ON SETS OF ALGEBRAIC INTEGERS WHOSE REMAINING CONJUGATES LIE IN THE UNIT CIRCLE(1)

BY

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1. **Introduction.** In this paper we investigate certain sets of algebraic integers which we call PV k-tuples and T k-tuples. These k-tuples are generalizations of the PV and T numbers defined in [9] and have many similar properties.

Let $(\theta_1, \theta_2, \cdots, \theta_k)$ be a k-tuple of distinct algebraic integers, each with absolute value greater than one. Let $P(z)$ be the polynomial of least degree with relatively prime integer coefficients which $\theta_1, \theta_2, \cdots, \theta_k$ all satisfy. If the remaining roots of $P(z)$ lie in the open (closed) unit circle then $(\theta_1, \theta_2, \cdots, \theta_k)$ is said to be a PV(T) k-tuple and $P(z)$ its defining polynomial.

Since $\theta_1, \theta_2, \cdots, \theta_k$ are all algebraic integers $P(z)$ is necessarily monic. Obviously every $\bar{\theta}$ belongs to the PV(T) k-tuple.

The PV 1-tuples are the PV numbers while the T 1-tuples are the PV and T numbers, together. These numbers have been studied extensively and so have the PV and T 2-tuples when $\theta_1 = \bar{\theta}_2$ [3; 10]. We show that many of the results obtained for these special cases extend to the general case, and we also obtain some results which are new even in these special cases.

In §2 we give definitions and state some previous results which we shall use. §3 is concerned with the basic properties of T and PV k-tuples. In §4 we investigate the relationship of PV and T k-tuples to power series with integer coefficients. Finally, in §5 we study the limits of PV and T k-tuples.

2. **Definitions and previous results.** We say that two PV(T) k-tuples are equal if they are equal as sets. If $S_n = (\theta_1^{(n)}, \theta_2^{(n)}, \cdots, \theta_k^{(n)})$, if $\lim_{n\to\infty}\theta_i^{(n)} = \theta_i$, and if $\theta_i^{(n)} \neq \theta_i$, then $(\theta_1^{(n)}, \theta_2^{(n)}, \cdots, \theta_k^{(n)})$ is said to have a limit and its limit consists of the set of distinct elements among $(\theta_1, \theta_2, \cdots, \theta_k)$. We shall use this definition only when $\{S_n\}$ is a sequence of PV (T) k-tuples. We say that a PV or T k-tuple is irreducible if its defining polynomial is irreducible.

For real x, $N(x)$ denotes the unique integer a satisfying $x - 1/2 < a \leqq x + 1/2$, $\{x\} = x - N(x)$, and $\| x \| = |x - N(x)|$. Thus $N(x)$ is the "nearest" integer to x, and $\| x \|$ is the distance from x to the "nearest" integer. It is easy to see that $\| x + y \| \leqq \| x \| + \| y \|$, and $\| x \| \leqq |x|$; it follows that if $x + y$ is an integer

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then $\|x\| \leqq |y|$. As usual $[x]$ denotes the integer part of x. We note that $N(x) = [x + 1/2]$. Suppose that $P(z)$ is a polynomial of degree n. We call $Q(z) = z^n P(1/z)$ its *reciprocal polynomial* and say that $P(z)$ is *reciprocal* if $P(z) = \mp Q(z)$.

We now quote some theorems which will be used in the sequel.

MINKOWSKI'S THEOREM ON COMPLEX LINEAR FORMS. *Suppose that n linear forms with real or complex coefficients, $L_p(x) = \sum_{q=1}^{n} a_{pq}x_q$, are given, such that the determinant $D = |a_{pq}|$ is not zero. In case some of the forms are not real suppose that their conjugate forms also appear among the $L_p(x)$. Finally let $c_1, c_2, \cdots, c_n$ with $c_1c_2 \cdots c_n \leqq |D|$, be positive quantities such that $c_i = c_j$ if the forms $L_i(x)$ and $L_j(x)$ are conjugates. Then there exist integers x_q, not all zero, such that $|L_p(x)| \leqq c_p$ for $1 \leqq p \leqq n$.*

For a proof of this theorem see [2]. We shall be concerned with the case when $a_{pq} = \theta_p^{q-1}$ where $\theta_1, \theta_2, \cdots, \theta_n$ are a complete conjugate set of algebraic integers. In this case for integer x, each linear form $L_p(x)$ is a polynomial $f(\theta_p)$ with integer coefficients. The determinant D is a Vandermondian of distinct quantities, hence nonzero. We obtain without difficulty the following corollary.

Let θ be one of the θ_i's. There exists a nonzero polynomial f of degree $< n$ with integer coefficients such that $|f(\theta_i)| < c < 1$ if $\theta_i \neq \theta$ and $\theta_i \neq \bar{\theta}$. Furthermore $\prod_{i=1}^{n} |f(\theta_i)|$ is a nonzero integer and hence $|f(\theta)| = |f(\bar{\theta})| > 1$.

In the next three theorems $f(z)$ denotes a function which is meromorphic in $|z| < 1$, whose Laurent series around the origin, $\sum_{n=-k}^{\infty} a_n z^n$, has integer coefficients.

(1) SALEM'S THEOREM [9]. *If there exist real numbers $\varepsilon > 0$, $\delta > 0$, and a complex number α such that $|f(z) - \alpha| > \varepsilon$ for $1 - \delta < |z| < 1$, then $f(z)$ is rational. Also if* $\limsup_{r \to 1-} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty$ *then $f(z)$ is rational.*

(2) THE PÓLYA-CARLSON THEOREM [7]. *If $f(z)$ can be extended analytically across an arc of the unit circle then $f(z)$ is rational.*

(3) THE FATOU-HURWITZ THEOREM [8]. *If $f(z)$ is rational it is of the form $A(z)/Q(z)$ where $A(z)$ and $Q(z)$ are polynomials with integer coefficients and $Q(0) = 1$.*

We remark, finally, that we will sometimes refer to a power series in a region where it diverges. In this case we mean an analytic extension of the function defined by the power series. We will do this only when such an analytic extension is unique.

3. **Basic properties.** Pisot has shown that in every real algebraic field there are PV numbers which generate the field [5]. The first two theorems of this section generalize this result.

THEOREM 3.1. *Let $S = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ be a (not necessarily complete) set of conjugate algebraic numbers with the property that if $\alpha \in S$ then $\bar{\alpha} \in S$. There exists a polynomial $f(z)$ with integer coefficients such that $(f(\alpha_1), f(\alpha_2), \cdots, f(\alpha_k))$ is a* PV *k-tuple.*

Proof. We asume that the α_i are algebraic integers, for otherwise there is an integer p such that $p\alpha_i$ is an algebraic integer for each i. Then if $f(z)$ is the polynomial constructed for the set $(p\alpha_1, p\alpha_2, \cdots, p\alpha_k)$, $f(pz)$ will be suitable for $(\alpha_1, \alpha_2, \cdots, \alpha_k)$.

We now rename and reorder the α_i so as to pair complex conjugate elements. Let $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2, \cdots, \beta_l, \bar{\beta}_l, \beta_{l+1}, \beta_{l+2}, \cdots, \beta_n$, $(n + l = k)$ be such a rearrangement with β_i complex for $1 \leqq i \leqq l$ and real for $l + 1 \leqq i \leqq n$. Let $\beta_{n+1}, \cdots, \beta_m$ be all other conjugates of the α_i. We construct polynomials $f_i(z)$ $(i = 1, \cdots, n)$ with integer coefficients which possess the following properties.

(1) $|f_i(\beta_j)| < 1/(2n)$ for $1 \leqq j \leqq m$, $j \neq i$.

(2) If β_i is complex, then $|\operatorname{Im} f_i(\beta_i)| > 2$, $1 \leqq i \leqq n$.

