SOME CALCULATIONS OF HOMOTOPY GROUPS OF SYMMETRIC SPACES

BY

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Introduction. We calculate the first few unstable homotopy groups of the symmetric spaces $\Gamma_n = SO_{2n}/U_n$ and $X_n = U_{2n}/Sp_n$ and of Sp_n. The homotopy groups of Γ_n are needed in studying the existence of almost complex structures and knowledge of the first unstable group $\pi_{2n-1}(\Gamma_n)$ is used in a paper of W. S. Massey [6]; in fact it was Professor Massey who first suggested to us the calculation of $\pi_{2n-1}(\Gamma_n)$ for $n \equiv 0 \pmod 4$ (the other three parities of n are worked out by him), and suggested to us the use of some fibrations involving Γ_n, or X_n, and spheres. Similarly, X_n is connected with "almost quaternion" structures. We rely heavily on Kervaire's calculations [4].

The space X_n possesses an involution σ, induced by the involutory automorphism of U_{2n} leaving Sp_n fixed. This automorphism of U_{2n} extends to an inner automorphism of SO_{4n} and so induces a map σ of period two on Γ_{2n}. We also study the effect of σ on homotopy groups; this is useful information, as shown in [2; 3].

The results are summarized in the following tables (the precise definition of σ and other notation will be given following the tables):

The groups $\pi_{2n+r}(\Gamma_n)$:

$r \diagdown n$	$4k$	$4k+1$	$4k+2$	$4k+3$	$(k>0)$
-1	$Z + Z_2$	$Z_{(n-1)!}$	Z	$Z_{(n-1)!/2}$	
0	$Z_2 + Z_2$	0	Z_2	0	
1	$Z_{n!} + Z_2$	Z	$Z_{n!}$ or $Z_{n!/2} + Z_2$	$Z + Z_2$	
3	Z				

If $n = 4k$ or $4k+2$, then σ is the identity except for the cases $r = 1$, $n = 4k$ or $4k+2$. The effect of σ on some of the other cases is also determined.

The groups $\pi_{4n+r}(X_n)$:

$r \diagdown n$	$2k$	$2k+1$	$(k>0)$
0	$Z_{(2n)!}$	$Z_{(2n)!/2}$	
1	Z_2		
5		Z_2	

$\sigma = -1$ in all cases (i.e., $\sigma(x) = -x$).

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The groups $\pi_{4n+r}(Sp_n)$:

$r \diagdown n$	$2k$	$2k+1$	$(k>0)$
2	$Z_{(2n+1)!}$	$Z_{2[(2n+1)!]}$	
3	Z_2	Z_2	
4		Z_2	

Notations. U_n is imbedded in SO_{2n} as the subset of matrices consisting of 2×2 blocks

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Let K_{2n} denote the $2n \times 2n$ matrix having alternately $+1$, -1, down the main diagonal, and zeros elsewhere. K_{2n} belongs to SO_{2n} if and only if n is even. Conjugation by K_{2n} induces an automorphism σ in SO_{2n}, and induces the complex conjugation map in U_n (if the 2×2 block represents the complex number $a + ib$). The induced map in $SO_{2n}/U_n = \Gamma_n$ is also written σ. SO_{2n} is imbedded in SO_{2n+r} as the upper left hand block. Conjugation by K_{2n+2} in SO_{2n+2} maps $U_n, U_{n+1}, SO_{2n}, SO_{2n+1}$ into themselves and induces σ in U_n, SO_{2n}. Denote by σ again the induced map of SO_{2n+1}. The induced map σ in $SO_{2n}/U_n = \Gamma_n$, SO_{2n+1}/U_n, $SO_{2n+2}/U_{n+1} = \Gamma_{n+1}$ is compatible with the natural maps

$$\Gamma_n \subset SO_{2n+1}/U_n \to \Gamma_{n+1}.$$

The natural map $SO_{2n+1}/U_n \to SO_{2n+2}/U_{n+1} = \Gamma_{n+1}$ is 1-1 and onto (the two manifolds having the same dimension) and will be used to identify these spaces. The fibration

$$SO_{2n}/U_n \to SO_{2n+1}/U_n \to S^{2n}$$

can then be written as $\Gamma_n \to \Gamma_{n+1} \to S^{2n}$. The induced map σ on S^{2n} is of degree $(-1)^n$.

Sp_m is the subset of U_{2m} of fixed points of the automorphism $\tau: A \to J^{-1}\bar{A}J$ where $\bar{A}$ denotes the complex conjugate matrix, and J is the $2m \times 2m$ matrix with blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

down the main diagonal and zeros elsewhere. Since $J \in U_{2m}$, this automorphism is homotopic to the complex conjugation automorphism σ. Extend τ to U_{2m+1} by the formula

$$B \to J_1^{-1}\bar{B}J_1 \text{ where } J_1 \text{ is the } (2m+1) \times (2m+1)$$

matrix consisting of J in the upper left hand block, 1 in the lower right hand corner, zeros elsewhere.

