ON RING EXTENSIONS FOR COMPLETELY
PRIMARY NONCOMMUTATIVE RINGS

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0. Introduction. It is the authors' purpose in this paper to initiate the study of ring extensions for completely $N$ primary noncommutative rings which satisfy the ascending chain condition for right ideals (A.C.C.). We begin here by showing that every completely $N$ primary ring $R$ with A.C.C. is properly contained in just such a ring. This is accomplished by first showing that $R[x]$, $x$ an indeterminate where $ax = xa$ for all $a \in R$, is $N$ primary and then constructing the right quotient ring $Q(R[x])$. The details of these results appear in §§1, 7 and 8. The corresponding results for the commutative case are given by E. Snapper in [7] and [8].

If $R \subseteq A$, where $A$ is completely $N$ primary with A.C.C. then, from the discussion in the preceding paragraph, it would seem natural to examine the structure of $R(\sigma)$ when $\sigma \in A$ and $ao = a\sigma$ for all $a \in R$ in the cases where $\sigma$ is algebraic or transcendental over $R$. These structures are determined in §§ 6 and 8 of the present paper.

The definitions and notations given in [2] will be used throughout this paper. As in [2], for a ring $R$, $N$ or $N(R)$ denotes the union of nilpotent ideals of $R$, $P$ or $P(R)$ denotes the set of nilpotent elements of $R$ and $J$ or $J(R)$ the Jacobson radical of $R$. The letter $H$ is used for the natural homomorphism from $R$ to $R/N = \bar{R}$. If $B$ is a subset of $R$ then $\bar{B}$ denotes the image of $B$ under $H$. If $N = P$ in $R$ and if $R'$ is a ring contained in $R$ then $N(R') = N \cap R'$ and $\bar{R}' = R'/N(R')$. Thus we consider the contraction of $H$ on $R'$ as the natural homomorphism from $R'$ onto $R'/N(R')$.

Unlike the commutative case, the results of this paper will at times depend on the three conditions (i), (ii) and (iii) of [2, §3]. Therefore, we make the following definition.

**Definition 0.1.** A ring $R$ with identity is called an extendable ring if it satisfies the three conditions:

(i) $P(q)$ is an ideal when $q$ is a right $P$ primary ideal.$^{(1)}$

(ii) $P(R) = N(R)$.

(iii) The nontrivial completely prime ideals of $R/N$ are maximal right ideals.

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$(1)$ Ideal shall always mean two-sided ideal.

$(2)$ See [2, §1] for the meaning of $P(q)$.
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1. Properties of $R[x]$. In [2] we defined a ring $R$ to be $N$ primary if $ab = 0$, $a \neq 0$ implies $b \in N$ and $b \neq 0$ implies $a \in N$. A ring $R$ with identity is completely $N$ primary if $R/N$ is a division ring. Note that if $R$ is $N$ primary or completely $N$ primary then $N$ is a completely prime ideal and $N = P$. For the rings considered in this paper $N$ and $P$ shall always be equal.

**Theorem 1.1.** If $R$ is $N$ primary then, in the polynomial ring $R[x]$, $N[x] = P[x] = N(R[x]) = P(R[x]) = J(R[x])$.

**Proof.** We may write $R[x]/N[x] = (R/N)[x]$ where $N[x] \subseteq J(R[x])$. Since $R/N$ is an integral domain it contains no nonzero nil ideals and hence, from Theorem 4 of [6, p. 12], $R[x]/N[x]$ is semisimple. Thus $J(R[x]) = N[x]$.

Certainly $N[x] = P[x] \subseteq P(P[x])$. We now show that $P(P[x]) \subseteq P[P[x]]$. If $f \in P(P[x])$ then $f^n = 0$ for some positive integer $n$. Hence in $R[x]/P[x] \cong (R/P)[x]$ we have $(f^n) = 0$. Since $(R/P)[x]$ has no divisors of zero it follows that $f \in P[x]$.

Next we show that $N(R[x]) = N[x]$. Clearly $N[R[x]] \subseteq P(R[x]) = N[x]$. If $f \in N[x]$ then $f = a_n x^n + \ldots + a_1 x + a_0$ for $a_i \in q_i$ where $q_i$ is an ideal of $R$ and $q_i = 0$ for positive integers $t_i$, $i = 0, 1, \ldots, n$. Thus $f$ is contained in the ideal $k = q_0[x] + q_1[x] + \ldots + q_n[x]$ and $k^{t_0 + t_1 + \ldots + t_n + 1} = 0$. Hence $f \in N(R[x])$. This completes the proof.

As a consequence of this theorem we have, for an $N$ primary ring $R$, that $R[x]/N(R[x]) = R[x]/N[x] \cong (R/N)[x]$. Thus we can consider $(R[x])^{-1}$ as $R[x]$. In this case, the natural homomorphism $H$ from $P[x]$ onto $P[x]$ maps the polynomial $\sum a_i x^i$ on $\sum a_i x^i$.

**Theorem 1.2.** Let $R$ be an $N$ primary ring. Then $f = a_n x^n + \ldots + a_0$ is a unit of $R[x]$ if and only if $a_0$ is a unit of $R$ and $a_1, \ldots, a_n \in N$.

**Proof.** If $f$ is of this form then $f$ is a unit of $R[x]$. Hence by 2a of [2] $f$ is a unit of $R[x]$.

Conversely, if $f = a_n x^n + \ldots + a_0$ is a unit of $R[x]$ then $f = \bar{a}_n x^n + \ldots + \bar{a}_0$ is a unit of $R[x]$. But since $\bar{R}$ is an integral domain it follows that $f = \bar{a}_0$ is a unit of $\bar{R}$ and $\bar{a}_1 = \ldots = \bar{a}_n = 0$. Thus $a_0$ is a unit of $R$ and $a_1, \ldots, a_n \in N(R)$.

**Theorem 1.3.** If $R$ is completely $N$ primary and $N^n = 0$ for some positive integer $n$, then $R[x]$ is $N[x]$ primary.