(3) $|f_1(\beta_1)| > 2$, $|f_i(\beta_i)| > 2 + |f_{i-1}(\beta_{i-1})|$, $2 \leqq i \leqq n$.

By the corollary to Minkowski's theorem we can construct polynomials $f_i(z)$ with integer coefficients which satisfy (1) and such that $|f_i(\beta_i)| > 1$. We modify these polynomials as follows for $i = 1$, then for $i = 2$, etc., in order to satisfy (2) and (3). For each i under consideration, we first replace $f_i(z)$ by $f_i(z)^{l_i}$ where l_i is a positive integer. By choosing l_i large enough (3) can be satisfied. If (2) is not yet satisfied we replace the new $f_i(z)$ by $zf_i(z)^{l_{i'}}$, and if $l_{i'}$ is suitably chosen (1), (2), and (3) will be satisfied.

Let $f(z) = \sum_{i=1}^{n} f_i(z)$. Then if $j \leqq n$,

$$f(\beta_j) = f_j(\beta_j) + \sum_{\substack{i=1 \\ i \neq j}}^{n} f_i(\beta_j);$$

hence

$$|f(\beta_j)| \geqq |f_j(\beta_j)| - \sum_{\substack{i=1 \\ i \neq j}}^{n} f_i(\beta_j) > 1.$$

Similarly, if $n + 1 \leqq j \leqq m$, $|f(\beta_j)| < 1/2$. Finally, if $\beta_j \neq \beta_n$ then $|f(\beta_j) - f(\beta_n)| > 1$, and if β_j is complex then $|f(\beta_j) - f(\bar{\beta}_j)| > 1$. It follows that $(f(\alpha_1), f(\alpha_2), \cdots, f(\alpha_k))$ is a PV k-tuple.

THEOREM 3.2. *Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ be a set of algebraic numbers with the property that if $\alpha \in S$, then $\bar{\alpha} \in S$. There exists a* PV *k-tuple $(\theta_1, \theta_2, \cdots, \theta_k)$ such that the field $R(\theta_i) = R(\alpha_i)$ for $1 \leqq i \leqq k$.*

Proof. We first rename and reorder the α_i so as to pair complex conjugate elements. Let $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2, \cdots, \beta_l, \bar{\beta}_l, \beta_{l+1}, \cdots, \beta_n$ be such a rearrangement with β_i complex for $1 \leqq i \leqq l$ and real for $l + 1 \leqq i \leqq n$. By Theorem 3.1 there exist

polynomials $g_i(z)$ with integer coefficients such that $(g_i(\beta_i), g_i(\bar{\beta}_i))$ is a PV 2-tuple for $1 \leqq i \leqq l$ and $(g_i(\beta_i))$ is a PV 1-tuple for $l+1 \leqq i \leqq n$. Let β_i be a real β and let $\gamma_1, \gamma_2, \cdots, \gamma_s$ be its conjugates. It follows that the conjugates of $g_i(\beta_i)$ are among $g_i(\gamma_1)$, $g_i(\gamma_2), \cdots, g_i(\gamma_s)$. Since they each have absolute value less than one, they are roots of the defining polynomial of $(g_i(\beta_i))$. Hence the fields generated by β_i and $g_i(B_i)$ have the same pegree. Now $R(g_i(\beta_i)) \subset R(\beta_i)$ and it follows that $R(g_i(\beta_i)) = R(\beta_i)$. If β_i is a complex β it follows, in the same way, that $R(g_i(\beta_i)) = R(\beta_i)$. Since $(g_1(\beta_1), g_1(\bar{\beta}_1), \cdots, g_l(\bar{\beta}_l), g_l(\beta_l), g_{l+1}(\beta_{l+1}), \cdots, g_n(\beta_n))$ is a PV k-tuple, the proof is complete.

LEMMA 3.3 *Let $P(z)$ be a monic polynomial with integer coefficients which has k distinct roots $(\theta_1, \theta_2, \cdots, \theta_k)$ outside the unit circle. Then $(\theta_1, \theta_2, \cdots, \theta_k)$ is a* T *k-tuple. If, moreover, $P(z)$ has no zeros on the unit circle then $(\theta_1, \theta_2, \cdots, \theta_k)$ is a* PV *k-tuple. In either case if $P(0) \neq 0$ and $P(z)$ has no cyclotomic polynomials as factors, then $P(z)$ is the defining polynomial of the* T *or* PV *k-tuple $(\theta_1, \theta_2, \cdots, \theta_k)$.*

Proof. Let $S(z)$ be the monic polynomial with integer coefficients of least degree which $\theta_1, \theta_2, \cdots, \theta_k$ all satisfy. Then $S(z)$ divides $P(z)$ so that all roots of $S(z)$ are roots of $P(z)$. Thus $(\theta_1, \theta_2, \cdots, \theta_k)$ is a T (PV) k-tuple if all other roots of $S(z)$ lie in the closed (open) unit circle. This proves the first half of the lemma. If $P(0) \neq 0$ then the polynomial $P(z)/S(z)$, if not constant, has all its roots in the closed unit circle and none of them is zero. Since their product is an integer they must all be on the circumference of the unit circle. Hence they are roots of cyclotomic polynomials [4]. Since this is impossible, by hypothesis, $P(z)/S(z)$ is constant, $P(z) \equiv S(z)$, and the lemma is proved.

The proof of the next theorem gives an easy method of constructing PV k-tuples of all possible degrees.

THEOREM 3.4. *There exist* PV *k-tuples whose defining polynomials have degree l whenever $l \geqq k$.*

Proof. Let $P(z)$ be a polynomial of degree $l-k$ with integer coefficients which has all its roots inside the unit circle. Such polynomials exist, e.g., $2z^{l-k} - 1$. Let m be an integer such that $|mP(z)| > 1$ if $|z| = 1$. Consider $z^l + mP(z)$. By Rouché's theorem it has the same number of roots in the unit circle as $P(z)$, namely $l-k$. It is of degree l and so has $l-(l-k) = k$ roots outside the unit circle, and by Lemma 3.3 is the defining polynomial of a PV k-tuple.

THEOREM 3.5. *Let $(\theta_1, \theta_2, \cdots, \theta_k)$ be a* PV *k-tuple. Let $f(z)$ be a polynomial with rational coefficients, and let ρ be any positive number. Suppose that $f(\theta_i)\theta_i^n$ is an algebraic integer for all large n, then $\sum_{n=0}^{\infty} \| \sum_{i=1}^{k} f(\theta_i)\theta_i^n \|^{\rho} < \infty$.*

Proof. Let $\theta_{k+1}, \theta_{k+2}, \cdots, \theta_m$ be the remaining roots of the defining polynomial

of $(\theta_1, \theta_2, \cdots, \theta_k)$. Let $\theta = \max |\theta_i|$ for $k+1 \leqq i \leqq m$, so that $\theta < 1$. Let $f = \max |f(\theta_i)|$ for $k+1 \leqq i \leqq m$. Now $\sum_{i=1}^m f(\theta_i)\theta_i^n$ is an integer for all large n, by the symmetric function theorem. Thus

$$\left\| \sum_{i=1}^{k} f(\theta_i)\theta_i^n \right\| \leqq \left| \sum_{i=k+1}^{m} f(\theta_i)\theta_i^n \right| \leqq (m-k) f \theta^n.$$

Hence

$$\sum_{n=0}^{\infty} \left\| \sum_{i=1}^{k} f(\theta_i)\theta_i^n \right\|^\rho \leqq \sum_{n=0}^{\infty} [(m-k) f \theta^n]^\rho$$
$$= [(m-k)f]^\rho (1 - \theta^{1/\rho}) < \infty.$$

This completes the proof.