If τ is defined on U_{2m+2} by the same formula as on U_{2m}, and $U_{2m} \to^{i} U_{2m+1} \to^{j} U_{2m+2}$ denote inclusions, then $\tau i = i\tau$ and τj is homotopic to $j\tau$.

Finally, τ induces involutions on $X_m = U_{2m}/Sp_m$, $X_{m+1} = U_{2m+2}/Sp_{m+1}$, and U_{2m+1}/Sp_m, and the natural maps between these spaces commute with τ up to homotopy. Just as for Γ_n, we have a natural homeomorphism $U_{2m+1}/Sp_m \to X_{m+1}$, and a fibration

$$X_m \to X_{m+1} \to S^{4m+1}.$$

The induced map τ on S^{4m+1} has degree (-1). In the future we shall not distinguish the various homotopic maps defined by τ.

Calculations of the groups $\pi_i(\Gamma_n)$. The first unstable homotopy group of Γ_n is $\pi_{2n-1}(\Gamma_n)$. For $i < 2n-1$, $\pi_i(\Gamma_n) \approx \pi_{i+1}(SO(l))$ (l large).

For convenience, *we will assume always that* $n \equiv 0 \pmod 4$, $n \neq 0$, and calculate the homotopy groups of Γ_{n+r}, $0 \leqq r \leqq 3$.

The only difficult calculation is the following:

THEOREM. $\pi_{2n-1}(SO_{2n}/U_n) = Z + Z_2$, *with* $\sigma =$ *identity* ($n \equiv 0 \pmod 4$, $n \neq 0$).

Proof. We need the following lemma (compare [5]):

LEMMA. *Let* $j: U_n \to SO_{2n}$ *be the inclusion described above, and* $k: SO_{2n} \to U_{2n}$ *the natural inclusion. Under the composite map* kj, *a generator of the group* $\pi_{2n-1}(U_n) = Z$ *goes into* twice *a generator of* $\pi_{2n-1}(U_{2n}) = Z$.

Proof of lemma. We will show that if A is an $n \times n$ matrix in U_n, then $kj(A)$ is conjugate to the $2n \times 2n$ matrix

$$\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}.$$

Recall that the map j consists of replacing the entries $a_{ij} = b_{ij} + (-1)^{1/2} c_{ij}$ of A by 2×2 blocks. If M denotes the matrix with entries

$$\begin{aligned} M_{ij} &= \delta_{2i-1,j} && \text{for } 1 \leqq i \leqq n, \\ &= \delta_{2(i-n),j} && \text{for } n < i \leqq 2n, \end{aligned}$$

and N the matrix

$$\frac{1}{2^{1/2}} \begin{pmatrix} I_n & -(-1)^{1/2} I_n \\ -(-1)^{1/2} I_n & I_n \end{pmatrix}$$

(both are in U_{2n}) then

$$NM(kj(A))M^{-1}N^{-1} = \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \in U_{2n}.$$

If i is the usual inclusion of U_n in U_{2n},

$$i(A) = \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}$$

and

$$i'(A) = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}$$

so that i' is homotopic to i, then $kj(A)$ is homotopic to $i(A)\overline{i'(A)}$ or to $i(A)\overline{i(A)}$. Thus if $x \in \pi_{2n-1}(U_n)$ then

$$kj(x) = i(x) + \sigma i(x).$$

But $\sigma i(x) = i(x)$, and $kj(x) = 2i(x)$, since $\pi_{2n-1}(U_{2n}) \to \pi_{2n-1}(SO_{4n})$ is a monomorphism ($\pi_{2n}(SO_{4n}/U_{2n}) = Z_2$ for $n \equiv 0 \pmod 4$)), and σ is inner in SO_{4n}.

Finally, $i: \pi_{2n-1}(U_n) \to \pi_{2n-1}(U_{2n})$ is an isomorphism, and the conclusion of the lemma follows. Q.E.D. for the lemma.

Next we consider the exact sequence

$$\pi_{2n-1}(SO_{2n-1}) \xrightarrow{i} \pi_{2n-1}(SO_{2n}) \xrightarrow{p} \pi_{2n-1}(S^{2n-1}) \to \pi_{2n-2}(SO_{2n-1}) = Z_2,$$

namely (see [4]),

$$0 \to Z \xrightarrow{i} Z + Z \xrightarrow{p} Z \to Z_2 \to 0.$$

Let x generate $\pi_{2n-1}(SO_{2n-1})$, y and z generate $Z + Z = \pi_{2n-1}(SO_{2n})$ and ι_{2n-1} generate $\pi_{2n-1}(S^{2n-1}) = Z$.

