ON COMMUTATIVE ALGEBRAS OF DEGREE TWO(i)

BY

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Let $\mathfrak{A}$ be a simple, commutative, power-associative algebra of degree 2 over an algebraically closed field $\mathbb{F}$ of characteristic not equal to 2, 3 or 5. The degree of $\mathfrak{A}$ is defined to be the number of elements in the maximal set of pairwise orthogonal idempotents in $\mathfrak{A}$. This algebra has a unit element 1 [1, Theorem 3]. The algebras $\mathfrak{A}$ of characteristic zero were considered by Kokoris [8] and found to be Jordan algebras. Kokoris also gave examples of algebras $\mathfrak{A}$ that were not Jordan [6]. This left the problem of determining those algebras $\mathfrak{A}$ that are not Jordan algebras.

Since $1 = e + f$ where $e$ and $f$ are primitive orthogonal idempotents, we have a decomposition $\mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0)$ where $x \in \mathfrak{A}_e(\lambda)$ if and only if $ex = Ax$. We have $\mathfrak{A}_e(\lambda) = \mathfrak{A}_e(1 - \lambda)$; $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1 - \lambda) + \mathfrak{A}_e(1/2)$ for $\lambda = 1,0$; and $\mathfrak{A}_e(1) = e\mathfrak{F} + \mathfrak{N}_1$, $\mathfrak{A}_e(0) = f\mathfrak{F} + \mathfrak{N}_0$ where $\mathfrak{N}_1$ and $\mathfrak{N}_0$ are nilideals of $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ respectively. If $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1/2)$ for $\lambda = 1,0$ we say that $e$ is a stable idempotent. If $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \subseteq \mathfrak{A}_e(1/2) + \mathfrak{N}_{1-\lambda}$ for $\lambda = 1,0$ we say that $e$ is a nilstable idempotent.

The results of Albert extend the characteristic zero case to include algebras of characteristic $p \neq 2,3,5$ for which every idempotent is stable [2]. He also characterized those algebras of characteristic $p \neq 2,3,5$ that have at least one stable idempotent [3; 4]. Recently Kokoris announced [9] that every simple, flexible, power-associative algebra over an algebraically closed field of characteristic $\neq 2,3$ that is of degree two and in which every idempotent is nilstable is a $J$-simple algebra.

It is the purpose of this paper to fill in the remaining gap by giving a characterization of those algebras $\mathfrak{A}$ that have an idempotent that is not nilstable. An example is also given of an algebra $\mathfrak{A}$ that does not have a stable idempotent.

1. Let $\mathfrak{A}$ be an algebra that is simple, commutative, power-associative, of degree two and whose base field $\mathbb{F}$ is an algebraically closed field of characteristic $p \neq 2,3,5$. Let $e$ be a primitive idempotent of $\mathfrak{A}$ that is not nilstable. Since $\mathfrak{A}$ is power-associative we have $x^2x^2 = x^4$ for all $x \in \mathfrak{A}$ and the linearization of this identity

$$P(x,y,s,t) = 4(xy)(st) + 4(xs)(yt) + 4(xt)(ys)$$

$$- x[y(xt) + s(yt) + t(xs)] - y[x(ts) + t(xs) + s(xt)]$$

$$- s[x(yt) + y(xt) + t(xy)] - t[x(ys) + y(xs) + s(xy)] = 0.$$
We will use $\mathcal{C}$ to represent the space $\mathfrak{A}(1) + \mathfrak{A}(0)$, $a_j$ to represent the $\mathfrak{A}(\lambda)$-component of $a$, $a_{10}$ to represent the $\mathcal{C}$-component of $a$, and $z$ to represent $e-f$.

We will make frequent use of some of the results of Albert on commutative power-associative algebras; namely, results (5), (6), (7), (8) of [1]. We state them as

\begin{align*}
(2) \quad [g(xy)_{1/2}] = [(g x_{1/2}) y_{1/2}]_{1/2} + [(g y_{1/2}) x_{1/2}]],
(3) \quad [g(xy)_{1-\lambda}] = 2[(g x_{1/2}) y_{1-\lambda}] + 2[(g y_{1/2}) x_{1-\lambda}],
(4) \quad [(g x_{1/2}) y_{1-\lambda}]_{1/2} = [(g y_{1-\lambda}) x_{1/2}]],
(5) \quad (g x_{1-\lambda}) y_{1/2} = 2[(g y_{1-\lambda}) x_{1/2}]],
\end{align*}

where $\lambda = 1, 0; g \in \mathfrak{A}(1/2)$ and $x$ and $y$ are in $\mathcal{C}$.

Two other relations

\begin{align*}
(6) \quad 2[(x_{1/2} g)_{1/2}]_k + [(x_{1/2} g)_{1-\lambda} g]_k = x_k g^2,
(7) \quad (x_{1/2} g)_{1/2} = (x_{1/2} g)_{1/2} \text{ implies } (x_{1/2} g)_{1/2} = (x_{1/2} g)_{1/2}
\end{align*}

for $x$ and $g$ as above will be useful. The first of these is obtained from $P(x, e, g, g) = 0$ while the second can be derived from (2) and (4).

**Theorem 1.** $\mathcal{C}$ is an associative subalgebra of $\mathfrak{A}$ with an element $c \in \mathcal{C}$ such that there is a $w \in \mathfrak{A}(1/2)$ with $z(cw) = 1$, $(c_1 w)_{1/2} = (c_0 w)_{1/2}$ and $(c_1^2 w)_0 = -2c_0$.

**Proof.** It is easily seen that the subset $\mathfrak{S}$ of $\mathfrak{A}(1)$ consisting of all elements of the form $(a_0 g)^1$ is an ideal of $\mathfrak{A}(1)$ where $g \in \mathfrak{A}(1/2)$ and $a_0$ is a fixed element of $\mathfrak{A}(0)$ because by (5) we have $b_1(a_0 g)^1 = 2[a_0(b_1 g)^1]_1$. The additive property of an ideal is immediate.

We now let $b_1$, $d_1$ be elements of $\mathfrak{A}(1)$, $g \in \mathfrak{A}(1/2)$ and $a_0 \in \mathfrak{A}(0)$ with $(a_0 g)^1 = a_1$. If we consider only the $\mathfrak{A}(1)$-components of each of the terms in $P(b_1, d_1, g, a_0) = 0$ we get $2(b_1 d_1 a_1) = b_1(d_1 a_1) + d_1(b_1 a_1)$. If $b_1$ is also in $\mathfrak{S}$ we can interchange $a_1$ and $b_1$ to get $a_1(d_1 b_1) = 2b_1(d_1 a_1) - d_1(b_1 a_1)$. Therefore $a_1(d_1 b_1) = (a_1 d_1) b_1$. Hence $\mathfrak{S}$ is associative.

It has been shown [1, Lemma 11] that if $(a_0 g) \in \mathfrak{S}$ for all $a_0 \in \mathfrak{A}(0)$ and $g \in \mathfrak{A}(1/2)$ then $(a_1 g)_0 \in \mathfrak{S}_0$ for all $a_1 \in \mathfrak{A}(1)$ and $g \in \mathfrak{A}(1/2)$. From this result and the assumption that $e$ is not nilstable we can conclude that there is an element $c_0 \in \mathfrak{A}(0)$ and an element $g$ in $\mathfrak{A}(1/2)$ such that $(c_0 g)^1$ is nonsingular. If $b_1$ is the inverse of $(c_0 g)^1$ in $\mathfrak{A}(1)$ then $[c_0(2b_1 g)^1]_1 = b_1(c_0 g)^1 = e$. We may also conclude that $\mathfrak{A}(1) = \mathfrak{S}$ is associative. In a similar manner we obtain the result that $\mathfrak{A}(0)$ is associative.

If we take $c_0 \in \mathfrak{A}(0)$ and $w \in \mathfrak{A}(1/2)$ such that $(c_0 w)^1 = e$ and let $2c_1 = (c_0^2 w)^1 = 4[c_0(c_0 w)^1]_1$ then we can quote the results of Kokoris [7, Lemma 4 and Identity 29] that $(c_1 w)_0 = -f$ or 0 and $(c_1 w)_{1/2} = (c_0 w)_{1/2}$. No generality will be lost if we also assume that $c_0$ is nilpotent because $\mathfrak{A}(0) = f\mathfrak{S} + \mathfrak{S}_0$ and
(c_0 w)_1 = [(a f + c_0 w)_1] for any \(a \in \mathfrak{F}\). To complete the proof of the theorem it remains only to show that \((c_1 w)_0 \neq 0\). We assume that \((c_1 w)_0 = 0\). If we examine the \(\mathfrak{A}(1)\)-components of the terms of the relation \(P(c_0, c_0, w, w) = 0\) we get
\[
8(c_0 w)_1 + 2[(c_0 w)^2]_{1/2} = 4[c_0(w_c_0)_{1/2}]_{1} + 2[w(c_0 w)^{1/2}]_{1} + 4[w(c_0 w)^{1/2}]_{1/2}.
\]
Using this relation together with (2), (6), (7) and \((c_0 w)_1 = e\), we get
\[
6e + 8[(c_1 w)^2]_{1/2} = 2w^2 c_1^2 = 2w[(c_1 w)_0]_{1/2}.
\]
But \((c_1 w)_0 = 4[c_1(c_1 w)^{1/2}]_{0} = 4[c_1(c_0 w)]_{0} = 2c_0(c_1 w)_0 = 0\). Therefore either \([c_1 w]_{1/2}\) or \(w^2 c_1^2\) must be nonsingular. If we again use (1) with \(P(c_1, c_1, w, w) = 0\) and examine the \(\mathfrak{A}(0)\)-components of the resulting terms we get
\[
8[(c_0 w)^2]_{1/2} = 2c_0^2 w^2.
\]
But \((c_0 w)^{1/2} = x_1 + n\) where \(n \in \mathfrak{A}_1 + \mathfrak{A}_0 [1, \text{Lemma 10}]\) we must also have \([c_1 w]^2_{1/2}\), nilpotent. Now by (6) we have
\[
2[(c_0 w)_1]_{1/2} = 2[(c_1 w)^{1/2}]_{1} = -[(c_1 w)_0 w]_{1/2} + c_1 w^2 = c_1 w^2.
\]
But \(2[(c_0 w)_1]_{1/2} = -[(c_0 w)_0]_{1/2} + c_0 w^2 = c_0 w^2\) is nilpotent. Therefore \(c_1 w^2\) and \(c_2 w^2\) are nilpotent. We have arrived at a contradiction. Hence \((c_1 w)_0 = -f\) and the theorem is proved.

**THEOREM 2.** There is an isomorphism \(T\) between \(\mathfrak{A}(1)\) and \(\mathfrak{A}(0)\) such that for \(b_1 \in \mathfrak{A}(1), T(b_1)\) is the unique element of \(\mathfrak{A}(0)\) satisfying \((b_1 w)^{1/2} = [T(b_1) w]_{1/2}\). The subset \(\mathcal{B}\) of \(\mathfrak{C}\) of all elements of the form \(b_1 + T(b_1)\) is an associative subalgebra of \(\mathfrak{C}\) isomorphic to both \(\mathfrak{A}(0)\) and \(\mathfrak{A}(1)\).