COROLLARY. *Let $(\theta_1, \theta_2, \cdots, \theta_k)$ be a* PV *k-tuple with defining polynomial $P(z)$. Let the remaining roots of $P(z)$ be $\theta_{k+1}, \theta_{k+2}, \cdots, \theta_m$. Let $f(z)$ be a polynomial with integer coefficients. Then $\lim_{n\to\infty} \| \sum_{i=1}^k f(\theta_i)\theta_i^n \| = 0$, and $N(\sum_{i=1}^k f(\theta_i)\theta_i^n) = \sum_{i=1}^m f(\theta_i)\theta_i^n$ for all large n.*

4. **Power series with integer coefficients.** In this section we shall be concerned with power series which define functions meromorphic in the unit circle whose coefficients are integers and depend upon PV or T k-tuples. We obtain conditions that these functions be rational, or equivalently, by the Pólya-Carlson theorem, that they can be extended analytically across an arc of the unit circle.

The first theorem in this section generalizes a result of Salem [9].

THEOREM 4.1. *Let $S = (\theta_1, \theta_2, \cdots, \theta_k)$ be a set of k distinct complex numbers such that if $\theta \in S$, then $\bar{\theta} \in S$ and $|\theta| > 1$. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be nonzero complex numbers, with $\lambda_i = \bar{\lambda}_j$ if $\theta_i = \bar{\theta}_j$. Let $\{a_n\}$ be a sequence of real numbers with $a_n = O((1+\varepsilon)^n)$ for all $\varepsilon > 0$. If the function $f(z) = \sum_{n=0}^\infty N(\sum_{i=1}^k \lambda_i\theta_i^n + a_n)z^n$ is rational then S is a* T *k-tuple and each λ_i is in the field $R(\theta_i)$.*

Proof. Suppose that $f(z)$ is rational. By the Fatou-Hurwitz theorem it is of the form $A(z)/Q(z)$ where $A(z)$ and $Q(z)$ have integer coefficients and $Q(0) = 1$. Let

$$\delta_n = N\left(\sum_{i=1}^{k} \lambda_i\theta_i^n + a_n\right) - \sum_{i=1}^{k} \lambda_i\theta_i^n,$$

so that $|\delta_n| \leqq |a_n| + 1/2$; hence $\delta_n = O((1+\varepsilon)^n)$ for all $\varepsilon > 0$. Thus $\sum_{n=0}^\infty \delta_n z^n$ is regular in the unit circle. Now

$$A(z)/Q(z) = f(z) = \sum_{n=0}^{\infty} \delta_n z^n + \sum_{i=1}^{k} \frac{\lambda_i}{1-\theta_i z};$$

consequently the zeros of $Q(z)$ in the unit circle are $1/\theta_1, 1/\theta_2, \cdots, 1/\theta_k$. Hence $S = (\theta_1, \theta_2, \cdots, \theta_k)$ is a T k-tuple. Since λ_i is the residue of $A(z)/Q(z)$ at $1/\theta_i$, it is in the field of θ_i, and the proof is complete.

THEOREM 4.2. *Let $(\theta_1, \theta_2, \cdots, \theta_k)$ be a* T *k-tuple with defining polynomial $P(z)$. Let the remaining roots of $P(z)$ be $\theta_{k+1}, \theta_{k+2}, \cdots, \theta_m$. Suppose that $\varepsilon > 0$ is given. Then there exist nonzero $\lambda_1, \lambda_2, \cdots, \lambda_m$ with $\lambda_i = \bar{\lambda}_j$ if $\theta_i = \bar{\theta}_j$, such that $\| \sum_{i=1}^k \lambda_i \theta_i^n \| < \varepsilon$ and $N(\sum_{i=1}^k \lambda_i \theta_i^n) = \sum_{i=1}^m \lambda_i \theta_i^n$ for all $n \geqq 0$.*

Proof. Assume, first, that $P(z)$ is irreducible. By Theorem 3.1. there exists a polynomial $f(z)$ with integer coefficients such that $(f(\theta_1), f(\theta_2), \cdots, f(\theta_k))$ is a PV k-tuple; thus $|f(\theta_i)| < 1$ if $k+1 \leqq i \leqq m$. By taking powers of $f(z)$, if necessary, we may assume that $\sum_{i=k+1}^m |f(\theta_i)| < \varepsilon$. Now $\sum_{i=1}^m f(\theta_i)\theta_i^n$ is an integer so that

$$\left\| \sum_{i=1}^k f(\theta_i)\theta_i^n \right\| \leqq \left| \sum_{i=k+1}^m f(\theta_i)\theta_i^n \right| \leqq \sum_{i=k+1}^m |f(\theta_i)| < \varepsilon .$$

We assume, without loss of generality, that $\varepsilon < 1/2$. It follows that

$$N\left(\sum_{i=1}^k f(\theta_i)\,\theta_i^n\right) = \sum_{i=1}^m f(\theta_i)\,\theta_i^n .$$

This completes the proof when $P(z)$ is irreducible. Suppose that $P(z) = \prod_{i=1}^s P_i(z)$, where the $P_i(z)$ are irreducible. We apply the construction above to the roots of each $P_i(z)$, replacing ε by ε/s. The theorem follows immediately.

THEOREM 4.3. *Let $(\theta_1, \theta_2, \cdots, \theta_k)$ be a* T *k-tuple. Let a be a real number with $\| a \| \neq 1/2$. There exist $\lambda_1, \lambda_2, \cdots, \lambda_k$, nonzero, such that*

$$\sum_{n=0}^{\infty} N\left(\sum_{i=1}^k \lambda_i \theta_i^n + a\right) z^n$$

is a rational function.

Proof. We assume, without loss of generality, that $|a| < 1/2$. Choose $\varepsilon > 0$ so that $|a| + |\varepsilon| < 1/2$. By Theorem 4.2 there exist $\lambda_1, \lambda_2, \cdots, \lambda_m$ so that $|\sum_{i=1}^k \lambda_i \theta_i^n - \sum_{i=1}^m \lambda_i \theta_i^n| < \varepsilon$ and $\sum_{i=1}^m \lambda_i \theta_i^n$ is an integer. Hence

$$\left| \sum_{i=1}^k \lambda_i \theta_i^n + a - \sum_{i=1}^m \lambda_i \theta_i^n \right| < 1/2 ,$$

and

$$N\left(\sum_{i=1}^k \lambda_i \theta_i^n + a\right) = \sum_{i=1}^m \lambda_i \theta_i^n .$$

Therefore

$$\sum_{n=0}^{\infty} N\left(\sum_{i=1}^k \lambda_i \theta_i^n + a\right) z^n = \sum_{n=0}^{\infty} \sum_{i=1}^m \lambda_i\, \theta_i^n z^n = \sum_{i=1}^m \frac{\lambda_i}{1 - \theta_i z}$$

which is rational.

THEOREM 4.4. *Let $(\theta_1, \theta_2, \cdots, \theta_k)$ be a* PV *k-tuple. Let $f(z)$ be a polynomial with rational coefficients. Let r be the least common multiple of the denominators*

of the coefficients of $f(z)$. Let a be a real number such that $\|ra\| \neq \|r/2\|$. Then $g(z) = \sum_{n=0}^{\infty} N(\sum_{i=1}^{k} f(\theta_i)\theta_i^n + a)z^n$ is rational.

We first establish two lemmas. By *recurrence relation* we shall mean a linear recurrence relation of finite order with constant coefficients (not all zero).

