CORRECTION TO "CONSTRUCTION OF AUTOMORPHIC FORMS ON H-GROUPS AND SUPPLEMENTARY FOURIER SERIES"

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In [2] there appear without proof two lemmas concerning the rearrangement of certain conditionally convergent double sums. Although these lemmas are quite crucial in [2], I gave no proofs since I was under the impression that they could be proved in exactly the same way as were similar lemmas in previous papers (cf. [1, pp. 272–277]). That this impression was a false one was pointed out by Dr. J. R. Smart. It is the purpose of this note to supply proofs of these lemmas.

The numbering of the lemmas will be as in [2].

LEMMA (2.10). *Let $\tau = iy$, with $y > 0$, and let $r > 0$ and ν a positive integer. Let*

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$$

with $\alpha < 0$, $\beta < 0$, $\gamma > 0$, $\delta > 0$, and put $t = (\alpha - 1/2\delta)\beta^{-1} > 0$. Let $\mathcal{T}_V(K)$ be the trapezoid in the $c - d$ plane bounded by the lines

$$c = 0, \quad \alpha c + \gamma d = \pm tK, \quad \beta c + \delta d = -K.$$

Then,

$$\begin{aligned} &\sum_{c \in C; c > 0} \lim_{N \to \infty} \sum_{d \in D^c; |d| \leq N} \frac{\varepsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu - \kappa) a/c\lambda\}}{c^{r+1}(c\tau + d)} \\ (1) \qquad &= \lim_{K \to \infty} \sum_{c \in C; c > 0} \sum\nolimits_{c} {}_{d \in D^c; (c,d) \in \mathcal{T}_V(K)} \frac{\varepsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu - \kappa) a/c\lambda\}}{c^{r+1}(c\tau + d)}. \end{aligned}$$

LEMMA (2.13). *Let τ, y, r, and ν be as above. Let ρ be any positive real number. Then*

$$\begin{aligned} &\sum_{c \in C: c > 0} \lim_{N \to \infty} \sum_{d \in D^c: |d| \leq N} \frac{\varepsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu - \kappa) a/c\lambda\}}{c^{r+1}(c\tau + d)} \\ (2) \qquad &= \lim_{K \to \infty} \sum_{c \in C: 0 < c \leq \rho K} \sum_{d \in D^c: |d| \leq K} \frac{\varepsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu - \kappa) a/c\lambda\}}{c^{r+1}(c\tau + d)}. \end{aligned}$$

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REMARKS. A careful examination will reveal that the above statement of Lemma (2.10) differs slightly from the statement in [2]. The changes are of a technical nature and of little importance. While the proof of (2.10) is by far the more complicated, the proof of (2.13) contains all the basic ideas. Hence we omit the proof of (2.10).

Proof of (2.13). We recall that

$$D^c = \left\{ d \mid \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \right\}$$

and

$$D_c = \left\{ d \mid \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, \text{ with } 0 \leqq -d < c\lambda \right\}.$$

Also we will make essential use of the following result due to Poincaré (cf. [3 p. 191]). If $r > 0$, then

$$\sum_{c \in C} \sum_{d \in D_c} |c\tau + d|^{-2-r}$$

converges. From this follows the convergence of

$$\sum_{c \in C; c > 0} \sum_{d \in D_c} c^{-r-2}.$$

Using this latter fact and the Lipschitz summation formula, we can proceed as in [1, p. 273] to show that the left-hand side of (2) converges.

The lemma can now be stated as follows:

$$\lim_{K \to \infty} \sum_{c \in C; 0 < c \leqq \rho K} \lim_{N \to \infty} \sum_{d \in D^c; K < |d| \leqq N} \frac{\varepsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu - \kappa) a/c\lambda\}}{c^{r+1}(c\tau + d)} = 0. \tag{3}$$

We replace d by $d - c\lambda q$ and thereby rewrite the inner sum as

$$\frac{1}{c^{r+1}} \sum_{d \in D_-} \varepsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda}$$

$$\times \left\{ \sum_{(d-N)/c\lambda \leqq q < (-K+d)/c\lambda} \left(\frac{e^{2\pi i q\kappa}}{c\tau + d - c\lambda q} \right) + \sum_{(d+K)/c\lambda < q \leqq (N+d)/c\lambda} (\quad) \right\}.$$

The argument is divided into two cases.

