THE ASYMPTOTIC DISTRIBUTION OF THE NUMBER OF ZERO FREE INTERVALS OF A STABLE PROCESS

BY

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1. **Introduction.** Let $\{X(t);\ t \geqq 0\}$ be the one-dimensional symmetric stable process of index α, $0 < \alpha \leqq 2$, that is, a process with stationary independent increments whose continuous transition density $f(t,y-x)$ relative to Lebesgue measure is given by

$$(1.1) \qquad f(t,x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-t|\xi|^{\alpha}} e^{ix\xi} d\xi.$$

We assume throughout this paper that $X(0) = 0$ and that the sample functions are normalized to be right continuous and have left-hand limits everywhere.

Let us also introduce the *stable subordinator*, $\{T(t): t \geqq 0\}$, of index β, $0 < \beta < 1$, that is, a process with stationary independent and *positive* increment whose continuous transition density, $g(t,y-x)$, is given by $g(t,x) = 0$ for $x \leqq 0$ and by

$$(1.2) \qquad e^{-ts^{\beta}} = \int_{0}^{\infty} e^{-sx} g(t,x)dx$$

for $x > 0$. We assume $T(0) = 0$ and that the sample functions are normalized to be right continuous and have left-hand limits everywhere. In addition almost all sample functions of T are *strictly* monotone increasing. Finally we assume that the processes X and T are completely independent and are defined over the same complete probability space, $(\Omega, \mathscr{F}, P)$. It is perhaps more reasonable to assume that X and T are defined over different (complete) probability spaces, but for notational convenience we prefer the above assumption.

Define

$$(1.3) \qquad \begin{aligned} A(\omega) &= \{t: 0 \leqq t \leqq 1,\ X(t,\omega) = 0 \text{ or } X(t-,\omega) = 0\}, \\ B(\omega) &= \{t: 0 \leqq t \leqq 1, T(\tau,\omega) = t \text{ or } T(\tau-,\omega) = t \text{ for some } \tau\}. \end{aligned}$$

If $0 < \alpha \leqq 1$ then $A(\omega) = \{0\}$ for almost all ω, and it is not difficult to see that for general α, $0 < \alpha \leqq 2$,

$$A(\omega) = \{t: 0 \leqq t \leqq 1,\ X(t,\omega) = 0\}$$

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for almost all ω. However we will not need this fact and so we omit its proof. In view of our regularity assumptions on the sample functions of X and T, it follows that $A(\omega)$ and $B(\omega)$ are compact subsets of $[0,1]$ for each ω. Therefore the complements of $A(\omega)$ and $B(\omega)$ in $[0,1]$ are relatively open subsets of $[0,1]$, and, as such, each can be written uniquely as the disjoint union of at most countably many (relatively) open subintervals of $[0,1]$. If $\varepsilon>0$, let $N_\alpha(\varepsilon)$ ($N^*_\beta(\varepsilon)$) be the number of such intervals in the complement of $A(\omega)$ ($B(\omega)$) which exceed ε in length. The following two theorems are the main results of the present paper.

THEOREM A. *$N_\alpha(\varepsilon)$ and $N^*_\beta(\varepsilon)$ are random variables and if $\beta = 1 - 1/\alpha$, $1 < \alpha \leqq 2$, then they have the same distribution for each fixed $\varepsilon > 0$.*

THEOREM B. *If $0 < \beta < 1$, then $\lim_{\varepsilon\downarrow 0} P[\Gamma(1-\beta)\varepsilon^\beta N^*_\beta(\varepsilon) \leqq x] = G_\beta(x)$ where $G_\beta(x)$ is a Mittag-Leffler distribution which is uniquely determined by its moments*

$$\int_0^\infty x^n dG_\beta(x) = n![\Gamma(1+n\beta)]^{-1}, \qquad n = 0,1,\ldots. \tag{1.4}$$

The definition of the distribution G_β and the fact that its moments are given by (1.4) *is contained in* [7]. *The fact that G_β is uniquely determined by its moments follows from the criterion on p.* 110 *of* [4].

An immediate consequence of Theorems A and B is the following corollary.

COROLLARY. *If $1 < \alpha \leqq 2$, then*

$$\lim_{\varepsilon\downarrow 0} P[\Gamma(1/\alpha)\varepsilon^{1-1/\alpha}N_\alpha(\varepsilon) \leqq x] = F_\alpha(x) = G_{1-1/\alpha}(x).$$

Of course, in the case $\alpha = 2$ these results are well known. Moreover the above corollary should be compared with the recent result of Kesten [5]. In [5] Kesten obtains the limiting distribution of the number, $N'_\alpha(\varepsilon)$, of intervals of *positivity* of X in $0 \leqq t \leqq 1$ for all α, $0 < \alpha \leqq 2$. We would like to thank Professor Kesten for making his manuscript available to us. In particular, we owe references [4;7] to him.