**Proof.** We use \(c_1, c_0\) and \(w\) as in Theorem 1. If we consider only the \(\mathfrak{A}(1/2)\)-components of the terms in \(P(c_0, b_1, w, w) = 0\) we get
\[
8[(c_0 w)^{1/2}(b_1 w)_0]_{1/2} = 2w(b_1^2 + [(b_1 w)_1]_{1})_{1/2} + 2w(c_0 w)^{1/2} + (b_1 w)_0]_{1/2} + 2(c_0 w)^{1/2}(w b_1)_0]_{1/2} + (b_1 w)_1].
\]
Using (5) and (2) on the terms \([c_0 w]_{1/2} + [b_1 w]_0\) and \([b_1 w]_{1/2} + (c_0 w)_0\) this relation reduces to
\[
2[(c_0 w)_1]_{1/2} + 2[(b_1 w)_0]_{1/2} + 2(c_0 w)^{1/2} + (b_1 w)_1]_{1/2}.
\]
We now consider the \(\mathfrak{A}(1/2)\)-component of each term in \(P(c_1, b_1, w, w) = 0\). We have
\[
-4(b_1 w)_1/2 + 8[(c_1 w)_1]_{1/2}(b_1 w)_0]_{1/2} = 2w[(c_1 b_1 w + c_1(b_1 w)_1/2 + b_1(c_1 w)_0]_{1/2} - (b_1 w)_1/2 + 2(c_1 w)^{1/2} + (b_1 w)_0]_{1/2}]
\]
This relation together with (2) and (4) gives us
\[
2[(c_1 w)_1]_{1/2} = (b_1 w)_1/2 + (w(c_1 b_1 w)_0]_{1/2} + 2[(c_1 w)_1]_{1/2} + (b_1 w)_1/2 + [c_0 w]_{1/2} + (b_1 w)_1/2]_{1/2}.
\]
Therefore \((b_1 w)_1/2 = (2[(c_1 w)_1/2] - [c_0 w]_{1/2} - [c_1 w]_{1/2})_{1/2} = -2[(b_1 w)_1/2]_{1/2}\). We can now define \(T(b_1) = -2[(b_1 w)_1/2]_{1/2}\) to be the element \(b_0\) in \(\mathfrak{A}(0)\) such that \((b_1 w)_1/2 = (b_0 w)_1/2\). To show that \(T\) is well-defined we assume \((a_0 w)_1/2 = 0\). We have \(a_0 = -a_0(c_1 w)_0 = -2[c_1(a_0 w)]_{1/2} = 0\) by (5). Therefore \((b_0 w)_1/2 = (b_0 w)_1/2\) implies \(b_0 = b_0\). Simply by changing the signs of \(c_1\) and \(c_0\) and interchanging 1 and 0 we can get a similar result for \(\mathfrak{A}(0)\); i.e., for every \(b_0 \in \mathfrak{A}(0)\) there is a unique \(b_1 = 2[[[b_0 w]_1/2]_{1/2}]_{1/2} = 0\) such that \((b_0 w)_1/2 = (b_1 w)_1/2\). Therefore \(T\) is onto \(\mathfrak{A}(0)\) and is a 1-1 correspondence between \(\mathfrak{A}(1)\) and \(\mathfrak{A}(0)\).
Now if \(a\) and \(b\) are elements of \(\mathcal{B}\) as defined in the theorem we have, with the help of (2) and (4), that
\[
[w(a_1b_1)]_{1/2} = [(wb_1)a_1 + (wa_1)b_1]_{1/2}
= [(wb_0)a_1 + (wa_0)b_1]_{1/2} = [(wa_1)b_0 + (wb_1)a_0]_{1/2}
= [(wa_0)b_0 + (wb_0)a_0]_{1/2} = [w(a_0b_0)]_{1/2}.
\]
Therefore \(T(a_1b_1) = a_0b_0\) and \(ab = a_1b_1 + a_0b_0\in \mathcal{B}\). Clearly \(\mathcal{B}\) is closed under addition and scalar multiplication.

Define \(S(b) = be\) for every \(b\in \mathcal{B}\). It follows immediately from the above results that \(S\) is a 1-1 correspondence of \(\mathcal{B}\) onto \(\mathbb{A}_e(1)\). From the definition we have \(S(ab) = (ae)(be) = S(a)S(b)\) and \(S(a + b) = S(a) + S(b)\) for all \(a\) and \(b\) in \(\mathcal{B}\). Therefore \(\mathcal{B}\) and \(\mathbb{A}_e(1)\) are isomorphic as rings and hence as algebras. In the same manner we show that \(\mathcal{B}\) is isomorphic to \(\mathbb{A}_e(0)\). We have shown also that \(T\) is an isomorphism. The associativity of \(\mathcal{B}\) follows from that of \(\mathbb{C}\).

From the definition of \(\mathcal{B}\) it is clear that \(c = c_1 + c_0\) is in \(\mathcal{B}\). From \(P(w,w,w,z) = 0\) it follows that \(w^2\) is in \(\mathcal{B}\). Theorem 2 also implies that \(\mathbb{C} = \mathcal{B} + \mathbb{B}z\).

**Theorem 3.** The mapping \(b \mapsto D(b) = (bw)z\) is a derivation of \(\mathcal{B}\) into \(\mathcal{B}\) such that \(D(c) = 1\).

**Proof.** Let \(a\) and \(b\) be arbitrary elements of \(\mathcal{B}\). Then \([(ab)w]_{10} = [(ab)_1w]_0 + [(ab)_0w]_1 = [(a_1b_1)w]_0 + [(a_0b_0)w]_1 = 2[a_1(b_1w)]_{1/2} + 2[b_1(a_1w)]_{1/2} + 2[a_0(b_0w)]_{1/2} + 2[b_0(a_0w)]_{1/2} = b_0(axw) + a_0(bxw) + bx(a_0w) + ax(b_0w) = b_0(axw) + a_0(bxw) + bx(a_0w) + ax(b_0w)
\]
by (3), (5) and the definition of \(\mathcal{B}\). If this relation is multiplied by \(z\) we have \(D(ab) = aD(b) + bD(a)\) and \(D\) is a derivation on \(\mathcal{B}\) into \(\mathbb{C}\).

To show that \(D(b)\) lies in \(\mathcal{B}\) for \(b = b_1 + b_0\), an element of \(\mathcal{B}\), we need several identities; the first of which is obtained from \(P(b_0,w,w,c_1) = 0\). We get
\[
8[(b_0w)_1(w_1c_1)_1/2]_0 + 8[(b_0w)_1/2 (w_1c_1)_1/2]_0 = 2(b_0c_0)w^2 + 2[c_1(b_0w)_1]_0\text{ after the usual simplifications using (2), (5), (6) and (c_1w)_1/2 = (c_0w)_1/2.}
8[(b_0w)_1/2(w_1c_1)_1/2]_0 = 2(b_0c_0)w^2 + 2[c_1(b_0w)_1]_0\text{ after the usual simplifications. We consider } P(b_0,w,w,c_0) = 0 \text{ next to get}
\]
\[
-3(b_0c_0)w^2 + 8[(b_0w)_1/2(wc_1)_1/2]_0 + 6[(b_0w)_1(c_0w)_1/2]_0
= -2[(b_0c_0w)_1/2]_0.
\]
Finally we obtain \(3(b_1w)_0 = 4[(b_1w)_1/2(c_0w)_1/2]_0 \in 2[w[(c_0b_0w)_1/2]_0\text{ from } P(b_1,w,w,c_1) = 0. \text{ Now, from the proof of Theorem 2 and from (3) we have}
\]
\[
6T[(b_0w)_1] = -12[(b_0w)_1]_0 - 6[(b_0w)_1]_0
= 12[(c_1w)_1/2(b_0w)_1]_0 - 6[(b_0w)_1]_0.
\]
By successively applying to this relation the three identities above in the order we
obtained them we get
\[6T[(b_0w)_{1/2}] = -12[(b_0w)(c_1w)_{1/2}]_0 - 24[(b_0w)(c_1w)_{1/2}]_0 + 6b_0c_0w^2 = -12[(b_0w)(c_0w)_{1/2}]_0 - 24[(b_0w)(c_0w)_{1/2}]_0 + 6b_0c_0w^2 = 4[(b_0c_0w)_{1/2}]_0 - 8[(b_0w)(c_0w)_{1/2}]_0 = -6(b_1w)_0.\]
Therefore we have
\[D(b)_0 = (b_0w)_0.\]
The fact that \(D(c) = 1\) follows immediately from the definition of \(c\).

**Theorem 4.** If \(a\) and \(b\) are elements of \(\mathcal{B}\) then
\[(wa)_{1/2}b_{1/2} = [w(ab)]_{1/2},\]
\\
\[(wa)_{1/2}b_{1/2}]_{10} = (wb)_{10}a\quad \text{and} \quad (wa)_{1/2}(wb)_{1/2} \in \mathcal{B}.\]

**Proof.** By (2) and (4) and the definition of \(\mathcal{B}\) we have
\[\begin{align*}
[(wa)_{1/2}b_{1/2}]_{1/2} & = 2[(wa)_{1/2}b_{1/2}]_{1/2} + 2[(wb)_{1/2}a_{1/2}]_{1/2} \\
& = [(wa)_{1/2}b_{1/2}]_{1/2} + 2[(wb)_{1/2}a_{1/2}]_{1/2} + [(wa)_{1/2}b_{1/2}]_{1/2} \\
& = [(wa)_{1/2}b_{1/2}]_{1/2}. \quad \text{By (5) we have} \\
[(wa)_{1/2}b_{1/2}]_{1/2} & = 2[(wa)_{1/2}b_{1/2}]_{1/2} + 2[(wa)_{1/2}b_{1/2}]_{1/2} \\
& = (wb)_{10}a + [(wb)_{1/2}a_{1/2}] = (wb)_{10}a. \quad \text{Now use} \quad P(w,w,a,b) \quad \text{to get} \\
4w^2ab + 8(wa)(wb) & = 2[w(ab)w + (aw)b + (bw)a] + a[w^2b + 2w(wb)] + b[w^2a + 2w(wa)]. \quad \text{If} \\
we consider only the \(G\)-components of each of the terms and if we use the facts \\
that \(wa)_{1/2} \in \mathcal{B}\) for all \(a \in \mathcal{B}\) and \([waz)]_{1/2} \in \mathcal{B}\) then \(8(wa)(wb)_{1/2} - 4w[(ab)w]_{1/2} = 2a[w(wb)]_{1/2} - 2b[(wa)]_{1/2}\) is in \(\mathcal{B}\). \quad \text{Now} \quad P(w,w,a,z,z) = 0 \\
implies \quad 2w^2a = -2D^2(a) + 2w[(wa)]_{1/2}. \quad \text{Since} \quad w^2 \quad \text{and} \quad D(a) \quad \text{are in} \quad \mathcal{B}, \quad \text{so also} \\
is \quad [w(ab)]_{1/2}. \quad \text{Hence} \quad 8(wa)_{1/2}(wb)_{1/2} \quad \text{is in} \quad \mathcal{B}.
\]

**Corollary.** If \(a \in \mathcal{C}\) and \(b \in \mathcal{B}\) then
\([(wa)_{1/2}b_{1/2}]_{1/2} = [w(ab)]_{1/2}.

**Proof.** We can write \(a = a' + a''z\) where \(a'\) and \(a''\) are in \(\mathcal{B}\). Since \([(a'z)]_{1/2} = [(a''z)]_{1/2} = 0\) we have
\[\begin{align*}
[(wa)_{1/2}b_{1/2}]_{1/2} & = [(wa')_{1/2}b_{1/2}]_{1/2} = [w(ab)]_{1/2} \\
& = [w(ab)]_{1/2}. \\
\end{align*}\]

We now define \(\mathcal{G}\) to be the set of all \(g \in \mathcal{U}_\mathcal{C}(1/2)\) such that \((gc)_{10}\) is in \(\mathcal{B}\).