LEMMA 4.5. *Let $\{d_n\}$ be a sequence of integers satisfying a recurrence relation, and let r be a positive integer. Then the sequence $\{[d_n/r]\}$ (where "$[x]$" is the integer part of x) satisfies a recurrence relation.*

Proof. Let $b_n = d_n - r[d_n/r]$. Then $b_n \equiv d_n \pmod{r}$, $0 \leqq b_n < r$, and the sequence $\{b_n\}$ satisfies a recurrence relation mod r, which we suppose to be of order s. Since there are only finitely many distinct sequences of the form b_l, $b_{l+1}, \cdots, b_{l+s-1}$, two of them must be equal. Suppose that $b_l, b_l+_1, \cdots, b_{l+s-1}$ is the same, term by term, as $b_m, b_{m+1}, \cdots, b_{m+s-1}$ with $l \neq m$. Then, by the recurrence relation, $b_{l+s} = b_{m+s}$, and inductively $b_{l+t} = b_m + t$ for $t = 0, 1, 2, 3, \cdots$. Consequently the sequence $\{b_n\}$ is periodic and therefore satisfies a recurrence relation. Since $[d_n/r] = \{(d_n - b_n)/r\}$ is the term by term difference of two recurrent sequences, it, too, is recurrent.

LEMMA 4.6. *Let r be an integer and ρ a real number. There exists an integer c such that $[(x + c)/r] = N((x + \rho)/r)$ for all integral x.*

Proof.
$$N\left(\frac{x+\rho}{r}\right) = \left[\frac{x+\rho}{r} + \frac{1}{2}\right] = \left[\frac{x+(\rho+r/2)}{r}\right].$$

Let $c = [\rho + r/2]$. Then

$$\left[\frac{x+c}{r}\right] = \left[\frac{x+(\rho+r/2)}{r}\right]$$

and the proof is complete.

Proof of Theorem 4.4. Let $d_n = N(\sum_{i=1}^{k} rf(\theta_i)\theta_i^n)$ and $\varepsilon_n = \sum_{i=1}^{k} rf(\theta_i)\theta_i^n - d_n$. We show first that $\sum_{n=0}^{\infty} d_n z^n$ is rational. By the Corollary to Theorem 3.5, $d_n = \sum_{i=1}^{m} rf(\theta_i)\theta_i^n$ for large n and $\varepsilon_n \to 0$ as $n \to \infty$. Thus $\sum_{n=0}^{\infty} d_n z^n$ differs from $\sum_{i=1}^{m} rf(\theta_i)/(1 - \theta_i z)$ by a polynomial and hence is rational. Now let $p = N(ra)$ and $\delta = ra - p$. Then

$$N\left(\sum_{i=1}^{k} f(\theta_i)\theta_i^n + a\right) = N\left(\frac{d_n + p + \delta + \varepsilon_n}{r}\right).$$

We shall show that $\|(d_n + p + \delta)/r\|$ is bounded away from $1/2$. Since this expression takes on at most r different values, it is enough to show that it is never $1/2$. If $\|(d_n + p + \delta)/r\| = 1/2$ then there is an integer q such that $(d_n + p + \delta)/r$

$-q = 1/2$. Hence $d_n + ra - rq = r/2$, or $\| ra \| = \| r/2 \|$ which violates the hypothesis. It follows that, for large n,

$$N\left(\frac{d_n + p + \delta + \varepsilon_n}{r}\right) = N\left(\frac{d_n + p + \delta}{r}\right),$$

since $\varepsilon_n \to 0$ as $n \to \infty$. Thus, by Lemma 4.6 there exists an integer c such that

$$\left[\left(\frac{d_n + c}{r}\right)\right] = N\left(\frac{d_n + p + \delta}{r}\right) = N\left(\frac{a_n + p + \delta + \varepsilon_n}{r}\right)$$

$$= N\left(\sum_{i=1}^{k} f(\theta_i)\theta_i^n + a\right)$$

for large n. Consequently $\sum_{n=0}^{\infty} N(\sum_{i=1}^{k} f(\theta_i)\theta_i^n + a)z^n$ differs by a polynomial from $\sum_{n=0}^{\infty}[(d_n + c)/r]z^n$. Now $\sum_{n=0}^{\infty} d_n z^n$ is rational, hence $\sum_{n=0}^{\infty}(d_n + c)z^n$ is also rational, and, by Lemma 4.5 so is $\sum_{n=0}^{\infty}[(d_n + c)/]z^n$. The proof is complete.

In the last two theorems we have considered rational functions with poles *in* the unit circle. We now consider the same problem when the poles lie *on* the unit circle. We allow multiple poles; but, for simplicity, restrict these to the point $z = 1$.

THEOREM 4.7. *Let $f_1(n), f_2(n), \cdots, f_k(n)$ be polynomials with integer coefficients whose derivatives are linearly independent. Suppose that $\{a_n\}$ is a sequence of real numbers with $a_n = o(n)$. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be real numbers. If the function $g(z) = \sum_{n=0}^{\infty} N(\sum_{i=1}^{k} \lambda_i f_i(n) + a_n)z^n$ is rational, then each λ_i is rational.*

Proof. We may incorporate the constant term of each polynomial into the sequence $\{a_n\}$ and so assume that $f_i(0) = 0$ for $1 \leqq i \leqq k$. It now follows that the f_i's are linearly independent and that no f_i is constant. We now define b's by $\sum_{i=1}^{m} b_i n^i = \sum_{i=1}^{k} \lambda_i f_i(n)$, where m is the largest degree among the f's. Since the f's are linearly independent the b's determine the λ's uniquely, and if the b's are rational so are the λ's. Now define $c_0, c_1, c_2, \cdots, c_m$ by

$$\sum_{i=1}^{m} b_i n^i = \sum_{i=0}^{m} c_i \binom{n+i}{i}.$$

If $c_1, c_2, \cdots, c_m$ are rational so are the b's, since $\sum_{i=0}^{m} c_i = 0$. Let $r_n = \sum_{i=1}^{k} \lambda_i f_i(n)$. Then

$$\sum_{n=0}^{\infty} r_n z^n = \sum_{i=0}^{m} \sum_{n=0}^{\infty} c_i \binom{n+i}{i} z^n = \sum_{i=0}^{m} c_i \sum_{n=0}^{\infty} \binom{n+i}{i} z^n = \sum_{i=0}^{m} \frac{c_i}{(1-z)^{i+1}}.$$

Let $\delta_n = a_n - N(r_n + a_n)$, so that $s_n = o(n)$, and $\sum_{n=0}^{\infty}(r_n - s_n)z^n = g(z)$. If $g(z)$ is rational, then by the Fatou-Hurwitz theorem, it is of the form $A(z)/Q(z)$ where $A(z)$ and $Q(z)$ are polynomials with integer coefficients and $Q(0) = 1$. Thus $g(z) = A(z)/Q(z)$ has the form $\sum_{i=1}^{p} d_i/(1-z)^i + h(z)$ where $h(z)$ is regular in a

neighborhood of 1. By calculating residues we see that each d_i is rational. Since $s_n = o(n)$, $(t-1)^2 \sum_{n=0}^{\infty} s_n t^n \to 0$ as $t \to 1-$, by [11]. Therefore

$$\lim_{t \to 1-} (t-1)^l \left(\sum_{n=0}^{\infty} r_n t^n - g(t) \right) = 0 \text{ if } l \geqq 2.$$

Consequently $d_i = c_{i-1}$ for $i \geqq 2$. Thus $c_1, c_2, \cdots, c_m$ are rational, and therefore the λ's are rational. The proof is complete.

THEOREM 4.8. *Let $f(n)$ be a polynomial with rational coefficients and let a be a real number. Then $\sum_{n=0}^{\infty} N(f(n)+a)z^n$ is rational.*

Proof. Let r be an integer such that $rf(z)$ has integer coefficients. By Lemma 4.6

$$N(f(n)+a) = N\left(\frac{rf(n)+ra}{r}\right) = \left[\frac{rf(n)+c}{r}\right].$$

But $\sum_{n=0}^{\infty}(rf(n)+c)z^n$ is rational, hence so is $\sum_{n=0}^{\infty}[(rf(n)+c)/r]z^n$, by Lemma 4.5.