2. **The proof of Theorem A.** Given a complete probability space $(\Omega, \mathscr{F}, P)$, a function A from Ω to subsets of the real line, R, is said to be a *random set* if

(i) $A(\omega)$ is compact for almost all ω,

(ii) $\{\omega: A(\omega) \subset E\} \in \mathscr{F}$ for all open subsets E of R.

Two random sets A and B (not necessarily defined over the same probability space) are *stochastically equivalent* if for every set E that is a *finite* union of open intervals

$$P\{\omega: A(\omega) \subset E\} = P\{\omega: B(\omega) \subset E\}. \tag{2.1}$$

These definitions were introduced in [2]. A random set, A, is *contained* in the closed interval $[a,b]$ if $A(\omega) \subset [a,b]$ for all ω. If a random set A is contained in $[a,b]$ then $[a,b] - A(\omega)$ is an open subset of $[a,b]$ for almost all ω, and, as such, can be written uniquely as the union of at most countably many disjoint (relatively) open subintervals of $[a,b]$. If $\varepsilon > 0$, let $N_A(\varepsilon)$ be the number of such intervals whose length is greater than ε. Clearly $N_A(\varepsilon)$ is defined and finite for almost all ω.

THEOREM 2.1. *If A is a random set contained in $[a,b]$, then $N_A(\varepsilon)$ is a random variable. If A and B are stochastically equivalent random sets contained in $[a,b]$ then $N_A(\varepsilon)$ and $N_B(\varepsilon)$ have the same distribution for each $\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$ be fixed, and let $k \geqq 1$ be an integer. Let E denote a finite disjoint union of *exactly k closed* intervals $I_1, \cdots, I_k$ each of which has rational end points, is contained in $[a,b]$, and has length greater than ε. Of course, if k is too large there will be no such E's. Let $E_1, E_2, \cdots$ be an enumeration of all such E's; then ($\emptyset$ denotes the empty set)

$$\{\omega: N_A(\varepsilon) \geqq k\} = \bigcup_{n=1}^{\infty} \Delta_n, \tag{2.2}$$

where

$$\Delta_n = \{\omega: E_n \cap A(\omega) = \emptyset\} = \{\omega: A(\omega) \subset E_n^c\}. \tag{2.3}$$

Here E_n^c is the complement of E_n in $[a,b]$ and hence is a finite union of (relatively) open subintervals of $[a,b]$. Clearly this implies that $N_A(\varepsilon)$ is a random variable. Moreover

$$P\{\omega: N_A(\varepsilon) \geqq k\} = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} \Delta_i\right), \tag{2.4}$$

and for fixed n the inclusion-exclusion formula implies that

$$P\left(\bigcup_{i=1}^{n} \Delta_i\right) = \sum P(\Delta_i) - \sum P(\Delta_i \cap \Delta_j) + \cdots.$$

Looking at a typical intersection we see that

$$P(\Delta_i \cap \ldots \cap \Delta_j) = P\{\omega: A(\omega) \subset E_i^c \cap \ldots \cap E_j^c\}.$$

Since $E_i^c \cap \ldots \cap E_j^c$ is a finite union of open intervals (not necessarily disjoint), it follows that if A and B are stochastically equivalent the left side of (2.4) is unchanged if A is replaced by B. Thus $N_A(\varepsilon)$ and $N_B(\varepsilon)$ have the same distribution.

Let a be a real number satisfying $0 \leqq a < 1$, and define

$$\begin{aligned} A_a(\omega) &= \{t: a \leqq t \leqq 1,\ X(t,\omega) = 0 \text{ or } X(t-,\omega) = 0\}, \\ B_a(\omega) &= \{t: a \leqq t \leqq 1,\ T(\tau,\omega) = t \text{ or } T(\tau-,\omega) = t \text{ for some } \tau\}, \end{aligned} \tag{2.5}$$

where X and T are the processes defined in §1. We assume from now on that the index, α, of X satisfies $1 < \alpha \leqq 2$. In particular, A_0 and B_0 are the sets A and B defined in (1.3). It was shown in [2, Proof of Theorem A] that A_a and B_a are stochastically equivalent random sets for each $a > 0$ provided $\beta = 1 - 1/\alpha$. Thus by Theorem 2.1 we see that $N_\alpha(a,\varepsilon) = N_{A_a}(\varepsilon)$ and $N^*_\beta(a,\varepsilon) = N_{B_a}(\varepsilon)$ have the same distribution for each fixed $a > 0$ and $\varepsilon > 0$ provided $\beta = 1 - 1/\alpha$. Since for almost all ω the sets $A(\omega)$ and $B(\omega)$ contain points arbitrarily close to 0 (this is an immediate consequence of Lemma 3.1 of [2]), it follows that $N_\alpha(a,\varepsilon) \to N_\alpha(\varepsilon)$ and $N^*_\beta(a,\varepsilon) \to N^*_\beta(\varepsilon)$ as $a \to 0$ for almost all ω. In fact for a sufficiently small (depending on ε and ω) we have $N_\alpha(a,\varepsilon,\omega) = N_\alpha(\varepsilon,\omega)$ and similarly for N^*. Thus $N_\alpha(\varepsilon)$ and $N^*_\beta(\varepsilon)$ have the same distribution if $\beta = 1 - 1/\alpha$, and Theorem A is proved.