**Theorem 5.** \(\mathcal{U}_\mathcal{C}(1/2)\) is the direct sum of the two subspaces \((w\mathcal{B})_{1/2}\) and \(\mathcal{G}\). Moreover \((Ga)_{1/2} \subseteq \mathcal{G}, \quad ([Gaz])_{1/2} \subseteq (w\mathcal{B})_{1/2}, \quad \text{and} \quad ([w\mathcal{B}]_{1/2}(az)]_{1/2} \subseteq \mathcal{G}, \quad \text{for all} \quad a \in \mathcal{B}.

**Proof.** If \(g\) is any element of \(\mathcal{U}_\mathcal{C}(1/2)\), let \((gc)_{10} = a + a'z\) where \(a\) and \(a'\) are in \(\mathcal{B}\). Since \([(a'z)]_{1/2} = [(a''z)]_{1/2} = 0\) we have
\[\begin{align*}
[(w(a')_{1/2}c)]_{10} & = a \in \mathcal{G} \quad \text{and} \\
[(w(a'')_{1/2}c)]_{10} & = a \in \mathcal{G}. \quad \text{If} \\
h \quad \text{lies in both} \quad (w\mathcal{B})_{1/2} \quad \text{and} \quad \mathcal{G} \quad \text{then} \quad (hc)_{10} \quad \text{lies in} \quad \mathcal{B} \quad \text{and} \quad \mathcal{B}. \quad \text{Hence} \\
(hc)_{10} & = 0. \quad \text{But} \\
[(wa)_{1/2}c)]_{10} & = az. \quad \text{Therefore if} \\
h = (wa)_{1/2} \quad \text{then} \quad a = (wa)_{1/2} = 0 \quad \text{and} \quad h = 0. \quad \text{Hence} \\
\mathcal{U}_\mathcal{C}(1/2) \quad \text{is the direct sum of} \quad \mathcal{G} \quad \text{and} \quad (w\mathcal{B})_{1/2}.
\]

Since \(D(c^2) = 2c\), the \(\mathcal{U}_\mathcal{C}(1/2)\)-components of the terms obtained from
\[P(c,c,w,g) = 0 \quad \text{with} \quad g \in \mathcal{G} \quad \text{yield the relation}
\]
\[8[(cw)_{1/2}(cg)]_{1/2} = \frac{2c[w(cg)]_{1/2}}{1/2} + \frac{w(c^2g)}{1/2} + \frac{2w[c(cg)]_{1/2}}{1/2} + \{2w[c(cg)]_{1/2} + 6g(cz)]_{1/2}.
\]
Using this relation, Theorem 4 and the property that \((c^g)_{10} \in \mathcal{B}\) it is easily seen that \([g(c^z)]_{1/2}\) is in \((w\mathcal{B})_{1/2}\). Therefore \([g(c^z)]_{1/2} c_{10} \in 2\mathcal{B}\). But
\[
[g(c^z)]_{1/2} c_{10} = [(g c_1)_{1/2} c_1 - (g c_0)_{1/2} c_1 - (g c_0)_{1/2} c_0 + (g c_1)_{1/2} c_0]_{10}
\]
\[
= [(1/4)(c^z)_{1/2} g - (1/4)(c^g)_{1/2} g - (1/2)(g c_1)_{0} c_0 + (1/2)(g c_0)_{0} c_1]_{10}
\]
\[
= -(1/4)(c^g)_{10} z + (1/2)(c^z)(c^g)_{10}.
\]
Therefore since \((c^g)_{10}\) is an element of \(\mathcal{B}\) we also have \((c^2 g)_{10}\) is an element of \(\mathcal{B}\). Similarly \([(c^g)]_{1/2} c_{10} = (1/4)(c^2 g)_{10} + (1/2)c(c^g)_{10}\) is in \(\mathcal{B}\). Therefore \((c^g)_{1/2}\) is in \(\mathcal{B}\). We now examine the \(\mathcal{U}_w(1/2)\)-components of the terms resulting from
\[P(a_1, c_1, \omega, g) = 0.\]
With the help of (3) and (4) we get
\[
[2(a_1 \omega)_{0} (c_1 g)_{1/2} + 2(a_1 \omega)_{1/2} (c_1 g)_{0} + 2(c_1 \omega)_{1/2} (a_1 g)_{0} + 2(c_1 \omega)_{0} (a_1 g)_{1/2}]_{1/2}
\]
\[= \{w[(a_1 c_1) g]_0 + g[(a_1 c_1) w]_0\}_{1/2}.
\]
Interchanging the subscripts 1 and 0 we obtain
\[
[2(a_0 \omega)_{1} (c_0 g)_{1/2} + 2(a_0 \omega)_{1/2} (c_0 g)_{1} + 2(c_0 \omega)_{1/2} (a_0 g)_{1} + 2(c_0 \omega)_{1} (a_0 g)_{1/2}]_{1/2}
\]
\[= \{w[(a_0 c_0) g]_1 + g[(a_0 c_0) w]_1\}_{1/2}.
\]
But
\[
\{g[(a_0 c_0) w]_1\}_{1/2} = \{2g[a_0 (c_i w)_{1/2} + c_i (a_0 w)_{1/2}]_1\}_{1/2}
\]
\[= \{2g[a_0 (c_i w)_{1/2}]_1 + g[a_1 (c_0 w)_{1}]_1\}_{1/2}
\]
\[= \{g[c_i (a_0 w)_{1}] + ga_1\}_{1/2}.
\]
Therefore
\[
[2(a_0 \omega)_{1} (c_0 g)_{1/2} + 2(a_0 \omega)_{1/2} (c_0 g)_{1} + 2(c_0 \omega)_{1/2} (a_0 g)_{1} + 2(c_0 \omega)_{1} (a_0 g)_{1/2}]_{1/2}
\]
\[= \{g[c_i (a_0 w)_{1}] + w[(a_0 c_0) g]_1 + [a_1 g]_{1/2}\}.
\]
Again consider only the \(\mathcal{U}_w(1/2)\)-components of the terms of \(P(a_0, c_1, \omega, g) = 0.\)
This relation together with (2), (3) and (4) gives us
\[
[2(a_0 \omega)_{1/2} (c_1 g)_{0} + 2(a_0 \omega)_{0} (c_1 g)_{1/2} + 2(a_0 g)_{1} (c_1 w)_{1/2} + 2(c_1 \omega)_{0} (a_0 g)_{1/2}]_{1/2}
\]
\[= \{w[a_0 (c_1 w)_{0}]_0 + g[c_1 (a_0 g)_{1}] + w[c_1 (a_0 g)_{1}]_0\}_{1/2}.
\]
Interchanging the subscripts 0 and 1 in (10) we obtain
\[
[2(a_1 \omega)_{1/2} (c_0 g)_{1} + 2(a_1 \omega)_{0} (c_0 g)_{1/2} + 2(a_1 g)_{0} (c_0 w)_{1/2} + 2(c_0 \omega)_{1/2} (a_1 g)_{1/2}]_{1/2}
\]
\[= \{w[a_1 (c_0 w)_{1}]_0 + g[c_0 (a_1 g)_{0}]_1 + w[c_0 (a_1 g)_{0}]_1\}_{1/2}.
\]
We now subtract the sum of identities (10) and (11) from the sum of the identities (8) and (9) and use the facts that \((a_1 \omega)_{1/2} = (a_0 \omega)_{1/2}, (c_1 w)_{0} = -f\) and \((c_0 \omega)_{1} = e\). We have \(\{2[(a_2) w]_{0} (c_2 g)_{1/2} + (a_0 g) - 2(a_1 g) - g[(a_1 c_1) w]_0 + g[c_0 (a_1 g)_{0}]_1\}_{1/2}\) is in \((w\mathcal{B})_{1/2}\). Therefore
\[ \{ -2D(a)[(cz)g]^{1/2} + (a_0g) - 2(a_1g) - 2g[(a_1c_1w)_{1/2}]_0 \]
\[ - 2g[a_1(c_1w)_{1/2}]_0 + 2g[a_1(c_0w)_{1/2}]_0 \}^{1/2} \]
\[ = \{ -2D(a)[(cz)g]^{1/2} + (a_0g) - 2(a_1g) - 2g[(a_1c_1w)_{1/2}]_0 \]
\[ - g(a_0(c_1w)_{1/2}) + 2g[a_1(c_1w)_{1/2}]_0 \}^{1/2} \]
\[ = \{ -2D(a)[(cz)g]^{1/2} - 2(a_0g) \}^{1/2} \]
is in \( \mathfrak{B} \). Since \( [(cz)g]^{1/2} \in (w\mathfrak{B})_{1/2} \) we have \( [(az)g]^{1/2} \in (w\mathfrak{B})_{1/2} \). To show that \( (ga)_{1/2} \in \mathfrak{B} \) we consider
\[ [(ga)_{1/2}]^{10} = [(ga_1)_{1/2}c_1 + (ga_0)_{1/2}c_0 + (ga_1)_{1/2}c_0 + (ga_0)_{1/2}c_0]^{10} \]
\[ = [2(ga_0)_{1/2}c_1 + [g(az)]_{1/2}c_1 + 2(ga_1)_{1/2}c_0 - [g(az)]_{1/2}c_0]^{10} \]
\[ = [(gc_1)_{1/2}a_0 + (gc_0)_{1/2}a_1 + g(az)]_{1/2}[(cz)_{1/2}]^{10} \]
\[ = (gc)_{10}a + [(g(az)]_{1/2}[(cz)]^{10}. \]
Since \( (gc)_{10} \in \mathfrak{B} \) so is \( (gc)_{10}a \). Also since \( [g(az)]_{1/2} \in (w\mathfrak{B})_{1/2} \) we have
\[ [(g(az)]_{1/2}[(cz)]^{10} \in \mathfrak{B}. \]
Hence \( [(ga)_{1/2}]^{10} \in \mathfrak{B} \) and \( (ga)_{1/2} \in \mathfrak{G} \). Finally if we take \( a, b \) and \( h \) in \( \mathfrak{B} \) we have \( \{[(wa)_1/(2b_1)_{1/2}]_0 h_1 = [(wa_0)_1/(2b_0)_{1/2}]_0 h_1 \}
\[ = \{[(wb_1)a_0]_{1/2} h_1 - (1/4)(wh_1)a_0 b_0 \}_0 = \{[(wb_0)a_0]_{1/2} h_1 - (1/4)(wh_1)a_0 b_0 \}_0 \]
\[ = (1/4)(wh_1)b_0 a_0 - (1/4)(wh_1)b_0 a_0 = 0. \]
Similarly \( \{(wa_0)(b_1)_{1/2}h_1 \}_0 = 1. \) By taking \( h = c \) we can see that the \( (w\mathfrak{B})_{1/2} \) component of \( [(wa)_{1/2}(bz)]_{1/2} \) is 0. Hence \( [(wa)_{1/2}(bz)]_{1/2} \) is in \( \mathfrak{G} \).

