SIMPLIFICATION OF TURNING POINT PROBLEMS FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS(1)

BY

WOLFGANG WASOW

1. **Introduction.** R. E. Langer and several other mathematicians have developed a powerful method for the treatment of so-called "turning point problems" for ordinary linear differential equations with a parameter. The essential idea of this method has been described in a particularly lucid manner by Langer in [4].

The purpose of the present paper is to show that an analogous procedure exists for a fairly wide class of *systems* of differential equations. The essential idea of Langer's method is to simplify the given differential equation by a sequence of transformations *valid in a closed domain containing the turning point* until it differs so little from some special differential equation with known asymptotic behavior that the two problems have solutions that can be shown to be asymptotically equal. Thus, this method leads to a complete asymptotic analysis only if such a simpler "comparison equation" is available. In view of the great variety and complexity of problems for systems of differential equations it is not surprising that even the simplest form of the differential systems attainable by such transformations is only in rare cases one that has been previously studied in the literature with regard to its asymptotic properties. The purpose of the present paper is therefore the simplification of turning point problems and not their complete solution. The precise nature of this transformation is summarized in §9.

The systems to be studied here can be written in the form

$$\varepsilon \frac{dy}{dz} = A(z,\varepsilon)y. \tag{1.1}$$

Here $A(z,\varepsilon)$ is an n by n matrix function of the two complex variables z and ε which in a region defined by inequalities of the form

$$|z| \leqq z_0, \quad |\varepsilon| \leqq \varepsilon_0, \quad |\arg \varepsilon| \leqq \theta, \tag{1.2}$$

possesses an asymptotic expansion of the form

$$A(z,\varepsilon) \sim \sum_{r=0}^{\infty} A_r(z)\varepsilon^r, \quad \varepsilon \to 0 \tag{1.3}$$

Presented to the Society, June 16, 1961; received by the editors, December 18, 1961.

(1) The work on this paper was partially supported by National Science Foundation Grant G-19925.

with holomorphic coefficients. The letter y designates an n-dimensional vector function of z and ε.

The most important case is, of course, the one in which the series in (1.3) converges, but the arguments are the same if only (1.3) is assumed.

2. **Preparatory transformations. Assumptions.** In [5], Sibuya proved the existence of a matrix $T(z,\varepsilon)$ with the following properties.

(a) In a region of the (z,ε) space defined by inequalities of the form

$$|z| \leqq z_1, \quad |\varepsilon| \leqq \varepsilon_1, \quad |\arg \varepsilon| \leqq \theta_1 \tag{2.1}$$

$T(z,\varepsilon)$ possesses the uniform asymptotic expansion

$$T(z,\varepsilon) \sim \sum_{r=0}^{\infty} T_r(z)\varepsilon^r, \quad \varepsilon \to 0,$$

where all $T_r(z)$ are holomorphic in $|z| \leqq z_1$, and $\det T_0(z) \neq 0$.

(b) The transformation

$$y = T(z,\varepsilon)y^* \tag{2.2}$$

changes (1.1) into

$$\varepsilon \frac{dy^*}{dz} = A^*(z,\varepsilon)y^*, \tag{2.3}$$

where $A^*(z,\varepsilon)$ is the direct sum of $p \leqq n$ matrices $A_j^*(z,\varepsilon)$ of order n_j, $j = 1,2,\dots,p$. Here n_j is the multiplicity of the jth distinct eigenvalue λ_j of $A(0,0)$ and $A_j^*(0,0)$ has λ_j as its only eigenvalue. The matrices $A_j^*(z,\varepsilon)$ possess asymptotic expansions with respect to ε with holomorphic coefficients uniformly valid in the region (2.1).

Sibuya's theorem reduces the asymptotic study of the original problem to the study of a set of p problems of a simpler nature. To simplify the notation we assume, without loss of generality, that the original differential equation (1.1) is already the result of such a reduction, so that we may assume, from the outset, that *the matrix* $A(0,0)$ *possesses only one distinct eigenvalue.*

The transformation

$$y = \exp\left\{\frac{1}{n\varepsilon}\int_0^z \operatorname{tr}(A_0(t))dt\right\}y^*, \tag{2.4}$$

where "tr" indicates the trace of the matrix, changes (1.1) into

$$\varepsilon\frac{dy^*}{dz} = \left(A(z,\varepsilon) - \frac{1}{n}\operatorname{tr}(A_0(z))I\right)y^*.$$

The matrix of this problem has, for $\varepsilon = 0$, the trace zero. Again we assume that this transformation has already been performed beforehand so that $A(z,\varepsilon)$ in (1.1) has the property

$$\operatorname{tr} A_0(z) \equiv 0. \tag{2.5}$$

Together with the preceding assumption this implies, in particular, that $A_0(0)$ *is nilpotent.*