3. **First passage times.** In this section we give a preliminary calculation that will be needed in the proof of Theorem B. Let $T = \{T(t);\ t \geqq 0\}$ be the stable subordinator of index β, $0 < \beta < 1$, and we assume in this section that $T(0) = 0$. Let us recall the Ito representation of T (see [3] or [6, §37]). In the present case ($T(t)$ strictly increasing) this is especially simple. For fixed ω let $p(dt, dx,\omega)$ be the measure on $[0,\infty) \times (0,\infty)$ defined by the relationship that

$$p((t_1,t_2], (x_1, x_2], \omega)$$

is the number of points τ, $t_1 < \tau \leqq t_2$ such that $[T(\tau,\omega) - T(\tau-,\omega)] \in (x_1, x_2]$. Here $0 \leqq t_1 < t_2$ and $0 < x_1 < x_2$. The measure, p, is called the Poisson measure of T. The random variable $p(dt,dx)$ has a Poisson distribution with expected value $dt\,\nu(dx)$ where $\nu(dx) = \beta[\Gamma(1 - \beta)x^{1+\beta}]^{-1}dx$ is the Lévy measure of T. See [1, §6]. Moreover

$$T(t,\omega) = \int_{0+}^{\infty} x\, p([0,t],\ dx, \omega), \tag{3.1}$$

where in this case the integral is just the countable sum of the jumps of $T(\tau,\omega)$ on the interval $0 \leqq \tau \leqq t$. Finally the random variable $p([t,s),dx), s > t$, is independent of $\mathscr{B}_{t-}$, the σ-algebra generated by $\{T(\tau)\colon \tau < t\}$, and if $A_1, \dots, A_n$ are disjoint Borel subsets of $\{(t,x)\colon t \geqq 0, x > 0\}$ which are at a positive distance from the t-axis, then $\int_{A_1} p(dt,dx), \dots, \int_{A_n} p(dt,dx)$ are independent random variables.

If $u > 0$ define

$$S(u,\omega) = \inf\{t\colon\ T(t,\omega) \geqq u\}. \tag{3.2}$$

Since $T(t)$ is strictly increasing, $S(u)$ is continuous and nondecreasing. It is the first passage time of T past u. We now state the main result of this section.

THEOREM 3.1. *For each $u > 0$ and integer $k \geqq 0$ we have*

$$E(S(u)^k) = k![\Gamma(1 + \beta k)]^{-1} u^{\beta k}.$$

Proof. For $\lambda > 0$ and $s > 0$, define

$$H_k(\lambda,s) = E \int_0^\infty e^{-\lambda S(u)^k} e^{-su} du.$$

Now $T(t,\omega)$ is a sum of jumps and so if we let t_n be the places where $T(t)$ jumps and $I_n = [T(t_n -), T(t_n))$ then $\bigcup I_n = [0,\infty)$ since $T(0) = 0$ and $T(t) \to \infty$ as $t \to \infty$. Of course, the t_n depend on ω. For notational convenience let us write $T^*(t)$ for $T(t-)$. Thus

$$\begin{aligned}\int_0^\infty e^{-\lambda S(u)^k} e^{-su} du &= \frac{1}{s} \sum_n e^{-\lambda t_n^k} [e^{-sT^*(t_n)} - e^{-sT(t_n)}] \\ &= \frac{1}{s} \sum_n e^{-\lambda t_n^k} e^{-sT^*(t_n)} [1 - e^{-s[T(t_n) - T^*(t_n)]}] \\ &= \frac{1}{s} \int_0^\infty \int_0^\infty e^{-\lambda t^k} e^{-sT^*(t)} [1 - e^{-sx}] \, p(dt,dx).\end{aligned}$$

But $T^*(t)$ and $p(dt,dx)$ are independent and

$$E[\exp(-sT^*(t))] = E[\exp(-sT(t))] = e^{-ts^\beta}$$

for fixed t. Therefore (these manipulations are easily justified by first approximating the integral by a sum and then passing to the limit)

$$\begin{aligned}H_k(\lambda,s) &= \frac{1}{s} \int_0^\infty \int_0^\infty e^{-\lambda t^k} e^{-ts^\beta} \left(1 - e^{-sx}\right) \frac{\beta dt dx}{\Gamma(1 - \beta) x^{1+\beta}} \\ &= s^{\beta - 1} \int_0^\infty e^{-\lambda t^k} e^{-ts^\beta} dt.\end{aligned}$$