**Proof.** This proof will be by induction on the smallest integer $n$ such that $N^n = 0$. If $n = 1$ obviously $R[x]$ is $N(R[x])$ primary. By Theorem 1.1 $N(R[x]) = N[x]$ and thus $R[x]$ is $N[x]$ primary.
Suppose the theorem is true for all rings $R$ where $N^n = 0$, $N^{n-1} \neq 0$. Let $R$ satisfy the conditions of the theorem and $N^{n+1} = 0$, $N^n \neq 0$. Consider the ring $R/N^n$. The radical of this ring is $N/N^n$ and $(N/N^n)^n$ is the zero coset. In addition since $(R/N^n)/(N/N^n) \cong R/N$ we have that $R/N^n$ is completely $N/N^n$ primary. Hence, by the induction hypothesis $(R/N^n)[x]$ is $(N/N^n)[x]$ primary.

Suppose in $R[x]$ we have $fg = 0$, $g \neq 0$ and $f \notin N[x]$. Since $(R/N)[x]$ is an integral domain $g \in N[x]$. If $g \notin N^n[x]$, then $\tilde{f}g = 0$ in $(R/N^n)[x]$ and from above $\tilde{f} \in (N/N^n)[x]$. Thus $f \in N[x]$, a contradiction. Suppose $g \in N^n[x]$. Let $a_i$ be the coefficient of the highest power of $x$ in $f$ which is not in $N$. Since $N^{n+1} = 0$ and $fg = 0$, we have $a_i b_i = 0$ where $g = b_0 x^i + \ldots + b_0, b_i \neq 0$. Since $R$ is completely $N$ primary $a_i \in N$. Thus in any case the assumption that $fg = 0$, $g \neq 0$, $f \notin N[x]$ leads to a contradiction. Hence if $g \neq 0$ we can only conclude that $f \in N[x]$. Similarly if $gf = 0$ and $f \neq 0$ then $g \in N[x]$. Therefore, $R[x]$ is $N[x]$ primary and the proof by induction is complete.

2. Extensions of rings. A ring $A$ is an extension of a ring $R$ if $R \subseteq A$. In the remainder of this paper, unless otherwise stated, we assume that if $A$ is an extension of $R$ then $R$ and $A$ have the same identity element. If $R \subseteq A$ then $A$ can be considered as an $R$ module with submodule $R$. Thus, in order to develop a theory for extensions of a ring, it is convenient to discuss some notions concerning modules.

Let $R$ be a ring with identity and $M$ a unital $R$ module. If $R_1$ and $M_1$ are nonempty subsets of $R$ and $M$ respectively, then $R_1 M_1$ denotes the set of all finite sums $\sum r_i m_i$ where $r_i \in R_1$ and $m_i \in M_1$. A subset $S$ of $M$ is called a generating system of a submodule $Q$ of $M$ if $Q = RS$. If $S$ contains a finite number of elements then $Q$ is said to be finitely generated. A finite generating system $S$ of $Q$ is called a basis of $Q$ if $Q$ does not have a generating system containing fewer elements than $S$. If $Q$ is finitely generated then the rank $\rho(Q)$ of $Q$ is the number of elements in a basis of $Q$.

The following theorem was proved in [7, p. 685] for commutative rings. The proof carries over immediately to the noncommutative case.

**Theorem 2.1.** Let $M$ be an $R$ module and $Q$ a submodule of $M$ such that $M - Q$ is finitely generated. If $q$ is any $J$ ideal of $R$ and $Q'$ any submodule of $M$ let $\tilde{S}$ denote the image of the subset $S$ of $M$ under the natural homomorphism from $M$ onto $M - qQ'$. Then $M = Q + RS$ if and only if $\tilde{M} = \tilde{Q} + R\tilde{S}$. Consequently $\rho(M - Q) = \rho(\tilde{M} - \tilde{Q})$ and if $\tilde{M} = \tilde{Q}$ then $M = Q$.

If a ring $A$ has finite rank as a module over a subring $R$ we call the rank the degree $[A : R]$ of the ring extension and say that $A$ is a finite extension of $R$. 

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(4) $R$ module shall always mean left $R$ module. For a discussion of left $R$ modules read Chapter I of [6].

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Theorem 2.2. Let $R$ and $A$ be rings where $R \subseteq A$ and suppose every right unit of $A$ is a left unit. Then $R = A$ if and only if $[A : R] = 1$.

Proof. If $R = A$ then the identity element of $A$ is a basis of $A$ and hence $[A : R] = 1$. Conversely if $[A : R] = 1$, let $z$ be a basis of $A$. Then since $1 \in A$ we have $az = 1$ for some $a \in R$. Thus $z$ is a right and left unit of $A$ and certainly regular. Then for $z^2 \in A$ we have $bz = z^2$ for some $b \in R$ which implies that $b = z \in R$. Thus $R = A$.

By statements 1.1 and 1.2 of [2] it follows that the conditions of Theorem 2.2 are satisfied when $A$ is a $J$ primary ring. In addition, if $R$ and $A$ have the same identity element and $A/N$ is a principal ideal domain, then by the discussion in §5 of [2] we know that Theorem 2.2 is valid.

If $R$ and $A$ are rings where $R \subseteq A$, the contraction $q_*$ of an ideal $q$ of $A$ is defined as the largest ideal of $R$ which is contained in $q$, i.e., $q_* = q \cap R$. The extension $q^*$ of an ideal $q$ of $R$ in $A$ is defined as the smallest ideal of $A$ which contains $q$, i.e., $q = AqA$, the set of all sums $\sum a_i q_i b_i$ where $q_i \in q$ and $a_i, b_i \in A$. Thus, by definition, $N(R)^* = AN(R)A$. Of particular importance to us is the case when $N(R)^* = N(R)A$ and in addition $N(R)^* = N(A)$. We make the following Definition 2.1. A ring $A$ is called a principal extension of a subring $R$ if $N(A) = N(R)^* = N(R)A$.

If $R$ is $N$ primary then, by Theorem 1.1, the ring $R[x]$ is a principal extension of $R$.

Theorem 2.3. Let $A$ be a finite principal extension of $R$. If $S$ is a subset of $A$ then $A = RS$ if and only if $A = RS$. Hence $[A : R] = [A : R]$ and if $R = A$ then $R = A$.

Proof. If $A$ is a principal extension of $R$ then $N(A) = N(R)A$. Consider $A$ as an $R$ module with $N(R) \subseteq J(R)$. The first part of the theorem then follows from Theorem 2.1 by setting $A = Q', M = 0$ and $q = N(R)$. If in Theorem 2.1 we let $A = Q' = M$, $Q = R$ and $q = N(R)$ it follows that $R = A$ implies $R = A$.