**Theorem 6.** \( [(w\mathfrak{B})_{1/2}(\mathfrak{B}_{1/2})_{1/2}] = 0. \)

**Proof.** Let \( a \) be a nilpotent element of \( \mathfrak{A}_1(1) \). There exists a \( \lambda \in \mathfrak{S} \) such that \( d = a + \lambda c \) has the property that \( (d_0w)_{1/2} \) is a nonsingular element \( b_1 \) of \( \mathfrak{A}_1(1) \). Then
\[ \{d(b^{-1}w)_{1/2}]_1 = b_1^{-1}(dw)_1 = e. \]
If we let \( b \) be the unique element of \( \mathfrak{B} \) whose \( \mathfrak{A}_1(1) \)-component is \( b_1 \) we have by the isomorphism established in Theorem 2 that
\[ d(b^{-1}w)_{1/2} = b^{-1}(d)_{1/2} = z. \]
For these elements \( d \in \mathfrak{C} \) and \( (wb^{-1})_{1/2} \in \mathfrak{A}_1(1/2) \) we get a \( \mathfrak{B} \subseteq \mathfrak{C} \) such that \( \mathfrak{B} + \mathfrak{B}_z = \mathfrak{C} \) and where \( \mathfrak{B} \) has the properties described for \( \mathfrak{B} \) in Theorems 2-5. Let \( t + sz \in \mathfrak{B}_z \) where \( t \) and \( s \in \mathfrak{B} \). We have
\[ \{(wb^{-1})_{1/2}(t + sz)]_{1/2} = 0. \]
Therefore \( (wb^{-1})_{1/2} + [(wb^{-1})_{1/2}(sz)]_{1/2} \) is 0. Since \( [(wb^{-1})_{1/2}(sz)]_{1/2} \in \mathfrak{G} \) we must have \( (wb^{-1})_{1/2} = 0 \) and \( b^{-1} t = 0 \). Therefore \( t = 0 \) and \( \mathfrak{B}_z \subseteq \mathfrak{B} \). If \( \mathfrak{B}_z \) is a proper subset of \( \mathfrak{B}_z \) then \( \mathfrak{B} \) is a proper subset of \( \mathfrak{B} \). But this would imply that \( \mathfrak{B} + \mathfrak{B}_z \) is a proper subset of \( \mathfrak{C} \) which is a contradiction. Therefore we must have \( \mathfrak{B}_z = \mathfrak{B} \) and \( [(wb^{-1})_{1/2}(\mathfrak{B}_z)]_{1/2} = 0. \) Now let \( \mathfrak{S} \) be the subset of \( \mathfrak{B} \) of all elements \( s \) such that \( [(ws)_{1/2}](\mathfrak{B}_z)]_{1/2} = 0 \). Let \( x, y \in \mathfrak{B} \). The re-
lation $P(y,x,w,z) = 0$ yields $[(wx)(yz)]_{1/2} + [(wy)(xz)]_{1/2} = 0$. Let $t \in \mathfrak{B}$, $s$ and $s' \in \mathfrak{S}$. Then we get $[(w(ss'))_{1/2}(tz)]_{1/2} = -[(wt)_{1/2}(ss'z)] = 0$ from $P(tw, s, s'z) = 0$. Hence $\mathfrak{S}$ is a subalgebra of $\mathfrak{B}$. If we let $b_{-1} = a + n$ where $b$ is as described above and $n$ is a nilpotent element of $\mathfrak{B}$ and $\alpha \in \mathfrak{S}$, then $n \in \mathfrak{S}$ and hence every power of $n$ is in $\mathfrak{S}$. But $b$ is the sum of a multiple of the identity and a linear combination of powers of $n$. Hence $b = \lambda + D(a) \in \mathfrak{S}$ and the derivative of every element of $\mathfrak{B}$ is in $\mathfrak{S}$. Now $a \in \mathfrak{B}$ implies $a = D(\alpha) - cD(a)$. Since $D(\alpha c) = (1/2)D(c^2)$ and $D(a)$ are in $\mathfrak{S}$ we have $\mathfrak{B} \subseteq \mathfrak{S}$ and $[(w\mathfrak{B})_{1/2}(\mathfrak{B}z)]_{1/2} = 0$.

At this point we have obtained partial results on the multiplications of $\mathfrak{A}$. However, the chief remaining gap in the characterization of $\mathfrak{A}$ lies with the products involving elements of $\mathfrak{S}$. To facilitate the determination of these products we shall introduce some symbols $Q_{\phi}, \phi_{\phi}, k_{\phi}, f_{\phi},$ and $h_{\phi}$ on $\mathfrak{B}$ into $\mathfrak{B}$ for every $g \in \mathfrak{S}$ by letting

$$(12) \; Q_{\phi} = [wQ_{\phi}(b)]_{1/2},$$

$$(13) \; (gb)_{10} = h_{\phi}(b) + k_{\phi}(b)z,$$

$$(14) \; [g(wb)]_{1/2} = f_{\phi}(b) + \phi_{\phi}(b)z$$

for every $b \in \mathfrak{B}$. In our subscripts we abbreviate $(ga)_{1/2}$ to $ga$.

From (2) and (3) and the definition of $\mathfrak{S}$ we have

$$[(ga)_{1/2}(bz)]_{1/2}c_1 = \{[(ga)_{1/2}b_1]_{1/2}c_0\}_{1} - \{[(ga)_{1/2}b_0]_{1/2}c_0\}_{1}$$

$$= \{(gb_0)_{1/2}a_1\}_{1/2}c_0 + \{[(wQ_{\phi}(a))_{1/2}b_1]_{1/2}c_0\}$$

$$- 2\{(ga_1)_{1/2}b_0\}_{1/2}c_0 - \{[(wQ_{\phi}(a))_{1/2}b_1]_{1/2}c_0\}$$

$$= (1/2)(gc_0)_{1/2}a_1 + (1/2)b_1Q_{\phi}(a) - (1/2)b_1Q_{\phi}(a)$$

$$- 2\{(gb_0)_{1/2}a_1\}_{1/2}c_0$$

$$= (1/2)(gc_0)_{1/2}a_1 - 2\{[(gb_1)_{1/2}a_1]_{1/2}c_0\}$$

$$+ 2\{[(wQ_{\phi}(b))_{1/2}a_1]_{1/2}c_0\}$$

$$= a_1Q_{\phi}(b).$$

Now $[(ga)_{1/2}(bz)]_{1/2} = [wQ_{\phi}(b)]_{1/2}$ and therefore $[(wQ_{\phi}(b))_{1/2}c_1]_{10} = Q_{\phi}(b)z$.

Hence

$$(15) \; Q_{\phi}(b) = aQ_{\phi}(b).$$

Consider $h_{\phi}(a) + k_{\phi}(a)z = [(gb)_{1/2}a_1]_{10} = [(gb)_{1/2}a_1]_{10} + (gb)_{1/2}a_0)_{1}$

$$= 2[(gb_0)_{1/2}a_1]_{10} + [(wQ_{\phi}(b))_{1/2}a_1]_{10} + 2[(gb_1)_{1/2}a_0]_{1} - [(wQ_{\phi}(b))_{1/2}a_0]_{1}$$

$$= b_0(ga_1)_{10} + b_1(ga_0)_{10} + Q_{\phi}(b)[(az)w]_{10} = b_0h_{\phi}(a) + bzk_{\phi}(a) - Q_{\phi}(b)D(a).$$

From this relation we obtain

$$(16) \; h_{\phi}(a) = bh_{\phi}(a) - Q_{\phi}(b)D(a),$$

$$(17) \; k_{\phi}(a) = bk_{\phi}(a).$$
We now consider the \( \mathbb{C} \)-components of the terms of \( P(a, a, g, z) = 0 \). We have
\[
3ahg(a)z + 3akg(a) - 5hga(a)z - 5kga(a) = Q_g(a)D(a)z - h_g(a^2)z - k_g(a^2).
\]
If we equate \( \mathbb{B} \)-components and \( \mathbb{B}z \)-components we have
\[
(18) \quad k_g(a^2) = 2ak_g(a), \\
(19) \quad h_g(a^2) = 2ah_g(a) - 4Q_g(a)D(a)
\]
by using (16) and (17).

We have proved that \( k_g \) is a derivation for every \( g \in \mathcal{G} \). We shall now prove that \( Q_g \) is a derivation for every \( g \in \mathcal{G} \). We have
\[
[wQ_g(ab)]_{1/2} = [g(abz)]_{1/2} - [g(ab)0]_{1/2}
\]
\[
= [(ga_1)_{1/2}b_1 + (gb_1)_{1/2}a_1 - (ga_0)_{1/2}b_0 - (gb_0)_{1/2}a_0]_{1/2}
\]
\[
= [(ga_1)_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1 - (ga_0)_{1/2}b_0]
\]
\[
- (gb_1)_{1/2}a_0 + (wQ_g(b))_{1/2}a_0]_{1/2}
\]
\[
= (ga_0)_{1/2}b_1 + (gb_0)_{1/2}a_1 + (wQ_g(b))_{1/2}a_1 - (ga_1)_{1/2}b_0 + (gb_1)_{1/2}a_0 + (wQ_g(b))_{1/2}a_0]_{1/2}
\]
\[
= (wQ_g(ab))_{1/2} + (wQ_g(b))_{1/2}a_1
\]
\[
= (gb)a_{1/2}.
\]
By (4) we have \( (wQ_g(ab))_{1/2} = [wQ_g(a)b + Q_g(b)a]_{1/2} \). Therefore
\[
(20) \quad Q_g(ab) = Q_g(a)b + Q_g(b)a.
\]
Next, we consider the \( \mathcal{G} \)-components of the terms of \( P(g, a, bz, z) = 0 \) to get
\[
4[(ga)b]_{1/2} = [3g(ab) + (gb)a]_{1/2}.
\]
However
\[
[(ga)b]_{1/2} = [2(ga_0)b_1 + (wQ_g(a))b_1 + 2(ga_1)b_0 - (wQ_g(a))b_0]_{1/2}
\]
\[
= 2[(gb)_{1/2}a_0 + (gb)_{1/2}a_1]_{1/2}
\]
\[
= [(gb)a_0 + (wQ_g(b))a_0 + (gb)a_1 - (wQ_g(b))a_1]_{1/2}
\]
\[
= [(gb)a]_{1/2}.
\]
If we combine the above two relations we have
\[
(21) \quad [(ga)b]_{1/2} = [g(ab)]_{1/2}.
\]
A similar computation using \( P(w, w, a, z) = 0 \) and \( P((wa)_{1/2}, w, a, z) = 0 \) gives us
\[
(22) \quad w(wa)_{1/2} = w^2a + D^2(a)
\]
\[
(23) \quad (wa)^2_{1/2} = w^2a^2 + 2aD^2(a) - D(a)(D(a)).
\]
If we consider the \( (w\mathbb{B})_{1/2} \)-components of the terms of \( P(z, (aw)_{1/2}, w, g) = 0 \) we have \( [wQ_g(w^2a) + wQ_g(D^2(a)) + w(a\phi_g(1) + w\phi_g(a))]_{1/2} = 0 \). By letting \( a = 1 \) we get
\[
(24) \quad \phi_g(1) = -\frac{1}{2} Q_g(w^2).
\]