Differentiating with respect to λ and letting $\lambda \to 0$ we obtain (the interchange of limit procedures is easily justified)

$$\begin{aligned}\int_0^\infty E[S(u)^k] e^{-su} du &= s^{\beta - 1} \int_0^\infty t^k e^{-ts^\beta} dt \\ &= k! s^{-1-\beta k} .\end{aligned}$$

Inverting this Laplace transform we obtain Theorem 3.1.

It is an easy consequence of this theorem that the distribution function of $S(u)$ is $G_\beta(u^{-\beta} x)$ where G_β is defined in (1.4).

4. **Proof of Theorem B.** Let $M(\varepsilon)$ be the number of jumps of $T(t)$ in the interval $0 \leqq t \leqq S(1)$ of length greater than ε. Thus if $Q(dt,\varepsilon) = p(dt,(\varepsilon,\infty))$ where p is the Poisson measure for $T(t)$, then $M(\varepsilon) = Q([0,S_1],\varepsilon)$. Of course, $S_1 = S(1)$

is defined in (3.2). Clearly $0 \leqq M(\varepsilon) - N_\beta^*(\varepsilon) \leqq 1$ for almost all ω. We will prove that

$$E\{[\Gamma(1-\beta)\varepsilon^\beta M(\varepsilon)]^k\} \to k![\Gamma(1+\beta k)]^{-1} \tag{4.1}$$

as $\varepsilon \to 0$ for each integer $k \geqq 0$. From this the corresponding relation with $M(\varepsilon)$ replaced by $N_\beta^*(\varepsilon)$ follows easily, and then Theorem B is a consequence of the moment convergence theorem for distributions [4, p. 115]. Thus the proof of Theorem B reduces to the proof of (4.1). Note that $M(\varepsilon) \leqq [\varepsilon^{-1}] + 2$, where $[\cdot]$ is the greatest integer function, and so all moments of $M(\varepsilon)$ exist.

It will be convenient to consider the subordinator $T(t)$ starting not only at 0 but also at any $x \geqq 0$. We will write P_x and E_x for probabilities and expectations when $T(0) = x$, and, as is usual in the general theory of Markov processes, $E_{T(t)}\{\ \}$ stands for the evaluation of the function $E_x\{\ \}$ at the point $x = T(t)$. Let $G(t,x,A)$ be the transition probability function for $T(t)$, and $U(x,A) = \int_0^\infty G(t,x,A)dt$ be the potential kernel. Since $U(x,A)$ is just the expected amount of time $T(t)$ spends in A when $T(0) = x$, we have

$$U(x,[0,y]) = E_x(S(y)) = E_0(S(y-x))$$

provided $y > x$. Thus Theorem 3.1 implies

$$U(x,dy) = \begin{cases} \beta[\Gamma(1+\beta)]^{-1}(y-x)^{\beta-1}dy & \text{if } y > x, \\ 0 & \text{if } y \leqq x. \end{cases} \tag{4.2}$$

We begin our calculations with several lemmas. Recall that $T^*(t) = T(t-)$ and $\mathscr{B}_t$ is the σ-algebra generated by $T(\tau)$ for $\tau \leqq t$.

Lemma 1. *Let g be a bounded Baire function with compact support; then*

$$\int_0^\infty E_x[g(T_t^*)]dt = \int_0^\infty E_x[g(T_t)]dt = \frac{\beta}{\Gamma(1+\beta)} \int_x^\infty (y-x)^{\beta-1}g(y)dy.$$

Proof. Since $T^*(t) = T(t)$ for almost all ω for each fixed t, we have using (4.2)

$$\begin{aligned}\int_0^\infty E_x[g(T_t^*)]dt &= \int_0^\infty E_x[g(T_t)]dt \\ &= \int_0^\infty dt \int G(t,x,dy)g(y) = \frac{\beta}{\Gamma(1+\beta)} \int_x^\infty (y-x)^{\beta-1}g(y)dy.\end{aligned}$$

It will be convenient to introduce the following conventions. Let $0 \leqq a < b < \infty$; then $\int_a^b$ and $\int_a^\infty$ are the integrals over the sets $(a,b]$ and (a,∞), respectively. Also let

$$h(u) = \begin{cases} 1 & \text{if } u \leqq 1, \\ 0 & \text{if } u > 1. \end{cases}$$

LEMMA 2. *Let ϕ be a bounded non-negative continuous function; then*

$$E_x \int_0^\infty h(T_u^*)\phi(T_u^*)Q(du,\varepsilon) = \lambda E_x \int_0^\infty h(T_u^*)\phi(T_u^*)du,$$

where $\lambda = [\varepsilon^\beta \Gamma(1-\beta)]^{-1}$.

