ON COMMUTATIVE ALGEBRAS OF DEGREE TWO

BY

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Let \( \mathfrak{A} \) be a simple, commutative, power-associative algebra of degree 2 over an algebraically closed field \( \mathbb{F} \) of characteristic not equal to 2, 3 or 5. The degree of \( \mathfrak{A} \) is defined to be the number of elements in the maximal set of pairwise orthogonal idempotents in \( \mathfrak{A} \). This algebra has a unit element 1 [1, Theorem 3]. The algebras \( \mathfrak{A} \) of characteristic zero were considered by Kokoris [8] and found to be Jordan algebras. Kokoris also gave examples of algebras \( \mathfrak{A} \) that were not Jordan [6]. This left the problem of determining those algebras \( \mathfrak{A} \) that are not Jordan algebras.

Since \( 1 = e + f \) where \( e \) and \( f \) are primitive orthogonal idempotents, we have a decomposition \( \mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0) \) where \( x \in \mathfrak{A}_e(\lambda) \) if and only if \( ex = \lambda x \). We have \( \mathfrak{A}_e(\lambda) = \mathfrak{A}_e(1 - \lambda) ; \mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1 - \lambda) + \mathfrak{A}_e(1/2) \) for \( \lambda = 1, 0 \); and \( \mathfrak{A}_e(1) = e\mathfrak{F} + \mathfrak{N}_1, \mathfrak{A}_e(0) = f\mathfrak{F} + \mathfrak{N}_0 \) where \( \mathfrak{N}_1 \) and \( \mathfrak{N}_0 \) are nilideals of \( \mathfrak{A}_e(1) \) and \( \mathfrak{A}_e(0) \) respectively. If \( \mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1/2) \) for \( \lambda = 1, 0 \) we say that \( e \) is a stable idempotent. If \( \mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1/2) + \mathfrak{N}_{1-\lambda} \) for \( \lambda = 1, 0 \) we say that \( e \) is a nilstable idempotent.

The results of Albert extend the characteristic zero case to include algebras of characteristic \( p \neq 2, 3, 5 \) for which every idempotent is stable [2]. He also characterized those algebras of characteristic \( p \neq 2, 3, 5 \) that have at least one stable idempotent [3; 4]. Recently Kokoris announced [9] that every simple, flexible, power-associative algebra over an algebraically closed field of characteristic \( \neq 2, 3 \) that is of degree two and in which every idempotent is nilstable is a \( J \)-simple algebra.

It is the purpose of this paper to fill in the remaining gap by giving a characterization of those algebras \( \mathfrak{A} \) that have an idempotent that is not nilstable. An example is also given of an algebra \( \mathfrak{A} \) that does not have a stable idempotent.

1. Let \( \mathfrak{A} \) be an algebra that is simple, commutative, power-associative, of degree two and whose base field \( \mathbb{F} \) is an algebraically closed field of characteristic \( p \neq 2, 3, 5 \). Let \( e \) be a primitive idempotent of \( \mathfrak{A} \) that is not nilstable. Since \( \mathfrak{A} \) is power-associative we have \( x^2x^2 = x^4 \) for all \( x \in \mathfrak{A} \) and the linearization of this identity

\[
P(x,y,s,t) = 4(xy)(st) + 4(xs)(yt) + 4(xt)(ys)
\]

\[
= x[y(x(t) + s(y(t)) + t(y(s)) - y(x(t)) + s(x(t)) + t(x(y))]
\]

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We will use $C$ to represent the space $\mathfrak{A}_e(1) + \mathfrak{A}_e(0)$, $a_j$ to represent the $\mathfrak{A}_e(\lambda)$-component of $a$, $a_{10}$ to represent the $C$-component of $a$, and $z$ to represent $e-f$.

We will make frequent use of some of the results of Albert on commutative power-associative algebras; namely, results (5), (6), (7), (8) of [1]. We state them as

\[(2) \quad [g(xy)_{\lambda}]_{1/2} = [g(x_{\lambda}y_{\lambda})]_{1/2} + [g(y_{\lambda})]_{1/2} x_{\lambda},\]

\[(3) \quad [g(xy)_{\lambda}]_{1-\lambda} = 2([g(x_{\lambda}y_{\lambda})]_{1-\lambda} + 2([g(y_{\lambda})]_{1-\lambda} x_{\lambda}),\]

\[(4) \quad [g(x_{\lambda}y_{\lambda})]_{1/2} = [g(y_{\lambda})]_{1/2} x_{\lambda},\]

\[(5) \quad (g_{\lambda}y_{\lambda})_{1/2} = 2([g_{\lambda}y_{\lambda}]_{1/2} x_{\lambda}),\]

where $\lambda = 1, 0; g \in \mathfrak{A}_e(1/2)$ and $x$ and $y$ are in $C$.

Two other relations

\[(6) \quad 2([x_{\lambda}g]_{1/2} x_{\lambda} + [x_{\lambda}g]_{1-\lambda} g_{\lambda}]_{\lambda} = x_{\lambda} g^2,\]

\[(7) \quad (x_{1/2}g)_{1/2} = (x_0g)_{1/2} \Rightarrow (x^2g)_{1/2} = (x_0g)_{1/2}\]

for $x$ and $g$ as above will be useful. The first of these is obtained from $P(x, e, g, g) = 0$ while the second can be derived from (2) and (4).

**Theorem 1.** $C$ is an associative subalgebra of $\mathfrak{A}$ with an element $c \in C$ such that there is a $w \in \mathfrak{A}_e(1/2)$ with $z(cw) = 1$, $(c_1w)_{1/2} = (c_0w)_{1/2} = (c_1^2w)_{1/2} = -2c_0$.

**Proof.** It is easily seen that the subset $\mathfrak{I}$ of $\mathfrak{A}_e(1)$ consisting of all elements of the form $(a_0g)_{1}$ is an ideal of $\mathfrak{A}_e(1)$ where $g \in \mathfrak{A}_e(1/2)$ and $a_0$ is a fixed element of $\mathfrak{A}_e(0)$ because by (5) we have $b_1(a_0g)_{1} = 2[a_0(b_1g)_{1/2}]_{1}$. The additive property of an ideal is immediate.

We now let $b_1, d_1$ be elements of $\mathfrak{A}_e(1)$, $g \in \mathfrak{A}_e(1/2)$ and $a_0 \in \mathfrak{A}_e(0)$ with $(a_0g)_{1} = a_1$. If we consider only the $\mathfrak{A}_e(1)$-components of each of the terms in $P(b_1, d_1, g, a_0) = 0$ we get $2(b_1d_1)a_1 = b_1(d_1a_1) + d_1(b_1a_1)$. If $b_1$ is also in $\mathfrak{I}$ we can interchange $a_1$ and $b_1$ to get $a_1(d_1b_1) = 2b_1(d_1a_1) - d_1(b_1a_1)$. Therefore $a_1(d_1b_1) = (a_1d_1)b_1$. Hence $\mathfrak{I}$ is associative.

It has been shown [1, Lemma 11] that if $(a_0g)_{1} \in \mathfrak{I}$ for all $a_0 \in \mathfrak{A}_e(0)$ and $g \in \mathfrak{A}_e(1/2)$ then $(a_1g)_{0} \in \mathfrak{N}$ for all $a_1 \in \mathfrak{A}_e(1)$ and $g \in \mathfrak{A}_e(1/2)$. From this result and the assumption that $e$ is not nilstable we can conclude that there is an element $c_0 \in \mathfrak{A}_e(0)$ and an element $g$ in $\mathfrak{A}_e(1/2)$ such that $(c_0g)_{1}$ is nonsingular. If $b_1$ is the inverse of $(c_0g)_{1}$ in $\mathfrak{A}_e(1)$ then $[c_0(2b_1g)_{1/2}]_{1} = b_1(c_0g)_{1} = e$. We may also conclude that $\mathfrak{A}_e(1) = \mathfrak{I}$ is associative. In a similar manner we obtain the result that $\mathfrak{A}_e(0)$ is associative.

If we take $c_0 \in \mathfrak{A}_e(0)$ and $w \in \mathfrak{A}_e(1/2)$ such that $(c_0w)_1 = e$ and let $2c_1 = (c_0w)_1 = 4[c_0(c_0w)_{1/2}]_{1}$ then we can quote the results of Kokoris [7, Lemma 4 and Identity 29] that $(c_1w)_0 = -f$ or 0 and $(c_1w)_{1/2} = (c_0w)_{1/2}$. No generality will be lost if we also assume that $c_0$ is nilpotent because $\mathfrak{A}_e(0) = f\mathfrak{N} + \mathfrak{N}_0$ and
(c_0 w)_1 = \left[(\alpha f + c_0 w)\right]_1 \text{ for any } \alpha \in \mathcal{F}. \text{ To complete the proof of the theorem it remains only to show that } (c_1 w)_0 \neq 0. \text{ We assume that } (c_1 w)_0 = 0. \text{ If we examine the } \mathcal{A}_x(1)-\text{components of the terms of the relation } P(c_0, c_0, w, w) = 0 \text{ we get } 8(c_0 w)^2 + 8[(c_0 w)^2]_1 = 4[c_0 w(w c_0)]_1/2_1 + 2[w(c_2 w)_1/2]_1 + 4[w[c_0 (c_0 w)_1/2]_1/2]. \text{ Using this relation together with (2), (6), (7) and } (c_0 w)_1 = e, \text{ we get } 6 e + 8[(c_1 w)^2]_1/2 = 2 w^2 c_1^2 - 2[w(c_1 w)_0]_1. \text{ But } (c_1^2 w)_0 = 4[c_1 (c_1 w)_1/2]_0 = 4[c_1 (c_0 w)_1/2]_0 = 2 c_0 (c_1 w)_0 = 0. \text{ Therefore either } [(c_1 w)^2]_1/2 \text{ or } w^2 c_1^2 \text{ must be nonsingular. If we again use (1) with } P(c_1, c_1, w, w) = 0 \text{ and examine the } \mathcal{A}_x(0)-\text{components of the resulting terms we get } 8[(c_0 w)^2]_1/2 = 2 c_0^2 w^2. \text{ But then } [(c_0 w)^2]_1/2 \text{ is nilpotent. Since } (c_0 w)^2_1/2 = \alpha 1 + n \text{ where } n \in \mathcal{G}_i + \mathcal{R}_0 \text{ [1, Lemma 10] we must also have } [(c_1 w)^2]_2, \text{ nilpotent. Now by (6) we have } 2[(c_0 w)_1/2 w]_1 = 2[c_1 (c_1 w)_1/2]_1 = - [(c_1 w)_0 w]_1 + c_1 w^2 = c_1 w^2. \text{ But } 2[(c_0 w)_1/2 w]_0 = - [(c_0 w)_1 w]_0 + c_0 w^2 = c_0 w^2 \text{ is nilpotent. Therefore } c_1 w^2 \text{ and } c_0^2 w^2 \text{ are nilpotent. We have arrived at a contradiction. Hence } (c_1 w)_0 = -f \text{ and the theorem is proved.}

