SOLUTIONS TO COOPERATIVE GAMES WITHOUT SIDE PAYMENTS

BY

BEZALEL PELEG

An extension of Von Neumann Morgenstern solution theory to cooperative games without side payments has been outlined in [1]. In this paper we revise some of the definitions given in [1] and prove that in the new theory every three-person constant sum game is solvable (see [1, Theorem 1]). Other results that were formulated in [1] had already been proved in [2]. [1; 2] are also necessary for a full understanding of the basic definitions of this paper.

1. Basic definitions. If $N$ is a set with $n$ members, we denote by $E^N$ the $n$-dimensional euclidean space the coordinates of whose points are indexed by the members of $N$. Subsets of $N$ will be denoted by $S$. If $x \in E^N$ and $i \in N$, $x^i$ will denote the coordinate of $x$ corresponding to $i$; $x^S$ will denote the set $\{x^i : i \in S\}$. The superscript $N$ will be omitted, thus we write $x$ instead of $x^N$. We write $x^S \geq y^S$ if $x^i \geq y^i$ for all $i \in S$; similarly for $>$ and $=$. $\emptyset$ denotes the empty set.

**Definition 1.1.** An $n$-person characteristic function is a pair $(N, v)$ where $N$ is a set with $n$ members, and $v$ is a function that carries each $S \subseteq N$ into a set $v(S) \subseteq E^N$ so that

1. $v(S)$ is closed,
2. $v(S)$ is convex,
3. $v(\emptyset) = E^N$,
4. if $x \in v(S)$ and $x^S \geq y^S$ then $y \in v(S)$.

**Definition 1.2.** An $n$-person game is a triad $(N, v, H)$, where $(N, v)$ is an $n$-person characteristic function and $H$ is a convex compact subset of $v(N)$.

We notice that this definition is not identical with that given in [1; 2]. In the first place $v$ is not assumed to be superadditive, i.e., the condition: $v(S_1 \cup S_2) \supseteq v(S_1) \cap v(S_2)$ for every pair of disjoint coalitions $S_1$ and $S_2$ is dropped. Secondly $H$ need not be a polyhedron.

2. Solutions. Let $G = (N, v, H)$ be an $n$-person game.

**Definition 2.1.** Let $x, y \in E^N$, $S \neq \emptyset$. $x$ dominates $y$ via $S$, written $x \triangleright_S y$, if $x \in v(S)$ and $x^S > y^S$.

**Definition 2.2.** $x$ dominates $y$, written $x \triangleright y$, if there is an $S$ such that $x \triangleright_S y$.

For $x \in E^N$ the following sets are defined: $\text{dom}_S x = \{y : x \triangleright_S y\}$ and $\text{dom} x = \{y : x \triangleright y\}$. Let $K \subseteq E^N$. We define $\text{dom}_S K = \bigcup_{x \in K} \text{dom}_S x$ and $\text{dom} K = \bigcup_{x \in K} \text{dom} x$.

Received by the editors June 28, 1961 and, in revised form, February 8, 1962.
Definition 2.3. \( V \) is \( K \)-stable if \( V = K - \text{dom } V \).

Definition 2.4. The \( K \)-core is the set \( K - \text{dom } K \).

We use the following abbreviation: P.S.O.—the proof, which is straightforward, will be omitted.

Proposition 2.5. Every \( K \)-stable set contains the \( K \)-core. P.S.O.

Proposition 2.6. If for each \( x \in K \cap \text{dom } K \) there is a \( y \in K - \text{dom } K \) such that \( y \geq x \) then the \( K \)-core is the only \( K \)-stable set. P.S.O.

We denote: \( u^i = \sup_{x \in \mathcal{v}(\{i\})} x^i \).

Definition 2.7. \( x \) is individually rational if \( x^i \geq u^i \) for all \( i \in N \).

Definition 2.8. \( x \) is group rational if there is no \( y \in H \) such that \( y > x \).

We denote: \( \mathcal{A} = \{x: x \in H, x \text{ is individually rational}\} \) and \( A = \{x: x \in \mathcal{A}, x \text{ is group rational}\} \).

Proposition 2.9. \( K \) is \( \mathcal{A} \)-stable if and only if \( \mathcal{A} \)-stable.