Proof. Since $T_u^* \leqq 1$ if and only if $u \leqq S_1$, we see that $\int_0^\infty h(T_u^*)\phi(T_u^*)Q(du,\varepsilon) \leqq M(\varepsilon)\sup\phi \leqq ([\varepsilon^{-1}]+2)\sup\phi$, and $\int_0^\infty h(T_u^*)\phi(T_u^*)du \leqq S_1 \sup\phi$. Thus both integrals exist for almost all ω and have finite expectations. The result now follows by a straightforward approximation argument making use of the easily verified facts that (i) T_u^* and $Q([u,v),\varepsilon)$, $v > u$, are independent, (ii) $h(T_u^*)\phi(T_u^*)$ is left continuous in u, and (iii) $E_xQ(du,\varepsilon) = E_xp(du,(\varepsilon,\infty)) = \lambda du$.

LEMMA 3. *For each $\varepsilon > 0$ let $Y^\varepsilon = \{Y_u^\varepsilon; u \geqq 0\}$ be a stochastic process defined on the sample space of $\{T(t);\ t \geqq 0\}$ with right continuous sample functions. Let $H_x(u,\omega)$ be a fixed version of $E_x\{Y_u^\varepsilon | \mathscr{B}_u\}$ defined for $u \geqq 0$ and all ω which is assumed to satisfy $H_x(u,\omega) = g(T_u) + O(\gamma(\varepsilon))$ where g is a bounded continuous function and the O-term is uniform in u,ω, and x. Finally assume that $0 \leqq Y_u^\varepsilon(\omega) \leqq M_\varepsilon < \infty$ for all u,ω. Under these conditions*

$$E_x \int_0^\infty h(T_u^*)Y_u^\varepsilon Q(du,\varepsilon) = E_x \int_0^\infty h(T_u^*)g(T_u)Q(du,\varepsilon) + O(\varepsilon^{-\beta}\gamma(\varepsilon))$$

where the O-term is uniform in x.

Proof. Let $0 < b < \infty$ and let ${}_Nt_j = jb/N$, $j = 0,1,\dots,N$, for each integer $N > 0$. Let $Z(u) = h(T_u^*)$, then since $Z(u)$ is left continuous and $Y(u) = Y_u^\varepsilon$ is right continuous we have for each fixed $\varepsilon > 0$ that

$$\begin{aligned} J(b) &= E_x \int_0^b Z(u)Y_u^\varepsilon Q(du,\varepsilon) \\ &= \lim_{N\to\infty} \sum_{j=0}^{N-1} E_x\{Z({}_Nt_j)Y({}_Nt_{j+1})Q(\Delta_N t_j,\varepsilon)\} \end{aligned}$$

where $\Delta_N t_j = ({}_Nt_j, {}_Nt_{j+1}]$. But $Z({}_Nt_j)$ and $Q(\Delta_N t_j,\varepsilon)$ are $\mathscr{B}({}_Nt_{j+1})$ measurable and so using our fixed version of $E_x\{Y_u^\varepsilon \mid \mathscr{B}_u\}$ we have

$$\begin{aligned} J(b) &= \lim_{N\to\infty} E_x \sum_{j=0}^{N-1} Z({}_Nt_j)[g(T({}_Nt_{j+1})) + O(\gamma(\varepsilon))]Q(\Delta_N t_j,\varepsilon) \\ &= E_x \int_0^b h(T_u^*)g(T_u)Q(du,\varepsilon) + O(\gamma(\varepsilon))E_x \int_0^b h(T_u^*)Q(du,\varepsilon). \end{aligned}$$

Using Lemmas 1 and 2 we see that

$$E_x \int_0^\infty h(T_u^*)Q(du,\varepsilon) = \lambda E_x \int_0^\infty h(T_u^*)du = \lambda[\Gamma(1+\beta)]^{-1}(1-x)^\beta h(x) = O(\varepsilon^{-\beta})$$

uniformly in x. Thus letting $b \to \infty$ in the above expression for $J(b)$ we obtain Lemma 3.

We are now ready to begin the proof of (4.1). Since $t \leqq S_1$ if and only if $T^*(t) \leqq 1$ we may write (Q assigns no mass to 0, so $\int_{0-}^{\infty}$ equals $\int_0^{\infty}$)

$$M(\varepsilon) = \int_0^{\infty} h(T^*(u))Q(du,\varepsilon). \tag{4.3}$$

Using Lemma 2 with $\phi \equiv 1$ and then Lemma 1 we obtain

$$E_y(M(\varepsilon)) = E_y\{p([0,S_1],(\varepsilon,\infty))\} = \frac{\lambda}{\Gamma(1+\beta)}(1-y)^{\beta}h(y). \tag{4.4}$$

Setting $y = 0$ and recalling that $\lambda = [\varepsilon^{\beta}\Gamma(1-\beta)]^{-1}$ we have established (4.1) when $k = 1$.

