(R) \) exists.

**Theorem 7.2.** If \( R \) is a completely \( N \) primary ring which satisfies the A.C.C. for right ideals then \( R[x] \) has a right quotient ring \( Q(R[x]) \).

**Proof.** From Theorem 1.3, for any integer \( n \), the divisors of zero of \( R/N^n[x] \) are contained in \( N/N^n[x] \). Hence \( R[x] \) satisfies the hypothesis of Theorem 7.1.

**Lemma 7.1.** If a ring \( R \) has a right quotient ring \( Q(R) \) and if \( R \) satisfies the A.C.C. for right ideals then \( Q(R) \) satisfies the A.C.C. for right ideals.

**Proof.** Let \( k \) and \( h \) denote right ideals of \( Q(R) \) where \( k \subseteq h \). As in the proof of Lemma 1.3 of [3] it follows that \( k_h \subset h_h \) in \( R \). The lemma is now immediate.
Lemma 7.2. If a ring $R$ with A.C.C. on right ideals has a right quotient ring $Q(R)$ and if the elements not in $N$ are regular in $R$ then $Q(R)$ is completely $N$ primary.

Proof. Let $T = \{a/b|a/b \in Q(R), a \in N\}$. For any elements $a/b$ and $c/d$ of $T$, there exist regular elements $b_1$ and $d_1$ in $R$ such that $m = db_1 = bd_1$. Using the rule for addition in $Q(R)$ we have $a/b - c/d = (a/d_1 - c/b_1)/m$ which is in $T$ since $a$ and $c$ are in $N$. Next, consider any elements $a/b$ of $T$ and $c/d$ of $Q(R)$. Let $c_1, b_1 \in R, b_1$ regular, such that $cb_1 = bc_1$. Then $(a/b)(c/d) = (ac_1/db_1)$ which proves that $TQ(R) \subseteq T$. Similarly, one can prove that $Q(R)T \subseteq T$. Thus $T$ is an ideal in $Q(R)$. It is easily seen that for any positive integer $n$, the product of $n$ elements of $T$ can be written in the form $a/b$ where $a \in N^n$ and $b$ is regular. Since $N$ is nilpotent, $T \subseteq N(Q(R))$. Also, the elements of $Q(R)$ which are not in $T$ are units and hence $Q(R)/T$ is a division ring. Then $T$ is maximal ideal and $T = N(Q(R))$. Thus $Q(R)$ is completely $N$ primary.

Lemma 7.3. If a ring $R$ has a right quotient ring $Q(R)$ which is completely $N$ primary then $Q(R)$ is the smallest completely $N$ primary ring containing $R$.

Proof. Let $R'$ be a completely $N'$ primary ring where $N' = N(R')$ and suppose $R \subseteq R'$. If $b \in R$ is regular then $b \notin N'$ and hence $b$ has an inverse $b^{-1}$ in $R'$. Thus $a, b \in R, b$ regular implies $ab^{-1} \in R'$; that is, $Q(R) \subseteq R'$.

Theorem 7.3. Let $R$ be a completely $N$ primary ring which satisfies the A.C.C. for right ideals and let $q$ be a $P$ primary ideal of $R[x]$ with $q \subseteq N(R[x])$. Then $Q(R[x]/q)$ exists and is completely $N$ primary. Moreover, $Q(R[x]/q)$ satisfies the A.C.C. for right ideals and is the smallest completely $N$ primary ring containing $R[x]/q$.

Proof. We know, by Theorem 7.2, that $Q(R[x])$ exists. Furthermore, as in the proof of Lemma 1.2 of [3], one can show that $q = q^* \cap R[x]$, where $q^*$ is the extension of $q$ to $Q(R[x])$. It follows that the mapping $f(x) + q \rightarrow f(x) + q^*$, $f(x) \in R[x]$, is an isomorphism of $R[x]/q$ into $Q(R[x])/q^*$. We shall identify $R[x]/q$ with the subring of $Q(R[x])/q^*$ which corresponds to $R[x]/q$ under this isomorphism.