Let $T: S^{2n-1} \to SO_{2n}$ be the characteristic map [7, §23], and R the automorphism of period 2 in SO_{2n} leaving SO_{2n-1} pointwise fixed and inducing a map R of degree -1 in S^{2n-1}. If $s \in S^{2n-1}$, $s = p(A)$ for $A \in SO_{2n}$, then $T(s) = AR(A)^{-1}$. Hence $RT(s) = T(s)^{-1}$ and $RT(\iota_{2n-1}) = -T(\iota_{2n-1})$.

Also $pT(\iota_{2n-1}) = 2T(\iota_{2n-1})$ generates the image of p in $\pi_{2n-1}(S^{2n-1})$. Thus $\pi_{2n-1}(SO_{2n})$ is the direct sum of Image i and the subgroup generated by $T(\iota_{2n-1})$, so we may take $y = i(x)$, $z = T(\iota_{2n-1})$ and so $R(z) = -z$, $R(y) = y$. We note that under $k: SO_{2n} \to U_{2n}$, z maps into zero, since R becomes inner in U_{2n}, and $\pi_{2n-1}(U_{2n}) = Z$, (for, $k(z) = Rk(z) = kR(z) = -k(z)$).

Now consider the (commutative) diagram

$$\begin{tikzcd} \pi_{2n-1}(U_n) \arrow[r, "j"] \arrow[dr, "p'"'] & \pi_{2n-1}(SO_{2n}) \arrow[d, "p"] \\ & \pi_{2n-1}(S^{2n-1}). \end{tikzcd}$$

We may choose the generator x of $\pi_{2n-1}(U_n)$ so that $p'(x) = (n-1)!\,\iota_{2n-1}$ (since $\pi_{2n-2}(U_{n-1}) = Z_{(n-1)!}$), and, if $j(x) = ry + sz$ then $s = (n-1)!/2$, (since $p(z) = 2\iota_{2n-1}$), $p(y) = 0$, $pj = p'$.

Next we show that $r = 2$; for, under

$$\pi_{2n-1}(U_n) \xrightarrow{j} \pi_{2n-1}(SO_{2n}) \xrightarrow{k} \pi_{2n-1}(U_{2n}),$$

$kj(x) = k(ry + (n-1)!/2z) = rk(y) =$ twice a generator of $\pi_{2n-1}(U_{2n})$; however $k(y)$ is a generator, and k is onto (since k followed by the isomorphism $\pi_{2n-1}(U_{2n}) \to \pi_{2n-1}(U_{2n+1})$ equals the composition of $\pi_{2n-1}(SO_{2n}) \to \pi_{2n-1}(SO_{2n+1})$, which is an epimorphism, and $\pi_{2n-1}(SO_{2n+1}) \to \pi_{2n-1}(U_{2n+1})$, which is also an epimorphism since the stable group $\pi_{2n-1}(U_{2n+1}/SO_{2n+1})$ is zero) so that $r = 2$.

Thus the cokernel of j is isomorphic to $Z + Z_2$. However $\pi_{2n-1}(SO_{2n}) \to \pi_{2n-1}(\Gamma_n)$ is onto, since $\pi_{2n-2}(U_n)$ is zero. Thus $\pi_{2n-1}(\Gamma_n)$ is isomorphic to the cokernel of j; further, $\sigma =$ identity on it, since $\sigma =$ identity on $\pi_i(SO_{2n})$ for even n. This concludes the proof.

The values for $\pi_{2n+1}(\Gamma_{n+1})$, $\pi_{2n+3}(\Gamma_{n+2})$, $\pi_{2n+5}(\Gamma_{n+3})$ are computed in [6], and it only remains to determine the value of σ on these groups (we do not settle the case $\pi_{2n+1}(\Gamma_{n+1})$).

For any integer e, we have an exact sequence

$$\pi_{2e-1}(SO_{2e}) \to \pi_{2e-1}(\Gamma_e) \to \pi_{2e-2}(U_e) = 0.$$

Hence it suffices to determine σ on $\pi_{2e-1}(SO_{2e})$. If e is even, $\sigma =$ identity. If $e = n + 3$, the exact sequence

$$\pi_{2n+5}(SO_{2n+6}) \xrightarrow{P} \pi_{2n+5}(S^{2n+5}) \to \pi_{2n+4}(SO_{2n+5})$$

or,

$$0 \to Z \xrightarrow{P} Z \to Z_2 \to 0$$

and the fact that σ on S^{2n+5} has degree -1, shows that $\sigma = -1$ on $\pi_{2n+5}(SO_{2n+6})$ and also on $\pi_{2n+5}(\Gamma_{n+3})$.

If $e = n + 1$, $\pi_{2n+1}(SO_{2n+2})$ is $Z + Z_2$ and σ sends the generator of Z into its negative *or* its negative + the element of order two.