Finally, by a further linear transformation of the dependent variable, this time one with constant coefficients, $A_0(0)$ can be reduced to its Jordan canonical form. This assumption, too, will be incorporated into the original formulation: $A_0(0)$ *is in Jordan canonical form.*

Two *restrictive* hypotheses, the same as in [6; 7], are now imposed in this article. Without them some weaker and more complicated results can doubtless be proved by similar methods:

ASSUMPTION A. *$A_0(0)$ has only one Jordan block, i.e.,*

$$A_0(0) = J = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1.. & 0 \\ . & . & \dots & . \\ 0 & . & \dots & 1 \\ 0 & . & \dots & 0 \end{bmatrix}. \tag{2.6}$$

ASSUMPTION B.

$$\left[\frac{d}{dz}(\det A_0(z))\right]_{z=0} \neq 0.$$

3. Some lemmas on matrices.

LEMMA 3.1. *Let $M(z)$ be an n by n matrix holomorphic for $|z| \leqq z_0$. Assume that all the eigenvalues of $M(z)$ are equal for $z = 0$, but distinct for $z \neq 0$. If $M(0)$ has only one elementary divisor, then any matrix pointwise similar to $M(z)$ in $|z| \leqq z_0$ is holomorphically similar to $M(z)$ at $z = 0$, i.e., the similarity relation can be satisfied by a matrix that is holomorphic at $z = 0$ and nonsingular.*

For a proof see example b) of [8].

Let X be any n by n matrix with complex elements. In the n^2-dimensional linear vector space of all such matrices consider the linear operator defined by

$$L_z X = M(z)X - XM(z), \tag{3.1}$$

where $M(z)$ satisfies the assumptions of Lemma 3.1.

LEMMA 3.2. *For every z in a neighborhood of $z = 0$ the n matrices $M^k(z)$ $k = 0, 1, \dots, n-1$ constitute a basis for the null-space of the operator L_z.*

Proof. By example b) of [8] and by [2; vol. I, p. 222] the dimension of the null-space of L_z is constant and equal to n. The lemma now follows from [2, p. 222]

LEMMA 3.3. *Let $A(z)$ be an m by m matrix holomorphic at $z = 0$ and of* *constant rank r, and let $b(z)$ be a vector function holomorphic at $z = 0$. If the equation*

$$A(z)w(z) = b(z) \tag{3.2}$$

for the vector function $w(z)$ possesses solutions at every point z of a neighborhood of $z = 0$, it possesses solutions that are holomorphic in a neighborhood of $z = 0$. This neighborhood depends on $A(z)$ only.

Proof. Without loss of generality it may be assumed that the minor formed by the first r rows and columns of $A(z)$ does not vanish in a neighborhood of $z = 0$. Let this minor be called $A_{11}(z)$ and partition the equation (3.2), in a self-explanatory manner, into the form

$$\begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{bmatrix} \begin{bmatrix} w_1(z) \\ w_2(z) \end{bmatrix} = \begin{bmatrix} b_1(z) \\ b_2(z) \end{bmatrix}. \tag{3.3}$$

The fact that the rank of $A(z)$ is identically equal to r implies (see [8, formula (4)]) that

$$-A_{21}(z)A_{11}^{-1}(z)A_{12}(z) + A_{22}(z) \equiv 0. \tag{3.4}$$

Hence, left multiplication of (3.3) by

$$\begin{bmatrix} I_r & 0 \\ -A_{21}(z)A_{11}^{-1}(z) & I_{m-r} \end{bmatrix} \tag{3.5}$$

leads to the relation

$$\begin{bmatrix} A_{11}(z)w_1(z) + A_{12}(z)w_2(z) \\ 0 \end{bmatrix} = \begin{bmatrix} b_1(z) \\ -A_{21}(z)A_{11}^{-1}(z)b_1(z) + b_2(z) \end{bmatrix}. \tag{3.6}$$

If (3.3) has a solution the identity

$$-A_{21}(z)A_{11}^{-1}(z)b_1(z) + b_2(z) \equiv 0$$

must therefore be satisfied. Then (3.6) is, in particular, satisfied by the holomorphic vector function

$$w(z) = \begin{bmatrix} A_{11}^{-1}(z)b_1(z) \\ 0 \end{bmatrix}$$

This must also be a solution of (3.3), because the matrix (3.5) is nonsingular.

COROLLARY. *If the n by n matrix $K(z)$ is holomorphic at $z = 0$, and if the equation*

$$L_z X(z) = K(z) \tag{3.7}$$

for the matrix $X(z)$ has solutions for every z in a neighborhood of $z = 0$, then (3.7) *has solutions that are holomorphic at $z = 0$.*

Proof. The equation (3.7) is of the form (3.2) with $m = n^2$. By Lemma 3.2 its rank is identically equal to n near $z=0$. Hence Lemma 3.3 can be applied.