Therefore

\begin{equation}
\phi_g(a) = \frac{1}{2} a Q_g(w^2) - Q_g(a w^2) - Q_g(D^2(a)).
\end{equation}

From (15) and (25) we have

\begin{equation}
\phi_{ga}(b) = a \phi_g(b).
\end{equation}

We now wish to express \( h_g \) in terms of \( Q_g \) and \( D \). We examine the \( Bz \)-components of \( P(w, g, c, a) = 0 \) and use (21) and (26) to get

\begin{equation}
3 \phi_g(c)a + 3 \phi_g(a)c + 3h_g(a) + 3aD(h_g(a)) + 3D(a)h_g(c) + 3cD(h_g(a)) - 4h_{gc}(D(a)) = 3 \phi_g(1)c + D(h_g(c)) + h_{gc}(a) + h_g(c)a + h_{ga}(c) + h_g(a)c + 3 \phi_g(1)c - 3h_g(D(ca)) - c \phi_g(D(a)) + \phi_g(D(a)).
\end{equation}

We simplify this relation using (25), (16) and the linearized form of (19) to get

\begin{equation}
-3Q_g(D^2(a))c + 3h_g(a) = -3Q_g(D^2(ca)) - Q_g(c)D^2(a) - 3D(Q_g(c))D(a) - 3D(Q_g(a)) - 3h_g(c)D(a) - 3h_g(D(a)).
\end{equation}

Since \( Q_g \) and \( D \) are derivations we have

\begin{equation}
3h_g(a) = -3D(Q_g(c))D(a) - 3D(Q_g(a)) - 3h_g(c)D(a) + Q_g(D(a)) - 4Q_g(c)D^2(a).
\end{equation}

If we let \( a = c \) in (27) we get \( h_g(c) = -D(Q_g(c)) \). Therefore (27) simplifies to

\begin{equation}
3h_g(a) = -3D(Q_g(a)) + Q_g(D(a)) - 4Q_g(c)D^2(a).
\end{equation}

We substitute the values obtained from (28) in \( h_g(ac) = ch_g(a) + ah_g(c) - 2Q_g(a) - 2Q_g(c)D(a) \), a linearized form of (19), to get

\begin{equation}
Q_g(a) = Q_g(c)D(a).
\end{equation}

If we use this relation in (28) we obtain

\begin{equation}
h_g(a) = -D(Q_g(c))D(a) - 2Q_g(c)D^2(a).
\end{equation}

We now investigate the behaviour of \( f_g \). Consider the \( Bz \)-components of the terms of \( P(wb, g, a, z) = 0 \). We have

\begin{equation}
2f_g(b)a = f_g(ab) + f_g(ab) - bD(k_g(a)) - bk_g(D(a)) - D(a)k_g(b)
\end{equation}

and when \( b = 1 \)

\begin{equation}
2f_g(1)a = f_g(1) + f_g(a) - D(k_g(a)) - k_g(D(a)).
\end{equation}

We define a new mapping \( T_g \) on \( B \) into \( B \) for each \( g \) by

\begin{equation}
T_g(a) + f_g(1)a - f_g(1) + D(k_g(a)).
\end{equation}

This definition together with (32) gives us \( f_g(a) = f_g(1)a + T_g(a) + k_g(D(a)) \) and \( f_{ga}(1) = f_g(1)a + T_g(a) + D(k_g(a)) \). Now \( f_g(ab) = -f_g(ab) + 2f_g(b)a + b(Dk_g + k_gD)(a) + k_g(b)D(a) \) and \( f_g(b) = -f_g(1) + 2f_g(1)b + a(Dk_g + k_gD)(b) + D(a)k_g(b) \) by (31) and (32). Substituting the values for \( f_g(ab) \), \( f_g(b) \), \( f_{ga}(1) \) and \( f_{ga}(1) \) expressed in terms of \( T_g \) in these relations and simplifying we have
(34) \[ T_{g}(ab) = T_{g}(a)b + T_{g}(b)a \]

and

(35) \[ f_{g}(b) = f_{g}(1)ab + T_{g}(b)a - bT_{g}(a) + ak_{g}(D(b)) + bD(k_{g}(a)) - k_{g}(a)D(b). \]

It follows readily that

(36) \[ T_{g}(b) = aT_{g}(b) - D(b)k_{g}(a). \]

We have already shown that \( \phi_{g}(a) = Q_{g}(c)[(1/2)aD(w^{2}) - D(w^{2}a) - D(a)] \). We also have that \( P(g,g,(aw)_{1/2};z) = 0 \) implies \([g\phi_{g}(a)]_{1/2} = 0. \) If we let \( a = c^{3} \) we have \( \phi_{g}(c^{3}) = Q_{g}(c)[(-1/2)c^{3}D(w^{2}) - 3c^{2}D(w^{2}) - 6]. \) Since the second factor on the right-hand side is nonsingular we have \([gQ_{g}(c)]_{1/2} = 0. \) Multiplying by \( cz \) and considering the \((w^{2})_{1/2}\)-component we get

(37) \[ Q_{g}(c)^{2} = 0. \]

Similarly we have

(39) \[ Q_{g}(c)k_{g}(a) = 0. \]

Now consider the element \( w' = [w - wD(Q_{g}(c))]_{1/2} + g \) of \( \mathbb{B}_{0}(1/2). \) We have \((c^{2}w')_{0} = -f. \) By Theorem 1 and its proof, \( c_{2} - (1/2)(c^{2}w')_{0} \) is an element \( a \) in \( \mathbb{C} \) such that \((aw')_{z} = 1. \) Also \((c^{2}w')_{0} = -2c_{0} - 4(Q_{g}(c))_{0}. \) Therefore \((aw')_{z} = [c + Q_{g}(c) - Q_{g}(c)z]w'_{0} = 1 - 2D(Q_{g}(c))^{2} - 2D(Q_{g}(c))D^{2}(Q_{g}(c))z + k_{g}(Q_{g}(c)) - 2Q_{g}(c)D^{2}(Q_{g}(c)) + k_{g}(Q_{g}(c))z. \) Simple properties of derivations and the fact that \( Q_{g}(c)^{2} = 0 \) gives us \((aw')_{z} = 1 + k_{g}(Q_{g}(c)) + k_{g}(Q_{g}(c))z. \) Therefore

(40) \[ k_{g}(Q_{g}(c)) = 0. \]

We also have from (35) and (36) that

(41) \[ T_{g}(Q_{g}(c)) = f_{g}(1)Q_{g}(c) \text{ and } T_{g}(b)Q_{g}(c) = 0 \]

for every \( b \in \mathbb{B}. \)

For \( w' \) and \( c' = c + Q_{g}(c) - Q_{g}(c)z \) we have a corresponding \( \mathbb{B}' \) and \( \mathbb{B}'z \) as described in Theorem 2. To determine these two subspaces we let \( a + bz \) be an element of \( \mathbb{C} \) with \( a, b \in \mathbb{B} \) and such that the \( 1/2 \)-component of \((w'(a + bz)) \) is 0. We obtain \( w = -wD(Q_{g}(c)) + qa + wQ_{g}(b) = 0. \) Therefore \( a[1 - D(Q_{g}(c))] = -Q_{g}(c)D(b). \) Solving for \( a \) we have \( a = -D(b)Q_{g}(c). \) Since \( \mathbb{B} + \mathbb{B}'z = \mathbb{C}, \) we can conclude from the above result that \( \mathbb{B}'z \) consists of all elements of the form \( a - Q_{g}(a)z. \) We note that the \( \mathbb{C} \)-component of the element \((a - Q_{g}(a)z)w' \) must be an element of \( \mathbb{B}'z \) by Theorem 3. If we calculate this element we obtain \( D(a)z - D(a)D(Q_{g}(c))z + Q_{g}(c)D^{2}(a) + k_{g}(a)z - D(Q_{g}(a))D(Q_{g}(c)) + D(Q_{g}(c))^{2} \cdot D(a)z. \) In order for this element to be in \( \mathbb{B}'z \) we must have \( Q_{g}(c)D^{2}(a) + D(Q_{g}(c))^{2}D(a) = Q_{g}(c)D[D(a) - D(a)D(Q_{g}(c)) + k_{g}(a) + D(Q_{g}(c))^{2}D(a)]. \) Therefore
We also have
\[(42) \quad Q_g(c)D(k_g(a)) = k_g(a)D(Q_g(c)) = 0.\]

We define \(t'\) to be the 1/2-component of
\[
w[-D(Q_g(c))D(Q_g(c)) + Q_g(c)D^2(Q_g(c)) - k_g(Q_g(c))] + t
\]
for \(t \in \mathcal{G}\). Then the \(\mathbb{C}\)-component of \((c + Q_g(c) - Q_g(c)z)t'\) is
\[(44) \quad -D(Q_g(c)) - D(Q_t(c))D(Q_g(c)) - 2Q_g(c)D^2(Q_g(c)) + k_g(Q_g(c)) + Q_g(c)D^2(Q_g(c))z
\]
since \(Q_g(c)D^2(Q_g(c)) + 2D(Q_g(c))D(Q_g(c)) + Q_g(c)D^2(Q_g(c)) = 0\) and
\[2D(Q_g(c))D(Q_g(c))D(Q_g(c)) = -Q_t(c)D^2(Q_g(c))D(Q_g(c)) = 3Q_g(c)Q_g(c)D^3(Q_g(c)) = 0.\]

Hence \(t'\) is in \(\mathcal{G}'\). We now compute \(D'\) and \(Q'_t\). We have simply that
\[(45) \quad D' : a - Q_g(a)z \rightarrow D(a) - D(Q_g(a))D(a) + D(Q_g(a))^2D(a) + k_g(a)
\]
\[\quad - [Q_g(a)D^2(a) + D(Q_g(a))^2D(a)]z,
\]
\[(46) \quad Q'_t : c + Q_g(c) - Q_g(c)z \rightarrow Q_t(c) + Q_g(c)D(Q_g(c)) - Q_g(c)D(Q_g(c))z.
\]