We note for future use that $\sigma = -1$ on $\pi_{4k}(U_{2k})$ and $\pi_{4k}(U_{2k-1})$, and $\sigma = +1$ on $\pi_{4k+2}(U_{2k+1})$: for the exact sequence

$$\pi_{4k+1}(S^{4k+1}) \to \pi_{4k}(U_{2k}) \to \pi_{4k}(U_{2k+1}) = 0$$

and the fact that σ has degree -1 on S^{4k+1}, shows that $\sigma = -1$ on $\pi_{4k}(U_{2k})$. Also, under inclusion $\pi_{4k}(U_{2k-1})$ maps monomorphically into $\pi_{4k}(U_{2k})$. Since $\sigma = +1$ on S^{4k+3}, $\sigma = +1$ on $\pi_{4k+2}(U_{2k+1})$.

The rest of the groups $\pi_i(\Gamma_n)$ now follow; we denote by n always a positive integer $\equiv 0 \bmod 4$.

1. $\pi_{2n}(\Gamma\ ) = Z_2 + Z_2$, $\sigma =$ identity.

Proof. The exact sequence

$$\pi_{2n+1}(S^{2n}) \to \pi_{2n}(\Gamma_n) \to \pi_{2n}(\Gamma_{n+1}) \to \pi_{2n}(S^{2n})$$

or

$$Z_2 \to \pi_{2n}(\Gamma_n) \to Z_2 \to Z$$

shows that $\pi_{2n}(\Gamma_n)$ has order 2 or 4. In the exact sequence

$$\pi_{2n}(U_n) \xrightarrow{i} \pi_{2n}(SO_{2n}) \xrightarrow{P} \pi_{2n}(\Gamma_n) \xrightarrow{\partial} \pi_{2n-1}(U_n)$$

$$Z_{n!} \xrightarrow{i} Z_2 + Z_2 + Z_2 \xrightarrow{P} \pi_{2n}(\Gamma_n) \xrightarrow{\partial} Z$$

the image of i is cyclic, hence 0 or Z_2. But ∂ is zero, hence $\pi_{2n}(\Gamma_n)$ has order 4 or 8. Finally $\pi_{2n}(\Gamma_n) = Z_2 + Z_2$, and $\sigma = +1$ (since $\sigma = +1$ on $\pi_{2n}(SO_{2n})$).

We note also that since i has image Z_2, $\partial: \pi_{2n+1}(\Gamma_n) \to \pi_{2n}(U_n)$ has cokernel Z_2, i.e., image of ∂ is $2Z_{n!}$.

2. $\pi_{2n+1}(\Gamma_n) = Z_{n!} + Z_2$.

Proof. From the exact sequence

$$\pi_{2n+2}(S^{2n}) \to \pi_{2n+1}(\Gamma_n) \to \pi_{2n+1}(\Gamma_{n+1}) \to \pi_{2n+1}(S^{2n})$$

$$Z_2 \to \pi_{2n+1}(\Gamma_n) \to Z_{n!} \to Z_2$$

we see that $\pi_{2n+1}(\Gamma_n)$ has order $\leqq 2(n!)$.

From the exact sequence

$$\pi_{2n+1}(U_n) \to \pi_{2n+1}(SO_{2n}) \xrightarrow{P} \pi_{2n+1}(\Gamma_n) \xrightarrow{\partial} \pi_{2n}(U_n)$$

and the remark at the end of 1, we get

$$Z_2 \to Z_2 + Z_2 + Z_2 \xrightarrow{P} \pi_{2n+1}(\Gamma_n) \to 2Z_{n!} \to 0.$$

Thus $\pi_{2n+1}(\Gamma_n)$ has order at least $2(n!)$, therefore exactly $2(n!)$. Furthermore it is not a cyclic group since image of $P = Z_2 + Z_2$ is not cyclic. Thus $\pi_{2n+1}(\Gamma_n) = Z_{n!} + Z_2$. Since $\sigma = -1$ on $\pi_{2n}(U_n)$ σ is, at least, different from the identity on $\pi_{2n+1}(\Gamma_n)$.

3. $\pi_{2n+2}(\Gamma_{n+1}) = 0 = \pi_{2n+6}(\Gamma_{n+3})$.

Proof. Let $m = n+1$ or $n+3$. The exact sequence

$$\pi_{2m+1}(S^{2m}) \to \pi_{2m}(SO_{2m}) \xrightarrow{j} \pi_{2m}(SO_{2m+1}) \to \pi_{2m}(S^{2m})$$

reduces to

$$Z_2 \to Z_4 \xrightarrow{j} Z_2 \to 0.$$

Consider next

$$\begin{tikzcd} & \pi_{2m+1}(S^{2m+1}) \arrow[r, "\partial"] \arrow[d, "\partial'"] & \pi_{2m}(U_m) = Z_{m!} \arrow[dl, "k"] \\ Z_4 = \pi_{2m}(SO_{2m}) \arrow[r, "j"] & \pi_{2m}(SO_{2m+1}) = Z_2. & \end{tikzcd}$$