If $f(x) + q$ is regular in $R[x]/q$ then $f(x) \notin N[x]$. By Theorem 1.3, $R[x]$ is $N[x]$ primary and consequently $f(x)$ is regular in $R[x]$. Hence $f(x)$ has an inverse $f(x)^{-1}$ in $Q(R[x])$ and we have $f(x)^{-1} + q^* = (f(x) + q^*)^{-1}$. Thus the regular elements of $R[x]/q$ have inverses in $Q(R[x])/q^*$. Now let $f(x)g(x)^{-1} + q^* \in Q(R[x])/q^*$ where $f(x), g(x) \in R[x], g(x)$ regular. Then $f(x)g(x)^{-1} + q^* = (f(x) + q^*)(g(x) + q^*)^{-1}$. This proves that $Q(R[x])/q^*$ is a right quotient ring for $R[x]/q$.

The remaining part of the theorem follows from Lemmas 7.1, 7.2 and 7.3.

8. Simple transcendental extensions. In this section, $R$ will always denote a completely $N$ primary ring which satisfies the A.C.C. for right ideals and $A$ will denote a completely $N$ primary ring which contains $R$. 
Let $\sigma \in A$ be transcendental over $R$. Then $R[\sigma]$ is an $N$ primary ring where $N(R[\sigma]) = N(A) \cap N[R[\sigma]]$. For an indeterminate $x$ we have the usual homomorphism $\theta$ of $R[x]$ onto $R[\sigma]$ defined by $f(x)\theta = f(\sigma)$. The defining ideal $q$ of $\sigma$ is then a nil, $P$ primary ideal of $R[x]$ and $q_* = q \cap R = 0$. Since $R[x]$ satisfies the A.C.C. for right ideals, $q$ is a nilpotent ideal and $q \subseteq N(R[x]) = N[x]$ by [2, §1]. By Theorem 7.3, $Q(R[x]/q)$ exists and, since $R[\sigma] \cong R[x]/q$, $Q(R[\sigma])$ exists. Moreover, $Q(R[\sigma])$ is the smallest completely $N$ primary ring containing $R$ and $\sigma$. Hence $Q(R[\sigma]) = R(\sigma)$, the simple transcendental extension of $R$ by $\sigma$. Also, $R(\sigma)$ satisfies the A.C.C. for right ideals. This establishes the first part of

**Theorem 8.1.** If $S = R(\sigma)$ is a simple transcendental extension of $R$ then $R(\sigma) = Q(R[\sigma])$ and $R(\sigma)$ satisfies the A.C.C. for right ideals. The division ring $(R(\sigma))^\sim = R(\sigma)$ is obtained by adjoining the transcendental element $\sigma$ to $R$. To every unit of $S$ a unique order can be associated.

To prove the last part of the theorem we observe from above that $(R[\sigma])^\sim \cong (R[x]/q)^\sim \cong (R/N)[x] = R[x]$, where $R$ is a division ring. To every nonzero element $\tilde{r}$ of $(R[\sigma])^\sim$ a unique degree is associated, namely the degree of the polynomial of $R[x]$ which is the image of $\tilde{r}$ under the isomorphism from $(R[\sigma])^\sim$ onto $R[x]$. If we extend this isomorphism to an isomorphism from $Q((R[\sigma])^\sim)$ onto $R(x) = Q(R[x])$ then to every nonzero element of $Q((R[\sigma])^\sim)$ a unique degree is associated, namely the degree of the corresponding (image) element of $R(x)$. (The degree of a fraction, by definition, is the maximum of the degrees of the numerator and denominator.) Since $(Q(R[\sigma]))^\sim \cong (Q(R[\sigma])^\sim)$ there is a unique degree associated with each element of $(Q(R[\sigma]))^\sim$. Now the set of not nilpotent elements of the completely $N$ primary ring $Q(R[\sigma]) = R(\sigma)$ coincides with the set of units of $Q(R[\sigma])$. We define the order of a unit $r$ of $R(\sigma)$ as the degree of the element $\tilde{r}$ onto which $r$ is mapped by the natural homomorphism from $Q(R[\sigma])$ onto $(Q(R[\sigma]))^\sim$. Thus, to each unit of $Q(R[\sigma]) = R(\sigma)$ a unique order is associated.(11)

**Theorem 8.2.** Every completely $N$ primary ring $R$ which satisfies the A.C.C. for right ideals is properly contained in just such a ring. Specifically, $R \subseteq Q(R[x])$, which is a completely $N$ primary ring satisfying the A.C.C. for right ideals.