Theorem 2.4. Let $R = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n$, where $A_i$ is a finite principal extension of $A_{i-1}$ of degree $[A_i : A_{i-1}] = r_i$ for $i = 1, 2, \ldots, n$. Then, if all the rings $R, A_1, \ldots, A_{n-1}$ are completely $N$ primary, $[A_n : R] = r_1 r_2 \ldots r_n$.

Proof. Since $N(A_n) = N(R)A_1 A_2 \ldots A_n = N(R)A_n$ it follows that $A_n$ is a finite principal extension of $R$. Hence $[A_n : R] = [A_n : R]$. Since $A_i$ is a finite principal extension of $A_{i-1}$, $r_i = [A_i : A_{i-1}]$. Consequently, since $R, A_1, \ldots, A_{n-1}$ are division rings we have $[A_n : R] = r_1 r_2 \ldots r_n$.

3. Degrees of ideals.

Definition 3.1. Let $R$ and $A$ be rings where $R \subseteq A$. An ideal $p$ of $A$ has finite degree $\operatorname{deg}(p)$ if the ring $A/p$ is a finite extension of $R/p_*$. If $p$ has finite degree then $\operatorname{deg}(p) = [A/p : R/p_*]$. 

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Definition 3.1 is equivalent to saying that if the rank of the $R$ module $A - p$ is finite then $\deg(p) = \rho(A - p)$.

**Theorem 3.1.** Let $A$ be a principal extension of $R$ and $p$ an ideal of $A$ of finite degree. If $S$ is a subset of $A$ then $A = p + RS$ if and only if $A = \bar{p} + \bar{R}S$. Hence $\deg(p) = \deg(\bar{p})$.

**Proof.** Apply Theorem 2.1 with $M = Q' = A$, $Q = p$ and $q = N(R)$.

The following two results are proved in [7] for commutative rings. Using the definitions and results listed above the proofs now carry over, without essential modification, to the noncommutative case.

**Statement 3.1.** An ideal $p$ of $R[x]$ has finite degree if and only if $p$ contains a monic polynomial\(^a\).

**Theorem 3.2.** If $R$ is $N$ primary, then an ideal $p$ of $R[x]$ has finite degree if and only if $\bar{p}$ has finite degree in $R[x]$. In this case if $B$ is a subset of $R[x]$ then $R[x] = RB + p$ if and only if $\bar{R}[x] = \bar{R}B + \bar{p}$. Consequently $\deg(p) = \deg(\bar{p})$.

Notice that if $R$ is a division ring then $R[x]$ is a principal ideal domain and if $\bar{p} = (f)$ then $\deg(\bar{p}) = D(f)$. (The symbol $D(f)$ denotes the degree of the polynomial $f$.)

**Definition 3.2.** The order of a regular polynomial $f$ of $R[x]$ is the minimal degree of the nonzero polynomials of $f/R[x]$. The order of $f$ is denoted by $O(f)$.

Note that $O(f) = D(f)$ when $R$ is an integral domain.

**Lemma 3.1.** If $R$ is completely $N$ primary and $N^n = 0$ for a positive integer $n$, then $O(f)$ is equal to the exponent of the highest power of $x$ in $f$ whose coefficient is a unit of $R$.

**Proof.** Let $f = a_0x^l + \ldots + a_m x^m + \ldots + a_0$ where $a_l, \ldots, a_{m+1} \in N$ and $a_m \notin N$. Certainly $O(f) \leq m$. If $N = 0$, the theorem is obviously true. Assume inductively that the theorem is true for rings $R$ where $N^r = 0$, $r \leq n$. We shall show that the theorem is true for rings $R$ where $N^{n+1} = 0$. Suppose $D(fg) < m$ where $g = b_0x^s + \ldots + b_0$, $b_s \neq 0$. Since $(R/N)[x]$ is an integral domain, we have $g \in N[x]$. Since $R$ is completely $N$ primary, $R/N^n$ is completely $N/N^n$ primary. Also, by Theorem 1.3, the coset of $f$ is regular in $R/N^n[x]$. If $g \in N[x]$ but $g \notin N^n[x]$ then, by the induction hypothesis, $D(fg)$ is not less than $m$. If, on the other hand, $g \in N^n[x]$ and $D(fg) < m$ then $a_m b_s = 0$ which is impossible because $a_m$ is a unit of $R$. Thus, in any case, the assumption $D(fg) < m$ leads to a contradiction. Consequently $O(f) = m$.

**Theorem 3.3.** Let $R$ be a completely $N$ primary ring with $N^n = 0$ for some positive integer $n$. If, for $f \in R[x]$, the principal ideal $(f) = fR[x] = R[x]f$ then $(f)$ is generated by a monic polynomial of degree $m$ if an only if $f = a_0x^l + \ldots + a_m x^m + \ldots + a_0$ where $a_l, \ldots, a_{m+1} \in N$ and $a_m$ is a unit of $R$.

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\(^a\) A nonzero polynomial of $R[x]$ is called monic if its leading coefficient is a unit element of $R$. 

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Proof. If \( f \) is of this form let \( B \) denote the set of elements \( x^{m-1}, \ldots, x, 1 \) of \( R[x] \). Then we may write \( R[x] = RB + (f) \). From Theorem 3.2 it follows that \( R[x] = RB + (f) \). Thus \( x^n = -b_{m-1}x^{m-1} - \ldots - b_0 + f_1 \) where \( f_1 \in (f) \) and \( b_i \in R, i = 0, 1, \ldots, m - 1 \). Hence \( g = x^n + b_{m-1}x^{m-1} + \ldots + b_0 \in (f) \). Now there exist polynomials \( h \) and \( k \) such that \( f = hg + k \) where \( k = 0 \) or \( D(k) < m \). By Lemma 3.1, \( O(f) = m \) and hence \( k = 0 \). Thus \( R[x] g = (f) \). Similarly \( gR[x] = (f) \). Thus \( (g) = (f) \) where \( g \) is monic.

Conversely, let \( (f) = (g) \) where \( g \) is monic of degree \( m \). Then \( O(f) = O(g) \) (by the definition of \( O(f) \)), while \( O(g) = m \) by Lemma 3.1 (because \( g \) is monic). Thus \( O(f) = m \), which means again by Lemma 3.1 that \( f \) is of the required form.