**Theorem 2.** There is an isomorphism } T \text{ between } \mathcal{A}_x(1) \text{ and } \mathcal{A}_x(0) \text{ such that for } b_1 \in \mathcal{A}_x(1), T(b_1) \text{ is the unique element of } \mathcal{A}_x(0) \text{ satisfying } (b_1 w)_1/2 = [T(b_1) w]_1/2. \text{ The subset } \mathcal{B} \text{ of } \mathcal{C} \text{ of all elements of the form } b_1 + T(b_1) \text{ is an associative subalgebra of } \mathcal{C} \text{ isomorphic to both } \mathcal{A}_x(0) \text{ and } \mathcal{A}_x(1). \text{ }

**Proof.** We use } c_1, c_0 \text{ and } w \text{ as in Theorem 1. If we consider only the } \mathcal{A}_x(1/2)-\text{components of the terms in } P(c_0, b_1, w, w) = 0 \text{ we get } 8[(c_0 w)_1/2 (b_1 w)_0]_1/2 + 4(b_1 w)_1/2 = 2[w[b_1 + [(b_1 w)_1/2 c_0]_1/2]_1/2 + 2[w[c_0 w]_1/2 b_1 + (b_1 w)_0 c_0]_0]_1/2 + 2[c_0 w(w b_1)_0]_1/2 + (b_1 w)_1/2. \text{ Using (5) and (2) on the terms } [(c_0 w)_1/2 b_1]_0, [(b_1 w)_1/2 c_1]_0 \text{ and } (c_0 w(w b_1)_0]_1/2 \text{ this relation reduces to } [(c_0 w)_1/2 (b_1 w)_0]_1/2 = (w[c_0 w]_1/2 b_1)_0]_1/2. \text{ We now consider the } \mathcal{A}_x(1/2)-\text{component of each term in } P(c_1, b_1, w, w) = 0. \text{ We have } (-4(b_1 w)_1/2 + 8[(c_1 w)_1/2 (b_1 w)_0]_1/2 = 2[w[c_1 b_1] w + c_1 (b_1 w)_1/2 + b_1 (c_1 w)]_1/2_1/2 + (b_1 w)_1/2 + 2[c_1 [w(w b_1)]_0]_1/2. \text{ This relation together with (2) and (4) gives us } 2[(c_1 w)_1/2 (b_1 w)_0]_1/2 = (b_1 w)_1/2 + 2[w[(c_1 b_1 w)]_0]_1/2. \text{ But } [(c_0 w)_1/2 (b_1 w)_0]_1/2 = (w[(c_0 w)_1/2 b_1]_0]_1/2 \text{ and } (c_0 w)_1/2 = (c_1 w)_1/2. \text{ Therefore } (b_1 w)_1/2 = (2[c_1 (b_1 w)_1 b_1]_0 - [c_1 (b_1 w)]_0]_1/2 = -2[(b_1 w)_1/2 c_1]_0]_1/2. \text{ We can now define } T(b_1) = -2[(b_1 w)_1/2 c_1]_0 \text{ to be the element } b_0 \text{ in } \mathcal{A}_x(0) \text{ such that } (b_1 w)_1/2 = (b_0 w)_1/2. \text{ To show that } T \text{ is well-defined we assume } (a_0 w)_1/2 = 0. \text{ We have } a_0 = -a_0 (c_1 w)_1/2 = -2[c_1 (a_0 w)_1/2]_0 = 0 \text{ by (5). Therefore } (b_1 w)_1/2 = (b_0 w)_1/2 \text{ implies } b_0 = b_1. \text{ Simply by changing the signs of } c_1 \text{ and } c_0 \text{ and interchanging } 1 \text{ and } 0 \text{ we can get a similar result for } \mathcal{A}_x(0); \text{ i.e., for every } b_0 \in \mathcal{A}_x(0) \text{ there is a unique } b_1 = 2[(b_0 w)_1/2 c_1]_0]_1/2 \text{ such that } (b_0 w)_1/2 = (b_1 w)_1/2. \text{ Therefore } T \text{ is onto } \mathcal{A}_x(0) \text{ and is a 1-1 correspondence between } \mathcal{A}_x(1) \text{ and } \mathcal{A}_x(0).
Now if $a$ and $b$ are elements of $\mathcal{B}$ as defined in the theorem we have, with the help of (2) and (4), that
\[
[w(a,b)]_{1/2} = [(wb_1)a_1 + (wa_1)b_1]_{1/2}
= [(wb_0)a_1 + (wa_0)b_1]_{1/2} = [(wa_1)b_0 + (wb_1)a_0]_{1/2}
= [(wa_0)b_0 + (wb_0)a_0]_{1/2} = [w(a_0b_0)]_{1/2}.
\]
Therefore $T(a,b) = a_0b_0$ and $ab = a_1b_1 + a_0b_0 \epsilon \mathcal{B}$. Clearly $\mathcal{B}$ is closed under addition and scalar multiplication.

Define $S(b) = be$ for every $b \in \mathcal{B}$. It follows immediately from the above results that $S$ is a 1-1 correspondence of $\mathcal{B}$ onto $\mathcal{A}_c(1)$. From the definition we have $S(ab) = (ab)e = (ae)(be) = S(a)S(b)$ and $S(a + b) = S(a) + S(b)$ for all $a$ and $b$ in $\mathcal{B}$. Therefore $\mathcal{B}$ and $\mathcal{A}_c(1)$ are isomorphic as rings and hence as algebras. In the same manner we show that $\mathcal{B}$ is isomorphic to $\mathcal{A}_c(0)$. We have shown also that $T$ is an isomorphism. The associativity of $\mathcal{B}$ follows from that of $\mathcal{C}$.

From the definition of $\mathcal{B}$ it is clear that $c = c_1 + c_0$ is in $\mathcal{B}$. From $P(w,w,w,z) = 0$ it follows that $w^2$ is in $\mathcal{B}$. Theorem 2 also implies that $\mathcal{C} = \mathcal{B} + \mathcal{B}z$.

**Theorem 3.** The mapping $b \rightarrow D(b) = (bw)z$ is a derivation of $\mathcal{B}$ into $\mathcal{B}$ such that $D(c) = 1$.

**Proof.** Let $a$ and $b$ be arbitrary elements of $\mathcal{B}$. Then
\[
[(ab)w]_{10} = [(ab_1)w]_0 + [(ab)_0w]_1 = [(a_1b_1)w]_0 + [(a_0b_0)w]_1 = 2[a_1(b_1w)]_{1/2} + 2[b_1(a_1w)]_{1/2} + 2[a_0(b_0w)]_{1/2} + 2[b_0(a_0w)]_{1/2}
+ 2[a_1(b_0w)]_{1/2} + 2[b_0(a_1w)]_{1/2} = 2[a_1(b_0w)]_{1/2} + 2[b_1(a_0w)]_{1/2} + 2[a_0(b_1w)]_{1/2} + 2[b_0(a_0w)]_{1/2}.
\]
This relation is multiplied by $z$ we have $D(ab) = aD(b) + bD(a)$ and $D$ is a derivation on $\mathcal{B}$ into $\mathcal{C}$.

To show that $D(b)$ lies in $\mathcal{B}$ for $b = b_1 + b_0$, an element of $\mathcal{B}$, we need several identities; the first of which is obtained from $P(b_0,w,w,cx) = 0$. We get
\[
8[(b_0w)_1(wc_1)_1]_0 + 8[(b_0w)_1(wc_1)_1]_0 = 2[(b_0c_0)_0w^2 + 2\{[c_1(b_0b_1)]_0w \}
+ 8[(b_0w)_1(wc_1)_1]_0 = 2[(b_0c_0)_0w^2 + 2\{[c_1(b_0b_1)]_0w \}
+ 8[(b_0w)_1(wc_1)_1]_0 = 2[(b_0c_0)_0w^2 + 2\{[c_1(b_0b_1)]_0w \}.
\]
We consider $P(b_0,w,w,c_0) = 0$ next to get
\[
-3(b_0c_0)_0w^2 + 8[(b_0w)_1(wc_1)_1]_0 + 6[(b_0w)_1(c_0w)_1]_0
+ 6T[b_0w]_1 = -12\{[(b_0w)_1]_1c_1\}_0.
\]
Finally we obtain $3(b_1w)_0 = 4[(b_1w)_1(c_0w)_1]_0 - 2w[(c_0b_0)_0w]_{1/2}$ from $P(b_1,w,w,c_1) = 0$. Now, from the proof of Theorem 2 and from (3) we have
\[
6T[b_0w]_1 = -12\{[(b_0w)_1]_1c_1\}_0
+ 12\{[(c_0w)_1]_1(b_0w)\}_0 - 6\{[(b_0w)_1]_1w\}_0.
\]
By successively applying to this relation the three identities above in the order we obtained them we get
\[ 6T[(b_0w)]_0 = -12[(b_0w)(c_1w)]_1/2 - 24[(b_0w)(c_1w)]_1/2 + 6c_0b_0w^2 = 4[(b_0w)(c_0w)]_1/2w_0 - 8[(b_0w)(c_0w)]_1/2w_0 = -6(b_1w)_0. \]
Therefore we have
\[ D(b)z = (b_0w)z = (b_0w)_1. \]
The fact that \( D(c) = 1 \) follows immediately from the definition of \( c \).

**Theorem 4.** If \( a \) and \( b \) are elements of \( \mathbb{B} \) then
\[ [(wa)]_{1/2} = [(wb)]_{1/2}, [(wa)_{1/2}b]_{1/2} = (wa)_{1/2}a \] and \( (wa)_{1/2}(wb)_{1/2} \in \mathbb{B} \).

**Proof.** By (2) and (4) and the definition of \( \mathbb{B} \) we have
\[ [w(ab)]_{1/2} = 2[w(ab)]_{1/2} = 2[(wa)]_{1/2}b_{1/2} + 2[(wb)]_{1/2}a_{1/2} = [(wa)]_{1/2}b_{1/2} + 2[(wb)]_{1/2}a_{1/2} = [(wa)]_{1/2}b_{1/2}. \]
By (5) we have
\[ [(wa)]_{1/2} = 2[(wa)]_{1/2}b_{1/2} + 2[(wa)]_{1/2}a_{1/2} = [(wa)]_{1/2}b_{1/2} + 2[(wa)]_{1/2}a_{1/2}. \]
Now use \( P(w, w, a, b) \) to get
\[ 4w^2a + 8(wa)(wb) = 2w[(2a)w + (aw)b + (bw)a] + a[w^2b + 2w(wb)] + b[w^2a + 2w(wa)]. \]
If we consider only the \( \mathbb{B} \)-components of each of the terms and if we use the facts that \( (wa)_{1/2} \in \mathbb{B} \) for all \( a \in \mathbb{B} \) and \( [w(a)]_{1/2} \in \mathbb{B} \) then
\[ 8[(wa)_{1/2}b]_{1/2} = 2[(wa)_{1/2}b]_{1/2} = [(wa)]_{1/2}b_{1/2}. \]
Hence \( 8[(wa)_{1/2}b]_{1/2} \in \mathbb{B} \).