Therefore
\[D'Q'_t : c + Q_g(c) - Q_g(c)z \rightarrow D(Q_t(c)) + D(Q_g(c))D(Q_g(c))
\[\quad + Q_g(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_t(c)) + k_g(Q_g(c)) - Q_g(c)D^2(Q_g(c))z.
\]

By (30) and (44) we have
\[D(Q_t(c)) + D(Q_g(c))D(Q_g(c)) + Q^2(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_g(c)) + k_g(Q_g(c))
\[= D(Q_t(c)) + D(Q_t(c))D(Q_g(c)) + 2Q_t(c)D(Q_g(c)) - k_t(Q_g(c)).
\]

Therefore \(Q_t(c)D^2(Q_g(c)) - D(Q_g(c))D(Q_g(c)) = 2Q_t(c)D^2(Q_g(c))\) and
\[(47) \quad Q_t(c)D^2(Q_g(c)) = -D(Q_g(c))D(Q_g(c)).
\]

Replacing \(t\) by \((ct)^{1/2}\) we have \(cQ_t(c)D^2(Q_g(c)) = -cD(Q_t(c))D(Q_g(c)) - Q_t(c)D(Q_g(c))\) and therefore
\[(48) \quad Q_t(c)D(Q_g(c)) = 0.
\]

We now examine the \(\mathcal{B}\)-components of the terms of \(P(g, t, a, z) = 0\) for \(g, t \in \mathcal{G}\) and \(a \in \mathcal{B}\). We have
\[(49) \quad m(1, a) + m(a, 1) = 2m(1, 1)a + 2D(Q_t(c))D(D(Q_g(c))D(a))
\[\quad + 2D(Q_g(c))D(D(Q_t(c))D(a)) + (k_gk_t + k_tk_g)(a)
\]
where \( m(a, b) \) denotes the \( \mathcal{B} \)-component of \((ga)_{1/2} \cdot (tb)_{1/2} \). Since \( m(a, b) \) does depend on \( g \) and \( t \) also, we will use \( m_{g,t}(a, b) \) for \( m(a, b) \) when there is any chance of confusion. Replacing \( t \) by \((tb)_{1/2} \) in (49) we obtain

\[
m(1, ab) + m(a, b) = 2m(1, b)a + 2bD(Q_g(c))D(D(Q_t(c))D(a))
+ 2bD(Q_t(c))D(D(Q_g(c))D(a)) + 2D(Q_t(c))D(b)D(a)
+ k_g(b)k_t(a) + b(k_gk_t + k_tk_g)(a).
\]

Define

\[
S_{g,t}(a) = m(1, a) - m(1, 1)a - 2D(Q_g(c))D(D(Q_t(c))D(a)) - k_gk_t(a)
\]
for all \( a \in \mathcal{B} \). If \( g = t \) the right-hand side of (51) reduces to identity (49) with \( g = t \). Therefore \( S_{g,t} \) is identically zero. A simple linearization gives us

\[
S_{g,t} = -S_{t,g}.
\]

Substituting (51) into (50) and letting \( a = b \) we have \( S_{g,t}(a^2) + 2L_gL_t(a^2)
\]

\[
+ m(a, a) + k_gk_t(a^2) = 2S_{g,t}(a)a + m(1, 1)a^2 + 4aL_gL_t(a) + 2ak_gk_t(a)
\]
where \( L_g = D(Q_g(c))D \) and \( L_t = D(Q_t(c))D \) are derivations. Interchanging \( g \) and \( t \) in this result and subtracting gives us

\[
2S_{g,t}(a^2) + 2L_gL_t(a^2) - 2L_tL_g(a^2) + (k_gk_t - k_tk_g)(a^2)
= 4S_{g,t}(a)a + 4a(L_gL_t - L_tL_g)(a) + 2a(k_gk_t - k_tk_g)(a).
\]
Since both \( L_gL_t - L_tL_g \) and \( k_gk_t - k_tk_g \) are derivations this relation reduces to \( S_{g,t}(a^2) = 2aS_{g,t}(a) \). Hence \( S_{g,t} \) is a derivation of \( \mathcal{B} \) into \( \mathcal{B} \).

We can now replace (50) by

\[
m(a, b) = m(1, 1)ab + aS_{g,t}(b) - bS_{g,t}(a) + 2aL_gL_t(b) + 2bL_tL_g(a)
\]

\[
- 2L_g(a)L_t(b) + ak_gk_t(b) + bk_tk_g(a) - k_g(a)k_t(b).
\]

By setting \( g = t, a = 1 \) and \( b = Q_g(c) \) in (53) we have

\[
m_{g,t}(1, 1)Q_g(c) = 0.
\]

An examination of the \((w\mathcal{B})_{1/2}\)-components of the terms of \( P(g, g, g, z) = 0 \) gives us

\[
Q_g(c)D(m_{g,t}(1, 1)) = 0.
\]

Finally we compute \( P((ga)_{1/2}, (tb)_{1/2}, w, z) = 0 \) to get

\[
n_{g,t}(a, b) = -aQ_g(f_g(1)b - T_t(b) + D(k_t(b)) - bQ_g(f_g(1)a - T_g(a)) + D(k_t(a))
\]
where \( n_{g,t}(a, b) \) is the \( \mathcal{B} \)-component of \((ga)_{1/2} \cdot (tb)_{1/2} \cdot \mathcal{B} \). Now \( P(g, g, (wa)_{1/2}, z) = 0 \)
Therefore \( n_{g,t}(1, 1)a + 2Q_g(f_g(1)a) + 2Q_g(T_g(a)) = 0 \). From (56) with \( g = t \) and
\( a = b = 1 \) we have

\[
Q_g(T_g(a)) = -Q_g(a)f_g(1).
\]

2. In the previous section we expressed the multiplications of \( \mathcal{A} \) in terms of constants and derivations. In this section we use these multiplicative properties to construct a simple power-associative algebra of degree two from an associative algebra.
Let \( \mathfrak{B} \) be an associative, commutative algebra over a field \( \mathbb{F} \) of characteristic \( p > 5 \). Also assume that \( \mathfrak{B} \) has a single nonzero idempotent \( 1 \) that is a unity quantity.

Let \( \mathfrak{B}_0, \ldots, \mathfrak{B}_{n-1} \) be \( n \) homomorphic images of the vector space \( \mathfrak{B} \). We let \( \mathfrak{L} \) be a sum of these \( n \) vector spaces, but not necessarily the vector space direct sum. We let \( z \mathfrak{B} \) be a one-dimensional module over \( \mathfrak{B} \). Clearly \( z \mathfrak{B} \) is a vector space over \( \mathbb{F} \) and we form the vector space direct sum \( \mathfrak{A} = \mathfrak{B} + \mathfrak{L} + z \mathfrak{B} \). We now extend the multiplication of \( \mathfrak{B} \) to \( \mathfrak{A} \) in such a way that \( \mathfrak{A} \) remains a commutative, power-associative algebra. First we define

\[
(58) \quad (za)(zb) = (zb)(za) = ab,
\]
\[
(59) \quad 1x = x,
\]
\[
(60) \quad zy = 0
\]

for every \( a \) and \( b \) in \( \mathfrak{B} \), every \( x \) in \( \mathfrak{A} \) and every \( y \) in \( \mathfrak{L} \). The element \( e = (1/2)(1+z) \) is an idempotent. We have already defined sufficient multiplicative properties to determine an idempotent decomposition of \( \mathfrak{A} \). Clearly \( \mathfrak{L} \subseteq \mathfrak{A}_e(1/2) \) and \( \mathfrak{B} + \mathfrak{B}z \subseteq \mathfrak{A}_e(1) + \mathfrak{A}_e(0) \). The second part of this statement follows by consideration of \( a + bz = (c + cz) + (d - dz) \) with \( 2c = a + b \) and \( 2d = a - b \). For each of the vector spaces \( \mathfrak{B}_i \) and the corresponding homomorphism of \( \mathfrak{B} \) onto \( \mathfrak{B}_i \) we define \( (g_i b)_{1/2} \) to be the image of \( b \). Since this notation is consistent with that of the decomposition of \( \mathfrak{A} \) with respect to \( e \) we will allow the confusion of the two notations.

In order to complete our definitions of the multiplications of \( \mathfrak{A} \) we choose elements \( b_{ij} \) and \( b_{i} \) of \( \mathfrak{B} \) and derivations \( D_{ij} \) and \( D_{i} \) on \( \mathfrak{B} \) into \( \mathfrak{B} \) for \( i, j = 0, 1, \ldots, n - 1 \) with the following restrictions:

\[
(61) \quad D_{ij} = -D_{ji}, \quad b_{ij} = b_{ji}, \quad b_{0} = 0
\]

for all values of \( i \) and \( j \) and

\[
(62) \quad b_{i}D_{0}(b_{j}) = (b_{i} + b_{j})D_{0}(b_{ij}) = D_{j}(b_{j}b_{i}) + D_{j}(b_{i}b_{j}) = 0,
\]

\[
(b_{i}g_{j} + b_{j}g_{i})_{1/2} = 0, \quad b_{i}b_{0}D_{0} = -b_{i}D_{i}D_{0}
\]

for all \( i \) and \( j \) different from \( 0 \) and all \( b \in \mathfrak{B} \). We now define

\[
(63) \quad (g_{i}a)_{1/2}b = [g_{i}(ab)]_{1/2} - D_{0}(ab_{i}b)D_{0}(b) - 2b_{i}aD_{0}^{2}(b) + aD_{i}(b)z,
\]
\[
(64) \quad (g_{i}a)_{1/2}(bz) = -[(g_{i}a)_{1/2}b]z + [g_{i}[aD_{0}(b)b_{i}]]_{1/2},
\]
\[
(65) \quad (g_{i}a)_{1/2}(g_{j}b)_{1/2} = abb_{j} + aD_{ij}(b) - bD_{ij}(a) + aD_{j}D_{i}(b) + bD_{i}D_{j}(a)
\]
\[
- D_{j}(b)D_{i}(a) + 2aL_{j}L_{i}(b) + 2bL_{j}L_{i}(a) - 2L_{j}(b)L_{i}(a)
\]
\[
+ ab_{l}[D_{0}(b) - b_{0}a - D_{0}(b)]z \cdot bb_{j}D_{0}[D_{0}(a) - b_{0}a - D_{0}(a)]z
\]
where $L_i = D_0(b_i)D_0$, $i,j = 0,\ldots,n-1$, and $a$ and $b \in \mathcal{B}$. Since we did not restrict $\mathcal{L}$ to be a direct sum of subspaces it is necessary to assume that our multiplications in $\mathcal{A}$, as defined above, are well-defined. We place two additional assumptions on $\mathcal{A}$. If $\mathcal{D}$ is the set of derivations consisting of $D_i$ and $D_{ij}$ for all $i$ and $j$ we assume, in the terminology of Albert [3], that $\mathcal{B}$ is $\mathcal{D}$-simple; i.e., there is no nontrivial ideal $\mathcal{I}$ of $\mathcal{B}$ such that $\mathcal{I}$ is $\mathcal{D}$-admissible. The second assumption is that for every element $g$ in $\mathcal{L}$ there is a $t$ in $\mathcal{L}$ such that $gt$ is not zero.