4. Primary ideals in \( R \), where \( R/N \) is a principal ideal domain. If \( R \) is a principal ideal domain then the A.C.C. holds for right ideals and hence \( P(q) \) is an ideal when \( q \) is a right \( P \) primary ideal of \( R \). Moreover, in this case \( P = N = 0 \) and the completely prime ideals of \( R/N \) are maximal right ideals. Hence, by Definition 0.1, a principal ideal domain is an extendable ring.

If \( q \) is a \( P \) primary ideal then \( P(q) = N(q) \) since \( P(q) \) is a completely prime ideal. Thus \( P(q) \) is also a maximal left ideal.

**Theorem 4.1.** If \( R \) is a principal ideal domain and \( q = (a) \) is a \( P \) primary ideal in \( R \) with \( P(q) = (b) \) then \( b \) is irreducible and \( a = vb^n = b^n u \) where \( u \) and \( v \) are units of \( R \).

**Proof.** If \( b = cd \) then \( c \) or \( d \) must be in \( (b) \). Suppose \( d \in (b) \), say \( d = eb(\delta) \). Then \( b = ceb \) and \( c \) is a unit of \( R \). If \( c \in (b) \), say \( c = be \), then \( b = bed \) and \( d \) is a unit of \( R \). Hence \( b \) is irreducible.

For the second part of the proof we have \( (a) \subseteq (b) \) where \( b \) is irreducible. Let \( a = bc \) where \( c \in R \). If \( b \not\in (a) \) then \( c^n \in (a) \) since \( (a) \) is \( P \) primary. Hence \( c \in P((a)) = (b) \). Thus \( c = bd \) and \( a = b^2d \). This process continues until \( a = b^v d' \) where \( b^v \in (a) \) and \( b^{v-1} \notin (a) \). Consequently, \( b^v = ae \) and \( a = aed' \). Thus \( d' \) is a unit and \( a = b^v = ub^n \) where \( u \) and \( v \) are units of \( R \).

**Lemma 4.1.** If \( R/N \) is a principal ideal domain and \( q \) is an ideal of \( R \), then there is an element \( a \) in \( q \) such that \( q = aR + N' = Ra + N' \) where \( N' = q \cap N \).

**Proof.** Let \( N' = q \cap N \). Since \( R \) is a principal ideal domain \( \bar{q} = (\bar{a}) = \bar{a}R = \bar{R} \bar{a} \) for some \( \bar{a} \in \bar{R} \). Let \( a \) be any element of \( q \) such that \( aH = \bar{a} \). Obviously \( aR + N' \subseteq q \). Moreover, if \( b \in q \) then \( bh = \bar{a} \bar{r} \) where \( \bar{r} \in R \). Thus \( b = ar + n \) where \( r \in R \) and \( n \in N \). Since \( n = b - ar \), \( n \) is also in \( q \) and therefore \( n \in N' \). Hence \( q = aR + N' \). Similarly \( q = Ra + N' \).

**Theorem 4.2.** Let \( R \) be an extendable ring such that \( R/N \) is a principal ideal domain. If \( q \) is a \( P \) primary, not nil, nontrivial ideal of \( R \) then

\[
q = (vn^k + n)R + N' = R(vn^k + n) + N'
\]

(9) Note that if \( b \) is any element of a principal ideal domain \( R \) such that \( bR \) is a left, whence two-sided ideal, then from [5, p. 37] \( bR = Rb \), which one may denote by \( (b) \).
where \( v \) is a unit of \( R \), \( n \in N \), \( \pi \) is an irreducible element of \( \mathcal{R} \) and \( N' = N \cap q \).

Moreover, \( P(q) = \pi R + N \) and \( R/P(q) \) is a division ring.

**Proof.** By Lemma 4.1, \( q = aR + N' = Ra + N' \) where \( a \in q \). From 3b of [2] the not nil ideal \( aR + N' \) is \( P \) primary in \( \mathcal{R} \) if and only if the ideal \((\overline{a})\) is \( P \) primary in \( \mathcal{R} \). From Theorem 4.1 we may write \( a = \pi u + n_1 = v\pi^k + n_2 \) where \( u \) and \( v \) are units of \( R \) and \( n_1, n_2 \in N \).

To find \( P(q) \), write \( q = (\nu \pi^k) = (\nu \pi) = \pi^k R \). Clearly \( \pi R \subseteq P(q) \). Moreover, since \( \pi \) is irreducible, the ideal \( \pi R \) is maximal in \( R \) and hence \( P(q) = \pi R \). From 2g of [2] we have \( (P(q))^--(\pi R)^- \) and hence \( P(q) = \pi R + N \). The fact that \( R/P(q) \) is a division ring is a consequence of statement 3.1 of [2].

**Theorem 4.3.** If \( R \) is a completely \( N \) primary ring which satisfies the A.C.C. for right ideals then \( R[x] \) is an extendable ring, \( \mathcal{R}[x] \) is a principal ideal domain and \( R[x] \) is \( \mathcal{N}[x] \) primary.

**Proof.** If \( R \) satisfies the A.C.C. then \( N \) is nilpotent and, from Theorem 1.3, \( R[x] \) is \( \mathcal{N}[x] \) primary. Since \( \mathcal{R} \) is a division ring, \( \mathcal{R}[x] \) is a principal ideal domain<sup>(7)</sup>. It remains to show that \( R[x] \) is an extendable ring. If \( q \) is a right \( P \) primary ideal in \( R[x] \) then, since \( R[x] \) satisfies the A.C.C., \( P(q) \) is an ideal of \( R[x] \) and, from Theorem 1.1, \( P(R[x]) = \mathcal{N}(R[x]) \). Finally, condition (iii) of Definition 0.1 holds since \( R[x]/N(R[x]) = R[x]/\mathcal{N}[x] = R[x] \) is a principal ideal domain.

5. **\( R[x] \), where \( R \) is a completely \( N \) primary ring.** In this section let \( R \) denote a completely \( N \) primary ring which satisfies the A.C.C. for right ideals. By Theorem 4.3, \( R[x] \) is an extendable ring, \( N(R[x]) = \mathcal{N}[x] \) and \( (R/N)[x] \) is a principal ideal domain. In addition, by [6, p. 199], \( (\mathcal{N}[x])^t = 0 \) for some integer \( t \).