**Corollary.** If \( a \in \mathbb{C} \) and \( b \in \mathbb{B} \) then
\[ [(wa)]_{1/2} = [(wb)]_{1/2}. \]

**Proof.** We can write \( a = a' + a''z \) where \( a' \) and \( a'' \) are in \( \mathbb{B} \). Since \( [(a'z)]_{1/2} = [(a'z)]_{1/2} = [(a'z)]_{1/2} = [(wa)]_{1/2}b_{1/2} = [(wa)]_{1/2}b_{1/2} \),
\[ [(a''z)]_{1/2} = 0 \]
we have
\[ [(wa)]_{1/2} = [(wa)]_{1/2}b_{1/2} = [(wa)]_{1/2}b_{1/2} = [(wa)]_{1/2}. \]
We now define \( \mathbb{G} \) to be the set of all \( g \in \mathbb{A}_1(1/2) \) such that \( (gc)_{10} \) is in \( \mathbb{B} \).

**Theorem 5.** \( \mathbb{A}_1(1/2) \) is the direct sum of the two subspaces \( (w\mathbb{B})_{1/2} \) and \( \mathbb{G} \). Moreover \( (Ga)_{1/2} \subseteq \mathbb{G} \), \( (Gz)_{1/2} \subseteq (w\mathbb{B})_{1/2} \), and \( [(w\mathbb{B})_{1/2}z]_{1/2} \subseteq \mathbb{G} \), for all \( a \in \mathbb{B} \).

**Proof.** If \( g \) is any element of \( \mathbb{A}_1(1/2) \), let \( (gc)_{10} = a + a'z \) where \( a \) and \( a' \) are in \( \mathbb{B} \). Since \( [(a'z)w]_{1/2} = [(a'z)w]_{1/2} = [(a'z)w]_{1/2} = 0 \), we have
\[ [(wa)]_{1/2} = [(wa)]_{1/2}b_{1/2} = [(wa)]_{1/2}b_{1/2} = [(wa)]_{1/2}. \]
We now define \( \mathbb{G} \) to be the set of all \( g \in \mathbb{A}_1(1/2) \) such that \( (gc)_{10} \) is in \( \mathbb{B} \).

Since \( D(c^2) = 2c \), the \( \mathbb{A}_1(1/2) \)-components of the terms obtained from \( P(c, c, w, g) = 0 \) with \( g \in \mathbb{G} \) yield the relation
\[ 8[(cw)_{1/2}(cg)]_{1/2} = 2c[(cw)(cg)]_{1/2} + w(c^2)g + 2w[c(cg)]_{1/2} + 2w[c(cg)]_{1/2} = 6g(cz)_{1/2}. \]
Using this relation, Theorem 4 and the property that \((cg)_{10} \in \mathcal{B}\) it is easily seen that \([g(cz)]_{1/2}\) is in \((\omega \mathcal{B})_{1/2}\). Therefore \([g(cz)]_{1/2}c_{10} \in \omega \mathcal{B}\). But

\[
[g(cz)]_{1/2}c_{10} = [(gc_1)_{1/2}c_1 - (gc_0)_{1/2}c_0 + (gc_0)_{1/2}c_0 + (gc_1)_{1/2}c_1]_{10}
\]

\[
= [(1/4)c_1^2g(1/4)c_2^2g - (1/2)(gc_1)_{0}c_0 + (1/2)(gc_0)_{1}c_1]_{10}
\]

\[
= -(1/4)(c_2^2g)_{10} z + (1/2)(c_2z)_{(cg)_{10}}.
\]

Therefore since \((cg)_{10}\) is an element of \(\mathcal{B}\) we also have \((c^2g)_{10}\) is an element of \(\mathcal{B}\). Similarly \([(cg)]_{1/2}c_{10} = (1/4)(c^2g)_{10} + (1/2)c(cg)_{10}\) is in \(\mathcal{B}\). Therefore \((cg)_{1/2}\)

\[
\text{is in } \mathcal{B}. \text{ We now examine the } \mathcal{A}(1/2)-\text{components of the terms resulting from } P(a_1, c_1, w, g) = 0. \text{ With the help of (3) and (4) we get}
\]

\[
[2(a_1w)_0(c_1g)_{1/2} + 2(a_1w)_{1/2}(c_1g)_0 + 2(c_1w)_{1/2}(a_1g)_0 + 2(c_1w)_0(a_1g)_{1/2}]_{1/2}
\]

\[
\text{(8)}
\]

\[
= \{w[(a_1c_1)g]_0 + g[(a_1c_1)w]_0\}_{1/2}.
\]

Interchanging the subscripts 1 and 0 we obtain

\[
[2(a_0w)_1(c_0g)_{1/2} + 2(a_0w)_{1/2}(c_0g)_1 + 2(c_0w)_{1/2}(a_0g)_1 + 2(c_0w)_1(a_0g)_{1/2}]_{1/2}
\]

\[
\text{(8)}
\]

\[
= \{w[(a_0c_0)g]_1 + g[(a_0c_0)w]_1\}_{1/2}.
\]

But

\[
\{g[(a_0c_0)w]_1\}_{1/2} = \{2g[a_0(c_0w)_{1/2} + c_0(a_0w)_{1/2}][\}_{1/2}
\]

\[
= \{2g[a_0(c_1w)_{1/2}] + g[a_1(c_0w)_1]\}_{1/2}
\]

\[
= \{g[c_1(a_0w)_1] + g(a_1g)\}_{1/2}.
\]

Therefore

\[
[2(a_0w)_1(c_0g)_{1/2} + 2(a_0w)_{1/2}(c_0g)_1 + 2(c_0w)_{1/2}(a_0g)_1 + 2(c_0w)_1(a_0g)_{1/2}]_{1/2}
\]

\[
\text{(8)}
\]

\[
= \{g[c_1(a_0w)_1] + w[(a_0c_0)g]_1 + [a_1g]\}_{1/2}.
\]

Again consider only the \(\mathcal{A}(1/2)-\text{components of the terms resulting from } P(a_0, c_1, w, g) = 0. \text{ This relation together with (2), (3) and (4) gives us}
\]

\[
[2(a_0w)_{1/2}(c_1g)_0 + 2(a_0w)_1(c_1g)_{1/2} + 2(a_0g)_1(c_1w)_{1/2} + 2(c_1w)_0(a_0g)_{1/2}]_{1/2}
\]

\[
\text{(9)}
\]

\[
= \{[a_0(c_1w)_0]g + g[c_1(a_0w)_1] + w[a_0(c_1g)_0] + w[c_1(a_0g)_1]\}_{1/2}.
\]

Interchanging the subscripts 0 and 1 in (10) we obtain

\[
[2(a_1w)_{1/2}(c_0g)_1 + 2(a_1w)_0(c_0g)_{1/2} + 2(a_1g)_0(c_0w)_{1/2} + 2(c_0w)_1(a_1g)_{1/2}]_{1/2}
\]

\[
\text{(9)}
\]

\[
= \{[a_1(c_0w)_1]g + g[c_0(a_1w)_0] + w[a_1(c_1g)_1] + w[c_0(a_1g)_0]\}_{1/2}.
\]

We now subtract the sum of identities (10) and (11) from the sum of the identities (8) and (9) and use the facts that \((a_1w)_{1/2} = (a_0w)_{1/2}, (c_1w)_0 = -f \text{ and } (c_0w)_1 = e. \text{ We have \{2[(aw)]_0(cz)g]_{1/2} + (a_0g) - 2(a_1g) - g[(a_1c_1)w]_0 + g[c_0(a_1w)]]_{1/2} is in } (\omega \mathcal{B})_{1/2}. \text{ Therefore}
\[-2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g) - 2g[(a_1c_1w)_{1/2}]_0
- 2g[c_1(a_1w)_{1/2}]_0 + 2g[a_1(c_0w)_{1/2}]_0\}_{1/2}

\[-2D(a)[(cz)g]_{1/2} + (a_0g) - 2(a_1g) - 2g[(a_1c_1w)_{1/2}]_0
- g[a_0(c_0w)]_0 + 2g[a_1(c_1w)_{1/2}]_0\}_{1/2}

\[-2D(a)[(cz)g]_{1/2} - 2(a_2g)_{1/2}\]

is in \(B\). Since \([(cz)g]_{1/2} \in (wB)_{1/2}\) we have \([(az)g]_{1/2} \in (wB)_{1/2}\). To show that \((ga)_{1/2} \in G\) for \(a \in B\) we consider

\[[ga]_{1/2} = \begin{cases} (ga_1)_{1/2} + (ga_0)_{1/2}c_1 + (ga_1)_{1/2}c_0 + (ga_0)_{1/2}c_0 \end{cases}
= \begin{cases} 2(ga_0)_{1/2}c_1 + [g(az)]_{1/2}c_1 + 2(ga_1)_{1/2}c_0 - [g(az)]_{1/2}c_0 \end{cases}

= \begin{cases} (gc)_{1/2} + (gc_1)_{1/2}(cz)_{1/2} \end{cases}

Since \((gc)_{1/2} \in B\) so is \((gc)_{1/2}a_0\). Also since \([g(az)]_{1/2} \in (wB)_{1/2}\) we have

\[[g(az)]_{1/2}(cz)_{1/2} \in G\]

Hence \([(ga)_{1/2}]_{10} \in B\) and \((ga)_{1/2} \in G\). Finally if we take \(a\), \(b\), and \(h\) in \(B\) we have

\[[wa]_{1/2}(bz)_{1/2} = [(wa_0)_{1/2}b_1]_{1/2}h_1 - [(wa_0)_{1/2}b_0]_{1/2}h_1 \]

\[[wb_1a_0]_{1/2}h_1 - (1/4)(wb_1a_0b_0)_{1/2} = [(wb_0a_0)_{1/2}h_1 - (1/4)(wb_1a_0b_0)_{1/2} = (1/4)(wb_1a_0b_0)_{1/2}h_1 - (1/4)(wb_1a_0b_0)_{1/2} = 0. Similarly [(wa_0)_{1/2}b_2]_{1/2}h_1 \] = 0. By taking \(h = c\) we can see that the \((wB)_{1/2}\) component of \([(wa)_{1/2}(bz)]_{1/2}\) is 0.

Hence \([(wa)_{1/2}(bz)]_{1/2} \in G\).