Proof. Let \( K \) be \( \mathcal{A} \)-stable. We show firstly that (1) \( \mathcal{A} - A \subset \text{dom } K \). If \( x \in \mathcal{A} - A \) then there is a \( y_0 \in \mathcal{A} \) such that \( y_0 > x \). Define \( f(y) = \min_{i \in N} (y^i - x^i) \). Since \( f \) is continuous and \( \mathcal{A} \) is compact \( f \) receives its maximum in \( \mathcal{A} \) at a point \( z \), which must be in \( A \). By 1.2 \( z \in \mathcal{v}(N) \). \( f(z) = f(y_0) \); therefore \( z > x \). We have that \( z \geq N x \) and if \( w \geq z \) then \( w \geq x \). If \( z \in K \) then \( x \in \text{dom } K \). If \( z \in \text{dom } K \) then there is a \( w_0 \in K \) and \( w_0 \geq z \) and therefore \( w_0 \geq x \), so \( x \in \text{dom } K \). From (1) it follows that \( \mathcal{A} - \text{dom } K = A - \text{dom } K \) and therefore \( K \) is \( \mathcal{A} \)-stable. Now, let \( K \) be \( \mathcal{A} \)-stable. If \( x \in \mathcal{A} - A \) we define \( z \) as before and we see that \( z \in A \subset K \cup \text{dom } K \) implies that \( x \in \text{dom } K \). We conclude that (1) holds and therefore \( K = A - \text{dom } K = \mathcal{A} - \text{dom } K \), i.e., \( K \) is \( \mathcal{A} \)-stable.

Definition 2.10. A solution of \( G \) is an \( \mathcal{A} \)-stable set.

If \( G \) has a solution we say that \( G \) is solvable.

Theorem 2.11. Every two person game has a unique solution, consisting of all of \( \mathcal{A} \). P.S.O.

Definition 2.12. \( G \) is constant-sum if \( H \) is contained in a plane \( \sum_{i \in N} x^i = e \).

3. Three-person constant sum games.

I. Auxiliary lemmas. We use the following abbreviations: 3-P.C.G.—three-person constant sum game, W. L. G.—without loss of generality.

Let \( G = (N, v, H) \) be 3-P.C.G. We denote the members of \( N \) by the first three positive integers and set \( S_i = N - \{i\} \) for \( i = 1, 2, 3 \). Let \( x \in H \). We denote: \( \sum_{i=1}^3 x_i = e \) and \( L = \{y: \sum_{i=1}^3 y_i = e\} \). We have that \( \mathcal{A} = \{x: x \in H, x_i \geq v_i, i = 1, 2, 3\} = A \). So \( A \) is a convex compact subset of \( L \). Domination between
points of \( A \) is possible only via the \( S_i \), i.e., if \( x, y \in A \) and \( x \leq_S y \) then \( S \) is one of the \( S_i \). For a subset \( B \) of \( L \) and \( i \in \mathbb{N} \) the following sets are defined: \( B^i = B \cap v(S_i) \), \( B^i = B^i \cap \text{dom } S_i \), \( B^i = B^i \cap \text{dom } S_i \), and \( B^i = B^i \cap \text{dom } S_i \).

**Lemma 1.1.** If \( B \) is convex then \( B^i \) is convex.

**Proof.** \( B^i = B \cap v(S_i) \) is convex. If \( x_1, x_2 \in B^i \) then there are \( y_1, y_2 \in B^i \) such that \( y_j \leq_S x_j \) for \( j = 1, 2 \). If \( 0 < t < 1 \), \( x = tx_1 + (1 - t)x_2 \) and \( y = ty_1 + (1 - t)y_2 \) then \( x, y \in B^i \) and \( y \leq_S x \), so \( x \in B^i \).

We remark that \( A^i \) is convex and compact, \( A^i \) is convex and \( A^i \) is compact.

Let \( x \in L \) and \( \varepsilon > 0 \). The set \( \{ y : y \in L, \sum_i (y_i - x_i^2) < \varepsilon^2 \} \) is denoted by \( S(x, \varepsilon) \). \( x \) is an interior point of a subset \( B \) of \( L \) if there is an \( \varepsilon > 0 \) such that \( S(x, \varepsilon) \subset B \).

**Lemma 1.2.** If \( B \subset L \) is convex and \( K = \hat{B}^i \cap \hat{B}^j \neq \emptyset, i \neq j \), then \( K \) contains an interior point.