Let a scalar product in the linear vector space of all n by n matrices be defined by

$$(X,Y) = \mathrm{tr}(XY^T), \tag{3.8}$$

where "tr" denotes the trace, and Y^T is the transpose of Y.

LEMMA 3.4. *The adjoint of the operator L_z is given by*

$$L_z^T X = M^T(z)X - XM^T(z).$$

Proof. Since

$$\mathrm{tr}(XY) = \mathrm{tr}(YX), \quad \mathrm{tr}X = \mathrm{tr}X^T,$$

we have

$$(L_z X,Y) = \mathrm{tr}[(MX - XM)Y^T] = \mathrm{tr}(Y^T MX - MY^T X) = (X, M^T Y - YM^T),$$

which proves the lemma.

4. Simplification of the leading term.

Let

$$\det[A_0(z) - \lambda I] \equiv (-1)^n[\lambda^n - a_1^*(z)\lambda^{n-1} - \dots - a_n^*(z)] \tag{4.1}$$

be the characteristic polynomial of the matrix $A_0(z)$. By Assumption A of §2 the characteristic polynomial of $A_0(0)$ is $(-1)^n\lambda^n$. Hence

$$a_j^*(z) = za_j(z) \ , \qquad j = 1,2,\dots,n$$

with $a_j(z)$ holomorphic in $|z| \leqq z_1$. Because of formula (2.5)

$$a_1^*(z) \equiv 0. \tag{4.2}$$

Furthermore, by Assumption B,

$$a_n(0) \neq 0. \tag{4.3}$$

This implies that the zeros of the polynomial (4.1) are distinct (see, e.g., [3, p. 239–40], if z is sufficiently small and different from zero. It follows from Lemma 3.1 that $A(z)$ is at $z = 0$ holomorphically similar to any matrix $B(z)$ to which it is pointwise similar in a neighborhood of $z = 0$. The simplest such matrix is

$$B(z) = \begin{bmatrix} 0\ 1\ 0 & \dots & 0 & 0 \\ 0\ 0\ 1 & \dots & 0 & 0 \\ .\ \ .\ \ . & \dots & & \\ 0\ \ .\ \ . & \dots & 0 & 1 \\ a_n(z)z & \dots & a_2(z)z & 0 \end{bmatrix}. \tag{4.4}$$

In other words, there exists a matrix $Q(z)$, holomorphic and not singular at $z = 0$ such that in a neighborhood of $z = 0$

$$Q^{-1}(z)A_0(z)Q(z) = B(z). \tag{4.5}$$

Let us assume that z_1 has been taken so small that $|z| \leqq z_1$ s such a neighborhood.

The transformation

$$y = Q(z)y^* \tag{4.6}$$

transforms the equation (1.1) into a problem with the same properties for which the leading matrix is $B(z)$ instead of $A_0(z)$. Again we assume without loss of generality that this transformation has already been performed, i.e., *we assume that*

$$A_0(z) = B(z) \tag{4.7}$$

and otherwise keep the same notations as before.

5. **Description of the method.** A transformation of the form

$$y = P(z,\varepsilon)u \tag{5.1}$$

changes the differential equation (1.1) into

$$\varepsilon\frac{du}{dz} = P^{-1}(AP - \varepsilon P')u \qquad \left(P' = \frac{dP}{dz}\right). \tag{5.2}$$

Our ultimate aim is to determine P in such a way that the matrix in the right member of the new equation (5.2) is independent of ε. More specifically, we wish to solve the equation

$$P^{-1}(AP - \varepsilon P') = B,$$

or equivalently, we wish to satisfy the equation

$$\varepsilon P'(z,\varepsilon) = A(z,\varepsilon)P(z,\varepsilon) - P(z,\varepsilon)B(z) \tag{5.3}$$

by a nonsingular matrix $P(z,\varepsilon)$. This transformation should be valid in the region (1.2) or, at least, in a subregion obtained by decreasing the constants z_1, ε_1, θ_1 in (1.2).