**Theorem 7.** Every commutative, power-associative, simple algebra of degree two over an algebraically closed field $\mathcal{F}$ of characteristic $p \neq 2, 3, 5$ is an algebra of the type described above.

**Proof.** We choose a set of elements $g_1, \ldots, g_{n-1}$ in $\mathcal{G}$ such that every element of $\mathcal{G}$ is expressible in the form $\sum (g_i a_i)_{1/2}$ where $a_i \in \mathcal{B}$. We translate the notation of §1 to the notation of this section by letting $\mathcal{L} = \mathcal{A}_1(1/2)$, $g_0 = w$, $D_0 = D$, $b_{00} = w^2$, $b_{0i} = f_{g_1}(1)$, $D_{0i} = T_g$, $D_i = k_{g_i}$, $b_i = Q_{g_1}(c)$, $b_{ij} = m_{g_i, g_j}(1, 1)$ and $D_{ij} = S_{g_i, g_j}$ where $i,j \neq 0$. Identities (25)–(57) give us the relations (61)–(65).

If $\mathcal{I}$ is a nontrivial ideal of $\mathcal{B}$ that is $\mathcal{D}$-admissible then if $a \in \mathcal{I}$ we have $Q_{g_1}(a), f_{g_1}(b), \phi_{g_1}(a), f_{g_1}(a)m_{g_1}(a,b)$ and $n_{g_1}(a,b) \in \mathcal{I}$. This is sufficient to guarantee that $\mathcal{I} + \mathcal{I}z + (w_3)_{1/2} + (\mathcal{G})_{1/2}$ is a proper ideal of $\mathcal{A}$. Since this contradicts the simplicity of $\mathcal{A}$ we have that $\mathcal{B}$ is $\mathcal{D}$-simple.

Let $(wa)_{1/2} + g$ be an element of $\mathcal{A}_1(1/2)$ such that there is no element $t$ in $\mathcal{A}_1(1/2)$ such that $(wa)_{1/2} + gt \neq 0$. Choosing $t$ to be successively $w, (wc)_{1/2}$ and $w^2(c^2)_{1/2}$ and considering only the $\mathcal{B}$-components of the resulting terms we have $w^2 a + D^2(a) + f_{g_1}(1) = w^2 ac + cd^2(a) - D(a) + f_{g_1}(1)c + T_g(c) = w^2 ac^2 + c^2 D^2(a) + 2a - 2cd(a) + f_{g_1}(1)c^2 + 2T_g(c) = 0$. Eliminating $w^2$ from these equations we have $-D(a) + T_g(c) = 2a - cd(a) + cT_g(c) = 0$. Hence $a = 0$ and $f_{g_1}(1) = T_g(c) = 0$. If we multiply $g$ by $(wb)_{1/2}$ for $b \in \mathcal{B}$ we have $f_{g_1}(b) = \phi_{g_1}(b) = 0$ by our assumption on $g$. By a previous result we had that $Q_{g_1}(c)$ was a multiple of $\phi_{g_1}(c^3)$. Hence $Q_{g_1}(c) = 0$. Now $f_{g_1}(b) = T_g(b) + k_{g_1}(D(b)) = 0$ for all $b \in \mathcal{B}$. If we substitute $bc$ for $b$ we have $cT_g(b) + c \phi_{g_1}(D(b)) + k_{g_1}(b) = 0$. Therefore $k_{g_1}(b) = 0$. We now have that $\mathcal{E} = \{ag_{1/2} : a \in \mathcal{B}\}$. With this choice of $g$ and for any $b \in \mathcal{B}$ we have $f_{g_1}(b) = 0$ by (35) and $\phi_{g_1}(b) = 0$ since $Q_{g_1}(c) = aQ_{g_1}(c)$. Also $m_{g_1}(a,b) = aS_{g_1}(b) - bS_{g_1}(a)$. But by the assumption on $g$ and (51) we have $S_{g_1} = 0$. Therefore $m_{g_1}(a,b) = 0$ for all $a$ and $b \in \mathcal{B}$. Combining this result with (56) we have $(ga)_{1/2}t = 0$ for all $a \in \mathcal{B}$ and all $t \in \mathcal{A}_1(1/2)$. Therefore the ideal generated by $g$ is $\{ag_{1/2} : a \in \mathcal{B}\}$. This contradicts the assumption of simplicity of $\mathcal{A}$. Hence for each $x \in \mathcal{A}_1(1/2)$ there is an element $t$ in $\mathcal{A}_1(1/2)$ such that $xt \neq 0$.

**Theorem 8.** An algebra $\mathcal{A}$ over a field $\mathcal{F}$ of characteristic $p \neq 2, 3, 5$ as described in identities (58)–(65) is a commutative, power-associative, simple algebra.
Proof. It follows readily from the definition of $\mathfrak{A}$ that $\mathfrak{B} + \mathfrak{B}z + (g_0 \mathfrak{B})_{1/2}$ is a subalgebra of $\mathfrak{A}$. We shall show that this subalgebra is power-associative by examining $P(x, y, s, t)$ for various values in $\mathfrak{B} + \mathfrak{B}z + (g_0 \mathfrak{B})_{1/2}$. If $P(x, y, s, t) = 0$ for all possible choices of the variables $x, y, s$ and $t$ in $\mathfrak{B}, \mathfrak{B}z$ or $(g_0 \mathfrak{B})_{1/2}$ we have $\mathfrak{B} + \mathfrak{B}z + (g_0 \mathfrak{B})_{1/2}$ power-associative. We examine the powers of $x = a + g_0$ for $a \in \mathfrak{B}$. We have $x^2 = a^2 + b_{00} + (a g_0)_{1/2} + 2D_0(a)z$, $x^3 = a^3 + 2ab_{00} - D_0^2(a)z + 5aD_0(a)z + D_0(b_{00})z + [(2a^2 + b_{00})g_0]_{1/2}$ and $x^2x^2 = x^3x$. The proof of this result depends on the properties

$$(66) \quad (a(bz)) = (ab)z;$$

$$(bz)(g_0a)_{1/2} = -aD_0(b),$$

$$b(g_0a)_{1/2} = [(ab)g_0]_{1/2} + aD_0(b)z,$$

$$(g_0a)_{1/2}(g_0b)_{1/2} = abb_{00} + aD_0^2(b) + bD_0^2(a) - D_0(a)D_0(b).$$

If $d \in \mathfrak{B}$ and if we replace $D_0$ by $dD_0$, $b_{00}$ by $b_{00}d^2 + 2dD_0(d) - D_0(d)^2$ and $g_0$ by $(g_0d)_{1/2}$ we see that relations similar to those expressed in (66) hold. Therefore we can conclude that $a + (g_0d)_{1/2}$ has a unique fourth power.

Next we investigate the fourth powers of $x = az + g_0$. We have $x^2 = a^2 + b_{00} - 2D_0(a)$, $x^3 = a^3z + b_{00}az + D_0(b_{00})z - 2D_0^2(a)z + a^2 + b_{00} - [2D_0(a)g_0]_{1/2}$ and $x^2x^2 = x^3x$. Again the only multiplicative properties used were those expressed in (66). Therefore $az + (g_0d)_{1/2}$ has a unique fourth power for all $a$ and $b \in \mathfrak{B}$. It is easily seen that $\mathfrak{B} + \mathfrak{B}z$ is associative. Hence $a + bz$ has a unique fourth power. The assumption on the characteristic and simple linearizations of these three fourth powers we have obtained give us the result that $P(x, y, s, t) = 0$ provided that in any evaluation the four values $x, y, s$, and $t$ are chosen from only two of the three subspaces $\mathfrak{B}, \mathfrak{B}z$ and $(g_0 \mathfrak{B})_{1/2}$. This leaves us those choices of $x, y, s$ and $t$ for which $x \in \mathfrak{B}$, $y \in \mathfrak{B}z$, $s \in (g_0 \mathfrak{B})_{1/2}$ and $t$ is arbitrary. Because of the linearization process we need only consider $P(a, bz, (g_0d)_{1/2}, a)$, $P(a, bz, (g_0d)_{1/2}, bz)$ and $P(a, bz, (g_0d)_{1/2}, (g_0d)_{1/2})$. Straightforward computations, which we omit, show that each of these relations is zero. Therefore $\mathfrak{B} + \mathfrak{B}z + (g_0 \mathfrak{B})_{1/2}$ is power-associative.

Now let $g = \Sigma(a_i)_{1/2}$ where $a_i \in \mathfrak{B}$. The index $i$, or indices $i$ and $j$, of this summation and all subsequent ones will run from 1 to $n - 1$. Define

$$b_g = \Sigma a_i b_i,$$

$$D_g = \Sigma a_i D_i,$$

$$b_{0g} = \Sigma a_i b_{0i} - \Sigma D_0(a_i) + \Sigma D_0 D_i(a_i),$$

$$D_{0g} = \Sigma a_i D_{0i} - \Sigma D_i(a_i) D_0,$$

$$b_{gg} = \Sigma b_{ij} a_i a_j + 2 \Sigma a_i D_i(j) a_j + 4 \Sigma a_j L_j L_i(a_i) - \Sigma D_i(a_i) D_j(a_j).$$

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From (62) and (67) we have
\[ b^2 = b_g b_{gg} = b_g D_0(b_{gg}) = D_g(b_g) = b_g D_0 D_g(b) = 0, \]
(68)
\[ b_g b_{00} D_0(a) = -b_g D_0 D_{00}(a), \]
\[ (gb_g)_{1/2} = 0. \]

From (65) we have \((ga)_{1/2} (ga)_{1/2} = b_{gg} + 2a D_0^2(a) - D_g(a)^2 + 4 \sum a_i L_i a_j L_j(a) - 2 \sum a_i L_i(a) a_j L_j(a). \) Now \( \sum a_i L_i(a) = \sum a_i D_0(b_i) D_0(a) = D_0(b_g) D_0(a) - \sum b_i D_0(a) D_0(a). \) Therefore \( \sum a_i L_i(a) a_j L_j(a) = L_g(a)^2 \) where \( L_g = D_0(b_g) D_0. \) Also
\[ \sum a_i L_i a_j L_j(a) = \sum L_g a_j L_j(a) - \sum b_i D_0(a) D_0 a_j L_j(a) = L_g^2(a). \]

Therefore
\[ (ga)_{1/2}^2 = b_{gg} + 2a D_0^2(a) - D_g(a)^2 + 4a L_g^2(a) - 2L_g(a)^2. \]

We also have
\[ b(ga)_{1/2} = g(ab)_{1/2} - D_0(ab) D_0(b) - 2b_g a D_0^2(b) + a D_g(b)^2, \]
(70)
\[ (b^2)(ga)_{1/2} = g_0(a D_0(b) b_g)_{1/2} - [(ga)_{1/2}]^2 \]
for all \( a \) and \( b \) in \( \mathcal{B}. \)