**Proof.** W.L.O.G. \( i = 1 \) and \( j = 2 \). We show firstly that \( K \neq \emptyset \) implies that \( B \) contains an interior point. If \( B \) has no interior points then there are points \( x_1 \) and \( x_2 \) such that every \( y \in B \) can be written as \( y = tx_1 + (1 - t)x_2 \), \( -\infty < t < \infty \). Let \( x \in K \). \( x = t_0 x_1 + (1 - t_0) x_2 \). There are \( y_l = t_l x_1 + (1 - t_l) x_2, y_l \leq_S x \) for \( l = 1, 2 \). We have \( y_l^2 > x^2 \) and \( y_l^3 > x^3 \), i.e., \( t_l x_1^2 + (1 - t_l) x_2^2 > t_0 x_1^2 + (1 - t_0) x_2^2 \) and \( t_1 x_1^3 + (1 - t_1) x_2^3 > t_0 x_1^3 + (1 - t_0) x_2^3 \). So \( (t_1 - t_0) (x_1^3 - x_2^3) > 0 \) and \( (t_1 - t_0) (x_1^3 - x_2^3) > 0 \). Therefore \( \text{sgn} (x_1^3 - x_2^3) = \text{sgn} (x_1^3 - x_2^3) \). In the same way \( y_2 \leq_S x \) implies that \( \text{sgn} (x_1^3 - x_2^3) = \text{sgn} (x_1^3 - x_2^3) \). So the three differences \( x_1 - x_2 \) have the same sign, which is impossible since \( \sum_i x_i^3 = \sum k x_k^3 \). Now, let \( z \) be an interior point of \( B \) and \( y \in K \). For small positive \( t \) the points \( tz + (1 - t)y \) are interior points of \( K \).

**Lemma 1.3.** If \( x \in O = \bigcap_i \hat{A}^i \) then \( x \) is an interior point of \( O \).

**Proof.** There are \( y_j \leq_S x \) for \( j = 1, 2, 3 \). We have: \( y_1^2 > x^2, y_1^3 > x^3, y_2^1 > x^1, y_2^2 > x^2 \), \( y_3^1 > x^3 \) and \( y_3^2 > x^2 \). There exist \( 0 < t_k < 1 \) such that \( z_k = t_k y_1 + (1 - t_k) y_k \) satisfy \( z_k^1 = x^1, k = 2, 3 \). Since \( z_2^2 > x^2 \) and \( z_3^2 > x^2 \) there is a \( 0 < t_1 < 1 \) such that \( x = t_1 z_2 + (1 - t_1) z_3 \). So \( x \) is an interior point of the convex hull of \( \{ y_1, y_2, y_3 \} \) and therefore of \( A \). But if \( x \in O \) is an interior point of \( A \) then \( x \) is also an interior point of \( O \).

**Lemma 1.4.** If \( B \subset L \) is convex, \( x_1, x_2 \in B \), \( x_1 = x_2, x_1^1 < x_2^1, x_1^1 > x_2^1 \) and \( y \) satisfies \( y^1 = x_2^1, y^j = x_1^1 \) and \( y^k = x_1^1 + x_1^k - x_2^1 \) then: \( y \notin B \) if and only if \( B \cap \{ z : z^S_k \geq y^S_k \} = \emptyset \). P.S.O.

**Lemma 1.5.** If \( x \in \hat{A}^i \) then there is a \( y \in A^i \) such that \( y \leq_S x \) and for every \( \varepsilon > 0 \), \( S(y, \varepsilon) \cap \hat{A}^i \neq \emptyset \).

**Proof.** Define \( f(z) = \min_{j \leq_S i} (z^j - x^j). f \) receives its maximum in \( A^i \) at a point
y which must be in \( a'. \) Since \( x \in A_t \) then \( y \succeq_{S_1} x. \) If \( 0 \leq t < 1 \) then \( ty + (1 - t)x \) is in \( A_t, \) therefore for every \( \varepsilon > 0 \) \( S(y, x) \cap A_t \neq \emptyset. \)

If \( x, y \in L \) then the set \( \{ z : z = tx + (1 - t)y, 0 \leq t \leq 1 \} \) is denoted by \([xy]\) and is called an interval. \( x \) and \( y \) are called the ends of \([xy]\) \( \cdot \) \([xy] = [xy] - \{x\} \cdot \{xy\} = [xy] - \{y\} \cdot \{xy\} = [xy] - (\{x\} \cup \{y\}). \) For \( i = 1, 2, 3 \) the following sets are defined: \( D^i = A^i \cap A^k, \) where \( S_i = (j, k), \) and \( F^i = \{ x : x \in D^i, x_t \succeq y^i \} \) for every \( y \in D^j. \) \( D^i \) is an interval.