As in most procedures in this field of mathematics, the argument is divided into two parts, one "formal," the other analytic. In the formal part we replace $P(z,\varepsilon)$ in (5.3) tentatively by a power series

$$\sum_{r=0}^{\infty} P_r(z)\varepsilon^r; \tag{5.4}$$

substitute for $A(z,\varepsilon)$ its asymptotic series from (1.3), multiply the series termwise and identify the coefficients of like powers of ε in the two members. The resulting recursion formulas can be written, in view of (4.7),

$$(5.5) \qquad BP_0 - P_0B = 0,$$

$$(5.6) \qquad BP_\mu - P_\mu B = P'_{\mu-1} - \sum_{\nu=1}^{\mu} A_\nu P_{\mu-\nu}, \qquad \mu > 0.$$

In the next two sections it will be shown that these equations for the $P_\mu(z)$ can be successively solved, even though the linear operator $BX - XB$ on X is singular. In fact, it will even be shown that all $P_\mu(z)$ can be chosen so as to be holomorphic in a fixed neighborhood of $z = 0$, and that $P_0(0)$ is nonsingular.

The series (5.4) formed with these matrices is, in general, divergent. The analytic part of the argument consists in proving that the series (5.4) represents asymptotically a function $P(z,\varepsilon)$ which actually solves (5.3). It will be given in a separate paper. With this choice of $P(z,\varepsilon)$ the differential equation (5.2) takes on the simple form

$$(5.7) \qquad \varepsilon \frac{du}{dz} = B(z)u.$$

The results of the present paper imply only the weaker form

$$(5.8) \qquad \varepsilon \frac{du}{dz} = [B(z) + \varepsilon^{m+1}\tilde{B}_m(z,\varepsilon)]u.$$

Here m is an arbitrarily large integer, and $\tilde{B}_m(z,\varepsilon)$ is bounded in the region (1.2).

For $n = 2$ a final simple change of variables reduces (5.7) to the system form of Airy's equation, so that a complete asymptotic theory in the neighborhood of $z = 0$ can be based on known properties of Bessel functions. For $n > 2$ the asymptotic solution of equations of the form (5.7) is still a hard, in general unsolved, problem.

6. **Calculation of** $P_0(z)$. The matrix $B(z)$ of (4.4) satisfies the assumptions on $M(z)$ in §3. Hence, by Lemma 3.2, any matrix of the form

$$(6.1) \qquad P_0(z) = \sum_{r=1}^{n} q_r(z)B^{n-r}(z),$$

where the $q_r(z)$ are scalar functions, satisfies equation (5.5).

On the other hand, equation (5.6) for $\mu = 1$ possesses solutions if and only if its right member, as a vector in the n^2-dimensional space with the scalar product defined by (3.8), is orthogonal to the null space of the operator $B^TX - XB^T$ adjoint to $BX - XB$ (see Lemma 3.4). Since the matrices $(B^T)^k$, $k = 0, 1, \ldots, n-1$ form, by Lemma 3.2, a basis for the null space of this adjoint operator, equation (5.6) for $\mu = 1$ will be soluble if and only if $P_0(z)$ satisfies the n equations

$$(6.2) \qquad ((P'_0 - A_1P_0),\ (B^T)^k) = 0, \qquad k = 0, 1, \ldots, n-1.$$

By means of (6.1) and (3.8) these conditions are seen to be equivalent with the differential equations

$$\sum_{r=1}^{n} \operatorname{tr} \left\{ B^{n-r+k} q_r' + [(B^{n-r})' B^k - B^{n-r+k} A_1] q_r \right\} = 0 \tag{6.3}$$

$$k = 0, 1, \ldots, n-1$$

for $q_r(z)$, $r = 1, 2, \ldots, n$. The equations (6.3) can be simplified with the help of the following lemmas.

LEMMA 6.1.

$$\operatorname{tr}(B^j(z)) = \begin{cases} n, & j = 0 \\ j a_j(0) z + O(z^2), & 1 \leqq j \leqq n \\ O(z^2), & j > n. \end{cases}$$

Proof. The case $j = 0$ is obvious. By (4.4) we can write

$$B(z) = J + zE \tag{6.4}$$

where J was defined in (2.6) and

$$E = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ \cdot & \ldots & \cdot & \cdot \\ 0 & \ldots & 0 & 0 \\ a_n(z) & \ldots & a_2(z) & a_1(z) \end{bmatrix}. \tag{6.5}$$

(We recall that $a_1(z) \equiv 0$, by (4.2).) Since the trace of a product of matrices is invariant under a cyclic permutation of these matrices, (6.4) implies

$$\operatorname{tr}(B^j) = j z \operatorname{tr}(J^{j-1} E) + O(z^2). \tag{6.6}$$

Since $J^k = 0$ for $k \geqq n$ the statement of the lemma for $j > n$ follows immediately from (6.6). Furthermore, $J^{j-1}E$ is the matrix obtained from E by shifting the last row into the $(j-1)$st row from the bottom and putting zeros everywhere else. The trace of this matrix for $1 \leqq j \leqq n$ is $a_j(z)$. Insertion of this result into (6.6) completes the proof of the lemma.