Here ∂' is onto, since $\pi_{2m}(SO_{2m+2}) = 0$ for $2m \equiv 2$ or $6 \bmod 8$; hence k is onto. However k factors:

$$\begin{tikzcd} & \pi_{2m}(U_m) = Z_{m!} \arrow[dl, "e"'] \arrow[dr, "k"] & \\ \pi_{2m}(SO_{2m}) \arrow[rr, "j"'] & & \pi_{2m}(SO_{2m+1}), \end{tikzcd}$$

$k = ej$. Since $\pi_{2m}(SO_{2m}) = Z_4$, $\pi_{2m}(SO_{2m+1}) = Z_2$ and j is onto, the fact that k is onto implies that e is also onto. Finally,

$$\pi_{2m}(U_m) \xrightarrow{e} \pi_{2m}(SO_{2m}) \to \pi_{2m}(\Gamma_m) \to \pi_{2m-1}(U_m) \to \pi_{2m-1}(SO_{2m})$$

gives

$$0 \to \pi_{2m}(\Gamma_m) \to Z = \pi_{2m-1}(U_m) \to \pi_{2m-1}(SO_{2m}).$$

However $\pi_{2m-1}(U_m) = Z \to \pi_{2m-1}(SO_{2m}) = Z$ or $Z + Z_2$ is a monomorphism, since $\pi_{2m-1}(\Gamma_m)$ is finite for $m \equiv 1$ or 3 mod 4. Hence $\pi_{2m}(\Gamma_m) = 0$ if $m = n + 1$ or $n + 3$.

4. $\pi_{2n+4}(\Gamma_{n+2}) = Z_2$.

Proof. From the exact sequence

$$\pi_{2n+5}(S^{2n+4}) = Z_2 \to \pi_{2n+4}(\Gamma_{n+2}) \to \pi_{2n+4}(\Gamma_{n+3})$$

and $\pi_{2n+4}(\Gamma_{n+3}) = \pi_{2n+5}(SO) = 0$, we see that $\pi_{2n+4}(\Gamma_{n+2}) = Z_2$ or 0. From

$$\pi_{2n+4}(U_{n+2}) \to \pi_{2n+4}(SO_{2n+4}) \to \pi_{2n+4}(\Gamma_{n+2})$$

$$Z_{(n+2)!} \to Z_2 + Z_2 \to \pi_{2n+4}(\Gamma_{n+2})$$

we see that $\pi_{2n+4}(\Gamma_{n+2})$ is not zero, hence is Z_2.

5. $\pi_{2n+3}(\Gamma_{n+1}) = Z$, $\pi_{2n+3}(\Gamma_n) = Z$, with $\sigma =$ identity on both.

Proof. In the exact sequence

$$\pi_{2n+4}(SO_{2n+2}) \xrightarrow{i} \pi_{2n+4}(SO_{2n+3}) \to \pi_{2n+4}(S^{2n+2}) \xrightarrow{\partial} \pi_{2n+3}(SO_{2n+2}),$$

namely,

$$Z_{12} \xrightarrow{i} Z_2 \to Z_2 \xrightarrow{\partial} Z.$$

∂ is zero, hence i is zero. Thus the composite map

$$j: \pi_{2n+4}(SO_{2n+2}) \xrightarrow{i} \pi_{2n+4}(SO_{2n+3}) \to \pi_{2n+4}(SO_{2n+4})$$

is also zero.

Next consider the commutative diagram

$$\begin{tikzcd}
\pi_{2n+4}(SO_{2n+2}) \arrow[r, "p"] \arrow[d, "j"] & \pi_{2n+4}(\Gamma_{n+1}) \arrow[r] \arrow[d, "k"] & \pi_{2n+3}(U_{n+1}) = 0. \\
\pi_{2n+4}(SO_{2n+4}) \arrow[r, "p'"] & \pi_{2n+4}(\Gamma_{n+2}) &
\end{tikzcd}$$

Image of $k =$ Image of kp (since p is onto) but $kp = p'j = 0$, so $k = 0$. Finally, the exact sequence

$$\pi_{2n+4}(\Gamma_{n+1}) \xrightarrow{k} \pi_{2n+4}(\Gamma_{n+2}) \to \pi_{2n+4}(S^{2n+2})$$
$$\to \pi_{2n+3}(\Gamma_{n+1}) \to \pi_{2n+3}(\Gamma_{n+2}) \to \pi_{2n+3}(S^{2n+2})$$

becomes

$$0 \to Z_2 \to Z_2 \to \pi_{2n+3}(\Gamma_{n+1}) \to \pi_{2n+3}(\Gamma_{n+2}) \to Z_2\,.$$

Thus $\pi_{2n+3}(\Gamma_{n+1})$ is a subgroup of $\pi_{2n+3}(\Gamma_{n+2}) = Z$, of index two. Since $\sigma = +1$ on $\pi_{2n+3}(\Gamma_{n+2})$, $\sigma = +1$ also on $\pi_{2n+3}(\Gamma_{n+1})$. The exact sequence $\pi_{2n+4}(S^{2n}) = 0 \to \pi_{2n+3}(\Gamma_n) \to \pi_{2n+3}(\Gamma_{n+1}) \to \pi_{2n+3}(S^{2n})$ shows that $\pi_{2n+3}(\Gamma_n) = Z$ with $\sigma = +1$.