(a) Suppose $\kappa \neq 0$. Then

$$\lim_{N \to \infty} \sum_{d \in D^c; K < |d| \leqq N} \frac{\varepsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu - \kappa) a/c\lambda\}}{c^{r+1}(c\tau + d)}$$

$$= \frac{1}{c^{r+1}} \sum_{d \in D_c} \varepsilon^{-1}(V_{c,d}) e^{-2\pi i(\nu-\kappa)a/c\lambda} (S_1 + S_2), \tag{4}$$

where

$$S_1 = \sum_{-\infty \leqq q < (-K+d)/c\lambda} \left(\frac{e^{2\pi i q\kappa}}{c\tau + d - c\lambda q} \right)$$

and

$$S_2 = \sum_{(d+K)/c\lambda < q \leqq \infty} \left(\frac{e^{2\pi i q\kappa}}{c\tau + d - c\lambda q} \right).$$

Let $\sigma_1 = (K - d)/c\lambda$, $\sigma_2 = (K + d)/c\lambda$, $E_q = \sum_{p=[\sigma_1]+1}^{q} e^{-2\pi i p\kappa}$, and $E'_q = \sum_{p=[\sigma_2]+1}^{q} e^{2\pi i p\kappa}$. Since $0 < \kappa < 1$ a simple calculation shows that, for all q,

$$|E_q| \leqq \frac{1}{|\sin \pi\kappa|}, \quad |E'_q| \leqq \frac{1}{|\sin \pi\kappa|}.$$

We write

$$S_1 = \sum_{q=[\sigma_1]+1}^{\infty} \frac{E_q - E_{q-1}}{(ciy + d + c\lambda q)} = \sum_{q=[\sigma_1]+1}^{\infty} E_q \left(\frac{1}{ciy + d + c\lambda q} - \frac{1}{ciy + d + c\lambda q + c\lambda} \right).$$

Therefore

$$|S_1| < \frac{c\lambda}{|\sin \pi\kappa|} \sum_{q=[\sigma_1]+1}^{\infty} (d + c\lambda q)^{-2}.$$

In the summation $q \geqq [\sigma_1] + 1 > \sigma_1 = (K - d)/c\lambda$, or $c\lambda q + d > K > 0$. We have that

$$|S_1| < \frac{c\lambda}{|\sin \pi\kappa|} \left\{ \frac{1}{(d + c\lambda([\sigma_1] + 1))^2} + \int_{[\sigma_1]+1}^{\infty} \frac{dx}{(d + c\lambda x)^2} \right\}$$

$$< \frac{1}{|\sin \pi\kappa|} \left\{ \frac{c\lambda}{K^2} + \frac{1}{K} \right\}.$$

Using the fact that we consider only those c in the range $0 < c \leqq \rho K$, and letting $s = \min(1, r/2)$, we find that

$$|S_1| < \frac{\lambda\rho + 1}{|\sin \pi\kappa|} K^{-1} \leqq \frac{(\lambda\rho + 1)\rho^{1-s}}{|\sin \pi\kappa| c^{1-s}} K^{-s}$$

In precisely the same fashion we obtain

$$|S_2| < \frac{(\lambda\rho + 1)\rho^{1-s}}{|\sin \pi\kappa| c^{1-s}} K^{-s}.$$

Going back to (4) we find that

$$\left| \lim_{N\to\infty} \sum_{d \in D^c;\, K < |d| \leqq N} \frac{\varepsilon^{-1}(V_{c,d}) \exp\{-2\pi i(\nu - \kappa) a/c\lambda\}}{c^{r+1}(c\tau + d)} \right|$$

$$< \frac{2(\lambda\rho + 1)\rho^{1-s}}{|\sin \pi\kappa|} \frac{1}{c^{2+s}} \sum_{d \in D_c} K^{-s}.$$

Therefore,

$$\left| \sum_{c \in C; 0 < c \leqq \rho K} \lim_{N \to \infty} \sum_{d \in D^c; K < |d| \leqq N} (\ ) \right| < \text{constant} \cdot K^{-s} \sum_{c \in C; 0 < c \leqq \rho K} \sum_{d \in D_c} \frac{1}{c^{2+s}},$$

so that

$$\lim_{K \to \infty} \left| \sum_{c \in C; 0 < c \leqq \rho K} \lim_{N \to \infty} \sum_{d \in D^c; K < |d| \leqq N} (\ ) \right|$$

$$\leqq \text{constant} \left(\sum_{c \in C; c > 0} \sum_{d \in D_c} \frac{1}{c^{2+s}} \right) \lim_{K \to \infty} K^{-s} = 0.$$

Hence (3) follows and the lemma is proved in this case.