**Theorem 6.** \([(wB)_{1/2}(Bz)]_{1/2} = 0.**

**Proof.** Let \(a\) be a nilpotent element of \(A_{1}(1)\). There exists a \(\lambda \in \mathbb{F}\) such that \(d = a + \lambda c\) has the property that \((d_0w)_{1/2}\) is a nonsingular element \(b_1\) of \(A_{1}(1)\). Then

\[d(2b_1^{-1}w)_{1/2} = b_1^{-1}(dw)_{1/2} = e.\]

If we let \(b\) be the unique element of \(B\) whose \(A_{1}(1)\)-component is \(b_1\) we have by the isomorphism established in Theorem 2 that

\[d(b^{-1}w)_{1/2} = b^{-1}(d)_{1/2} = z.\]

For these elements \(d \in C\) and \((wb^{-1})_{1/2} \in A_{1}(1/2)\) we get a \(B \subseteq C\) such that \(B + \mathbb{B} = C\) and where \(B\) has the properties described for \(B\) in Theorems 4–5. Let \(t + sz \in Bz\) where \(t\) and \(s \in B\). We have

\[[wb^{-1}]_{1/2}(t + sz)]_{1/2} = 0.\]

Therefore \((wb^{-1})_{1/2} + [(wb^{-1})_{1/2}(sz)]_{1/2} = 0.\) Since \([wb^{-1}]_{1/2}(sz)]_{1/2} \in G\) we must have \((wb^{-1})_{1/2} = 0\) and \(b^{-1} t = 0.\) Therefore \(t = 0\) and \(Bz \subseteq Bz\). If \(Bz\) is a proper subset of \(Bz\) then \(B\) is a proper subset of \(B\). But this would imply that \(B + Bz\) is a proper subset of \(C\) which is a contradiction. Therefore we must have \(Bz = Bz\) and \([(wb^{-1})_{1/2}(Bz)]_{1/2} = 0.\) Now let \(G\) be the subset of \(B\) of all elements \(s\) such that \([(ws)_{1/2}(Bz)]_{1/2} = 0.\) Let \(x, y \in B\). The re-
lation \( P(y, x, w, z) = 0 \) yields \([[(wx)(yz)]_{1/2} + [(wy)(xz)]_{1/2}] = 0 \). Let \( t \in \mathfrak{B}, s \) and \( s' \in \mathfrak{S} \). Then we get \( [[w(s's')(tz)]_{1/2}] = -[[w(t)]_{1/2}(ss')z] = 0 \) from \( P(tw, s, s'z) = 0 \). Hence \( \mathfrak{S} \) is a subalgebra of \( \mathfrak{B} \). If we let \( b_{-1} = \alpha + n \) where \( b \) is as described above and \( n \) is a nilpotent element of \( \mathfrak{B} \) and \( \alpha \in \mathfrak{S} \), then \( n \in \mathfrak{S} \) and hence every power of \( n \) is in \( \mathfrak{S} \). But \( b \) is the sum of a multiple of the identity and a linear combination of powers of \( n \). Hence \( b = \lambda + D(a) \in \mathfrak{S} \) and the derivative of every element of \( \mathfrak{B} \) is in \( \mathfrak{S} \). Now \( a \in \mathfrak{B} \) implies \( a = D(ca) - cD(a) \). Since \( D(ca) \), \( c = (1/2)D(c^2) \) and \( D(a) \) are in \( \mathfrak{S} \) we have \( \mathfrak{B} \subseteq \mathfrak{S} \) and \( [[(w\mathfrak{B})]_{1/2}(\mathfrak{B}z)]_{1/2} = 0 \).

At this point we have obtained partial results on the multiplications of \( \mathfrak{A} \). However, the chief remaining gap in the characterization of \( \mathfrak{A} \) lies with the products involving elements of \( \mathfrak{S} \). To facilitate the determination of these products we shall introduce some symbols \( Q_{g}, \phi_{g}, k_{g}, f_{g}, h_{g} \) on \( \mathfrak{B} \) into \( \mathfrak{B} \) for every \( g \in \mathfrak{S} \) by letting

\[
(12) \quad [g(bz)]_{1/2} = [wQ_{g}(b)]_{1/2},
\]

\[
(13) \quad (gb)_{10} = h_{g}(b) + k_{g}(b)z,
\]

\[
(14) \quad [g(wb)]_{1/2}_{10} = f_{g}(b) + \phi_{g}(b)z
\]

for every \( b \in \mathfrak{B} \). In our subscripts we abbreviate \((ga)_{1/2}\) to \( ga \).

From (2) and (3) and the definition of \( \mathfrak{S} \) we have

\[
\begin{align*}
&\{[[ga)]_{1/2}(bz)]_{1/2}c_{1}\} = \{[[ga)]_{1/2}b_{1}_{1/2}c_{0}\}_{1} - \{[[ga)]_{1/2}b_{0}_{1/2}c_{0}\}_{1} \\
&= \{2[[ga)]_{1/2}b_{1}_{1/2}c_{0} + [[wQ_{g}(a)]_{1/2}b_{1}_{1/2}c_{0} \\
&- 2[[ga)]_{1/2}b_{0}_{1/2}c_{0} - [[wQ_{g}(a)]_{1/2}b_{0}_{1/2}c_{0}\}_{1} \\
&= (1/2)(gc_{0})a_{1}b_{1} + (1/2)b_{1}Q_{g}(a) - (1/2)b_{1}Q_{g}(a) \\
&- 2\{[[gb)]_{1/2}a_{1}_{1/2}c_{0}\}_{1} \\
&= (1/2)(gc_{0})a_{1}b_{1} - 2\{[[gb)]_{1/2}a_{1}_{1/2}c_{0}\}_{1} \\
&+ 2\{[[wQ_{g}(b)]_{1/2}a_{1}_{1/2}c_{0}\}_{1} \\
&= a_{1}Q_{g}(b).
\end{align*}
\]

Now \([[(ga)]_{1/2}(bz)]_{1/2} = [wQ_{g}(b)]_{1/2}\) and therefore \( [[(wQ_{g}(b))]_{1/2}c_{1}]_{10} = Q_{g}(b)z\). Hence

\[
(15) \quad Q_{g}(b) = aQ_{g}(b).
\]

Consider \( h_{g}(a) + k_{g}(a)z = [(gb)]_{1/2}a_{1}_{10} = [(gb)]_{1/2}a_{1} + (gb)_{1/2}a_{0}\}_{1} \\
= 2[(gb)]_{1/2}a_{1}_{10} + [[wQ_{g}(b)]_{1/2}a_{1}_{10} + 2[(gb)]_{1/2}a_{0}\] \\
= b_{0}(ga)_{0} + b_{1}(ga)_{1} + Q_{g}(a)[(az)w]_{10} = bh_{g}(a) + bzk_{g}(a) - Q_{g}(b)D(a).
\]

From this relation we obtain

\[
(16) \quad h_{g}(a) = bh_{g}(a) - Q_{g}(b)D(a),
\]

\[
(17) \quad k_{g}(a) = bk_{g}(a).
\]
We now consider the $\mathbb{C}$-components of the terms of $P(a,a,g,z) = 0$. We have

$$3ahg(a)z + 3akg(a) - 5hga(a)z - 5kga(a) = Q_g(a)D(a)z - h_g(a^2)z - k_g(a^2).$$

If we equate $\mathcal{B}$-components and $\mathcal{B}z$-components we have

$$k_g(a^2) = 2ak_g(a),$$

$$h_g(a^2) = 2ah_g(a) - 4Q_g(a)D(a)$$

by using (16) and (17).

We have proved that $k_g$ is a derivation for every $g \in \mathbb{G}$. We shall now prove that $Q_g$ is a derivation for every $g \in \mathbb{G}$. We have

$$[wQ_g(ab)]_{1/2} = [g(abz)]_{1/2} - [g(ab)z]_{1/2}$$

$$= [(ga_1)_{1/2}b_1 + (gb_1)_{1/2}a_1 - (ga_0)_{1/2}b_0 - (gb_0)_{1/2}a_0]_{1/2}$$

$$= [(ga_1)_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1 - (ga_0)_{1/2}b_0$$

$$- (gb_1)_{1/2}a_0 + (wQ_g(b))_{1/2}a_0]_{1/2}$$

$$= [(ga_0)_{1/2}b_1 + (wQ_g(a))_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1$$

$$- (ga_1)_{1/2}b_0 + (wQ_g(a))_{1/2}b_0 - (gb_1)_{1/2}a_0 + (wQ_g(b))_{1/2}a_0]_{1/2}$$

$$= [(ga_0)_{1/2}b_1 - (gb_1)_{1/2}a_0 + (gb_0)_{1/2}a_1 - (ga_1)_{1/2}b_0$$

$$+ wQ_g(ab) + wQ_g(b)a]_{1/2}.$$ 

By (4) we have $[wQ_g(ab)]_{1/2} = [wQ_g(a)b + Q_g(b)a]_{1/2}$. Therefore

$$Q_g(ab) = Q_g(a)b + Q_g(b)a.$$

Next, we consider the $\mathcal{G}$-components of the terms of $P(g,a,bz,z) = 0$. However

$$[(ga)b]_{1/2} = [2(ga_0)b_1 + (wQ_g(a))b_1 + 2(ga_1)b_0 - (wQ_g(a))b_0]_{1/2}$$

$$= 2[(gb_1)a_0 + (gb_0)a_1]_{1/2}$$

$$= [(gb)a_0 + (wQ_g(b))a_0 + (gb)a_1 - (wQ_g(b))a_1]_{1/2}$$

$$= [(gb)a]_{1/2}.$$ 

If we combine the above two relations we have

$$[(ga)b]_{1/2} = [g(ab)]_{1/2}.$$ 

A similar computation using $P(w,w,a,z) = 0$ and $P((wa)_{1/2},w,a,z) = 0$ gives us

$$w(wa)_{1/2} = w^2a + D^2(a)$$

$$= w^2a^2 + 2aD^2(a) - D(a)D(a).$$ 

If we consider the $(w\mathcal{B})_{1/2}$-components of the terms of $P(z,(aw)_{1/2},w,g) = 0$ we have $[wQ_g(w^2a) + wQ_g(D^2(a)) + w(a\phi_g(1)) + w\phi_g(a)]_{1/2} = 0$. By letting $a = 1$ we get

$$\phi_g(1) = -\frac{1}{2} Q_g(w^2).$$
Therefore

\[ \phi_g(a) = \frac{1}{2} a Q_g(a^2) - Q_g(a)Q_g(D^2(a)). \]

From (15) and (25) we have

\[ \phi_{ga}(b) = a\phi_g(b). \]