6. $\pi_{2n+7}(\Gamma_{n+3}) = Z + Z_2$, $\sigma =$ identity.

Proof. In the exact sequence

$$\pi_{2n+7}(SO_{2n+6}) = Z \xrightarrow{i} \pi_{2n+7}(SO_{2n+7}) = Z \xrightarrow{p} \pi_{2n+7}(S^{2n+6})$$

p is zero [4, Theorem 1], so i is an isomorphism.

Writing $\Gamma_{n+4} = SO_{2n+7}/U_{n+3}$, $\Gamma_{n+3} = SO_{2n+6}/U_{n+3}$, we have a commutative diagram

$$\begin{tikzcd}
\pi_{2n+7}(SO_{2n+6}) \arrow[r, "p_1"'] \arrow[d, "i"'] & \pi_{2n+7}(\Gamma_{n+3}) \arrow[r] \arrow[d, "j"] & \pi_{2n+6}(U_{n+3}) \arrow[d, equal] \\
\pi_{2n+7}(SO_{2n+7}) \arrow[r, "p'"'] \arrow[d] & \pi_{2n+7}(\Gamma_{n+4}) \arrow[r] \arrow[d] & \pi_{2n+6}(U_{n+3}) \\
\pi_{2n+7}(S^{2n+6}) \arrow[r, equal] & \pi_{2n+7}(S^{2n+6})\,. &
\end{tikzcd}$$

p_1, p' are monomorphisms, since $\pi_{2n+3}(U_{n+1}) = 0$, and i is an isomorphism, thus j is a monomorphism. Since $\pi_{2n+7}(\Gamma_{n+4}) = Z + Z_2$, the subgroup $\pi_{2n+7}(\Gamma_{n+3})$ is either Z or $Z + Z_2$.

From the exact sequence

$$\pi_{2n+7}(SO_{2n+7}) \xrightarrow{p'} \pi_{2n+7}(\Gamma_{n+4}) \xrightarrow{\partial'} \pi_{2n+6}(U_{n+3})$$
$$\to \pi_{2n+6}(SO_{2n+7}) = Z_2 \to \pi_{2n+6}(\Gamma_{n+4}) = Z$$

and the fact that image of $\partial' = 2Z_{(n+3)!}$, we see that under p', a generator u of $\pi_{2n+7}(SO_{2n+7})$ maps into $((n+3)!/2)x + y$, where x, y generate Z, Z_2 in $\pi_{2n+7}(\Gamma_{n+4}) = Z + Z_2$. From the diagram

$$\begin{tikzcd}
\pi_{2n+7}(SO_{2n+7}) \arrow[r, "p'"] \arrow[dr, "p"'] & \pi_{2n+7}(\Gamma_{n+4}) \arrow[d, "q"] \\
& \pi_{2n+7}(S^{2n+6}) = Z_2
\end{tikzcd}$$

where $p = 0$ (as remarked at the beginning of the proof) we have $qp'(u) = p(u) = 0$, but

$$qp'(u) = q((n+3)!/2x + y) = q(y)$$

(since $(n+3)!/2$ is even), so finally $q(y) = 0$ and the element y of order 2 is in the image of j: $\pi_{2n+7}(\Gamma_{n+3}) \to \pi_{2n+7}(\Gamma_{n+4})$. Thus $\pi_{2n+7}(\Gamma_{n+3})$ has an element of order 2, and must be $Z + Z_2$. $\sigma = +1$ on it since $\sigma = +1$ on $\pi_{2n+7}(\Gamma_{n+4})$. This concludes the proof of 6.

7. $\pi_{2n+5}(\Gamma_{n+2}) = Z_{(n+2)!}$ *or* $Z_{(n+2)!/2} + Z_2$.

Proof. From the exact sequence

$$\pi_{2n+6}(\Gamma_{n+3}) \to \pi_{2n+6}(S^{2n+4}) \to \pi_{2n+5}(\Gamma_{n+2}) \to \pi_{2n+5}(\Gamma_{n+3})$$
$$\to \pi_{2n+5}(S^{2n+4}) \to \pi_{2n+4}(\Gamma_{n+2}) \to \pi_{2n+4}(\Gamma_{n+3})$$

and $\pi_{2n+6}(\Gamma_{n+3}) = 0 = \pi_{2n+4}(\Gamma_{n+3})$, $\pi_{2n+4}(\Gamma_{n+2}) = Z_2$ we get

$$0 \to Z_2 \to \pi_{2n+5}(\Gamma_{n+2}) \to \pi_{2n+5}(\Gamma_{n+3}) = Z_{(n+2)!/2} \to 0;$$

further, $\sigma = -1$ on $\pi_{2n+5}(\Gamma_{n+3})$, so $\sigma \neq 1$ on $\pi_{2n+5}(\Gamma_{n+2})$.