LEMMA 6.2.

$$\operatorname{tr}(B^j(z) B'(z)) = \begin{cases} a_{j+1}(0) + O(z), & 0 \leqq j < n, \\ O(z), & j \geqq n. \end{cases}$$

Proof. For $j = 0$ this follows immediately from (6.4). For $j > 0$ we find

$$\operatorname{tr}(B^j B') = \operatorname{tr}(J^j E) + O(z)$$

from which the result follows by an argument similar to that in the preceding Lemma.

Returning to (6.3) we observe that

$$\operatorname{tr}[(B^{n-r})' B^k] = (n-r) \operatorname{tr}(B^{n-r-1+k} B'). \tag{6.6}$$

We insert (6.6) into (6.3) and apply the Lemmas 6.1 and 6.2 to (6.3). If we set, for abbreviation, $q = q(z)$ for the vector with the components $q_r(z)$, $r = 1, 2, \ldots, n$, the expressions

$$\sum_{r=1} \operatorname{tr}(B^{n-r+k})q_r', \qquad k = 1, 2, \ldots, n-1, 0$$

(in this order!) can be combined into the vector

$$S(z)q'(z),$$

where $S(z)$ is a matrix which, in view of Lemma 6.1, has the form

$$S(z) = \begin{bmatrix} na_n(0)z & (n-1)a_{n-1}(0)z & \ldots & 2a_2(0) & a_1(0)z \\ 0 & na_n(0)z & \ldots & 3a_3(0)z & 2a_2(0)z \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \ldots & na_n(0)a & (n-1)a_{n-1}(0)z \\ (n-1)a_{n-1}(0)z & (n-2)a_{n-2}(0)z & \ldots & a_1(0)z & n \end{bmatrix} + O(z^2).$$

Hence, if $D(z)$ is the n-dimensional diagonal matrix

$$D(z) = \operatorname{diag}(1, 1 \ldots 1, za_n(0)), \tag{6.7}$$

we have, since $a_n(0) \neq 0$,

$$D(z)S(z) = zK(I + O(z)), \tag{6.8}$$

where K is the constant, upper triangular matrix

$$K = \begin{bmatrix} na_n(0) & (n-1)a_{n-1}(0) & \ldots & 2a_2(0) & a_1(0) \\ 0 & na_n(0) & \ldots & 3a\ (0) & 2a_2(0) \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \ldots & na_n(0) & (n-1)a_{n-1}(0) \\ 0 & 0 & \ldots & 0 & na_n(0) \end{bmatrix}. \tag{6.9}$$

Now we write (6.3) in the form

$$D(z)S(z)q'(z) = D(z)T(z)q(z), \tag{6.10}$$

where the matrix $T(z)$ has the form

$$T(z) = T_1(z) + T_2(z) \tag{6.11}$$

with

$$T_1(z) = -\{\operatorname{tr}[(B^{n-r})'B^k]\}, \quad T_2 = \{\operatorname{tr}(B^{n-r+k}A_1)\}. \tag{6.12}$$

In (6.12) it is understood that k determines the rows and assumes the values $k=1,2,\ldots,n-1,0$ (in this order), and that r indicates the columns $r=1,2,\ldots,n$.

An application of Lemma 6.2 shows that

$$T_1(z) = -\{(n-r)\operatorname{tr}(B^{n-r+k-1}B')\}$$

$$(6.13)\qquad = -\begin{bmatrix} (n-1)a_n(0) & (n-2)a_{n-1}(0) & \cdots & a_2(0) & 0 \\ 0 & (n-2)a_n(0) & \cdots & a_3(0) & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & a_n(0) & 0 \\ (n-1a_{n-1}(0) & (n-2)a_{n-2}(0) & \cdots & a_1(0) & 0 \end{bmatrix} + O(z).$$

Also

$$(6.14)\qquad T_2(z) = \{\operatorname{tr}(J^{n-r+k}A_1(0))\} + O(z).$$

All elements with $n-r+k \geqq n$ of the first matrix in the right member of (6.14) are zero. Hence, all elements on and below the main diagonal of this first matrix are zero except possibly those in the last row.

From (6.7), (6.11), (6.13) and (6.14), it follows that

$$(6.15)\qquad D(z)T(z) = \begin{bmatrix} (n-1)a_n(0) & & & & \\ 0 & (n-2)a_n(0) & & * & \\ 0 & 0 & \cdot & & \\ & & & \cdot & \\ \cdot & \cdot & & \cdot & \\ 0 & 0 & & a_n(0) & \\ 0 & 0 & \cdots & \cdot & 0 \end{bmatrix} + O(z).$$

The elements above the main diagonal are irrelevant and are only indicated by an asterisk.