THEOREM 4.9. *Let $S = (\theta_1, \theta_2, \cdots, \theta_k)$ be a set of k complex numbers such that if $\theta \in S$, then $\bar{\theta} \in S$ and $|\theta| > 1$. A necessary and sufficient condition that S be a T k-tuple is that there exist nonzero $\lambda_1, \lambda_2, \cdots, \lambda_k$ with $\lambda_i = \bar{\lambda}_j$ if $\theta_i = \bar{\theta}_j$, such that $h(z) = \sum_{n=0}^{\infty}\{\sum_{i=1}^{k} \lambda_i \theta_i^n\} z^n$ has its real part bounded above for $|z| < 1$.*

Proof. Suppose that $\sum_{n=0}^{\infty}\{\sum_{i=1}^{k} \lambda_i \theta_i^n\} z^n$ has its real part bounded from above. Now

$$\sum_{n=0}^{\infty}\left(\sum_{i=1}^{k} \lambda_i \theta_k^n\right) z^n = \sum_{i=1}^{k} \frac{\lambda_i}{1-\theta_i z}$$

is analytic in $|z| > 1-\delta$ if $\delta > 0$ is small enough, and hence its real part is bounded in $1 > |z| > 1-\delta$. Thus

$$\sum_{n=0}^{\infty} N\left(\sum_{i=1}^{k} \lambda_i \theta_i^n\right) z^n = \sum_{n=0}^{\infty}\left(\sum_{i=1}^{k} \lambda_i \theta_i^n\right) z^n - \sum_{n=0}^{\infty}\left\{\sum_{i=1}^{k} \lambda_i \theta_i^n\right\} z^n$$

has its real part bounded from below in $1 > |z| > 1-\delta$, and by Salem's theorem is rational. By Theorem 4.1, S is a T k-tuple.

Conversely, suppose that S is a T k-tuple. Let $P(z)$ be its defining polynomial. Suppose first that $P(z)$ is irreducible. If $(\theta_1, \theta_2, \cdots, \theta_k)$ is a PV k-tuple the theorem follows from Theorem 3.5. We assume, therefore, that $P(z)$ has at least one root, necessarily complex, on the unit circle. If this root is α then $\bar{\alpha} = 1/\alpha$ is also a root. Consequently $P(z)$ is a reciprocal polynomial and its roots are algebraic units. Let $\alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2, \cdots, \alpha_r, \bar{\alpha}_r$ be the roots of $P(z)$ on the unit circle. Let $\sigma_i = \theta_i + 1/\theta_i$ and $\rho_i = \alpha_i + 1/\alpha_i = \alpha_i + \bar{\alpha}_i$, so that the ρ_i are real. Now $\sigma_1, \sigma_2, \cdots, \sigma_k$, $\rho_i, \rho_2, \cdots, \rho_r$ are conjugate algebraic integers which satisfy the polynomial ob-

tained from $P(z)$ by setting $y = z + 1/z$. By Theorem 3.1 we can find a polynomial with integer coefficients, $f(z)$, such that $(f(\sigma_1), \cdots, f(\sigma_k))$ is a PV k-tuple. By replacing $f(z)$ by $f(z)^{2h}$ where h is a large integer we may assume that $\sum_{i=1}^{r} f(\rho_i) < 1/8$ and that $f(\rho_i) > 0$. Let $\lambda_i = f(\sigma_i)$ and $\gamma_i = f(\rho_i)$. Let

$$g(z) = \sum_{i=1}^{k} \left(\frac{\lambda_i}{1 - \theta_i z} + \frac{\lambda_i}{1 - \theta_i^{-1} z} \right) + \sum_{i=1}^{r} \left(\frac{\gamma_i}{1 - \alpha z} + \frac{\gamma_i}{1 - \alpha_i^{-1} z} \right).$$

In a neighborhood of the origin

$$g(z) = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{k} (\lambda_i \theta_i^n + \lambda_i \theta_i^{-n}) + \sum_{i=1}^{r} (\gamma_i \alpha_i^n + \gamma_i \alpha_i^{-n}) \right) z^n = \sum_{n=0}^{\infty} b_n z^n$$

where the b_n are integers. Since $\lambda_1 \theta_i^{-n} \to 0$ as $n \to \infty$ and $\left| \sum_{i=1}^{r} (\gamma_i \alpha_i^n + \gamma_i \alpha_i^{-n}) \right| < 1/4$ we see that $b_n = N(\sum_{i=1}^{k} \lambda_i \theta_i^n)$ for all large n. So, if $\mathrm{Re}(g(z))$ is bounded from below in $1 - \delta < |z| < 1$, then so is $\mathrm{Re}(\sum_{n=0}^{\infty} N(\sum_{i=1}^{k} \lambda_i \theta_i^n) z^n)$, since the two functions differ by a polynomial. The functions

$$\sum_{i=1}^{k} \lambda_i/(2 - \theta_i z) \text{ and } \sum_{i=1}^{k} \lambda_i/(1 - \theta_i^{-1} z)$$

are both regular in a region of the form $1 - \delta \leqq |z| \leqq 1 + \delta$, and consequently their real parts are bounded there, By elementary properties of conformal mapping $\mathrm{Re}(\lambda_i/(1 - \alpha_i z)) \geqq \gamma_i/2$ in $|z| < 1$, and so is $\sum_{n=0}^{\infty} N(\sum_{i=1}^{k} \lambda_i \theta_i) z_n$. It now follows that

$$\mathrm{Re}\, h(z) = \mathrm{Re}\left(\sum_{n=0}^{\infty} \left(\left(\sum_{i=1}^{\infty} \lambda_i \theta_i^n \right) - N\left(\sum_{i=1}^{k} \lambda_i \theta_i^n \right) z^n \right) \right)$$

is bounded from above in $1 - \delta \leqq |z| < 1$. Since $h(z)$ is regular in $|z| \leqq 1 - \delta$ its real part is bounded from above, there, also, hence in $|z| < 1$. Finally, if $P(z)$ is reducible, one applies the above construction to each of its irreducible factors, and adds resulting series together.

5. **Limit properties of PV and T k-tuples.** In [9] Salem has shown that the PV 1-tuples are closed, and in [3] Kelly has shown that the PV 2-tuples, with the restriction that $\theta_1 = \bar{\theta}_2$, have as limits other PV 2-tuples of the same type, or PV 1-tuples. We have been unable to prove the analogue of these theorems for the PV k-tuples and unable to find a counter-example. Instead we prove the following weaker version, whose proof is a generalization of that of Kelly's theorem, and which includes, as special cases, both Kelly's and Salem's results.

THEOREM 5.1. *Let $S_p = (\theta_1^{(p)}, \theta_2^{(p)}, \cdots, \theta_k^{(p)})$, $p = 1, 2, 3, \cdots$, be the pth term of a sequence of* PV *k tuples with limit $(\theta_1', \theta_2, \cdots, \theta_{k'}')$ and with $|\theta_i'| \neq 1$, for $1 \leqq i \leqq k'$. Then some nonvoid subset of $(\theta_1', \theta_2', \cdots, \theta_{k'}')$ containing k'' elements is a* PV *k'' tuple with $1 \leqq k'' \leqq k'$.*

We first establish some rather special properties of PV k-tuples. Let $(\theta_1, \theta_2, \cdots, \theta_l)$ be a PV l-tuple, $l \leqq k$, with defining polynomial $P(z)$, and let $Q(z)$ be the reciprocal polynomial of $P(z)$. Then $|P(z)/Q(z)| \equiv 1$ if and only if $P(z)$ is a reciprocal polynomial, hence of degree $2l$. Since there are only a finite number of such polynomials whose roots lie in any bounded region, the PV l-tuples they define play no role in the limit problem considered. We assume, therefore, in the remainder of this proof, that the defining polynomials of the PV k-tuples, S_p, have no reciprocal polynomials as factors. Let $(\theta_1, \theta_2, \cdots, \theta_k)$ be a PV k-tuple from $\{S_p\}$ with defining polynomial $P(z)$ and reciprocal polynomial $Q(z)$. Now $R(z) = P(z)/Q(z)$ is a rational function, and its Taylor series has integer coefficients. Let $R(z) = \sum_{i=0}^{\infty} c_i z^i$ and let $c_i = 0$ if $i < 0$. Define $s_{i,r}$ by $\sum_{i=0}^{k} s_{i,r} z^{ir} = \prod_{i=1}^{k} (1 - \theta_i^r z^r)$ so that $s_{0,r} = 1$ and $s_{i,r}$ is the ith symmetric function of the rth power of the θ_i. We now establish several lemmas.