**Lemma 1.6.** Let \( S_k = (i, j). \) If \( x^i \) receives its maximum in \( F^k \) at a point \( x, \) then:

\( x \notin a^i \) if and only if \( A^i \supset A^i. \)

**Proof.** W.L.G. \( i = 2 \) and \( j = 3. \) If \( A^3 \supset A^2 \) then \( x \in F^1 \subset D^1 \subset A^2 \subset A^3. \) If \( x \notin a^3 \) then there is an \( x \in A^3 \) such that \( x \succeq_{S_3} x. \) There is an \( \varepsilon > 0 \) such that \( U = S(x, x) \cap A^3. \) Now we show that if \( y \in A^3 \) then \( y^1 \preceq x^i \). If there is a \( z \in A^3 \) such that \( z^1 > x^1 \) then for a small positive \( tu = tz + (1 - t)x \) satisfies \( u^t > x^1 \) and \( u \in U. \) So we have \( u \in D^1 \) and \( u^t > x^1 \) which is impossible. Next we show that if \( y \in A^3 \) implies that \( y^2 \preceq x^2. \) Suppose that there is a \( z \in A^2 \) such that \( z^2 > x^2. \) If \( z^1 = x^1 \) then for a small positive \( tu = tz + (1 - t)x \) satisfies \( u \in U, u^1 = x^1 \) and \( u^2 > x^2 \) which is impossible. If \( z^1 < a^1 \) then there is a \( 0 < t < 1 \) such that \( u = tz + (1 - t)x \) satisfies \( w^t = a^1 \) and \( w^2 > x^2 \) \( \cdot \) \( a^3 \geq w^3 \) therefore \( w \in A^2, \) but this is impossible as we have already shown. We have shown that every \( y \in A^2 \) satisfies \( y^i \preceq x^i. \) Since \( x \in \text{dom}_{S_3} \) we have \( A^2 \subset A \cap \text{dom}_{S_3} \subset A^3. \)

The sets \( \{ A^1, A^2 \}, \{ A^2, A^3 \} \) and \( \{ A^3, A^1 \} \) will be called pairs.

**Definition 1.7.** The pair \( \{ A^i, A^j \} \) intersects maximally if:

1. \( A^i \cap A^j \neq \emptyset. \)
2. \( A^i \cap A^j \neq \emptyset. \)

The number of pairs that intersect maximally will be denoted by \( m(G). \)

**Lemma 1.8.** Let \( i \neq j \) and \( A^i \cap A^j \neq \emptyset. \) \( A^i \supset A^l \) and \( A^j \supset A^l \) if and only if \( \{ A^i, A^l \} \) intersects maximally.

**Proof.** W.L.G. \( i = 2 \) and \( j = 3. \) If \( A^3 \supset A^2 \) or \( A^2 \supset A^3 \) then \( a^2 \cap a^3 = \emptyset \) and therefore \( \{ A^2, A^3 \} \) does not intersect maximally. Now suppose that \( A^2 \supset A^3 \) and \( A^3 \supset A^2. \) Let \( x^2 \) and \( x^3 \) receive their maxima in \( F^1 \) at the points \( \alpha \) and \( \beta \) respectively. By 1.6: \( A^2 \supset A^3 \) implies that \( \alpha \in a^3 \) and \( A^3 \supset A^2 \) implies that \( \beta \in a^2. \)

We have \( \{ F^1 \cap a^2 \cap a^3 \). Since \( F^1 \) is connected and \( F^1 \cap a^2 \) and \( F^1 \cap a^3 \) are closed we must have \( (F^1 \cap a^2) \cap (F^1 \cap a^3) = F^1 \cap a^2 \cap a^3 \neq \emptyset. \)

From the proof of 1.8 we can conclude that: (I.9) if \( \{ A^i, A^l \} \) intersects maximally then \( F^k \cap a^l \cap A^l \neq \emptyset \) where \( \{ k \} = N - \{ i, j \}. \)

---

(1) Otherwise there is \( x \in F^1 \cap A^2 \cap A^3. \) Let \( z \succeq_{S_3} x. \) For small \( t > 0, u = tz + (1 - t)x \) satisfy \( u^l \succeq x^1 \) and \( u \in D^1, \) which is impossible.
Lemma 1.10. If \( i \neq j, x, y \in A_i, x \neq y \) and \( x^i = y^i \) then every \( z \in A_i \) satisfies \( z^i \leq x^i \) and \( \{ u : u \in A_i, u^i = x^i \} \subseteq A_i \). P.S.O.