Now we insert (6.8) and (6.15) into (6.10) and obtain the differential equation for $q(z)$ in the form

$$(6.16)\qquad zq' = H(z)q,$$

where

$$H(z) = (I + O(z))K^{-1}D(z)T(z).$$

By (6.9) and (6.15) the eigenvalues of $H(0)$ are $0, 1/n, \ldots, (n-1)/n$, and the last row of $H(0)$ is zero. By means of the theory of regular singular points, we conclude that (6.16) has exactly one linearly independent solution that is holomorphic at $z=0$. The leading term $q(0)$ of this solution is determined from the equation $H(0)q(0)=0$. Since the last row of $H(0)$ is zero, $q_n(0)$ can be chosen arbitrarily. Let us set $q_n(0)=1$. With this choice of the vector $q(z)$ the *matrix* $P_0(z)$ *of* (6.1) *is holomorphic and nonsingular in a neighborhood of* $z=0$, *say* $|z|\leqq z_1$.

7. **Calculation of $P_r(z)$, $r>0$.** with the matrix $P_0(z)$ determined as described in §6, the equation (5.6) with $\mu=1$, i.e.,

$$BP_1 - P_1B = P_0' - A_1P_0 \tag{7.1}$$

can be solved for P_1, in $|z|\leqq z_1$. By Lemma 3.2 the rank of the operator $B(z)X - XB(z)$ is constant in a neighborhood of $z=0$ and it follows from Lemma 3.3 that (7.1) possesses a particular solution $\tilde{P}_1(z)$ holomorphic in a neighborhood of $z=0$ depending on $B(z)$ only. Let z_1 be chosen so small that $|z|\leqq z_1$ is such a neighborhood. The general solution of (7.1) is then

$$P_1(z) = \tilde{P}_1(z) + \sum_{r=1}^{n} B^{-r}(z)q_r(z) \tag{7.2}$$

where the functions $q_r(z)$ are arbitrary. In the interest of economy of notation the letter q has been used again, although these $q_r(z)$ are in general not the functions so designated in §6.

If the matrices $P_0(z), P_1(z),\dots,P_{\mu-2}(z)$ have already been determined so as to be holomorphic in $|z|\leqq z_1$ the equation (5.6) can be written

$$BP_\mu - P_\mu B = P_{\mu-1}' - A_1P_{\mu-1} - F_\mu, \tag{7.3}$$

where

$$F_\mu = F_\mu(z) = \begin{cases} 0, & \mu=1 \\ \sum_{\nu=2}^{\mu} A_\nu(z)P_{\mu-\nu}(z), & \mu>1 \end{cases} \tag{7.4}$$

is a known holomorphic function. As the first step of a proof by induction we determine the $q_r(z)$ in (7.2) so that the compatibility conditions

$$(P_{\mu-1}' - A_1P_{\mu-1} - F_\mu, (B^T)^k) = 0, \quad k=1,2,\dots,n-1,0 \tag{7.5}$$

are satisfied for $\mu=2$. Inserting (7.2) into (7.5) we obtain the conditions

$$\sum_{r=1}^{n} \operatorname{tr}[B^{n-r+k}q_r' + ((B^{n-r})'B^k - B^{n-r+k}A_1)q_r] = \phi_{2k}(z), \tag{7.6}$$

where

$$\phi_{2k}(z) = \operatorname{tr}[(F_2 - \tilde{P}_1' + A_1\tilde{P}_1)B^k], \qquad k=1,2,\dots,(n-1),0, \tag{7.7}$$

are n known functions holomorphic in $|z|\leqq z_1$.

Comparison with (6.3) shows that (7.6) is a nonhomogeneous linear system of differential equations whose homogeneous part is the same as for (6.3). Hence, in the notation of §6, and with particular reference to (6.16), the condition (7.6) can be written

$$zq' = H(z)q + \psi_2(z) \tag{7.8}$$

where $\psi_2(z)$ is a vector function holomorphic in $|z| \leqq z_1$ of the form

$$\psi_2(z) = (I + O(z))K^{-1}D(z)\phi_2(z). \tag{7.9}$$

Here $\phi_2(z)$ is the vector with components $\phi_{2k}(z)$, $k = 1, 2, \ldots, n-1, 0$. Now, the last row of $D(0)$ is zero and K^{-1} is upper-triangular. Hence, *the last component of $\psi_2(0)$ is zero.*

The last mentioned fact enables us to satisfy (7.8) formally by a power series. In fact, set

$$q(z) = \sum_{r=0}^{\infty} p_r z^r, \quad \psi_2(z) = \sum_{r=0}^{\infty} \psi_{2r} z^r, \quad H(z) = \sum_{r=0}^{\infty} H_r z^r. \tag{7.10}$$

Then insertion into (7.8) and comparison of coefficients leads to the recursion formulas

$$H_0 p_0 + \psi_{20} = 0, \tag{7.11}$$

$$(H_0 - rI)p_r + \psi_{2r} + \sum_{\mu=1}^{r} H_\mu p_{r-\mu} = 0, \qquad r > 0. \tag{7.12}$$

The last row of H_0 as well as the last component of ψ_{20} are zero. Hence (7.11) possesses nontrivial solutions. For $r > 0$ the matrix $H_0 - rI$ is nonsingular, because the eigenvalues of H_0 are ν/n, $\nu = 0, 1, \ldots, n-1$, as was proved in §6. Therefore the p_r can be calculated successively from (7.12).