We now wish to express \( h_g \) in terms of \( Q_g \) and \( D \). We examine the \( \mathbb{B}z \)-components of \( P(w, g, c, a) = 0 \) and use (21) and (26) to get

\[ 3\phi_g(a)c + 3\phi_g(c)a + 3h_g(a) + 3D(h_g(c)) + 3D(a)h_g(c) + 3eD(h_g(a)) = 3\phi_g(1 c) + D(h_g(ca)) + h_g(a) + h_g(c)a + \phi_g(D(a)). \]

We simplify this relation using (25), (16) and the linearized form of (19) to get

\[ -3Q_g(D^2(a))c + 3h_g(a) = -3Q_g(D^2(c)a) - Q_g(c)D^2(a) - 3D(Q_g(c))D(a) - 3D(Q_g(a)) - 3h_g(c)D(a) + lQ_g(D(a)). \]

Since \( Q_g \) and \( D \) are derivations we have

\[ 3h_g(a) = -3D(Q_g(c))D(a) + 3h_g(c)D(a) + Q_g(D(a)) - 4Q_g(c)D^2(a). \]

If we let \( a = c \) in (27) we get \( h_g(c) = -D(Q_g(c)) \). Therefore (27) simplifies to

\[ 3h_g(a) = -3D(Q_g(c))D(a) + Q_g(D(a)) - 4Q_g(c)D^2(a). \]

We substitute the values obtained from (28) in \( h_g(a) = ch_g(c) + ah_g(c) - 2Q_g(a) - 2Q_g(c)D(a) \), a linearized form of (19), to get

\[ Q_g(a) = Q_g(c)D(a). \]

If we use this relation in (28) we obtain

\[ h_g(a) = -D(Q_g(c))D(a) - 2Q_g(c)D^2(a). \]

We now investigate the behaviour of \( f_g \). Consider the \( \mathbb{B}z \)-components of the terms of \( P((w^2g),(a, z) = 0 \). We have

\[ 2f_g(b)a = f_g(a) + f_g(ab) - D(k_g(a)) - bk_g(D(a)) \]

and when \( b = 1 \)

\[ 2f_g(1)a = f_g(1) + f_g(a) - D(k_g(a)) - k_g(D(a)). \]

We define a new mapping \( T_g \) on \( \mathbb{B}z \) into \( \mathbb{B}z \) for each \( g \) by

\[ T_g(a) + f_g(1)a - f_g(1) + D(k_g(a)). \]

This definition together with (32) gives us \( f_g(a) = f_g(1)a + T_g(a) + k_g(D(a)) \) and

\[ f_{ga}(1) = f_g(1)a - T_g(a) + D(k_g(a)). \]

Now, \( f_{ga}(b) = -f_g(ab) + 2f_g(b)a + b(Dk_g + k_gD)(D(a)) + k_g(b)D(a) \) and

\[ f_{ga}(b) = -f_{ga}(1) + 2f_{ga}(1) + a(Dk_g + k_gD)(b) + D(a)k_g(b) \]

by (31) and (32). Substituting the values for \( f_g(ab), f_g(b), f_{ga}(1) \) and \( f_{ga}(1) \) expressed in terms of \( T_g \) in these relations and simplifying we have
\[ T_g(ab) = T_g(a)b + T_g(b)a \]

and

\[ f_g(b) = f_g(1)ab + T_g(b)a - bT_g(a) + ak_g(D(b)) + bD(k_g(a)) - k_g(a)D(b) \]

It follows readily that

\[ T_g(b) = aT_g(b) - D(b)k_g(a). \]

We have already shown that \( \phi_g(a) = Q_g(c)[(1/2)aD(w^2) - D(w^2a) - D^4(a)] \). We also have that \( P(g, g(a), (aw)_{1/2}) = 0 \) implies \[ g_g(a) \] \( 1/2 = 0 \). If we let \( a = c^3 \) we have \( \phi_g(c^3) = Q_g(c)[-(1/2)c^3D(w^2) - 3c^2D(w^2) - 6] \). Since the second factor on the right-hand side is nonsingular we have \[ [g_g(Q_g(c))]_{1/2} = 0 \). Multiplying by \( cz \) and considering the \( (w^2B)_{1/2} \)-component we get

\[ Q_g(c)^2 = 0. \]

Similarly we have

\[ Q_g(c)k_g(a) = 0. \]

Now consider the element \( w' = [w - wD(Q_g(c))]_{1/2} + g \) of \( \mathfrak{B}_{1/2} \). We have \( (c^2w')_0 = -f \). By Theorem 1 and its proof, \( c_2 - (1/2)(c_2w')_0 \) is an element \( a \) in \( \mathfrak{C} \) such that \( (aw') = 1 \). Also \( (c_2w')_0 = -2c_0 - 4(Q_g(c)_0) \). Therefore \( (aw')z = \{[c + Q_g(c) - Q_g(c)z]w']z = 1 - 2D(Q_g(c))^2 - 2D(Q_g(c))^2z - 2Q_g(c)^2D^2(Q_g(c))z + k_g(Q_g(c)) - 2Q_g(c)^2D^2(Q_g(c)) + k_g(Q_g(c))z \}. Simple properties of derivations and the fact that \( Q_g(c)^2 = 0 \) gives us \( (aw') = 1 + k_g(Q_g(c)) + k_g(Q_g(c))z \). Therefore

\[ k_g(Q_g(c)) = 0. \]

We also have from (35) and (36) that

\[ T_g(Q_g(c)) = f_g(1)Q_g(c) \text{ and } T_g(b)Q_g(c) = 0 \]

for every \( b \in \mathfrak{B} \).

For \( w' \) and \( c' = c + Q_g(c) - Q_g(c)z \) we have a corresponding \( \mathfrak{B}' \) and \( \mathfrak{B}'z \) as described in Theorem 2. To determine these two subspaces we let \( a + bz \) be an element of \( \mathfrak{C} \) with \( a, b \in \mathfrak{B} \) and such that the \( 1/2 \)-component of \( w'(a + bz) \) is 0. We obtain \( wa - wD(Q_g(c)a + ga + wQ_g(b))_{1/2} = 0 \). Therefore \( a[1 - D(Q_g(c))] = -Q_g(c)D(b) \). Solving for \( a \) we have \( a = -D(b)Q_g(c) \). Since \( \mathfrak{B}' + \mathfrak{B}'z = \mathfrak{C} \), we can conclude from the above result that \( \mathfrak{B}' \) consists of all elements of the form \( a - Q_g(a)z \). We note that the \( \mathfrak{C} \)-component of the element \( (a - Q_g(a)z)w' \) must be an element of \( \mathfrak{B}'z \) by Theorem 3. If we calculate this element we obtain

\[ D(a)z - D(a)D(Q_g(c)z) + Q_g(c)D^2(a) + k_g(a)z - D(Q_g(a))D(Q_g(c)) + D(Q_g(c))^2 \cdot D(a)z. \]

In order for this element to be in \( \mathfrak{B}'z \) we must have \( Q_g(c)D^2(a) + D(Q_g(c))^2D(a) = Q_g(c)D[Da - D(a)D(Q_g(c)) + k_g(a) + D(Q_g(c))^2D(a)] \) by the definition of \( \mathfrak{B}'z \). Therefore
We also have

\[ Q_g(c)k_i(b) = 0 \]

for any \( t \in \mathfrak{g} \) and any \( b \in \mathfrak{b} \) since \( Q_g(c)k_i(b) = k_i(Q_g(c)b) - k_i(Q_g(c))b = k_i(Q_g(b)) - k_i(Q_g(c))b = 0 \).

We define \( t' \) to be the 1/2-component of

\[ w[-D(Q_g(c))D(Q_g(c)) + Q_g(c)D^2(Q_g(c)) - k_i(Q_g(c))] + t \]

for \( t \in \mathfrak{g} \). Then the \( \mathfrak{c} \)-component of \( (c + Q_g(c) - Q_e(c)z)t' \) is

\[ -D(Q_t(c)) - D(Q_t(c))D(Q_g(c)) - 2Q_t(c)D^2(Q_g(c)) + k_i(Q_g(c)) + Q_g(c)D^2(Q_g(c)) \]

since \( Q_t(c)D^2(Q_g(c)) + 2D(Q_g(c))D(Q_t(c)) + Q_g(c)D^2(Q_t(c)) = 0 \) and
\[ 2D(Q_g(c))D(Q_g(c)) = 0. \]

Hence \( t' \) is in \( \mathfrak{g}' \). We now compute \( D' \) and \( Q' \). We have simply that

\[ D' = a - Q_g(a)z - D(Q_g(c))D(a) + D(Q_g(c))D^2(a) + k_g(a) \]

\[ + Q_g(c)D^2(a) + D(Q_g(c))^2D(a), \]

\[ Q'_r : c + Q_g(c) - Q_g(c)z \rightarrow Q_r(a) + Q_g(c)D(Q_g(c)) - Q_g(c)D(Q_g(c))z. \]

Therefore

\[ D'Q'_r : c + Q_g(c) - Q_g(c)z \rightarrow D(Q_t(c)) + D(Q_g(c))D(Q_g(c)) \]

\[ + Q_g(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_g(c)) + k_g(Q_g(c)) - Q_g(c)D^2(Q_g(c))z. \]

By (30) and (44) we have

\[ D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) + Q_g(c)D^2(Q_g(c)) = D(Q_g(c))D(Q_t(c)) + k_g(Q_g(c)) \]

\[ = D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) + 2Q_t(c)D(Q_g(c)) + Q_g(c)D^2(Q_g(c)) - k_t(Q_g(c)). \]

Therefore

\[ Q_g(c)D^2(Q_g(c)) - Q_g(c)D(Q_g(c)) = 2Q_g(c)D^2(Q_g(c)) \]

and

\[ Q_g(c)D^2(Q_g(c)) = -D(Q_g(c))D(Q_g(c)). \]

Replacing \( t \) by \( (ct)_1/2 \) we have \( cQ_g(c)D^2(Q_g(c)) = -cD(Q_t(c))D(Q_g(c)) - Q_t(c)D(Q_g(c)) \) and therefore

\[ Q_g(c)D(Q_g(c)) = 0. \]

We now examine the \( \mathfrak{b} \)-components of the terms of \( P(g, t, a, z) = 0 \) for \( g, t \in \mathfrak{g} \) and \( a \in \mathfrak{b} \). We have

\[ m(1, a) + m(a, 1) = 2m(1, 1)a + 2D(Q_t(c))D(D(Q_g(c))D(a)) \]

\[ + 2D(Q_g(c))D(D(Q_t(c))D(a)) + (k_gk_t + k_tk_g)(a) \]
where \( m(a, b) \) denotes the \( \mathfrak{B} \)-component of \((ga)_{1/2} \cdot (tb)_{1/2}\). Since \( m(a, b) \) does depend on \( g \) and \( t \) also, we will use \( \hat{m}_{\mathfrak{g}, \mathfrak{i}}(a, b) \) for \( m(a, b) \) when there is any chance of confusion. Replacing \( t \) by \((tb)_{1/2}\) in (49) we obtain

\[
m(1, ab) + m(a, b) = 2m(1, b)a + 2bD(Q_{g}(c))D(D(Q_{\mathfrak{g}}(c))D(a))
\]