Lemma 1.11. Let \( S_k = \{ i, j \} \). If \( \{ A_i, A_j \} \) intersects maximally and \( x^i \) and \( x^j \) take their maxima in \( F^k \) at the points \( \alpha \) and \( \beta \) respectively then \( F^k = [\alpha \beta] \) and one of the following possibilities holds:

(a) \( \alpha = \beta, \quad \alpha \in a_i \cap a_j \)
(b) \( \alpha \neq \beta, \quad [\alpha \beta] \subseteq a_i \cap a_j \)
(c1) \( \alpha \neq \beta, \quad [\alpha \beta] \subseteq a_i \cap \hat{A}_j, \quad \alpha \in a_i \cap a_j \)
(c2) \( \alpha \neq \beta, \quad [\alpha \beta] \subseteq a_j \cap \hat{A}_i, \quad \beta \in a_i \cap a_j \).

Proof. W.L.G. \( i = 2 \) and \( j = 3 \). We saw in the proof of 1.8 that \( \alpha \in F^1 \cap a^3 \) and \( \beta \in F^1 \cap a^2 \). If \( \alpha = \beta \) then (a) holds. If \( \alpha \neq \beta \) we have the following possibilities for the relative positions of \( a^3 \) and \( F^1 \):

(1) There is no \( x \neq \alpha \) in \( a^3 \cap F^1 \), i.e., \( [\alpha \beta] \subseteq \hat{A}^3 \).
(2) There is an \( x \neq \alpha \) in \( a^3 \cap F^1 \), and therefore, by I.10, \( F^1 \subseteq a^3 \). And similarly for \( a^2 \) and \( F^1 \):

(3) There is no \( y \neq \beta \) in \( a^2 \cap F^1 \), i.e., \( [\alpha \beta] \subseteq \hat{A}^2 \).
(4) There is a \( y \neq \beta \) in \( a^2 \cap F^1 \) and therefore \( F^1 \subseteq a^2 \).

Since \( F^1 \subseteq a^2 \cup a^3 \) (1) and (3) cannot hold together. If (2) and (4) hold together then we have (b). If (1) and (4) hold together, then we have (c1). If (2) and (3) hold together then we have (c2). We say that \( F^k \) has a-shape if (a) holds; similarly for (b), (c1) and (c2).

For \( x \in A \) the following sets are defined: \( Q_i(x) = \{ y : y \in A, y^i \geq x^i \} \), \( T_i(x) = \{ y : y \in A, y^i < x^i \} \) and \( R_i(x) = A - T_i(x) \). We remark that:

(I.12) \( \text{dom } Q_i(x) \cap R_i(y) = \text{dom } s_i Q_i(x) \cap R_i(x) \),
(I.13) \( x \notin \hat{A}_i \) if and only if \( x \in Q_i(x) \cap \text{dom } Q_i(x) \),
(I.14) \( x \notin \hat{A}_i \) if and only if \( Q_i(x) \cap \hat{A}_i = \emptyset \).

Lemma 1.15. If \( x \in A - A^i \), \( y \neq x, y, z \in A^i \cap Q_i(x) \) then there is a \( j \in S_i \) such that every \( z \in A^i \) satisfies \( z^j \leq x^j \). P.S.O.

Lemma 1.16. Let \( S_k = \{ i, j \} \). We denote the ends of \( F^k \) by \( \alpha \) and \( \beta \) such that \( \alpha^i \geq \beta^i \). If \( \gamma \in A \) satisfies \( \gamma^i = \alpha^i \) and \( \gamma^j < \beta^j \) then \( Q_i(\gamma) \cap A^i = \emptyset \). P.S.O.

Lemma 1.17. Let \( S_k = \{ i, j \} \). If \( \gamma \in F^k \cap a_i \cap a_j \) and \( x \in R_i(\gamma) - F^k \) then \( \text{dom } x \cap Q_i(\gamma) = \emptyset \).