LEMMA 5.2. $\sum_{n=0}^{\infty} (\sum_{i=0}^{k} c_{n-ir} s_{i,r})^2 = \sum_{i=0}^{k} s_{i,r}^2$.

Proof. On $|z| = 1$, $|R(z)| = 1$, so that, if $z = e^{i\phi}$

$$\frac{1}{2\pi}\int_0^{2\pi} \left| R(z) \prod_{i=1}^{k} (1 - \theta_i^r z^r) \right|^2 d\phi = \frac{1}{2\pi}\int_0^{2\pi} \left| \prod_{i=1}^{k} (1 - \theta_i^r z^r) \right|^2 d\phi .$$

The proof is completed by an application of Parseval's identity.

LEMMA 5.3. *At least one of $c_r, c_{2r}, \cdots, c_{kr}$ is nonzero.*

Proof. Suppose that $c_r = c_{2r} = \cdots = c_{kr} = 0$. Let us take the terms corresponding to $n = 0, r, 2r, \cdots, kr$ in the left member of Lemma 6. We obtain $c_0^2 \sum_{i=0}^{k} s_{i,r}^2 \leqq \sum_{i=0}^{k} s_{i,r}^2$ with equality only if all other terms are zero. But since c_0 is a nonzero integer, $c_0 = \pm 1$ and all these other terms are indeed zero. Consequently $R(z) = 1$, but this case has been ruled out and the lemma is proved.

LEMMA 5.4. *If $\max_{1 \leqq i \leqq k} |\theta_i| \leqq M$, then $R(z) \leqq (2 + 2M)^k$ in $|z| < 1/(2M)$.*

Proof. $R(z) \prod_{i=1}^{k} (1 - \theta_i z)$ is regular in $|z| \leqq 1$ and by the maximum principle $|R(z) \prod_{i=1}^{k} (1 - \theta_i z)| \leqq (1 + M)^k$ in $|z| \leqq 1$, since $|R(z)| = 1$ and $|1 - \theta_i z| \leqq 1 + M$ on $|z| = 1$. Now $|1 - \theta_i z| \geqq 1 - |\theta_i z| \geqq 1/2$ in $|z| \leqq 1/(2M)$. Hence in $|z| \leqq 1/(2M)$, $|R(z)/2^k| \leqq |R(z) \prod_{i=1}^{k} (1 - \theta_i z)| \leqq (1 + M)^k$. Multiplying through by 2^k completes the proof of the lemma.

LEMMA 5.5. *Let $S = (\theta_1, \theta_2, \cdots, \theta_k)$ be a set of k complex numbers such that if $\theta \in S$, then $\bar{\theta} \in S$ and $|\theta| > 1$. Let $\{a_n\}$ be a sequence of integers with infinitely many nonzero elements. Define ε_n by $\sum_{n=0}^{\infty} \varepsilon_n z^n = (\sum_{n=0}^{\infty} a_n z^n) \prod_{i=1}^{k} (1 - \theta_i z)$. If $\sum_{n=0}^{\infty} \varepsilon_n^2 < \infty$, then $a_n z^n$ is rational and some nonvoid subset of S is a PV k''-tuple, $k'' \leqq k$.*

Proof. $|\sum_{n=0}^{\infty} a_n z^n| = |\sum_{n=0}^{\infty} \varepsilon_n z^n| / \prod_{i=1}^{k} |1 - \theta_i z|$. If $|z|$ is sufficiently close

to 1 the denominator of the above expression is bounded away from zero. For such a z, $\left|\sum_{n=0}^{\infty} a_n z^n\right| < M\left|\sum_{n=0}^{\infty} \varepsilon_n z^n\right| < \infty$. Thus $\sum_{n=0}^{\infty} a_n z^n$ is H_2 and hence by Salem's theorem, a rational function. By the Fatou-Hurwitz theorem this function is of the form $A(z)/Q(z)$ where $A(z)$ and $Q(z)$ have integer coefficients and $Q(0) = 1$. Now $A(z)/Q(z)$ has some poles in the unit circle and these must be among $1/\theta_1, 1/\theta_2, \cdots, 1/\theta_k$. The $1/\theta_i$ which are poles are roots of $Q(z)$, and by Lemma 3.3 their reciprocals form a PV k''-tuple.

We now proceed to the proof of Theorem 16. Let $\theta_i = \lim_{p\to\infty}, \theta_i^{(p)}$, $1 \leqq i \leqq k$, so that $(\theta_1', \theta_2', \cdots, \theta_{k'}')$ is the set of distinct numbers in $(\theta_1, \theta_2, \cdots, \theta_k)$. With each PV k-tuple S_p we associate the corresponding rational function

$$R_p(z) = \sum_{i=0}^{\infty} c_{i,p} z^i.$$

Since the S_p have limit S there exists a number M so that $|\theta_{i,p}| < M$ for all p $(i = 1, 2, \cdots, k)$. By Lemma 5.4 the $R_p(z)$ are uniformly bounded in $|z| \leqq 1/2M$ and hence form a normal family. Consequently there exists a subsequence of $R_p(z)$ which converges to an analytic function $S(z)$. We may assume, without loss of generality, that this subsequence is the entire sequence. Now $S(z) = \sum_{i=0}^{\infty} c_i z^i$ where $c_i = \lim_{p\to\infty} c_{i,p}$; hence each c_i is an integer. We now show that infinitely many of the c_i are nonzero. It is clear that any subsequence of the sequence $\{R_p\}$ also converges to $S(z)$. By Lemma 5.3, for any positive integer r, at least one of $c_{r,p}, c_{2r,p}, \cdots, c_{kr,p}$ is nonzero, and hence there exists a subsequence of $\{R_p\}$ and an integer $j, 1 \leqq j \leqq k$, such that $c_{jr,p}$ is never zero in this subsequence. Thus c_{jr} is a nonzero integer. Consequently there are infinitely many nonzero c_k. Let $\prod_{i=1}^{k}(1 - \theta_{i,p}^r z^r) = \sum_{i=0}^{k} s_{i,r,p} z^{ir}$. Clearly

$$\sum_{i=0}^{k} c_{n-ir,p} s_{i,r,p})^2 \to \sum_{i=0}^{k} c_{n-ir} s_{i,r})^2$$

as $n \to \infty$. Hence by Fatou's lemma and Lemma 5.2 $\sum_{n=0}^{\infty}(\sum_{i=0}^{k} c_{n-ir} s_{i,r})^2 \leqq \liminf_{p\to\infty} \sum_{n=0}^{\infty}(\sum_{i=0}^{k} c_{n-ir,p} s_{i,r})^2 < \infty$. Since $\sum_{n=0}^{\infty}(\sum_{i=0}^{k} c_{n-ir} s_{i,r}) z^n = S(z)\prod_{i=1}^{k}(1 - \theta_i z)$, an application of Lemma 5.5 completes the proof.