(50)

\[
+ 2bD(Q_{g}(c))D(D(Q_{\mathfrak{g}}(c))D(a)) + 2D(Q_{g}(c))D(b)D(a)
\]

\[
+ k_{\mathfrak{g}}(b)k_{i}(a) + b(k_{\mathfrak{g}}k_{i} + k_{i}k_{\mathfrak{g}})(a).
\]

Define

\[
S_{\mathfrak{g}, \mathfrak{i}}(a) = m(1, a) - m(1, 1)a - 2D(Q_{g}(c))D(D(Q_{\mathfrak{g}}(c))D(a)) - k_{\mathfrak{g}}k_{i}(a)
\]

for all \( a \in \mathfrak{B} \). If \( g = t \) the right-hand side of (51) reduces to identity (49) with \( g = t \). Therefore \( S_{\mathfrak{g}, \mathfrak{i}} \) is identically zero. A simple linearization gives us

\[
S_{\mathfrak{g}, \mathfrak{i}} = -S_{\mathfrak{i}, \mathfrak{g}}.
\]

Substituting (51) into (50) and letting \( a = b \) we have \( \hat{S}_{\mathfrak{g}, \mathfrak{i}}(a^{2}) + 2L_{g}L_{t}(a^{2}) + m(a, a) + k_{\mathfrak{g}}k_{i}(a^{2}) = 2S_{\mathfrak{g}, \mathfrak{i}}(a)a + m(1, 1)a^{2} + 4aL_{g}L_{t}(a) + 2ak_{\mathfrak{g}}k_{i}(a) \) where \( L_{g} = D(Q_{g}(c))D \) and \( L_{t} = D(Q_{\mathfrak{g}}(c))D \) are derivations. Interchanging \( g \) and \( t \) in this result and subtracting gives us \( 2S_{\mathfrak{g}, \mathfrak{i}}(a^{2}) + 2L_{g}L_{t}(a^{2}) - 2L_{t}L_{g}(a^{2}) + (k_{\mathfrak{g}}k_{i} - k_{i}k_{\mathfrak{g}})(a^{2}) = 4S_{\mathfrak{g}, \mathfrak{i}}(a)a + 4a(L_{g}L_{t} - L_{t}L_{g})(a) + 2a(k_{\mathfrak{g}}k_{i} - k_{i}k_{\mathfrak{g}})(a) \). Since both \( L_{g}L_{t} - L_{t}L_{g} \) and \( k_{\mathfrak{g}}k_{i} - k_{i}k_{\mathfrak{g}} \) are derivations this relation reduces to \( S_{\mathfrak{g}, \mathfrak{i}}(a^{2}) = 2aS_{\mathfrak{g}, \mathfrak{i}}(a) \).

Hence \( S_{\mathfrak{g}, \mathfrak{i}} \) is a derivation of \( \mathfrak{B} \) into \( \mathfrak{B} \).

We can now replace (50) by

\[
m(a, b) = m(1, 1)a + aS_{\mathfrak{g}, \mathfrak{i}}(b) - bS_{\mathfrak{g}, \mathfrak{i}}(a) + 2aL_{g}L_{t}(b) + 2bL_{t}L_{g}(a)
\]

(53)

\[
- 2L_{g}(a)L_{t}(b) + ak_{\mathfrak{g}}k_{i}(b) + bk_{\mathfrak{i}}k_{\mathfrak{g}}(a) - k_{\mathfrak{g}}(a)k_{i}(b).
\]

By setting \( g = t, a = 1 \) and \( b = Q_{g}(c) \) in (53) we have

\[
m_{\mathfrak{g}, \mathfrak{i}}(1, 1)Q_{g}(c) = 0.
\]

An examination of the \((w\mathfrak{B})_{1/2}\)-components of the terms of the \( P(g, g, g, z) = 0 \) gives us

\[
Q_{g}(c)D(m_{\mathfrak{g}, \mathfrak{i}}(1, 1)) = 0.
\]

Finally we compute \( P((ga)_{1/2}, (tb)_{1/2}, w, z) = 0 \) to get

\[
n_{\mathfrak{g}, \mathfrak{i}}(a, b) = -aQ_{g}(f_{\mathfrak{g}}(1)b - T_{t}(b) + D(k_{\mathfrak{i}}(b) - bQ_{g}(f_{\mathfrak{g}}(1)a - T_{g}(a) + D(k_{\mathfrak{g}}(a))
\]

(56)

where \( n_{\mathfrak{g}, \mathfrak{i}}(a, b) \) is the \( \mathfrak{B} \)-component of \((ga)_{1/2} \cdot (tb)_{1/2}\). Now \( P(g, g, (wa)_{1/2}, z) = 0 \) Therefore \( n_{\mathfrak{g}, \mathfrak{i}}(1, 1)a + 2Q_{g}(f_{\mathfrak{g}}(1)a) + 2Q_{g}(T_{g}(a)) = 0 \). From (56) with \( g = t \) and \( a = b = 1 \) we have

\[
Q_{g}(T_{g}(a)) = -Q_{g}(a)f_{\mathfrak{g}}(1).
\]

2. In the previous section we expressed the multiplications of \( \mathfrak{B} \) in terms of constants and derivations. In this section we use these multiplicative properties to construct a simple power-associative algebra of degree two from an associative algebra.
Let $\mathcal{B}$ be an associative, commutative algebra over a field $\mathbb{F}$ of characteristic $p > 5$. Also assume that $\mathcal{B}$ has a single nonzero idempotent $1$ that is a unity quantity.

Let $\mathcal{B}_0, \ldots, \mathcal{B}_n$ be $n$ homomorphic images of the vector space $\mathcal{B}$. We let $\mathcal{L}$ be a sum of these $n$ vector spaces, but not necessarily the vector space direct sum. We let $z\mathcal{B}$ be a one-dimensional module over $\mathcal{B}$. Clearly $z\mathcal{B}$ is a vector space over $\mathbb{F}$ and we form the vector space direct sum $\mathcal{A} = \mathcal{B} + \mathcal{L} + z\mathcal{B}$. We now extend the multiplication of $\mathcal{B}$ to $\mathcal{A}$ in such a way that $\mathcal{A}$ remains a commutative, power-associative algebra. First we define

\begin{align*}
(58) & \quad (za)(zb) = (zb)(za) = ab, \\
(59) & \quad 1x = x, \\
(60) & \quad zy = 0
\end{align*}

for every $a$ and $b$ in $\mathcal{B}$, every $x$ in $\mathcal{A}$ and every $y$ in $\mathcal{L}$. The element $e = \frac{1}{2}(1 + z)$ is an idempotent. We have already defined sufficient multiplicative properties to determine an idempotent decomposition of $\mathcal{A}$. Clearly $\mathcal{L} \equiv \mathcal{L}_e(1/2)$ and $\mathcal{B} + \mathcal{B}z \equiv \mathcal{L}_e(1) + \mathcal{L}_e(0)$. The second part of this statement follows by consideration of $a + bz = (c + cz) + (d - dz)$ with $2c = a + b$ and $2d = a - b$. For each of the vector spaces $\mathcal{B}_i$ and the corresponding homomorphism of $\mathcal{B}$ onto $\mathcal{B}_i$ we define $(g_ib)_{1/2}$ to be the image of $b$. Since this notation is consistent with that of the decomposition of $\mathcal{A}$ with respect to $e$ we will allow the confusion of the two notations.

In order to complete our definitions of the multiplications of $\mathcal{A}$ we choose elements $b_{ij}$ and $b_i$ of $\mathcal{B}$ and derivations $D_{ij}$ and $D_i$ on $\mathcal{B}$ into $\mathcal{B}$ for $i, j = 0, 1, \ldots, n - 1$ with the following restrictions:

\begin{align*}
(61) & \quad D_{ij} = - D_{ji}, \quad b_{ij} = b_{ji}, \quad b_0 = 0
\end{align*}

for all values of $i$ and $j$ and

\begin{align*}
(62) & \quad b_i b_j = (b_i + b_j) b_{ij} = 0, \\
& \quad b_i D_0(b_j) = (b_i + b_j) D_0(b_{ij}) = D_i(b_j) + D_j(b_i) = 0, \\
& \quad (b_i g_j + b_j g_i)_{1/2} = 0, \quad b_i b_0 D_0 = - b_i D_0 D_{0i}
\end{align*}

for all $i$ and $j$ different from 0 and all $b \in \mathcal{B}$. We now define

\begin{align*}
(63) & \quad (g_ia)_{1/2} b = [g(ab)]_{1/2} - D_0(ab) D_0(b) - 2b a D_0^2(b) + a D_i(b) z, \\
(64) & \quad (g_ia)_{1/2} (bz) = - [(g_ia)_{1/2} b] z + \{g_0[a D_0(b) b_i]_{1/2},
(65) & \quad (g_ia)_{1/2} (g_j b)_{1/2} = abb_i + a D_{ij}(b) - b D_j(a) + a D_j D_i(b) + b D_i D_j(a) \\
& \quad - D_j(b) D_i(a) + 2 a L_j L_i(b) + 2 b L_i L_j(a) - 2 L_j(b) L_i(a) \\
& \quad + a b_i [D_0(b) - b_0 b - D_0 D_j(b)] z \cdot b_j D_0 [D_0(a) - b_0 a - D_0 D_i(a)] z
\end{align*}
where $L_i = D_0(b_i)D_0$, $i,j = 0, \ldots, n-1$, and $a$ and $b \in \mathfrak{B}$. Since we did not restrict $\mathfrak{L}$ to be a direct sum of subspaces it is necessary to assume that our multiplications in $\mathfrak{A}$, as defined above, are well-defined. We place two additional assumptions on $\mathfrak{A}$. If $\mathfrak{D}$ is the set of derivations consisting of $D_i$ and $D_{ij}$ for all $i$ and $j$ we assume, in the terminology of Albert [3], that $\mathfrak{B}$ is $\mathfrak{D}$-simple; i.e., there is no nontrivial ideal $\mathfrak{I}$ of $\mathfrak{B}$ such that $\mathfrak{I}$ is $\mathfrak{D}$-admissible. The second assumption is that for every element $g$ in $\mathfrak{L}$ there is a $t$ in $\mathfrak{L}$ such that $gt$ is not zero.

**Theorem 7.** Every commutative, power-associative, simple algebra of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic $p \neq 2, 3, 5$ is an algebra of the type described above.