In order to fix the ideas let us consider the case $k = 2$ before proceeding to the general case. Now

$$\begin{aligned} E_0(M(\varepsilon)^2) &= 2E_0 \int_0^{\infty} h(T_t^*)Q(dt,\varepsilon) \int_t^{\infty} h(T_u^*)Q(du,\varepsilon) \\ &\quad + E_0 \iint_{0 \leqq t = u} h(T_t^*)h(T_u^*)Q(dt,\varepsilon)Q(du,\varepsilon). \end{aligned} \tag{4.5}$$

But for each ω, Q is a purely discrete measure assigning mass one to each point of a countable set (depending on ω) and $h^2 = h$. Therefore the second term in (4.5) reduces to $E_0(M(\varepsilon)) = O(\varepsilon^{-\beta})$ by (4.4). Let us consider the first term.

Define $Y_t^{\varepsilon} = \int_t^{\infty} h(T_u^*)Q(du,\varepsilon)$; then Y_t^{ε} is right continuous since the integral is over the open interval (t,∞), and is bounded by $M(\varepsilon) \leqq [\varepsilon^{-1}] + 2$. Since $T(t)$ is a Markov process we note that

$$E_x\{Y_t^{\varepsilon} \mid \mathscr{B}_t\} = E_{T(t)} \int_0^{\infty} h(T_u^*)Q(du,\varepsilon)$$

for almost all $\omega(P_x)$. Thus using Lemmas 2 and 1 we can take as our fixed version of $E_x\{Y_t^{\varepsilon} \mid \mathscr{B}_t\}$ the expression $\lambda q_1(T_t)$ where it is understood that $T(0) = x$, and $q_1(x) = [\Gamma(1+\beta)]^{-1}(1-x)^{\beta}h(x)$. Therefore the hypotheses of Lemma 3 are satisfied with $\gamma(\varepsilon) = 0$. Applying Lemma 3 with $x = 0$ the first term in (4.5) becomes

$$\begin{aligned} &2\lambda E_0 \int_0^{\infty} h(T_t^*)q_1(T_t)Q(dt,\varepsilon) \\ &\quad = 2\lambda E_0 \int_0^{\infty} h(T^*)q_1(T_t^*)Q(dt,\varepsilon) - 2\lambda E_0 \int_0^{\infty} h(T_t^*)[q_1(T_t^*) - q_1(T_t)]Q(dt,\varepsilon). \end{aligned}$$

Call the first term above J_1 and the second J_2. Using Lemmas 2 and 1, and then the definition of q_1, we find

$$\begin{aligned} J_1 &= 2\lambda^2\beta[\Gamma(1+\beta)]^{-1}\int_0^1 y^{\beta-1}q_1(y)dy \\ &= 2\lambda^2[\Gamma(1+2\beta)]^{-1}. \end{aligned}$$

Concerning J_2, we note that q_1 is bounded by $c = [\Gamma(1+\beta)]^{-1}$ and satisfies $0 \leqq q_1(x) - q_1(x+y) \leqq cy^\beta$ for all $x \geqq 0$ and $y \geqq 0$. Moreover there can be at most one jump of $T(t)$ of magnitude greater than one in the interval $0 \leqq t \leqq S_1$ and so

$$|J_2| \leqq 4c\lambda + 2\lambda c E_0 \int_0^{S_1} (T_t - T_t^*)^\beta p(dt,(\varepsilon,1]).$$

But this last integral is just $E_0\int_\varepsilon^1 x^\beta p([0,S_1],dx)$, and (4.4) implies that

$$E_0\{p([0,S_1],dx)\} = \beta[\Gamma(1+\beta)\Gamma(1-\beta)]^{-1}x^{-\beta-1}dx. \tag{4.6}$$

Thus using a simple approximation procedure we have that

$$|J_2| \leqq c_1\lambda + c_2\lambda \int_\varepsilon^1 x^\beta x^{-1-\beta}dx = O(\varepsilon^{-\beta}|\log\varepsilon|).$$

Combining these estimates we finally obtain $E_0(M(\varepsilon)^2) = 2\lambda^2[\Gamma(1+2\beta)]^{-1} + O(\varepsilon^{-\beta}|\log\varepsilon|)$, and this implies (4.1) in the case $k = 2$.