(b) If $\kappa = 0$, write

$$\lim_{N \to \infty} \sum_{d \in D^c; K < |d| \leqq N} (\,) = \frac{1}{c^{r+1}} \sum_{d \in D_c} \varepsilon^{-1}(V_{c,d}) e^{-2\pi i v a / c\lambda} (S_1' + S_2' + S_3'),$$

where

$$S_1' = \lim_{N \to \infty} \sum_{(K-d)/c\lambda \leqq |q| \leqq (N+d)/c\lambda} (ciy + d - c\lambda q)^{-1}$$

$$S_2' = \lim_{N \to \infty} \sum_{(-N+d)/c\lambda \leqq q < (-N-d)/c\lambda} (ciy + d - c\lambda q)^{-1}$$

and

$$S_3' = \sum_{(K+d)/c\lambda < q \leqq (K-d)/c\lambda} (ciy + d - c\lambda q)^{-1}.$$

Now,

$$S_1' = \sum_{q = [\sigma_1]+1}^{\infty} \left\{ \frac{1}{ciy + d - c\lambda q} + \frac{1}{ciy + d + c\lambda q} \right\}$$

$$= 2 \sum_{q = [\sigma_1]+1}^{\infty} \left\{ \frac{ciy + d}{(ciy + d)^2 - c^2\lambda^2 q^2} \right\}.$$

Therefore,

$$|S_1'| \leqq 2(c^2 y^2 + d^2)^{1/2} \sum_{q = [\sigma_1]+1}^{\infty} |c^2(\lambda^2 q^2 + y^2) - d^2|^{-1}.$$

In this summation $q > \sigma_1$; this can be rewritten as $0 \leqq -d < c\lambda q$. Hence $c^2\lambda^2 q^2 - d^2 > 0$, and we obtain

$$|S_1'| < 2c(y^2 + \lambda^2)^{1/2} \sum_{q = [\sigma_1]+1} (c^2\lambda^2 q^2 - d^2)^{-1}.$$

But, $c\lambda q + d > 0$ implies that $(c\lambda q)^2 - d^2 \geqq (c\lambda q + d)^2$, so that we have

$$|S_1'| < 2c(y^2 + \lambda^2)^{1/2} \sum_{q = [\sigma_1]+1}^{\infty} (c\lambda q + d)^{-2},$$

and, as in the previous case,

$$|S_1'| < \frac{2(y^2+\lambda^2)^{1/2}(\lambda\rho+1)\rho^{1-s}}{\lambda c^{1-s}\,K^s}\,.$$

Also,

$$|S_2'| \leqq \lim_{N\to\infty} \sum_{(-N+d)/c\lambda \leqq q < (-N-d)/c\lambda} |c\lambda q - d|^{-1}.$$

Here q is in an interval of length $(-2d)/c\lambda < 2$, so that there are at most two terms in the sum. In each term $|c\lambda q - d| > N + 2d$, so that $S_2' = 0$.

In S_3', q is again in an interval of length $(-2d)/c\lambda < 2$, so that there are again at most two terms. Also in S_3' we have $|ciy + d - c\lambda q|^{-1} \leqq |c\lambda q - d|^{-1} < 1/K$, so we obtain $|S_3'| < 2/K < 2\rho^{1-s}/c^{1-s}K^s$. We have made use of the fact that $0 < c \leqq \rho K$.

We now use these estimates for S_1', and S_3' to derive (3) as in the previous case.

The following corrections in [2] should also be noted.

$$a_m(\nu', r, \varepsilon') = -\overline{a_{-m-1}}(\nu, r, \varepsilon), \qquad \text{for } m \geqq 0, \tag{4.13}$$

$$a_m(\nu', r, \varepsilon') = -\overline{a_{-m}}(\nu, r, \varepsilon), \qquad \text{for } m \geqq 0. \tag{4.14}$$

$$\begin{aligned} \sum_{\nu=1}^{\mu} b_\nu a_{-m}(\nu, r, \varepsilon) &= b_m, &\qquad \text{for } 1 \leqq m \leqq \mu,\\ &= 0, &\qquad \text{for } m \geqq \mu + 1. \end{aligned} \tag{4.16}$$

$$F^*(\tau) \equiv 0. \tag{4.20}$$

References

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