**Proof.** We choose a set of elements $g_1, \ldots, g_{n-1}$ in $\mathfrak{G}$ such that every element of $\mathfrak{G}$ is expressible in the form $\sum a_i g_i$ where $a_i \in \mathfrak{B}$. We translate the notation of §1 to the notation of this section by letting $\mathfrak{L} = \mathfrak{A}_{1/2}$, $g_0 = w$, $D_0 = D$, $b_{00} = w^2$, $b_{0i} = g_i(1)$, $D_{0i} = T_g$, $D_i = k_{gi}$, $b_i = Q_g(c)$, $b_{ij} = m_{g,gi}(1,1)$ and $D_{ij} = S_{g,gi}$ where $i,j \neq 0$. Identities (25)-(57) give us the relations (61)-(65).

If $\mathfrak{I}$ is a nontrivial ideal of $\mathfrak{B}$ that is $\mathfrak{D}$-admissible then if $a \in \mathfrak{I}$ we have $Q_g(a), f_g(a), \phi_g(a), f_{g,ci}(a, b)$ and $n_{g,ci}(a, b) \in \mathfrak{I}$. This is sufficient to guarantee that $\mathfrak{I} + \mathfrak{I}z + (w\mathfrak{I})_{1/2} + (\mathfrak{I}z)_{1/2}$ is a proper ideal of $\mathfrak{A}$. Since this contradicts the simplicity of $\mathfrak{A}$ we have that $\mathfrak{B}$ is $\mathfrak{D}$-simple.

Let $(wa)_{1/2} + g$ be an element of $\mathfrak{A}_{1/2}$ such that there is no element $t$ in $\mathfrak{A}_{1/2}$ such that $(wa)_{1/2} t + gt \neq 0$. Choosing $t$ to be successively $w, (wc)_{1/2}$ and considering only the $\mathfrak{B}$-components of the resulting terms we have $w^2 a + D^2(a) + f_g(1) = w^2 ac + c D^2(a) - D(a) + f_g(1)c + T_g(c) = w^2 ac^2 + c^2 D^2(a) + 2a - 2 c D(a) + f_g(1)c^2 + 2c T_g(c) = 0$. Eliminating $w^2$ from these equations we have $-D(a) + T_g(c) = 2a - c D(a) + c T_g(c) = 0$. Hence $a = 0$ and $f_g(1) = T_g(c) = 0$. If we multiply $g$ by $(wb)_{1/2}$ for $b \in \mathfrak{B}$ we have $f_g(b) = \phi_g(b) = 0$ by our assumption on $g$. By a previous result we had that $Q_g(c)$ was a multiple of $\phi_g(c)$. Hence $Q_g(c) = 0$. Now $f_g(b) = T_g(b) + k_g(D(b)) = 0$ for all $b \in \mathfrak{B}$. If we substitute $bc$ for $b$ we have $c T_g(b) + c k_g(D(b)) + k_g(b) = 0$. Therefore $k_g(b) = 0$. We now have that $\mathfrak{G} = \{(ag)_{1/2} : a \in \mathfrak{B}\}$. With this choice of $g$ and for any $b \in \mathfrak{B}$ we have $f_g(b) = 0$ by (35) and $\phi_g(b) = 0$ since $Q_g(c) = a Q_g(c)$. Also $m_{g,ci}(a,b) = a S_{g,ci}(b) - b S_{g,ci}(a)$. But by the assumption on $g$ and (51) we have $S_{g,ci} = 0$. Therefore $m_{g,ci}(a,b) = 0$ for all $a$ and $b \in \mathfrak{B}$. Combining this result with (56) we have $(ga)_{1/2} t = 0$ for all $a \in \mathfrak{B}$ and all $t \in \mathfrak{B}_{1/2}$. Therefore the ideal generated by $g$ is $\{(ag)_{1/2} : a \in \mathfrak{B}\}$. This contradicts the assumption of simplicity of $\mathfrak{A}$. Hence for each $x \in \mathfrak{A}_{1/2}$ there is an element $t$ in $\mathfrak{A}_{1/2}$ such that $xt \neq 0$.

**Theorem 8.** An algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ of characteristic $p \neq 2, 3, 5$ as described in identities (58)-(65) is a commutative, power-associative, simple algebra.
Proof. It follows readily from the definition of \( \mathfrak{A} \) that \( \mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2} \) is a subalgebra of \( \mathfrak{A} \). We shall show that this subalgebra is power-associative by examining \( P(x, y, s, t) \) for various values in \( \mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2} \). If \( P(x, y, s, t) = 0 \) for all possible choices of the variables \( x, y, s \) and \( t \) in \( \mathfrak{B}, \mathfrak{B}z \) or \( (g_0\mathfrak{B})_{1/2} \) we have \( \mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2} \) power-associative. We examine the powers of \( x = a + g_0 \) for \( a \in \mathfrak{B} \). We have \( x^2 = a^2 + b_{00} + (ag_0)_{1/2} + 2D_0(a)z \), \( x^3 = a^3 + 2ab_{00} - D_2(a)z + 5aD_0(a)z + D_2(b_{00})z + [(2a^2 + b_{00})g_0]_{1/2} \) and \( x^2x^2 = x^3x \). The proof of this result depends on the properties

\[
\begin{align*}
(a(bz)) &= (ab)z, \\
(az)(bz) &= ab, \\
(bz)(g_0a)_{1/2} &= -aD_0(b), \\
b(g_0a)_{1/2} &= [(ab)g_0]_{1/2} + aD_0(b)z, \\
(g_0a)_{1/2}(g_0b)_{1/2} &= abb_{00} + aD_2(b) + bD_2(a) - D_0(a)D_0(b).
\end{align*}
\]

If \( d \in \mathfrak{B} \) and if we replace \( D_0 \) by \( dD_0 \), \( b_{00} \) by \( b_{00}d^2 + 2dD_1(d) - D_0(d)^2 \) and \( g_0 \) by \( (g_0d)_{1/2} \) we see that relations similar to those expressed in (66) hold. Therefore we can conclude that \( a + (g_0d)_{1/2} \) has a unique fourth power.

Next we investigate the fourth powers of \( x = az + g_0 \). We have \( x^2 = a^2 + b_{00} - 2D_0(a) \), \( x^3 = a^3z + b_{00}az + D_0(b_{00})z - 2D_2(a)z + a^2 + b_{00} - [2D_0(a)g_0]_{1/2} \) and \( x^2x^2 = x^3x \). Again the only multiplicative properties used were those expressed in (66). Therefore \( az + (g_0b)_{1/2} \) has a unique fourth power for all \( a \) and \( b \in \mathfrak{B} \). It is easily seen that \( \mathfrak{B} + \mathfrak{B}z \) is associative. Hence \( a + bz \) has a unique fourth power. The assumption on the characteristic and simple linearizations of these three fourth powers we have obtained give us the result that \( P(x, y, s, t) = 0 \) provided that in any evaluation the four values \( x, y, s, \) and \( t \) are chosen from only two of the three subspaces \( \mathfrak{B}, \mathfrak{B}z \) and \( (g_0\mathfrak{B})_{1/2} \). This leaves us those choices of \( x, y, s \) and \( t \) for which \( x \in \mathfrak{B}, \ y \in \mathfrak{B}z, \ s \in (g_0\mathfrak{B})_{1/2} \) and \( t \) is arbitrary. Because of the linearization process we need only consider \( P(a, bz, (g_0d)_{1/2}, a), \ P(a, bz, (g_0d)_{1/2}, bz) \) and \( P(a, bz, (g_0d)_{1/2}, (g_0d)_{1/2}) \). Straightforward computations, which we omit, show that each of these relations is zero. Therefore \( \mathfrak{B} + \mathfrak{B}z + (g_0\mathfrak{B})_{1/2} \) is power-associative.

Now let \( g = \sum (g_ia_i)_{1/2} \) where \( a_i \in \mathfrak{B} \). The index \( i, \) or indices \( i \) and \( j, \) of this summation and all subsequent ones will run from 1 to \( n - 1 \). Define

\[
\begin{align*}
b_g &= \sum a_i b_i, \\
D_g &= \sum a_i D_i, \\
b_{0g} &= \sum a_i b_{0i} - \sum D_0(a_i) + \sum D_0 D_i(a_i), \\
D_{0g} &= \sum a_i b_{0i} - \sum D_i(a_i)D_0, \\
b_{gg} &= \sum b_{ij} a_i a_j + 2 \sum a_i D_i(a_j) + 4 \sum a_i L_i L_j(a_i) \\
&- \sum D_i(a_i) D_j(a_j).
\end{align*}
\]
From (62) and (67) we have

\[ b_g^2 = b_g b_{gg} = b_g D_0(b_{gg}) = b_g D_g(b) = D_g(b_g) = b_g D_0 D_g(b) = 0, \]

(68)

\[ b_g b_{00} D_0(a) = -b_g D_0 D_{0g}(a), \]

\[ (g b_g)_{1/2} = 0. \]

From (65) we have \((ga)_{1/2}(ga)_{1/2} = b_{gg} = 2a D_g^2(a) - D_g(a)^2 + 4 \sum a_i L_i a_j L_j(a) - 2 \sum a_i L_i(a) a_j L_j(a).\) Now \(\sum a_i L_i(a) = \sum a_i D_0(b_i) D_0(a) = D_0(b_g) D_0(a) - \sum b_i D_0(a_i) D_0(a).\) Therefore \(\sum a_i L_i(a) a_j L_j(a) = L_g(a)^2\) where \(L_g = D_0(b_g) D_0.\) Also

\[ \sum a_i L_i(a) L_j(a) = \sum L_g a_j L_j(a) - \sum b_i D_0(a_i) D_0 a_j L_j(a) \]

\[ = L_g^2(a) - \sum L_g b_i D_0(a_i) D_0(a) - \sum b_i D_0(a_i) D_0 a_j L_j(a) \]

\[ = L_g^2(a) - \sum D_0(b_i) D_0(b_j) a_i D_0(a_j) D_0(a) - \sum b_i D_0(b_j) a_j D_0(a_i) D_0(a) \]

\[ = L_g^2(a). \]

Therefore

\[ (ga)^2_{1/2} = b_{gg} + 2a D_g^2(a) - D_g(a)^2 + 4a L_g^2(a) - 2L_g(a)^2. \]

We also have

\[ b(ga)_{1/2} = g(ab)_{1/2} - D_0(ab_g) D_0(b) - 2b_g a D_0^2(b) + a D_0(b) z, \]

(70)

\[ (bz)(ga)_{1/2} = g_0(a D_0(b) b_g)_{1/2} - [(ga)_{1/2} b] z \]

for all \(a\) and \(b\) in \(\mathcal{B}\).