In order to attack the general case it will be necessary to introduce some notation. As above let $q_1(x) = E_x\int_0^\infty h(T_t^*)dt = [\Gamma(1+\beta)]^{-1}(1-x)^\beta h(x)$, and define

$$q_n(x) = E_x \int_0^\infty h(T_t^*)q_{n-1}(T_t^*)dt \tag{4.7}$$

for $n \geqq 2$. Let us show by induction that

$$q_n(x) = [\Gamma(1+n\beta)]^{-1}(1-x)^{n\beta}h(x). \tag{4.8}$$

This is true when $n = 1$, assuming it for $n-1$ and using Lemma 1 we find

$$\begin{aligned} q_n(x) &= \frac{\beta}{\Gamma(1+\beta)}\int_x^\infty (y-x)^{\beta-1}q_{n-1}(y)dy \\ &= [\Gamma(1+n\beta)]^{-1}(1-x)^{n\beta}h(x), \end{aligned}$$

and so (4.8) is established.

Secondly letting $q_0(x) \equiv 1$ we define

$$L_n(t,\omega) = \lambda^{n-1}E_{T(t)}\int_0^\infty h(T_u^*)q_{n-1}(T_u)Q(du,\varepsilon) \tag{4.9}$$

for $n \geqq 1$. Of course, L_n depends on ε. Next we show that

$$L_n(t) = \lambda^n q_n(T_t) + O(\varepsilon^{-(n-1)\beta}|\log \varepsilon|) \tag{4.10}$$

for $n \geqq 1$, where the O-term is uniform in t and ω. If $n = 1$, applying Lemma 2 and the definition of q_1, we have

$$L_1(t,\omega) = E_{T(t)} \int_0^\infty h(T_u^*)Q(du,\varepsilon) = \lambda q_1(T_t),$$

which certainly implies (4.10) when $n = 1$. For $n \geqq 2$ we can write

$$\begin{aligned} L_n(t) &= \lambda^{n-1} E_{T(t)} \int_0^\infty h(T_u^*) q_{n-1}(T_u^*) Q(du,\varepsilon) \\ &\quad - \lambda^{n-1} E_{T(t)} \int_0^\infty h(T_u) [q_{n-1}(T_u^*) - q_{n-1}(T_u)] Q(du,\varepsilon) \\ &= J_1 - J_2. \end{aligned}$$

From Lemma 2 and (4.7) we see that $J_1 = \lambda^n q_n(T_t)$. Concerning J_2, we note q_{n-1} is bounded, say by M, and that $0 \leqq q_{n-1}(x) - q_{n-1}(x+y) \leqq cy^\beta$ if $n \geqq 2$, since $0 < \beta < 1$. Thus exactly as in the case $k = 2$ we obtain

$$J_2 \leqq 2\lambda^{n-1}M + \lambda^{n-1} E_{T(t)} \int_\varepsilon^1 x^\beta p([0,S_1],dx).$$

But from (4.4) with $y = T(t)$ we see that

$$E_{T(t)} p([0,S_1],dx) = c_1(1 - T_t)^\beta h(T_t) x^{-1-\beta} dx,$$

and so $J_2 = O(\varepsilon^{-(n-1)\beta}|\log \varepsilon|)$ uniformly in t and ω since $(1 - T_t)^\beta h(T_t) \leqq 1$. Combining these results we obtain (4.10).

Finally we will show by induction that

$$L_n(t) = E_{T(t)} \int_0^\infty h(T_{t_1}^*)Q(dt_1,\varepsilon) \int_{t_1}^\infty \cdots \int_{t_{n-1}}^\infty h(T_t^*)Q(dt_n,\varepsilon) + O(\varepsilon^{-(n-1)\beta}|\log \varepsilon|) \tag{4.11}$$

provided $n \geqq 1$, where again the O-term is uniform in t and ω. Denote the first term on the right-hand side on (4.11) by $J_n(t)$. If $n = 1$, (4.11) is immediate from the definition of L_1. Assume (4.11) for $n - 1$. Define

$$Y^\varepsilon(t_1) = \int_{t_1}^\infty h(T_{t_2}^*)Q(dt_2,\varepsilon) \int_{t_2}^\infty \cdots \int_{t_{n-1}}^\infty h(T_{t_n}^*)Q(dt_n,\varepsilon).$$