COROLLARY. *The* PV 1-*tuples are closed.*

COROLLARY. *The limit points of the* PV 2-*tuples with* $\theta_1 = \bar{\theta}_2$ are either PV 2-*tuples of the same type or* PV 1-*tuples.*

The next two theorems generalize and are proved similarly to, a result of Dufresnoy and Pisot [6].

THEOREM 5.6. *Let* $S_p = (\theta_{1,p}, \theta_{2,p}, \cdots, \theta_{k,p})$ *be a sequence of* PV k-*tuples with limit* $(\theta_1', \theta_2', \cdots, \theta_{k'}')$ *with* $|\theta_i'| > 1, 1 \leqq i \leqq k'$. *Let* $R_p(z)$ *be the corresponding rational function so that* $\lim_{p\to\infty} R_p(z) = S(z)$. *Then* $|S(z)| \leqq 1$ *on* $|z| = 1$ *with equality at only a finite number of points.*

Proof. Let $\theta_i = \lim_{p\to\infty}\theta_{i,p}$ for $1 \leqq i \leqq k$. We first show that $|S(z)| \leqq 1$ on $|z| = 1$. By elementary properties of conformal mapping

$$\left| R_p(z) \prod_{i=1}^{k} \left(\frac{1-\theta_{i,p}\, z}{z-\theta_{i,p}} \right) \right| \leqq 1$$

in $|z| \leqq 1$. Letting p tend to infinity gives

$$\left| S(z) \prod_{i=1}^{k} \left(\frac{1-\theta_i z}{z-\theta_i} \right) \right| \leqq 1$$

in $|z| \leqq 1$ or $|S(z)| \leqq 1$ on $|z| \leqq 1$. We now show that $|S(z)| \not\equiv 1$ on $|z| = 1$ We must first define several quantities. Let

$$\prod_{i=1}^{l} (1-\theta_i z)^2 = \sum_{i=0}^{2l} s_i z^i,$$

$$\prod_{i=1}^{l} (1-\theta_i z)(1-\theta_{i,p} z) = \sum_{i=0}^{2l} s_{i,p} z^i,$$

$$\begin{aligned}
\alpha &= \frac{1}{2\pi} \int_0^{2\pi} \left| S(z) \prod_{i=1}^{l} (1-\theta_i z)^2 \right|^2 d\phi, \\
\beta_p &= \frac{1}{2\pi} \int_0^{2\pi} \left| S(z) \prod_{i=1}^{l} (1-\theta_i z)(1-\theta_{i,p} z) \right|^2 d\phi, \\
\gamma_p &= \frac{1}{2\pi} \int_0^{2\pi} \left| R_p(z) \prod_{i=1}^{l} (1-\theta_i z)(1-\theta_{i,p} z) \right|^2 d\phi, \\
A_n &= \sum_{i=0}^{2l} c_{n-i} s_i, \\
B_{n,p} &= \sum_{i=0}^{2l} c_{n-1} s_{i,p}, \\
C_{n,p} &= \sum_{i=0}^{2l} c_{n-i,p} s_{i,p}.
\end{aligned}$$

From Parseval's identity we find that

$$\begin{aligned}
\alpha &= \sum_{n=0}^{\infty} A_n^2, \\
\beta_p &= \sum_{n=0}^{\infty} B_{n,p}^2, \\
\gamma_p &= \sum_{n=0}^{\infty} C_{n,p}^2 = \sum_{i=0}^{2l} s_{i,p}^2.
\end{aligned}$$

It is clear that $\beta_p \to \beta$ and $\gamma_p \to \sum_{i=0}^{2l} s_i^2$ as $p \to \infty$. Choose $\varepsilon > 0$. Let N be large enough so that $\sum_{n=N+1}^{\infty} A_n^2 < \varepsilon$. Let q be large enough so that if $p \geqq q$ then $|\alpha - \beta_p| < \varepsilon$ and $|\sum_{n=0}^{N} A_n^2 - \sum_{n=0}^{N} B_{n,p}^2| < \varepsilon$. It follows that $\sum_{n=N+1}^{\infty} B_{n,p}^2 < 3\varepsilon$ if $p \geqq 1$. Furthermore for large enough q there exists an integer $m > N$ such that $c_{n,q} = c_n$ for $n < m$ and $c_{m,q} \neq c_m$. Then, since they are integers, $|c_{m,q} - c_m| \geqq 1$; it follows that $|B_{m,q} - C_{m,q}| = 1$. Since $|B_{m,q}| \leqq \sqrt{3\varepsilon}$, we have $|C_{m,q}| \geqq 1 - \sqrt{3\varepsilon}$. Now $\gamma_q = \sum_{n=0}^{\infty} C_{n,q}^2 = \sum_{n=0}^{m-1} (B_{n,q})^2 + \sum_{n=m}^{\infty} (C_{n,q})^2$. But $|\sum_{n=0}^{m-1} B_{n,q}^2 - \alpha| < 3\varepsilon$ so that $\gamma_q \geqq \alpha - 3\varepsilon - \sqrt{3\varepsilon} + 1$. Since ε is arbitrary $\gamma_q \geqq \alpha + 1$, and this is true for all large q. If we had $|S(z)| \equiv 1$ on $|z| = 1$ then $\alpha = \sum_{i=0}^{2l} s_i^2$, which is impossible since $\lim_{q\to\infty} \gamma_q = \sum_{i=0}^{2l} s_i^2$. We now show that $|S(z)| = 1$ on $|z| = 1$ only a finite number of times. We already know that $S(z)$ is of the form $A(z)/Q(z)$ where $A(z)$ and $Q(z)$ are polynomials with integer coefficients. Now on $|z| = 1$, $|S(z)|^2 = (A(z)\bar{A}(z)/Q(z)\bar{Q}(z)) = A(z)A(1/z)/Q(z)Q(1/z)$. If this is equal to 1 infinitely often then $A(z)A(1/z) = Q(z)Q(1/z)$ and either $A(z) = Q(z)$ or $A(z) = P(z)$, where $P(z)$ is the reciprocal polynomial of $Q(z)$. In either case $|S(z)| = |A(z)/Q(z)| \equiv 1$ on $|z| = 1$. This contradiction completes the proof of the theorem.

THEOREM 5.7. *Let $S = (\theta_1, \theta_2, \cdots, \theta_k)$ be an irreducible* PV *k-tuple with defining polynomial $P(z)$. Then S is a limit of* PV *k-tuples if and only if there exists a polynomial $A(z)$, with integer coefficients, such that $|A(z)| \leqq |P(z)|$ on $|z| = 1$, with equality at only a finite number of points.*

Proof. Suppose that S is the limit of PV k-tuples. Let $Q(z)$ be the reciprocal polynomial of $P(z)$. Since $Q(z)$ is irreducible the rational function $S(z)$ in Theorem 5.6 is of the form $A(z)/Q(z)$, and since $|S(z)| \leqq 1$ with equality at only a finite number of points, the first half of this theorem is proved. Conversely, suppose that there exists $A(z)$, as above. Let $z_1, z_2, \cdots, z_r$ be the points on the unit circle for which $|A(z)| = |P(z)|$. Suppose that $P(z)$ is of degree m and let $0 < \rho < 1$. By Rouché's theorem $z^n P(z) + \rho A(z)$ has $n + m - k$ roots in the unit circle, for all large n. Letting ρ tend towards 1 we see that $P_n(z) = z^n p(z) + A(z)$ has $n + m - k$ roots in the closed unit circle and k roots outside the unit circle. The only possible roots on the unit circle are $z_1, z_2, \cdots, z_r$. Suppose that $z_j, 1 \leqq j \leqq r$, is such a root. Let $z_j = e^{2\pi i \rho_j}$. If ρ_j is rational then z_j is a root of unity, hence of a cyclotomic polynomial which must be a factor of $P_n(z)$. We may divide out all cyclotomic polynomial factors and obtain $P_n^*(z)$ which still has the same k roots outside the unit circle as $P(z)$. We now consider the case when ρ_j is irrational. In this case $z_j^n P(z_j) + A(z_j)$ equals zero at most once as n varies. Thus for large n, $P_n^*(z)$ defines a PV k-tuple. Since $P_n(z) \sim z^n P(z)$, for $|z| > 1$, as $n \to \infty$, the roots outside the unit circle of $P_n(z)$, hence of $P_n^*(z)$, approach those of $P(z)$. This completes the proof.