Proof. W.L.G. \( i = 2 \) and \( j = 3 \). We denote the ends of \( F^1 \) by \( \alpha \) and \( \beta \) such that \( \alpha^2 \geq \beta^2 \). Let \( x \in R_i(\gamma) - F^1 \) and \( y \in Q_i(\gamma) \). \( x^2 + x^3 \geq \gamma^2 + \gamma^3 \geq y^2 + y^3 \) so \( x \not\sim s_i y \) is impossible. If \( x^1 > \gamma^1 \) and \( x \not\sim s_2 y \) or \( x \not\sim s_3 y \) then \( x \not\sim s_2 y \) or \( x \not\sim s_3 y \) respectively, which is impossible. If \( x^1 = \gamma^1 \) then either \( x^2 < \beta^2 \) or \( x^3 < \alpha^3 \). If \( x^1 < \alpha^3 \) then, by I.16, \( x \notin A^3 \). Since \( y^3 \geq y^2 \geq x^3 \), if \( x \not\sim y \) then \( x \not\sim s_i y \), but this is impossible. Similarly if \( x^2 < \beta^2 \) then \( x \not\sim y \) is impossible.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Lemma 1.18. If a \( B \subseteq A \) is convex and compact then \((N,v,B)\) is 3-P.C.G., \( B' = A^i \cap B \) and \( \bar{B}' = A^i \cap B \). P.S.O.

If \( B \subseteq A \) is convex and compact we say that \( B \) is solvable or that \( B \) has a solution if \((N,v,B)\) is solvable. We also write \( m(B) \) instead of \( m((N,v,B)) \).

Lemma 1.19. If \( x \in D^k \) and \( l \in S_k \) then \( \bar{R}_l^k(x) = A^l \cap R_k(x) \).

Proof. If \( y \in A^l \cap R_k(x) \) then \( y^k \geq x^k \) and there is a \( z \in A^l \) such that \( z \succsim_s y \).
Since \( z^k > y^k \) \( z \in R_k(x) \). So we have \( z \in R_k(x) \cap A^l = R_l^k(x) \) and therefore \( y \in R_l^k(x) \). We have shown that \( R_l^k(x) \supseteq A^l \cap R_k(x) \). By 1.18 \( R_l^k(x) \subseteq A^l \cap R_k(x) \), so \( R_l^k(x) = A^l \cap R_k(x) \).

Lemma 1.20. If \( x \in D^k - A^k, l \in S_k \) and \( \{A^k, A^l\} \) does not intersect maximally, then \( \{R_l^k(x), R_l^k(x)\} \) does not intersect maximally.

Proof. Since \( \{A^k, A^l\} \) does not intersect maximally, by 1.8 at least one of the the following possibilities holds: \( A^k \cap A^l = \emptyset \) or \( A^k \supset A^l \). If \( A^k \cap A^l = \emptyset \) then \( R_l^k(x) \cap R_l^k(x) = A^k \cap A^l \cap R_k(x) = \emptyset \). If \( A^l \supset A^k \) then \( R_l^k(x) = A^l \cap R_k(x) \supset A^k \cap R_k(x) = R_k(x) \). \( A^k \supset A^l \) is impossible since \( x \in A^l - A^k \).

Definition 1.21. Let \( B_1, \ldots, B_l \) be convex compact subsets of \( A \). \( B_1, \ldots, B_l \) are called independent if there exist solutions \( V_1, \ldots, V_l, V_i \) solution of \( B_i \) respectively, such that \( \text{dom } V_i \cap \bigcup_{j=1}^l V_j = \emptyset \) for \( k = 1, \ldots, l \).

Lemma 1.22. If \( B_1, \ldots, B_l \) are independent then there exist solutions \( V_1, \ldots, V_l, V_i \) solution of \( B_i \) for \( i = 1, \ldots, l \), such that \( \bigcup_{j=1}^l V_j \) is \( \bigcup_{j=1}^l B_j \)-stable. P.S.O.

In the following three subsections we shall prove:

Theorem. Every 3-P.C.G. \( G \) is solvable.

The proof will be by induction on \( m(G) \).

II. First part: \( m(G) = 0 \). In this subsection we show that every 3-P.C.G. \( G \) for which \( m(G) = 0 \) is solvable. We also prove some additional auxiliary lemmas:

Lemma II.1. Let \( G \) be 3-P.C.G. If \( A^1 \cap A^2 = A^2 \cap A^3 = A^3 \cap A^1 = \emptyset \) then the A-core is the solution of \( G \).