Clearly $Y^\varepsilon(t_1)$ is right continuous since the integral is over (t_1,∞), and $0 \leqq Y^\varepsilon(t_1) \leqq ([\varepsilon^{-1}] + 2)^{n-1}$. Also since $T(t)$ is a Markov process it is easy to see that for each y a version of $E_y(Y^\varepsilon(t_1) \mid \mathscr{B}_{t_1})$ is given by (T starts from y)

$$\begin{aligned} E_{T(t_1)} \int_0^\infty h(T_{t_2}^*)Q(dt_2,\varepsilon) \int_{t_2}^\infty \cdots \int_{t_{n-1}}^\infty h(T_t^*)Q(dt_n,\varepsilon) \\ = L_{n-1}(t_1) + O(\varepsilon^{-(n-2)\beta}|\log \varepsilon|) \\ = \lambda^{n-1} q_{n-1}(T_{t_1}) + O(\varepsilon^{-(n-2)\beta}|\log \varepsilon|) \end{aligned}$$

where we have used the induction hypothesis and (4.10). Applying Lemma 3 whose hypotheses are satisfied we find that

$$\begin{aligned} J_n(t) &= E_{T(t)} \int_0^\infty h(T^*_{t_1}) Y^\varepsilon(t_1) Q(dt_1,\varepsilon) \\ &= \lambda^{n-1} E_{T(t)} \int_0^\infty h(T^*_{t_1}) q_{n-1}(T_{t_1}) Q(dt_1,\varepsilon) + O(\varepsilon^{-(n-1)\beta} \,|\log \varepsilon|), \end{aligned}$$

which is (4.11) in view of the definition (4.9) of $L_n(t)$.

We are now ready to prove (4.1) for general values of $k \geqq 2$. Assume (4.1) for $0, 1, \ldots, k-1$; then

$$\begin{aligned} E_0(M(\varepsilon)^k) &= E_0\left(\int_0^\infty h(T^*_u) Q(du,\varepsilon)\right)^k \\ &= k! E_0 \int_0^\infty h(T^*_{t_1}) Q(dt_1,\varepsilon) \int_{t_1}^\infty \cdots \int_{t_{k-1}}^\infty h(T^*_{t_k}) Q(dt_k,\varepsilon) \\ &\quad + \text{error term.} \end{aligned} \tag{4.12}$$

Now the error term consists of a finite sum of integrals over lower dimensional hyperplanes obtained by identifying a subset of the variables $t_1, \ldots, t_k$. Exactly as in the case $k = 2$ each such integral reduces to $E_0(M(\varepsilon)^{k-n+1})$ if n of variables are identified, $n = 2, \ldots, k$. Thus by the induction hypothesis the error term is $O(\varepsilon^{-(k-1)\beta})$. But arguing exactly as in the proof of (4.11) and making use of (4.11) and (4.10) the first term, J, on the right-hand side of (4.12) is just

$$J = k!\lambda^{k-1} E_0 \int_0^\infty h(T^*_u) q_{k-1}(T_u) Q(du,\varepsilon) + k! O(\varepsilon^{-(k-2)\beta} |\log \varepsilon|) E_0 \int_0^\infty h(T^*_u) Q(du,\varepsilon).$$

Writing $q_{k-1}(T_u) = q_{k-1}(T^*_u) - [q_{k-1}(T^*_u) - q_{k-1}(T_u)]$ and using an argument that is by now familiar we find

$$J = k!\lambda^k E_0 \int_0^\infty h(T^*_u) q_{k-1}(T^*_u) du + O(\varepsilon^{-(k-1)\beta} |\log \varepsilon|).$$

Finally combining this with (4.7) and (4.8) we have

$$E_0(M(\varepsilon)^k) = k!\lambda^k [\Gamma(1 + \beta k)]^{-1} + O(\varepsilon^{-(k-1)\beta} |\log \varepsilon|)$$

and this implies (4.1). Thus Theorem B is, at long last, established.

Note added in proof. The results of this paper and those of [2] are valid for general stable processes of index α, $1 < \alpha \leqq 2$. The proofs in the general case are exactly the same as in the symmetric case once Lemma 3.1 of [2] is established for general stable processes X of index α. However, a careful examination of the proof of this lemma in the symmetric case (which goes back to Kac) reveals that exactly the same argument works in the general case. See also a forthcoming paper of C. J. Stone in the Illinois Journal of Mathematics.

References

1. R. M. Blumenthal and R. K. Getoor, *Sample functions of stochastic processes with stationary independent increments*, J. Math. Mech. **10** (1961), 493–516.

2. ———, *The dimension of the set of zeros and the graph of a symmetric stable process*, Illinois J. Math. **6** (1962), 308–316.

3. K. Ito, *On stochastic processes*, Japan. J. Math. **18** (1942), 261–301.

4. M. G. Kendall and A. Stuart, *The advanced theory of statistics*, Vol. I, Hafner, Princeton, N.J., 1958.

5. H. Kesten, *Positivity intervals of a stable process*, J. Math. Mech. (to appear).

6. M. Loève, *Probability theory*, 2nd ed., Van Nostrand, 1960.

7. H. Pollard, *The completely monotonic character of the Mittag-Leffler function* $E_a(-x)$, Bull. Amer. Math. Soc. **54** (1948), 1115–1116.

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