We shall now give an example showing that the proof of Theorem 5.1 fails to

establish that the closure of the set of all PV k-tuples is contained in the set of all PV k'-tuples ($k' = 1, \cdots, k$). Let $P(z)$ be the defining polynomial of a limit PV l-tuple, $(\theta_1, \theta_2, \cdots, \theta_l)$, $1 \leqq l < k$, so that there exists a polynomial $A(z)$ which has integer coefficients and $|P(z)| \geqq |A(z)|$ on $|z| = 1$, with equality at only a finite number of points. Let $S(z)$ define a PV h-tuple $(\phi_1, \phi_2, \cdots, \phi_h)$, $h + l = k$. Let $Q(z)$ and $T(z)$ be, respectively, the reciprocal polynomials of $P(z)$ and $S(z)$. Now consider $U_n(z) = z^n P(z)S(z) + A(z)T(z)$. Clearly $|P(z)S(z)| \geqq |A(z)T(z)|$ on $|z| = 1$, with equality at only a finite number of points. We suppose that no θ_j equals any ϕ_i. Then $P(z)S(z)$ defines a PV k-tuple $(\theta_1, \theta_2, \cdots, \theta_l, \phi_1, \phi_2, \cdots, \phi_h)$ and by Theorem 5.7, $U_n(z)$ defines a PV k-tuple $(\theta_1^{(n)}, \theta_2^{(n)}, \cdots, \theta_l^{(n)}, \phi_1^{(n)}, \cdots, \phi_h^{(n)})$. Since $U_n(z) \sim z^n P(z)S(z)$ if $|z| > 1$. We may assume the θ_is and ϕ_is ordered so that $\theta_i^{(n)} \to \theta_i$ and $\phi_i^{(n)} \to \phi_i$ as $n \to \infty$. We now compute the rational function $R_n(z)$ associated with $U_n(z)$. Let $B(z)$ be the reciprocal polynomial of $A(z)$. Suppose that $A(z)$ is of degree a; $P(z)$, p; and $S(z)$, s. Then $U_n(z)$ is of degree $n + p + s$, if n is large, and

$$R^n(z) = U_n(z) \Big| \left(z^{n+p+s} U_n\left(\frac{1}{z}\right)\right) = \frac{z^n P(z)S(z) + A(z)T(z)}{z^{n+p-b} B(z)S(z) + Q(z)T(z)}.$$

It is clear that $\lim_{n\to\infty} R_n(z) = A(z)/Q(z)$ in $|z| < 1$, and that the poles $1/\phi_1^{(n)}$, $1/\phi_2^{(n)}, \cdots, 1/\phi_h^{(n)}$ have disappeared. We notice that limit of the sequence of PV k-tuples $(\theta_1^{(n)}, \theta_2^{(n)}, \cdots, \theta_l^{(n)}, \phi_1^{(n)}, \cdots, \phi_h^{(n)})$ is the PV k-tuple $(\theta_1, \theta_2, \cdots, \theta_l, \phi_1, \phi_2, \cdots, \phi_h^{(n)})$, even though the proof of Theorem 5.1 fails to show it. (We do conjecture that the limits of PV k-tuples are PV k'-tuples, $k' \leqq k$.)

We now give some sufficient conditions for a PV k-tuple to be a limit of PV k-tuples.

THEOREM 5.8. *Let $S = (\theta_1, \theta_2, \cdots, \theta_k)$ be a* PV *k-tuple with defining polynomial $P(z)$. If $\alpha_1, \alpha_2, \cdots, \alpha_t$ are the complex roots of $P(z)$ and if $\prod_{i=1}^t (1 - |\alpha_i|)^2 > \prod_{i=1}^t |\alpha_i|$ then S is a limit of* PV *k-tuples.*

Proof. Let $\beta_1, \beta_2, \cdots, \beta_r$ be the positive real roots of $P(z)$ and let $\gamma_1, \gamma_2, \cdots, \gamma_s$ be the negative real roots. It is clear that on $|z| = 1$, $|(z - \beta_i)/(z - 1)|$ takes its minimum value when $z = -1$ so that $|(z - \beta_i)/(z - 1)| \geqq (1 + \beta_i)/2 > \sqrt{\beta_i}$. Similarly,

$$\left|\frac{z - \gamma_i}{z + 1}\right| \geqq \frac{1 + |\gamma_i|}{2} > \sqrt{|\gamma_i|}.$$

On $|z| = 1$,

$$\begin{aligned} |P(z)/(z-1)^r(z+1)^s| &= \prod_{i=1}^r \left|\frac{z - \beta_i}{z - 1}\right| \prod_{i=1}^s \left|\frac{z - \gamma_i}{z + 1}\right| \prod_{i=1}^t |z - \alpha_i| \\ &> \prod_{i=1}^r \sqrt{|\beta_i|} \prod_{i=1}^s \sqrt{|\gamma_i|} \prod_{i=1}^t \big|1 - |\alpha_i|\big| \\ &\geqq \prod_{i=1}^r \sqrt{|\beta_i|} \prod_{i=1}^s \sqrt{|\gamma_i|} \prod_{i=1}^t \sqrt{\alpha_i|} \geqq 1 \end{aligned}$$

since the product of all the roots of $P(z)$ is an integer. Thus

$$|P(z)| > |z-1|^r |z+1|^s$$

on $|z| = 1$, and by Theorem 5.7 the proof is complete.

COROLLARY. *Let $S=(\theta_1, \theta_2, \cdots, \theta_k)$ be a* PV *k-tuple. Suppose that all conjugates of the θs in the unit circle are real, and that all complex θ in S satisfy $|\theta| > (3+\sqrt{5})/2$, then S is a limit of* PV *k-tuples.*

Proof. If $|\theta| > (3+\sqrt{5})/2$, then $|\theta|^2 - 3|\theta| + 1 > 0$ or $(1-|\theta|)^2 > |\theta|$. We now investigate the limits of the T k-tuples.

THEOREM 5.9. *The limits of the* T *k-tuples include all* PV *k-tuples whose defining polynomials have no reciprocal polynomials as factors.*

Proof. Let $(\theta_1, \theta_2, \cdots, \theta_k)$ be such a PV k-tuple and let $P(z)$ be its defining polynomial. Let $Q(z)$ be the reciprocal polynomial of $P(z)$ so that, on $|z| = 1$, $|P(z)| = |Q(z)|$. Consequently, if $0 < \rho < 1$, $z^m P(z) + \rho Q(z)$ has the same number of roots in the unit circle as $z^m P(z)$, and therefore has k roots outside the unit circle. Letting ρ tend towards 1 we see that for large m the polynomial $C_m(z) = z^m P(z) + Q(z)$ has k roots outside the unit circle, and the remaining roots in the closed unit circle. Thus $C_m(z)$ defines a T k-tuple. Now $C_m(z) \sim z^m P(z)$ in $|z| > 1$, and so the roots outside the unit circle of $C_m(z)$ approach those of $P(z)$. But $C_m(\theta_i) = Q(\theta_i) \neq 0$ since $P(z)$ and $Q(z)$ are relatively prime by hypothesis. This completes the proof.

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