The groups $\pi_i(X_m)$ **and** $\pi_i(Sp_m)$. For $i < 4k$, $\pi_i(X_k) = \pi_{i+2}(SO_l)$, l large. m will denote an *even* integer, $\geqq 2$. The involution τ described above will be denoted by σ here.

1. $\pi_{4m}(X_m) = Z_{(2m)!}$, with $\sigma = -1$. $\pi_{4m+1}(X_m) = Z_2$.

Proof. From the exact sequence

$$\pi_{4m-1}(Sp_m) \xrightarrow{i} \pi_{4m-1}(U_{2m}) \to \pi_{4m-1}(X_m)$$
$$Z \xrightarrow{i} Z \to Z_2$$

i is a monomorphism.

Hence the sequence

$$\pi_{4m}(Sp_m) \to \pi_{4m}(U_{2m}) \to \pi_{4m}(X_m) \to \pi_{4m-1}(Sp_m) \xrightarrow{i}$$

becomes $0 \to Z_{(2m)!} \to \pi_{4m}(X_m) \to 0$. Thus $\pi_{4m}(X_m) = Z_{(2m)!}$, and $\sigma = -1$ on it, since $\sigma = -1$ on $\pi_{4m}(U_{2m})$.

The exact sequence

$$\pi_{4m+1}(Sp_m) \to \pi_{4m+1}(U_{2m}) \to \pi_{4m+1}(X_m) \to \pi_{4m}(Sp_m)$$
$$0 \to Z_2 \to \pi_{4m+1}(X_m) \to 0$$

shows $\pi_{4m+1}(X_m) = Z_2$.

2. $\pi_{4m+4}(X_{m+1}) = Z_{[2(m+1)]!/2}$, with $\sigma = -1$.

Proof. From the fibrations

$$U_{2m+2} \to U_{2m+3} \to S^{4m+5}$$
$$X_{m+1} \to X_{m+2} \to S^{4m+5}$$

we get the diagram

$$\begin{array}{ccccc}
\pi_{4m+5}(U_{2m+3}) = Z & \xrightarrow{p} & \pi_{4m+5}(S^{4m+5}) & \xrightarrow{\partial} & \pi_{4m+4}(U_{2m+2}) \\
\downarrow p_1 & & \| & & \downarrow \\
\pi_{4m+5}(X_{m+2}) = Z & \xrightarrow{p'} & \pi_{4m+5}(S^{4m+5}) & \xrightarrow{\partial'} & \pi_{4m+4}(X_{m+1}) \\
\downarrow \partial_1 & & & & \\
\pi_{4m+4}(Sp_{m+1}) = Z_2. & & & &
\end{array}$$

∂, ∂' are onto since $\pi_{4m+4}(U_{2m+3}) = 0 = \pi_{4m+4}(X_{m+2})$, ∂_1 is onto since $\pi_{4m+4}(U_{2m+3}) = 0$, so that if u generates $\pi_{4m+5}(U_{2m+3})$, $p_1(u) = 2v$, where v generates $\pi_{4m+5}(X_{m+2})$. If w is a generator of $\pi_{4m+5}(S^{4m+5})$ then

$$p'p_1(u) = p(u) = (2m+2)!w$$

so $2p'(v) = (2m+2)!w$ or $p'(v) = [(2m+2)!/2]w$, and it is clear that $\pi_{4m+4}(U_{2m+2}) \to \pi_{4m+4}(X_{m+1})$ is onto, with kernel Z_2.

Since $\sigma = -1$ on $\pi_{4m+4}(U_{2m+2})$, $\sigma = -1$ on $\pi_{4m+4}(X_{m+1})$ also.

3. $\pi_{4m+6}(Sp_{m+1}) = Z_{2[(2m+3)!]}$, $\pi_{4m+2}(Sp_m) = Z_{(2m+1)!}$.

Proof. Consider the fibrations $Sp_{m+2}/Sp_{m+1} = S^{4m+7} = U_{2m+4}/U_{2m+3}$ and associated diagram

$$\begin{array}{ccccc}
\pi_{4m+7}(Sp_{m+2}) & \xrightarrow{p} & \pi_{4m+7}(S^{4m+7}) & \xrightarrow{\partial} & \pi_{4m+6}(Sp_{m+1}) \\
{\scriptstyle i}\downarrow & & \| & & \downarrow \\
\pi_{4m+7}(U_{2m+4}) & \xrightarrow{p'} & \pi_{4m+7}(S^{4m+7}) & \xrightarrow{\partial'} & \pi_{4m+6}(U_{2m+3}).
\end{array}$$