We now let \( g_0 = g_0 + g \) and \( a' = a - b_g D_0(a)z \) for \( a \in \mathcal{B}. \) We define a derivation \( D'_0(a') = [D_0(a) + D_0(b_g)^2 D_0(a) + D_g(a)]' \) and let \( t = b_{00} + 2b_{gg} - b_g D_0(b_{00})z + b_{gg} - 2b_g D_0(b_{gg})z. \) Now \( (D_0 + D_0(b_g)^2 D_0 + D_g)^2 = (D_0 + D_g)^2 + 2L_g^2. \) Therefore \( a'D'_0^2(a') = a(D_0 + D_g)^2(a) + 2aL_g^2(a) - b_g[aD_0^2(a) + aD_0^2 D_0(a) + D_0(a) D_0^2(a)]z \) since \( 3b_g D_0 L_g^2(a) = 3b_g D_0^2 b_g D_0^2(b_g) D_0(a) = -3D_0(b_g) D_0(b_g) D_0^2(b_g) D_0(a) = 2D_0(b_g) b_g D_0^2(b_g) D_0(a) = 0. \) Also \([D_0 + D_0(b_g)^2 D_0 + D_g(a)]^2 = [(D_0 + D_g(a)]^2 + 2L_g^2(a)^2. \) Therefore \([D'_0(a')]^2 = [(D_0 + D_g(a)]^2 + 2L_g^2(a)^2 - 2b_g D_0^2(a) D_0(a)z. \)

We have, using these results, that \( (g'_0 a')_{1/2}^2 = (g_0 a')_{1/2}^2 = b_{00} a'^2 + 2a D_0^2(a) - D_0(a)^2 + 2a^2 b_{00} + 2a D_0(a) + 2a D_0(a) - 2D_g(a) D_0(a) - ab_g a D_0(b_{00})z - 2b_{00} D_0(a) ab_g - 2a D_0^2(a) z + b_{gg} a'^2 + 4a L_g^2(a) - 2L_g(a)^2 + 2a D_0^2(a) - D_g(a)^2 - 2b_g a^2 D_0(b_{00})z - 2ab_g b_{00} D_0(a) + 2a b_g D_0 D_0(a) z - 2b_g a D_0^2 D_0(a) z = t(a'^2) + 2a' D_0^2(a') - D'(a')^2 + 2b_g b_{00} a D_0(a) + 2a b_g a D_0(a) = t(a'^2) + 2a' D_0^2(a') - D'(a')^2. \) Since \( t = g_0^2 \) we have
\[ (g'_0 a')_{1/2}^2 = g_0^2 + 2a' D_0^2(a) - D'_0(a')^2. \]

From (68) and (70) we have
\[ a'(b'z) = (a'b')z = (ab)'z, \]
\[ (a'z)(b'z) = a'b' = (ab), \]
\[ (b'z)(g_0a')_{1/2} = -a'D_0(b'), \]
\[ b'(g_0a')_{1/2} = [(a'b')g_0']_{1/2} + a'D_0(b')z. \]

If \( \mathcal{B}' \) is the set of all elements of the form \( a' \) where \( a \in \mathcal{B} \) then \( \mathcal{B}' + \mathcal{B}'z + (g_0\mathcal{B}')_{1/2} \) is a subalgebra with multiplications similar to those expressed in (66). Hence we can conclude that this subalgebra is power-associative and that \( a' + b'z + (g_0d')_{1/2} \) has a unique fourth power for every \( a', b' \) and \( d' \in \mathcal{B}' \). But \( \mathcal{C} = \mathcal{B}' + \mathcal{B}'z \).

Therefore \( a + bz + (g_0d + gd)_{1/2} \) has a unique fourth power for every \( a, b, d \in \mathcal{B} \) and every \( g \). If \( d \) is nonsingular then \( d \) can be absorbed in the coefficients \( a_i \) of \( g_i \) in the expression for \( g \). Hence \( a + bz + (g_0d)_{1/2} + g \) has a unique fourth power if \( d \) is nonsingular. We can restate this as \( x = g_0 + \alpha(a + bz) + \beta(g_0d)_{1/2} + yg \) has a unique fourth power for \( d \) a singular element of \( \mathcal{B}, a, b \in \mathcal{B}, g = (a'g_0)^{1/2} \) and \( a, \beta \in \mathcal{Y} \). The characteristic is sufficiently high so that the attached polynomials of the expression \( x^2x^2 - x^4 \) are all zero [6]. The sum of those polynomials with a coefficient \( x^j \beta^k \) where \( i + j + k = 4 \) is of course also equal to zero. But by replacing \( \alpha, \beta \) and \( y \) by 1 in this sum we get \( x^2y^2 - x^4 = 0 \) where \( x = (a + bz + (g_0d)_{1/2} + g) \). Hence any element of \( \mathcal{A} \) has a unique fourth power and \( \mathcal{A} \) is power-associative.

To complete the proof it remains only to show the simplicity of \( \mathcal{A} \). Let \( \mathfrak{I} \) be a proper ideal of \( \mathcal{A} \) with the nonzero element \( a + bz + t \) where \( a, b \in \mathcal{B} \) and \( t \in \mathcal{L} \). Since \( z\mathfrak{I} \subseteq \mathfrak{I} \) we have \( az + b \in \mathfrak{I} \). Now multiply \( az + b \) by \( g_0 \) to get \( (ag_0)_{1/2} + D_0(a)z = D_0(b) \). By the above \( (ag_0)_{1/2} \in \mathfrak{I} \). Multiplying this element by \( cz \) we get \( a \in \mathfrak{I} \) and therefore \( b, t, D(a) \) and \( D(b) \in \mathfrak{I} \). Let \( \mathfrak{P} \) be the set of all elements of \( \mathcal{B} \) that are in \( \mathfrak{I} \). Clearly, \( \mathfrak{P} \) is a proper ideal of \( \mathcal{B} \). Since \( \mathfrak{P}\mathfrak{L} \subseteq \mathfrak{I} \) and \( (\mathfrak{P}\mathfrak{L})_{1/2} \subseteq \mathfrak{I} \) it can be easily shown that \( \mathfrak{P} \) is \( \mathcal{D} \)-admissible. Hence \( \mathfrak{P} = 0 \) and the only nonzero elements that could be in \( \mathfrak{I} \) are of the form \( t \) where \( t \in \mathcal{L} \). But by the assumption on \( \mathcal{A} \) there is an \( x \in \mathcal{L} \) such that \( gx \neq 0 \). Since \( gx \in \mathcal{B} + \mathcal{B}z \) and \( \mathfrak{I} \cap (\mathcal{B} + \mathcal{B}z) = 0 \) we must have \( \mathfrak{I} = 0 \). Therefore \( \mathcal{A} \) is simple.

To further characterize the algebra \( \mathcal{A} \) and its subalgebra \( \mathcal{B} \) we quote a result of Harper [5, Theorem 1].

**Theorem 9.** Let \( \mathcal{B} \) be a commutative, associative algebra with unity 1 over an algebraically closed field \( \mathbb{F} \), and let \( \mathcal{B} \) be \( \mathcal{D} \)-simple relative to a set of derivations \( \mathcal{D} \) over \( \mathbb{F} \). Then \( \mathcal{B} = \mathbb{F}[x_1, x_2, \ldots, x_n] \) is an algebra with generators \( x_1, \ldots, x_n \) over \( \mathbb{F} \) which are independent except for the relations \( x_1^2 = \ldots = x_n^p = 0 \) where \( p \) is the characteristic of \( \mathbb{F} \).

3. Let \( p \) be a prime \( \neq 2, 3, 5 \) and let \( \mathcal{B} \) be the associative commutative algebra of all polynomials \( \sum_{i=0}^{p-1} a_i c^i \) in \( c \) with \( c^p = 0 \) and \( c^0 = 1 \), the identity of \( \mathcal{B} \). Let \( \mathcal{L} \) be \( \{ (g_0a)_{1/2} : a \in \mathcal{B} \} \). Then \( \mathcal{A} = \mathcal{B} + \mathcal{B}z + (g_0\mathcal{B})_{1/2} \). Let \( b_{00} = 0 \) and \( D_0 \) be ordinary polynomial differentiation; i.e., \( D_0(c) = 1 \). Assume that \( u = a + bz \)
+ (g_0d)_{1/2}, where a, b, d \in B, is an idempotent of \mathcal{A} that is not in \mathbb{C}. Then
\[ a^2 + b^2 + 2dD_0(d) - D_0(d)^2 - 2dD_0(b) + 2abz + 2D_0(a)z + 2(g_0(da))_{1/2} = a + bz + (g_0d)_{1/2}. \]
Therefore \(d(2a - 1) = 0\) and \(2ab + 2dD_0(a) = b\). If \(d = 0\) then \(u \in \mathbb{C}\). By our assumptions \(d \neq 0\) and we must have \(2a - 1\) is singular. Therefore we can write \(a = 1/2 + c's\) where \(s\) is a nonsingular element of \(B\) and \(t \geq 1\). We have \(dc' = 0\) and \(c'b + tc'^{-1}d = 0\). Hence \(c'^{t+1}b = 0\). Since

\[(73) \quad a^2 + b^2 + 2dD_0(d) - D_0(d)^2 - 2dD_0(b) = a\]

it follows that \(a^t c'^{t+1} = ac'^{t+2}\). But this implies that \(c'^{t+1} = 2c'^{t+1}\). Hence \(t + 1 \geq p\). Assume \(t = p - 1\); then \(c^{p-1}b = c^{p-2}d\). Now if \(b = \sum_{0}^{p-1} \alpha_i c^i\) and \(d = \sum_{0}^{p-1} \beta_i c^i\) then we must have \(\alpha_0 = 0\) and \(\beta_0 = \alpha_1\). From (73) we must also have \(\beta_0^2 - \alpha_1^2 = 1/4\) which is a contradiction. Therefore \(t + 1 > p\) and \(a = 1/2\).

Let \(x' = a' + b'z + (g_0d')_{1/2}\) be an arbitrary element of \(\mathcal{A}\). By considering the product \(x'u\) we see that a necessary and sufficient condition that \(x' \in \mathcal{A}_u(1)\) is that

\[(74) \quad 2a'd = d', \quad 2b'a' + 2D_0(a')d = b'.\]

The correspondence \(a' \rightarrow a' + 2a'bz + 2D_0(a')dz + 2[g_0(a'd)]_{1/2}\) is clearly a \(1-1\) correspondence between \(B\) and \(\mathcal{A}_u(1)\) preserving the vector space operations. Therefore \(\mathcal{A}_u(1)\) is of dimension \(p\).

If \(u\) is a stable idempotent then Albert has shown [3; 4] that \(\mathcal{A} = \mathcal{A}_u(1) + \mathcal{A}_u(0) + wB\) where \(B = \mathcal{A}_u(1) + \mathcal{A}_u(0)\) and \(wB + wB = \mathcal{A}_u(1/2)\). Albert also showed that the dimensions of \(\mathcal{A}_u(1)\), \(\mathcal{A}_u(0)\) and \(wB\) are all equal. Therefore \(wB = 0\). A further result of Albert's is that \(\mathcal{A}_u(1) + \mathcal{A}_u(0) + wB\) is associative. This implies that \(\mathcal{A}\) is a simple, associative algebra and hence we must have \(c = 0\). We can conclude that our example contains no stable idempotents.

**Bibliography**


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