Now it must be shown that the series for $q(z)$ so obtained converges. This follows from the following lemma.

LEMMA 7.1. *Let $C(z)$ be an n-by-n matrix and $f(z)$ a vector, both holomorphic for $|z| < z_1$. If the vectorial power series*

$$\sum_{r=0}^{\infty} u_r z^r \tag{7.13}$$

satisfies the differential equation

$$zu' = C(z)u + f(z) \tag{7.14}$$

formally, then the series (7.13) *converges in $|z| < z_1$ and represents there a solution of* (7.14).

Proof. The Lemma is essentially a corollary of well-known properties of linear systems with a regular singular point, for which we refer to [1, pp. 116–129]. The homogeneous equation possesses a fundamental matrix of the form $P(z)z^R$, where $P(z)$ is holomorphic and nonsingular in $|z| < z_1$ and R is a constant matrix. Hence, all solutions of (7.14) are of the form

$$P(z)z^R\left[c + \int_\alpha^z t^{-R-I}P^{-1}(t)f(t)dt\right], \tag{7.15}$$

where α is an arbitrary complex number with $0 < |\alpha| < z_1$ and c an arbitrary constant vector. The components of the vector (7.15) can be expanded in convergent "logarithmic sums" (cf. [1, p. 116]). Let $v(z)$ be a particular vector function of the form (7.15) and denote by $\hat{v}(z)$ the logarithmic series representing $v(z)$. Then termwise subtraction of the series $\sum_{r=0}^{\infty} u_r z^r$ from the series $\hat{v}(z)$ yields a formal logarithmic series $\hat{w}(z)$ that satisfies the homogeneous differential equation in the formal sense. According to [1, p. 117] the series $\hat{w}(z)$ is convergent and represents a solution $w(z)$ of the homogeneous equation. It follows that $v(z) - w(z)$ is a solution of (7.14) with a convergent series expansion which is obtained by subtracting the series $\hat{w}(z)$ termwise from the series $\hat{v}(z)$. But this gives us back the series $\sum_{r=0}^{\infty} u_r z^r$. Thus the lemma is proved.

The particular holomorphic solution $q(z)$ of (7.8), when inserted into (7.2) gives us a solution of (7.3) for $\mu = 1$ such that (7.3) is soluble for $\mu = 2$. The further procedure, by induction, is now quite straightforward, so that only a sketchy description is necessary: Assume that $P_0, P_1, \dots, P_{s-1}$ have been determined as holomorphic functions in $|z| < z_1$ so that equations (7.3) are satisfied for $\mu = 1, 2, \dots, s-1$ and the compatibility conditions (7.5) for $\mu = 1, 2, \dots, s$. If $\tilde{P}_s(z)$ is a particular solution of (7.3) for $\mu = s$, holomorphic in $|z| < z_1$, the general solution of (7.3) is of the form

$$P_s(z) = \tilde{P}_s(z) + \sum_{r=1}^{n} B^{n-r}(z) q_r(z)$$

with arbitrary scalars $q_r(z)$. The $q_r(z)$ can be determined as holomorphic functions so that the compatibility conditions for $\mu = s + 1$ are satisfied, and the induction is completed.

8. **Transformations of the independent variable.** Here we shall show that no generality is lost if we assume that

$$a_n(z) \equiv 1. \tag{8.1}$$

Consider the differential equation (1.1) with a leading matrix $A_0(z)$ of the form (4.4). Let $\omega(z)$, $\alpha(z)$ be scalar functions holomorphic in $|z| \leqq z_1$ such that

$$\omega(0) \neq 0, \qquad \alpha(0) = 0, \qquad \alpha'(0) \neq 0, \tag{8.2}$$

and perform the transformation

$$y = \operatorname{diag}[1, \omega(z), \omega^2(z), \dots, \omega^{n-1}(z)]v, \tag{8.3}$$

$$t = \alpha(z). \tag{8.4}$$

The resulting differential equation

$$\varepsilon \frac{dv}{dt} = C(t,\varepsilon)v \tag{8.5}$$

has a coefficient matrix that possesses an asymptotic expansion

$$C(t,\varepsilon) \sim \sum_{r=0}^{\infty} C_r(t)\varepsilon^r, \qquad \text{as} \quad \varepsilon \to 0. \tag{8.6}$$