We now let \(g'_0 = g_0 + g\) and \(a' = a - b_g D_0(a) z\) for \(a \in \mathcal{B}\). We define a derivation \(D'_0(a') = [D_0(a) + D_0(b_g)^2 D_0(a) + D_g(a)]'\) and let \(t = b_{00} + 2b_{0g} - b_g D_0(b_{00}) z + b_{gg} - 2b_g D_0(b_{gg}) z.\) Now \((D_0 + D_0(b_g)^2 D_0 + D_g)^2 = (D + D_g)^2 + 2L_g^2.\) Therefore \(a' D_0^2(a') = a D_0 + D_g^2(a) + 2a L_g^2(a) - b_g [a D_0^2(a) + a D_0^2 D_0(a) + D_0(a) D_0^2(a)] \]

since \(3 b_g D_0 L_g^2(a) = 3 b_g D_0 b_g D_0(b_g) D_0(a) = -3 D_0(b_g) D_0(b_g) D_0^2(b_g) D_0(a) = 2 D_0(b_g) b_g D_0(b_g) D_0(a) = 0.\) Also \([(D_0 + D_0(b_g)^2 D_0 + D_g(a))^2 = [(D_0 + D_g(a))^2 + 2L_g(a)^2.\) Therefore \([D'_0(a')]^2 = [(D_0 + D_g(a))^2 + 2L_g(a)^2 - 2b_g D_0^2(a) D_0(a) z.\]

We have, using these results, that \((g'_0 a')_{1/2} = (g_0 a)_{1/2} = b_{00} a^2 + 2a D_0^2(a) - D_0(a)^2 + 2a^2 b_{0g} + 2a D_0(a) D_0(a) + 2a D_g(a) D_0(a) - b_g a D_0(a) b_{00} z - 2b_{00} D_0(a) a b_g z - 2a b_g D_0(a) z + b_g a^2 + 4a L_g^2(a) - 2 L_g(a)^2 + 2a D_0^2(a) - D_0(a)^2 - 2b_g a^2 D_0(a) z - 2ab_g b_{00} D_0(a) + 2a b_g D_0 D_0(a) D_0(a) z - b_g a D_0 D_0 D_0(a) z = t(a^2)^2 + 2a D_0^2(a)^2 - D_0(a)^2 + 2 b_g b_{00} D_0(a) + 2 b_g a D_0 D_0(a) = t(a^2)^2 + 2a D_0^2(a)^2 - D_0(a)^2.\) Since \(t = g_0^2\) we have

\[ (g'_0 a')_{1/2} = g_0^2 + 2a D_0^2(a) - D_0(a)^2. \]

(71)

From (68) and (70) we have
\[ a'(b'z) = (a'b')z = (ab)'z, \]
\[ (a'z)(b'z) = a'b' = (ab)', \]
\[ (b'z)(g_0a')_{1/2} = -a'D_0(b'), \]
\[ b'(g_0a')_{1/2} = [(a'b')g_0]_{1/2} + a'D_0(b')z. \]

If \( \mathcal{B}' \) is the set of all elements of the form \( a' \) where \( a \in \mathcal{B} \) then \( \mathcal{B}' + \mathcal{B}'z + (g_0 \mathcal{B}')_{1/2} \) is a subalgebra with multiplications similar to those expressed in (66). Hence we can conclude that this subalgebra is power-associative and that \( a' + b'z + (g_0d')_{1/2} \) has a unique fourth power for every \( a', b' \) and \( d' \in \mathcal{B}' \). But \( \mathcal{C} = \mathcal{B}' + \mathcal{B}'z \). Therefore \( a + bz + (g_0d + gd)_{1/2} \) has a unique fourth power for every \( a, b, d \in \mathcal{B} \) and every \( g \). If \( d \) is nonsingular then \( d \) can be absorbed in the coefficients \( a_i \) of \( g_i \) in the expression for \( g \). Hence \( a + bz + (g_0d)_{1/2} + g \) has a unique fourth power if \( d \) is nonsingular. We can restate this as \( x = g_0 + a(a + bz) + \beta(g_0d)_{1/2} + yg \) has a unique fourth power for \( d \) a singular element of \( \mathcal{B}, a, b \in \mathcal{B}, \ g \). The characteristic is sufficiently high so that the attached polynomials of the expression \( x^2x^2 - x^4 \) are all zero [6]. The sum of those polynomials with a coefficient \( \alpha \beta^j k \) where \( i + j + k = 4 \) is of course also equal to zero. But by replacing \( \alpha, \beta \) and \( y \) by 1 in this sum we get \( y^2y^2 - y^4 = 0 \) where \( y = (a + bz + (g_0d)_{1/2} + g) \). Hence any element of \( \mathcal{A} \) has a unique fourth power and \( \mathcal{A} \) is power-associative.

To complete the proof it remains only to show the simplicity of \( \mathcal{A} \). Let \( \mathcal{I} \) be a proper ideal of \( \mathcal{A} \) with the nonzero element \( a + bz + t \) where \( a, b \in \mathcal{B} \) and \( t \in \mathcal{I} \). Since \( z \mathcal{I} \subseteq \mathcal{I} \) we have \( az + b \in \mathcal{I} \). Now multiply \( az + b \) by \( g_0 \) to get \( (ag_0)_{1/2} + D_0(a)(b) \in \mathcal{I} \). By the above \( (ag_0)_{1/2} \in \mathcal{I} \). Multiplying this element by \( cz \) we get \( a \in \mathcal{I} \) and therefore \( b, t, D(a) \) and \( D(b) \in \mathcal{I} \). Let \( \mathcal{B} \) be the set of all elements of \( \mathcal{B} \) that are in \( \mathcal{I} \). Clearly, \( \mathcal{B} \) is a proper ideal of \( \mathcal{B} \). Since \( \mathcal{B} \cap \mathcal{I} \subseteq \mathcal{I} \) and \( (\mathcal{B} \cap \mathcal{I})_{1/2} \subseteq \mathcal{I} \) it can be easily shown that \( \mathcal{B} \) is \( \mathcal{D} \)-admissible. Hence \( \mathcal{B} = 0 \) and the only nonzero elements that could be in \( \mathcal{I} \) are of the form \( t \) where \( t \in \mathcal{I} \). But by the assumption on \( \mathcal{A} \) there is an \( x \in \mathcal{I} \) such that \( gx \neq 0 \). Since \( gx \in \mathcal{B} + \mathcal{B}z \) and \( \mathcal{I} \cap (\mathcal{B} + \mathcal{B}z) = 0 \) we must have \( \mathcal{I} = 0 \). Therefore \( \mathcal{A} \) is simple.

To further characterize the algebra \( \mathcal{A} \) and its subalgebra \( \mathcal{B} \) we quote a result of Harper [5, Theorem 1].

**Theorem 9.** Let \( \mathcal{B} \) be a commutative, associative algebra with unity 1 over an algebraically closed field \( \mathbb{F} \), and let \( \mathcal{B} \) be \( \mathcal{D} \)-simple relative to a set of derivations \( \mathcal{D} \) over \( \mathbb{F} \). Then \( \mathcal{B} = \mathbb{F}[1, x_1, \ldots, x_n] \) is an algebra with generators \( x_1, \ldots, x_n \) over \( \mathbb{F} \) which are independent except for the relations \( x_1^{p} = \ldots = x_n^{p} = 0 \) where \( p \) is the characteristic of \( \mathbb{F} \).

3. Let \( p \) be a prime \( \neq 2, 3, 5 \) and let \( \mathcal{B} \) be the associative commutative algebra of all polynomials \( \sum_{i=0}^{p-1} a_i c_i \) in \( c \) with \( c^p = 0 \) and \( c^0 = 1 \), the identity of \( \mathcal{B} \). Let \( \mathcal{B} \) be \( \{ (g_0a)_{1/2} : a \in \mathcal{B} \} \). Then \( \mathcal{A} = \mathcal{B} + \mathcal{B}z + (g_0 \mathcal{B})_{1/2} \). Let \( b_{00} = 0 \) and \( D_0 \) be ordinary polynomial differentiation; i.e., \( D_0(c) = 1 \). Assume that \( u = a + bz \)
+ \sqrt{g_0 d)}$, where $a, b, d \in \mathcal{A}$, is an idempotent of $\mathcal{A}$ that is not in $\mathcal{C}$. Then 
\[ a^2 + b^2 + 2dD_0(d) - D_0(d)^2 - 2dD_0(b) + 2abz + 2dD_0(a)z + 2(g_0(da))_{1/2} = a + bz + (g_0 d)_{1/2}. \]
Therefore $d(2a - 1) = 0$ and $2ab + 2dD_0(a) = b$. If $d = 0$ then $a = 1/2$.

By our assumptions $d \neq 0$ and we must have $2a - 1$ is singular. Therefore we can write $a = 1/2 + e's$ where $s$ is a nonsingular element of $\mathcal{B}$ and $t \geq 1$. We have $dc = 0$ and $c' + tc'^{-1}d = 0$. Hence $c't + b = 0$. Since

\[ (73) \quad a^2 + b^2 + 2dD_0(d) - D_0(d)^2 - 2dD_0(b) = a \]

it follows that $a^2 + b^2 + 2dD_0(d) - D_0(d)^2 - 2dD_0(b) = a$.

Assume $t = p - 1$; then $c^{p-1}b = c^{p-2}d$. Now if $b = \sum_{0}^{p-1} \beta_i c_i$ and $d = \sum_{0}^{p-1} \alpha_i c_i$ then we must have $\alpha_0 = 0$, $\beta_0 = \alpha_1$. From $(73)$ we must also have $\beta_0^2 - \alpha_1^2 = 1/4$ which is a contradiction. Therefore $t + 1 \geq p$ and $a = 1/2$.

Let $x' = a' + b'z + (g_0 d')_{1/2}$ be an arbitrary element of $\mathcal{A}$. By considering the product $x'u$ we see that a necessary and sufficient condition that $x' \in \mathcal{A}u(1)$ is that

\[ 2a'd = d', \]

\[ 2fa' + 2D_0(a')d = fe'. \]

The correspondence $a' \rightarrow a' + 2a'bz + 2D_0(a')dz + 2[g_0(a'd)]_{1/2}$ is clearly a 1–1 correspondence between $\mathcal{B}$ and $\mathcal{A}u(1)$ preserving the vector space operations. Therefore $\mathcal{A}u(1)$ is of dimension $p$.

If $u$ is a stable idempotent then Albert has shown [3; 4] that $\mathcal{A} = \mathcal{A}u(1) + \mathcal{A}u(0) + wC$ where $C = \mathcal{A}u(1) + \mathcal{A}u(0)$ and $wC$. Albert also showed that the dimensions of $\mathcal{A}u(1)$, $\mathcal{A}u(0)$ and $wC$ are all equal. Therefore $\mathcal{G} = 0$. A further result of Albert’s is that $\mathcal{A}u(1) + \mathcal{A}u(0) + wC$ is associative. This implies that $\mathcal{A}$ is a simple, associative algebra and hence we must have $c = 0$. We can conclude that our example contains no stable idempotents.

**Bibliography**


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