Certainly the not nil ideals of \( R[x] \) are the regular ideals of \( R[x] \)<sup>(8)</sup>. If \( q \) is a regular ideal in \( R[x] \) then, by §4, \( q = (f) \) in \( R[x] \) where \( f \) is regular in \( R[x] \). Hence all the regular ideals of \( R[x] \) are of the form \( q = fR[x] + N' \) where \( f \) is regular in \( R[x] \) and \( N' = q \cap N[x] \). By §3 we have \( O(f) = D(f) \).

**Theorem 5.1.** An ideal \( q \) of \( R[x] \) has finite degree if and only if \( q \) is a regular ideal. In this case, \( \deg(q) = O(f) = D(f) \).

**Proof.** By Theorem 3.2, \( q \) has finite degree if and only if \( \overline{q} \) has finite degree, i.e., if and only if \( q \subseteq \mathcal{N}[x] \). Thus a necessary and sufficient condition that \( q \) have finite degree is that \( q \) be regular. Again, by Theorem 3.2, the degree of \( q \) is the same as the degree of \( \overline{q} = (f) \). Since \( \mathcal{R} \) is a division ring \( D(f) = O(f) \).

<sup>(7)</sup> See [5, Chapter 3].

<sup>(8)</sup> An ideal \( q \) of a ring \( R \) is called regular if it contains at least one regular element. Thus if \( R \) is \( N \) primary then it contains only regular ideals and nil ideals, for if an ideal \( q \) is not regular then every element of \( q \) is a divisor of zero and hence \( q \subseteq N \).
If \( q \) is a regular, nontrivial \( N \) primary ideal of \( R[x] \), then by Theorem 4.2, 
\[
q = (v(x)p(x)^k + n(x))R[x] + N'
\]
where \( v(x) \) is a unit of \( R[x] \), \( \bar{p}(x) \) is an irreducible polynomial of \( \bar{R}[x] \), \( n(x) \in N[x] \) and \( N' = N[x] \cap q \). The radical \( N(q) = p(x)R[x] + N[x] \). Then \( \deg(q) = k \deg(\bar{p}) \) and \( \bar{R}[x]/(\bar{p}(x)) \) is a division ring. Thus \( R[x]/q \) is a completely \( N \) primary ring.

6. Simple algebraic extensions. In this section let \( R \subseteq A \) where \( R \) and \( A \) are completely \( N \) primary rings which satisfy the A.C.C. If \( \sigma \in A \), where \( a\sigma = \sigma a \) for all \( a \in R \), the symbol \( R[\sigma] \) shall denote the smallest subring of \( A \) containing \( R \) and \( \sigma \). The symbol \( R(\sigma) \) shall denote the smallest completely \( N \) primary ring containing \( R \) and \( \sigma \). In the latter case, \( R(\sigma) \) is called a simple extension of \( R \).

Certainly \( R[\sigma] \subseteq R(\sigma) \) and, if \( x \) is an indeterminate, \( R[\sigma] \) is the homomorphic image of the polynomial ring \( R[x] \) under the homomorphism \( f(x) \to f(\sigma) \). Since \( R[\sigma] \) is a subring of a completely \( N \) primary ring we know that \( R[\sigma] \) is \( P \) primary. The kernel of the homomorphism must then be a \( P \) primary ideal \( q \) of \( R[x] \) and \( R[x]/q \cong R[\sigma] \). In addition \( q_* = q \cap R \) is the zero ideal since \( q \) is the set of polynomials which have \( \sigma \) as a root. As in §5, \( q \) is either a regular ideal or a nil ideal.

**DEFINITION 6.1.** Let \( R \subseteq A \) and let \( \sigma \in A \) where \( a\sigma = \sigma a \) for all \( a \in R \). If \( \sigma \) satisfies at least one regular polynomial of \( R[x] \) then \( \sigma \) is called central algebraic with respect to \( R \). If \( \sigma \) satisfies only nilpotent polynomials of \( R[x] \) then \( \sigma \) is called central transcendental with respect to \( R(\sigma) \). The ideal \( q \) consisting of the polynomials of \( R[x] \) which have \( \sigma \) as a root is called the defining ideal of \( \sigma \).

We call \( R(\sigma) \) a simple algebraic extension of \( R \) if \( \sigma \) is algebraic with respect to \( R \) and \( R(\sigma) \) is a simple transcendental extension of \( R \) if \( \sigma \) is transcendental with respect to \( R \).

Let \( S = R(\sigma) \) be a simple algebraic extension of \( R \) and let \( q \) be the defining ideal of \( \sigma \). Then \( q \) is a not nil, nontrivial \( P \) primary ideal of \( R[x] \). By §§4 and 5 we may write \( q = (v(x)p(x)^k + n(x))R[x] + N' \) where the symbols have the same meanings as before. Then \( R[x]/q \) is a completely \( N \) primary ring whose residue class ring is isomorphic to \( \bar{R}[x]/(\bar{p}(x)) \) and \( R[x]/q \) is an extension of degree \( kD(\bar{p}(x)) \) of \( R/q_* = R \). Since \( R[x]/q \cong R[\sigma] \) and \( R[\sigma] \) satisfies the A.C.C. we have

**THEOREM 6.1.** If \( S = R(\sigma) \) is a simple algebraic extension of \( R \), the defining ideal \( q \) of \( \sigma \) has the form \( q = (v(x)p(x)^k + n(x))R[x] + N' \) where \( v(x) \) is a unit of \( R[x] \), \( \bar{p}(x) \) is irreducible in \( \bar{R}[x] \), \( n(x) \in N[x] \) and \( N' = q \cap N[x] \). Then \( S = R(\sigma) = R[\sigma] \) which satisfies the A.C.C. Moreover, \( S \) is a finite extension of \( R \) where \( [S:R] = kD(\bar{p}(x)) \). The division ring \( S = \bar{R}(\tilde{\sigma}) \) is obtained from \( \bar{R} \) by the adjunction of the zero \( \tilde{\sigma} \) of the irreducible polynomial \( \bar{p}(x) \in \bar{R}[x] \) and hence \( [S:R] = k[S:\bar{R}] \).

(9) Hereafter we shall refer to central algebraic (central transcendental) elements as algebraic (transcendental) elements.
Next we shall prove

**Theorem 6.2.** Let \( S = R(\sigma) \) be a simple algebraic extension of \( R \) where \([S : R] = k [S : R]\). Then, \( S \) is a principal extension of \( R \) if and only if \( k = 1 \). For any \( k \), \( N(S) = p(\sigma)R[\sigma] + N[\sigma] \) and hence there exists a positive integer \( h \) such that \( N(S)^h = 0 \).