∂, ∂' are onto since $\pi_{4m+6}(Sp_{m+2}) = 0 = \pi_{4m+6}(U_{2m+4})$. The groups $\pi_{4m+7}(Sp_{m+2})$, $\pi_{4m+7}(U_{2m+4})$, $\pi_{4m+7}(S^{4m+7})$ are all Z, with generators x, y, z. From

$$\begin{aligned}
\pi_{4m+7}(Sp_{m+2}) \xrightarrow{i} \pi_{4m+7}(U_{2m+4}) \to \pi_{4m+7}(X_{m+2}) &= Z_2 \\
\to \pi_{4m+6}(Sp_{m+2}) &= 0
\end{aligned}$$

we see that $i(x) = 2y$, so that $p'i(x) = p(x) = 2p'(y) = 2[(2m+3)!]z$. Hence $\pi_{4m+6}(Sp_{m+1}) = Z_{2[(2m+3)!]}$ and

$$0 \to Z_2 \to \pi_{4m+6}(Sp_{m+1}) \to \pi_{4m+6}(U_{2m+3}) \to 0$$

is an exact sequence.

For $\pi_{4m+2}(Sp_m)$ we use the diagram

$$\begin{array}{ccccc}
\pi_{4m+3}(Sp_{m+1}) & \to & \pi_{4m+3}(S^{4m+3}) & \xrightarrow{\partial} & \pi_{4m+2}(Sp_m) \\
\downarrow{\scriptstyle i} & & \| & & \downarrow{\scriptstyle j} \\
\pi_{4m+3}(U_{2m+2}) & \to & \pi_{4m+3}(S^{4m+3}) & \xrightarrow{\partial'} & \pi_{4m+2}(U_{2m+1}).
\end{array}$$

Again ∂, ∂' are epimorphisms since $\pi_{4m+2}(Sp_{m+1}) = 0 = \pi_{4m+2}(U_{2m+2})$. i is actually an isomorphism since $\pi_{4m+3}(X_{m+1}) = 0$, so j is also an isomorphism.

4. $\pi_{4m+7}(Sp_{m+1}) = Z_2 = \pi_{4m+8}(Sp_{m+1})$.

Proof. Consider the diagram

$$\begin{array}{ccccc}
\pi_{4m+8}(U_{2m+3}) & \xrightarrow{p} & \pi_{4m+8}(X_{m+2}) & \xrightarrow{\partial} & \pi_{4m+7}(Sp_{m+1}) \\
& {\scriptstyle i}\searrow & & \nearrow{\scriptstyle p'} & \\
& & \pi_{4m+8}(U_{2m+4}). & &
\end{array}$$

∂ is onto since $\pi_{4m+7}(U_{2m+3}) = 0$, and p' is an isomorphism, by 1. i is a monomorphism with cokernel Z_2[4, p. 164], so p is a monomorphism with cokernel $Z_2 = \pi_{4m+7}(Sp_{m+1})$.

From the exact sequence

$$\pi_{4m+9}(Sp_{m+2}) \to \pi_{4m+9}(S^{4m+7}) \xrightarrow{\partial} \pi_{4m+8}(Sp_{m+1}) \to \pi_{4m+8}(Sp_{m+2})$$

and the (stable) values $\pi_{4m+9}(Sp_{m+2}) = 0 = \pi_{4m+8}(Sp_{m+2})$ we see that

$$\partial: \pi_{4m+9}(S^{4m+7}) = Z_2 \xrightarrow{\approx} \pi_{4m+8}(Sp_{m+1}).$$

5. $\pi_{4m+9}(X_{m+1}) = Z_2$.

Proof. In the homotopy sequence of the fibration $X_{m+2}/X_{m+1} = S^{4m+5}$, we have $\pi_{4m+9}(S^{4m+5}) = 0 = \pi_{4m+10}(S^{4m+5})$ and $\pi_{4m+9}(X_{m+2}) = Z_2$ (from 1).

6. $\pi_{4m+3}(Sp_m) = Z_2$.

Proof. We have the commutative diagram

$$\begin{array}{ccccc}
\pi_{4m+4}(U_{2m+1}) & \xrightarrow{p} & \pi_{4m+4}(X_{m+1}) & \xrightarrow{\partial} & \pi_{4m+3}(Sp_m) \\
\downarrow i & & \| & & \downarrow \\
\pi_{4m+4}(U_{2m+2}) & \xrightarrow{p'} & \pi_{4m+4}(X_{m+1}) & \to & \pi_{4m+3}(Sp_{m+1}) = Z.
\end{array}$$

Since $\pi_{4m+4}(X_{m+1}) = Z_{(2m+2)!/2}$ is finite, p' is an epimorphism; i is a monomorphism with cokernel Z_2, hence $p = p'i$ has cokernel Z_2. But ∂ is an epimorphism since $\pi_{4m+3}(U_{2m+1}) = 0$, so $\pi_{4m+3}(Sp_m) = Z_2$.

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