**Proof.** See Theorem 7.2, Lemma 7.1 and Lemma 7.2.

**Lemma 8.1.** Let $S$ be a principal extension of $R$ and let $N(R) = P(R)$ and $N(S) = P(S)$. Let $n$ be an ideal in $N(S)$ and suppose that $N(S/n) = N(S)/n$. Then $S/n$ is a principal extension of $R/n^*$, where $n^* = n \cap R$.

(11) Note that for $\sigma$ algebraic we have $R[\sigma] = R(\sigma)$ and hence in this case it is also true that $Q(R[\sigma]) = R(\sigma)$. 

**E. H. FELLER AND E. W. SWOKOWSKI**

[November]
Proof. As in [2, §2], we may assume that $\bar{R} = R/n_\ast \subseteq S/n$ where, for any set $B \subseteq S$, $\bar{B}$ denotes the image of $B$ under the natural homomorphism from $S$ to $S/n$. Let $v$ be an element of $S$ such that $\bar{v} \in N(S/n) = N(S)/n$. Then $v \in N(S)$ and, since $S$ is a principal extension of $R$ we may write $v = \sum v_i \sigma_i$ where $v_i \in N(R)$ and $\sigma_i \in S$. Hence $v = \sum \bar{v}_i \bar{\sigma}_i$ where $\bar{v}_i \in N(R(n)/n_\ast)$ and $\bar{\sigma}_i \in S/n$. Thus $S/n$ is a principal extension of $R/n_\ast$.

Theorem 8.3. If $S$ is a simple transcendental extension of $R$ then $S$ is a principal extension of $R$.

Proof. Let $S = R(\sigma)$ and let $n$ denote the defining ideal of the transcendental element $\sigma$. Applying Lemma 8.1 to $R[x]$ and $R$, and noting that $n_\ast = n \cap R = 0$, we have that $R[x]/n$ is a principal extension of $R/n_\ast = R$. Thus $R[\sigma] = R[x]/n$ is a principal extension of $R$. Now $Q = Q(R[\sigma])$ is a principal extension of $R[\sigma]$ since $N(Q)$ consists of elements of the form $a/b = ab^{-1}$ where $a \in N(R[\sigma])$ and $b^{-1} \in Q(R[\sigma])$. Hence $N(Q) = N(R[\sigma]) \cdot Q(R[\sigma]) = N(R) \cdot R[\sigma] \cdot R(\sigma) = N(R) \cdot R(\sigma)$, i.e., $Q(R[\sigma]) = R(\sigma)$ is a principal extension of $R$.

Example 8.1. Let $R$ be a completely $N$ primary ring which satisfies the A.C.C. for right ideals. Let $x$ be an indeterminate and let $n$ be any nil, $P$ primary ideal of $R[x]$ such that $n_\ast = 0$. Setting $\sigma = \bar{x}$, where $\bar{x}$ is the coset of $x$ in $R[x]/n$, then $\sigma$ is transcendental over $R$ with defining ideal $n$.

Example 8.2. Let $R$ and $R^*$ be as in Example 6.2 and let $F$ be the field of rational numbers. Then for the transcendental number $\pi$ we have $R(\pi) = D'[x]/(x^n)$ where $D'$ is the ring of quaternions with coefficients of the form $p(\pi)/q(\pi)$ where $p(\pi)$ and $q(\pi)$ are elements of $F[\pi]$, $q(\pi) \neq 0$.

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University of Wisconsin-Milwaukee,
Milwaukee, Wisconsin
Marquette University,
Milwaukee, Wisconsin