**Proof.** Writing \( q \) in the form stated in Theorem 6.1 we have, as in §4, \( N(q) = p(\sigma)R[x] + N(R[x]) \). Since \( R[x] \) is a principal extension of \( R \) this can be written \( N(q) = p(x)R[x] + N(R[x]) \). It follows from 2h of [2] that \( N(R[x]/q) = N(q)/q = (p(x)R[x] + N[R[x]])/q \). The isomorphism from \( R[x]/q \) onto \( S = R(\sigma) \) maps \( p(x)R[x] + N(R[x])/q \) onto \( p(\sigma)R[\sigma] + N[\sigma] \) and hence \( N(S) = p(\sigma)R[\sigma] + N[\sigma] \). Now \( N(q)/q \) is a nil ideal in \( R[x]/q \) and, since the A.C.C. holds, the ideal \( N(q) \) of \( R \) is nilpotent modulo \( q \). Thus there is a positive integer \( h \) such that \( N(S)^h = 0 \). Finally, \( S \) is a principal extension of \( R \) if and only if \( p(\sigma) \in N[\sigma] \); i.e., if and only if \( p(x) \in q' \), where \( q' = p(x)^k R[x] + N[R[x]] \). If \( p(x) \in q' \) then \( q' \) contains a regular polynomial of degree \( D(p(x)) \). However, from §3, the minimal degree of the regular polynomials in \( q' \) is \( D(p(x)^k) = kD(p(x)) \). Hence \( kD(p(x)) \leq D(p(x)) \) and therefore \( k = 1 \). Conversely if \( k = 1 \), the extension is clearly principal.

An element \( \sigma \) of a ring \( A \) is called principal with respect to a subring \( R \) if \( R(\sigma) \) is a principal extension of \( R \). It follows from Theorem 6.2 that an algebraic element \( \sigma \) is principal if and only if it is a root of a nontrivial fundamental irreducible of \( R[x] \).

**Example 6.1.** Let \( R \) be a completely \( N \) primary ring satisfying the A.C.C. If \( x \) is an indeterminate, the ring \( R[x] \) is \( N \) primary. Let \( q \) be a regular, non-trivial \( N \) primary ideal of \( R[x] \) such that \( q_* = 0 \). As above, we have that \( q = (p(x)R[x] + N[R[x]])/q \) is a completely \( N \) primary ring which contains \( R \). Setting \( \sigma = \bar{x} \), where \( \bar{x} \) is the coset of \( x \) in \( R[x]/q \), then \( \sigma \) is algebraic over \( R \) with defining ideal equal to \( q \).

**Example 6.2.** Let \( D \) be the division ring of quaternions with coefficients in the rational numbers and \( D^* \) the division ring of quaternions with coefficients in the real numbers. For an indeterminate \( x \), \( R = D[x]/(x^*) \) is completely \( N \) primary and is contained in the completely \( N \) primary ring \( R^* = D^*[x]/(x^*) \). If \( \sigma = \sqrt{2} \), then \( R_1 = R(\sqrt{2}) = D_1[x]/(x^*) \) where \( D_1 \) is the ring of quaternions with coefficients in the set of all real numbers of the form \( a + b\sqrt{2} \) where \( a \) and \( b \) are rational numbers. Thus \( R_1 \) is a simple algebraic extension of \( R \) of degree 2.

For an indeterminate \( y \), the ring \( R[y] \) is \( N \) primary. Then \( y^2 - 2 \) is a minimal degree polynomial satisfied by \( \sqrt{2} \). One can use the division algorithm to show that the defining ideal of \( \sqrt{2} \) is \( q = (y^2 - 2)R[y] \). Then \( q \) is \( N \) primary and \( R[y]/q \cong R_1 \) where \( R_1 \) is an extension of \( R \) of degree 2. Similarly, we could adjoin

\[ \text{(10) See [2,2e] for the meaning of this term.} \]
to $R_1$ the element $\sqrt{3}$. Thus $R_2 = R_1(\sqrt{3}) = D_2[x]/(x^n)$ where $D_2$ is the ring of quaternions with coefficients in the set of all real numbers of the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ where $a, b, c$ and $d$ are rational numbers. Then $R_2$ is an extension of $R_1$ of degree 2 and an extension of $R$ of degree 4.

7. **Quotient rings.** If a ring $R$ has a right quotient ring as described in [5, p. 118] we shall denote this ring by $Q(R)$. Then $Q(R)$ is a ring containing $R$ such that every regular element of $R$ has an inverse in $Q(R)$ and any element of $Q(R)$ may be written in the form $ab^{-1} = a/b$ where $a, b \in R$ and $b$ is regular. A necessary and sufficient condition for the existence of $Q(R)$ is that for any pair of elements $a, b$ in $R$, $b$ regular, there exists a common right multiple $m = ab_1 = ba_1$ such that $b_1$ is regular. We shall use this criterion to establish the following theorem, which generalizes a result of A. W. Goldie [4, p. 592].

**Theorem 7.1.** Let $R$ be a ring with identity which satisfies the A.C.C. for right ideals and suppose the elements not in $N(R/N^k)$ are regular in $R/N^k$ for all positive integers $k$. Then $Q(R)$ exists.

**Proof.** Since the elements not in $N$ are regular in $R$, $R/N$ is an integral domain. Hence, if $N = 0$, then by Theorem 1 of [4], $Q(R)$ exists. We proceed by induction on the smallest integer $n$ such that $N^n = 0$. Assume that the theorem is true when $N^k = 0$ and $N^{k-1} \neq 0$. In $R$ suppose $N^{k+1} = 0$ and $N^k \neq 0$. Then the ring $\tilde{R} = R/N^k$ satisfies the hypothesis of the theorem and, by the induction hypothesis, $Q(\tilde{R})$ exists.