In particular,

$$C_0(t) = \frac{dz}{dt}\begin{bmatrix} 0 & \omega(z) & 0 & \dots & 0 \\ 0 & 0 & \omega(z) & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & \omega(z) \\ d_n(t) & d_{n-1}(t) & d_{n-2}(t) & \dots & 0 \end{bmatrix}$$

where

$$d_j(t) = \omega^{1-j}(z)\, z\, a_j(z), \qquad j = 2, \dots, n. \tag{8.7}$$

Hence, if $\omega(z)$ and $\alpha(z)$ can be determined so that

$$\omega^{1-n}(z)\, z\, a_n(z)\frac{dz}{dt} = t \tag{8.8}$$

and

$$\frac{dz}{dt}\omega(z) = 1, \tag{8.9}$$

$C_0(t)$ takes the form

$$C_0(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 \\ t & tc_{n-1}(t) & tc_{n-2}(t) & \dots & tc_2(t) & 0 \end{bmatrix}$$

with the $c_j(t)$ holomorphic at $t = 0$. If $\alpha(z)$ and $\omega(z)$ satisfy (8.9), formula (8.8) becomes the differential equation

$$z\, a_n(z) = t\left(\frac{dt}{dz}\right)^n$$

for the function $t = \alpha(z)$. It is seen that

$$t = \alpha(z) = \left[\frac{n+1}{n}\int_0^z \left(\xi a_n(\xi)\right)^{1/n} d\xi\,\right]^{n/(n+1)} \tag{8.10}$$

is a particular solution.

Since $a_n(0) \neq 0$, the integrand is of the form

$$\xi^{1/n}[a_n(0)]^{1/n}\phi(\xi)$$

with $\phi(\xi)$ holomorphic at $\xi = 0$, and $\phi(0) = 1$. It follows that

$$\alpha(z) = [a_n(0)]^{1/(n+1)} z\psi(z),$$

with $\psi(z)$ holomorphic at $z = 0$, and $\psi(0) = 1$. Thus, the function in the right member of (8.10) does not have a branch point at $z = 0$ but describes n distinct holomorphic functions that are all constant multiples of each other. With any one of these determinations, $\alpha(z)$, as defined by (8.10), together with $\omega(z) = \alpha'(z)$, satisfies the conditions (8.8), (8.9).

9. **Summary.** By virtue of a theorem of Sibuya [5], the asymptotic theory of the differential equation (1.1) can always be reduced to the case that $A(0,0)$ is a nilpotent Jordan matrix. This is achieved by means of the transformations (2.2) and (2.4).

If Assumptions A and B of §2 are satisfied, then a transformation

$$y = \left(\sum_{r=0}^{m} P_r(z)\varepsilon^r\right)u, \quad \det P_0(z) \neq 0 \text{ in } |z| \leqq z_1$$

can be found which reduces (1.1) to the simpler form (5.8). The matrix $B(z)$ is defined by formula (4.4). The $P_r(z)$ are certain solutions of (5.5) and (5.6), holomorphic in $|z| \leqq z_1$. The positive integer m is arbitrary.

By means of the transformations (8.8) and (8.9) with $\alpha(z)$ and $\omega(z)$ defined by (8.10) and (8.9) one can always achieve that $a_n(z) = z$.

Bibliography

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

2. F. R. Gantmacher, *The theory of matrices*, Chelsea, New York, 1959.

3. E. Goursat, *A course in mathematical analysis*, V. II, Part I, Functions of a complex variable, Ginn and Co., Boston, 1916.

4. R. E. Langer, *Turning points in linear asymptotic theory*, Bol. Soc. Mat. Mexicana (2) **5** (1960), 1–12.

5. Y. Sibuya, *Sur réduction analytique d'un système d'équations différentielles ordinaires linéaires contenant un paramètre*, J. Fac. Sci. Univ. Tokyo. Sect. I, **7** (1958), 525–540.

6. W. Wasow, *Turning point problems for systems of linear differential equations.* I. *The formal theory*, Comm. Pure Appl. Math. **14** (1961), 657–673.

7. ———, *Turning point problems for certain systems of linear differential equations*, MRC Technical Summary Report No. 184, Mathematics Research Center, U. S. Army, University of Wisconsin, 1960. (This is a more detailed version of [6].)

8. ———, *On holomorphically similar matrices*, J. Math. Anal. and Appl. **4** (1962), 202–206.

University of Wisconsin,
Madison, Wisconsin