Let $a \neq 0$ and $b \notin N$. If $a \in N^k$ we consider the right ideals $I_n = aR + baR + \ldots + b^naR, n = 0, 1, 2, \ldots$. By the A.C.C., $I_t = I_{t+1}$ for some integer $t$ and we may write $b^{t+1}a = ar_0 + bar_1 + \ldots + b^t ar_t, r_i \in R$. Since $b^{t+1}a \neq 0$ and $N^{k+1} = 0$, not all the $r_i$ are in $N$. Let $i = h$ be the first subscript for which $r_i \notin N$. It follows that $b^{t+1-h}a = ar_h + \ldots + b^{t-h}ar_t$ and hence $b(b^{t-h}a - ar_h + \ldots - b^{t-h}a r_t)$ $= ar_h$ where $r_h$ is regular. Thus $a$ and $b$ have a common right multiple. If, on the other hand, $a \notin N^k$, then, since $Q(\tilde{R})$ exists, we have $\tilde{a}c = \tilde{b}d$ in $\tilde{R} = R/N^k$ where $c \notin N$. Thus $ac = bd + e$ with $e \in N^k$. If $e = 0$, we stop. Otherwise, as above, write $bf = eg$ where $g \notin N$, whence $acg = b(df + f)$ with $cg$ regular. Thus $Q(R)$ exists.

**Theorem 7.2.** If $R$ is a completely $N$ primary ring which satisfies the A.C.C. for right ideals then $R[x]$ has a right quotient ring $Q(R[x])$.

**Proof.** From Theorem 1.3, for any integer $n$, the divisors of zero of $R/N^n[x]$ are contained in $N/N^n[x]$. Hence $R[x]$ satisfies the hypothesis of Theorem 7.1.

**Lemma 7.1.** If a ring $R$ has a right quotient ring $Q(R)$ and if $R$ satisfies the A.C.C. for right ideals then $Q(R)$ satisfies the A.C.C. for right ideals.

**Proof.** Let $k$ and $h$ denote right ideals of $Q(R)$ where $k \subset h$. As in the proof of Lemma 1.3 of [3] it follows that $k_* \subset h_*$ in $R$. The lemma is now immediate.
Lemma 7.2. If a ring $R$ with A.C.C. on right ideals has a right quotient ring $Q(R)$ and if the elements not in $N$ are regular in $R$ then $Q(R)$ is completely $N$ primary.

Proof. Let $T = \{a/b | a/b \in Q(R), a \in N\}$. For any elements $a/b$ and $c/d$ of $T$, there exist regular elements $b_1$ and $d_1$ in $R$ such that $m = db_1 = bd_1$. Using the rule for addition in $Q(R)$ we have $a/b - c/d = (ad_1 - cb_1)/m$ which is in $T$ since $a$ and $c$ are in $N$. Next, consider any elements $a/b$ of $T$ and $c/d$ of $Q(R)$. Let $c_1$, $b_1 \in R$, $b_1$ regular, such that $cb_1 = bc_1$. Then $(a/b)(c/d) = (ac_1/db_1)$ which proves that $TQ(R) \subseteq T$. Similarly, one can prove that $Q(R)T \subseteq T$. Thus $T$ is an ideal in $Q(R)$. It is easily seen that for any positive integer $n$, the product of $n$ elements of $T$ can be written in the form $a/b$ where $a \in N^n$ and $b$ is regular. Since $N$ is nilpotent, $T \subseteq N(Q(R))$. Also, the elements of $Q(R)$ which are not in $T$ are units and hence $Q(R)/T$ is a division ring. Then $T$ is maximal ideal and $T = N(Q(R))$. Thus $Q(R)$ is completely $N$ primary.

Lemma 7.3. If a ring $R$ has a right quotient ring $Q(R)$ which is completely $N$ primary then $Q(R)$ is the smallest completely $N$ primary ring containing $R$.

Proof. Let $R'$ be a completely $N'$ primary ring where $N' = N(R')$ and suppose $R \subseteq R'$. If $b \in R$ is regular then $b \not\in N'$ and hence $b$ has an inverse $b^{-1}$ in $R'$. Thus $a, b \in R$, $b$ regular implies $ab^{-1} \in R'$; that is, $Q(R) \subseteq R'$.

Theorem 7.3. Let $R$ be a completely $N$ primary ring which satisfies the A.C.C. for right ideals and let $q$ be a $P$ primary ideal of $R[x]$ with $q \subseteq N(R[x])$. Then $Q(R[x]/q)$ exists and is completely $N$ primary. Moreover, $Q(R[x]/q)$ satisfies the A.C.C. for right ideals and is the smallest completely $N$ primary ring containing $R[x]/q$.

Proof. We know, by Theorem 7.2, that $Q(R[x])$ exists. Furthermore, as in the proof of Lemma 1.2 of [3], one can show that $q = q^* \cap R[x]$, where $q^*$ is the extension of $q$ to $Q(R[x])$. It follows that the mapping $f(x) + q \rightarrow f(x) + q^*$, $f(x) \in R[x]$, is an isomorphism of $R[x]/q$ into $Q(R[x])/q^*$. We shall identify $R[x]/q$ with the subring of $Q(R[x])/q^*$ which corresponds to $R[x]/q$ under this isomorphism.

If $f(x) + q$ is regular in $R[x]/q$ then $f(x) \not\in N[x]$. By Theorem 1.3, $R[x]$ is $N[x]$ primary and consequently $f(x)$ is regular in $R[x]$. Hence $f(x)$ has an inverse $f(x)^{-1}$ in $Q(R[x])$ and we have $f(x)^{-1} + q^* = (f(x) + q^*)^{-1}$. Thus the regular elements of $R[x]/q$ have inverses in $Q(R[x])/q^*$. Now let $f(x)g(x)^{-1} + q^* \in Q(R[x])/q^*$ where $f(x)$, $g(x) \in R[x]$, $g(x)$ regular. Then $f(x)g(x)^{-1} + q^* = (f(x) + q^*)(g(x) + q^*)^{-1}$. This proves that $Q(R[x])/q^*$ is a right quotient ring for $R[x]/q$.

The remaining part of the theorem follows from Lemmas 7.1, 7.2 and 7.3.

8. Simple transcendental extensions. In this section, $R$ will always denote a completely $N$ primary ring which satisfies the A.C.C. for right ideals and $A$ will denote a completely $N$ primary ring which contains $R$. 


Let \( \sigma \in A \) be transcendental over \( R \). Then \( R[\sigma] \) is an \( N \) primary ring where \( N(R[\sigma]) = N(A) \cap R[\sigma] \). For an indeterminate \( x \) we have the usual homomorphism \( \theta \) of \( R[x] \) onto \( R[\sigma] \) defined by \( f(x) \theta = f(\sigma) \). The defining ideal \( q \) of \( \sigma \) is then a nil, \( P \) primary ideal of \( R[x] \) and \( q_f = q \cap R = 0 \). Since \( R[x] \) satisfies the A.C.C. for right ideals, \( q \) is a nilpotent ideal and \( q \subseteq N(R[x]) = N[x] \) by [2, §1]. By Theorem 7.3, \( Q(R[x]/q) \) exists and, since \( R[\sigma] \cong R[x]/q, Q(R[\sigma]) \) exists. Moreover, \( Q(R[\sigma]) \) is the smallest completely \( N \) primary ring containing \( R \) and \( \sigma \). Hence \( Q(R[\sigma]) = R(\sigma) \), the simple transcendental extension of \( R \) by \( \sigma \). Also, \( R(\sigma) \) satisfies the A.C.C. for right ideals. This establishes the first part of

**Theorem 8.1.** If \( S = R(\sigma) \) is a simple transcendental extension of \( R \) then \( R(\sigma) = Q(R[\sigma]) \) and \( R(\sigma) \) satisfies the A.C.C. for right ideals. The division ring \( (R(\sigma))^- = (R\sigma) \) is obtained by adjoining the transcendental element \( \overline{\sigma} \) to \( R \). To every unit of \( S \) a unique order can be associated.

To prove the last part of the theorem we observe from above that \( (R[\sigma])^- \cong (R[x]/q)^- \cong (R/N)[x] = R[x] \), where \( R \) is a division ring. To every nonzero element \( \hat{r} \) of \( R[\sigma]^- \) a unique degree is associated, namely the degree of the polynomial of \( R[x] \) which is the image of \( \hat{r} \) under the isomorphism from \( (R[\sigma])^- \) onto \( R[x] \). If we extend this isomorphism to an isomorphism from \( Q((R[\sigma])^-) \) onto \( R(x) = Q(R[x]) \) then to every nonzero element of \( Q((R[\sigma])^-) \) a unique degree is associated, namely the degree of the corresponding (image) element of \( R(x) \). (The degree of a fraction, by definition, is the maximum of the degrees of the numerator and denominator.) Since \( Q((R[\sigma]))^- \cong Q((R[\sigma])^-) \) there is a unique degree associated with each element of \( (Q(R[\sigma]))^- \). Now the set of not nilpotent elements of the completely \( N \) primary ring \( Q(R[\sigma]) \) coincides with the set of units of \( Q(R[\sigma]) \). We define the order of a unit \( \sigma \) of \( R(\sigma) \) as the degree of the element \( \hat{r} \) onto which \( \sigma \) is mapped by the natural homomorphism from \( Q(R[\sigma]) \) onto \( Q((R[\sigma])^-) \). Thus, to each unit of \( Q(R[\sigma]) \) = \( R(\sigma) \) a unique order is associated (11).

**Theorem 8.2.** Every completely \( N \) primary ring \( R \) which satisfies the A.C.C. for right ideals is properly contained in just such a ring. Specifically, \( R \subseteq Q(R[x]) \), which is a completely \( N \) primary ring satisfying the A.C.C. for right ideals.

**Proof.** See Theorem 7.2, Lemma 7.1 and Lemma 7.2.

**Lemma 8.1.** Let \( S \) be a principal extension of \( R \) and let \( N(R) = P(R) \) and \( N(S) = P(S) \). Let \( n \) be an ideal in \( N(S) \) and suppose that \( N(S/n) = N(S)/n \). Then \( S/n \) is a principal extension of \( R/n \), where \( n = n \cap R \).

(11) Note that for \( \sigma \) algebraic we have \( R[\sigma] = R(\sigma) \) and hence in this case it is also true that \( Q(R[\sigma]) = R(\sigma) \).
Proof. As in [2, §2], we may assume that $\bar{R} = R/n_\ast \subseteq S/n$ where, for any set $B \subseteq S$, $\bar{B}$ denotes the image of $B$ under the natural homomorphism from $S$ to $S/n$. Let $v$ be an element of $S$ such that $\bar{v} \in N(S/n) = N(S)/n$. Then $v \in N(S)$ and, since $S$ is a principal extension of $R$ we may write $v = \sum v_i \sigma_i$ where $v_i \in N(R)$ and $\sigma_i \in S$. Hence $\bar{v} = \sum \bar{v}_i \bar{\sigma}_i$ where $\bar{v}_i \in N(R/n_\ast)$ and $\bar{\sigma}_i \in S/n$. Thus $S/n$ is a principal extension of $R/n_\ast$.

Theorem 8.3. If $S$ is a simple transcendental extension of $R$ then $S$ is a principal extension of $R$.

Proof. Let $S = R(\sigma)$ and let $n$ denote the defining ideal of the transcendental element $\sigma$. Applying Lemma 8.1 to $R[x]$ and $R$, and noting that $n_\ast = n \cap R = 0$, we have that $R[x]/n$ is a principal extension of $R/n_\ast = R$. Thus $R[\sigma]/R$ is a principal extension of $R$. Now $Q = Q(R[\sigma])$ is a principal extension of $R[\sigma]$ since $N(Q)$ consists of elements of the form $a/b = ab^{-1}$ where $a \in N(R[\sigma])$ and $b^{-1} \in Q(R[\sigma])$. Hence $N(Q) = N(R[\sigma]) \cdot Q(R[\sigma]) = N(R) \cdot R[\sigma] \cdot R(\sigma) = N(R) \cdot R(\sigma)$, i.e., $Q(R[\sigma]) = R(\sigma)$ is a principal extension of $R$.

Example 8.1. Let $R$ be a completely $N$ primary ring which satisfies the A.C.C. for right ideals. Let $x$ be an indeterminate and let $n$ be any nil, $P$ primary ideal of $R[x]$ such that $n_\ast = 0$. Setting $\sigma = x$, where $x$ is the coset of $x$ in $R[x]/n$, then $\sigma$ is transcendental over $R$ with defining ideal $n$.

Example 8.2. Let $R$ and $R^\ast$ be as in Example 6.2 and let $F$ be the field of rational numbers. Then for the transcendental number $\pi$ we have $R(\pi) = D'[x]/(x^n)$ where $D'$ is the ring of quaternions with coefficients of the form $p(\pi)/q(\pi)$ where $p(\pi)$ and $q(\pi)$ are elements of $F[\pi]$, $q(\pi) \neq 0$.

Bibliography


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