Proof. Denote \( C = A - \text{dom } A \). If \( x \in A - C \) then there is a \( y \in A \) that dominates it. There is an \( i \) such that \( y \succsim_s x, \) i.e., \( x \in A^i \). By 1.5 there is a \( z \in a^i \) such that \( z \succsim_s x \). If \( z \notin C \) then \( z \in A^l \) where \( l \neq i \). There is an \( e > 0 \) such that \( S(z, e) \cap A \subseteq A^l \). But \( S(z, e) \cap A^i \neq \emptyset \); therefore \( A^i \cap A^l \neq \emptyset \) which is impossible. We have shown that for every \( x \in A - C \) there is a \( z \in C \) such that \( z \succsim x \). By 2.6 \( C \) is the only A-stable set.

Lemma II.2. Let \( G \) be 3-P.C.G. If \( m(G) = 0 \) then the A-core is the solution of \( G \).

Proof. If \( A^1 \cap A^2 = A^2 \cap A^3 = A^3 \cap A^1 = \emptyset \) then by II.1 the A-core is the solution of \( G \). If it is not the case then, W.L.G., we assume that \( A^2 \cap A^3 \neq \emptyset \). Since \( \{A^2, A^3\} \) does not intersect maximally we have that either \( A^3 \supset A^2 \) or \( A^2 \supset A^3 \).
W.L.G. we suppose that $A^3 \supset A^2$. There are three possibilities for the relative position of $A^1$ and $A^3$: (a) $A^1 \cap A^3 = \emptyset$, (b) $A^3 \supset A^1$ or (c) $A^1 \supset A^3$. In each case we show that $C$, the $A$-core, is the solution of $G$. (a) $A^1 \cap A^3 = \emptyset$. If $x \in A - C$ then there is a $y \in A$ such that $y \prec_{s_i} x$ for some $i$. Since $A^3 \supset A^2$ we may assume that $i \in S_2$. There is a $z \in a^i$ such that $z \prec_{S_2} x$. If $i = 1$ then, since $A^1 \cap A^j = \emptyset$ for $j = 2, 3$, $z \in C$. If $i = 3$ then, since $a^3 \cap A^2 = \emptyset$ and $A^1 \cap A^3 = \emptyset$, $z \in C$. By 2.6 $C$ is the solution of $G$. (b) $A^3 \supset A^1$. If $x \in A - C$ then there is a $y \in A$ such that $y \prec_{S_2} x$. So there is a $z \in a^3$ such that $z \prec_{S_2} x$. Since $a^3 \cap (A^1 \cup A^3) = \emptyset$, $z \in C$. So $C$ is the solution of $G$. (c) $A^1 \supset A^3$. The proof in this case parallels that in case (b).

Let $G = (N, v, H)$ be 3-P.C.G.

**Lemma II.3.** If $x \in D^k - \hat{A}^k$, $\eta \in D^k \cap Q_k(\xi)$ and $U$ is a solution of $Q_k(\eta)$ then $V = U \cup [\xi_\eta]$ is a solution of $Q_k(\xi)$.

**Proof.** W.L.G. $k = 1$. $\eta^1 \leq \xi^1$, $\eta^2 \geq \xi^2$ and $\eta^3 \geq \xi^3$; therefore $(\text{dom}_{S_1}[\xi_\eta] \cup \text{dom}_{S_2}[\xi_\eta] \cap [\xi_\eta]) = \emptyset$. So $Q_1(\xi) \cap \hat{A}^1 = \emptyset$. Now we show (2) $Q_1(\xi) - [\xi_\eta] - Q_1(\eta) \subset \text{dom}[\xi_\eta]$. Let $x \in Q_1(\xi) - [\xi_\eta] - Q_1(\eta)$. If $x^1 \geq \xi^1$ then there is a $y \in [\xi_\eta]$ such that $y^1 = x^1$. If $x^1 < \xi^1$ then, since $x \notin Q_1(\eta)$, either $x^2 < \eta^2$ or $x^3 < \eta^3$ and therefore $x \in \text{dom}_{S_3}\eta \cup \text{dom}_{S_2}\eta$. We now prove (3) $U \cap [\xi_\eta] = \emptyset$ and $\text{dom}[\xi_\eta] \cap U = \emptyset$. Let $x \in [\xi_\eta]$ and $y \in U$. $y^1 \leq x^1$, $y^2 \geq x^2$ and $y^3 \geq x^3$ therefore $x \prec_{s} y$ is impossible and if $y \prec_{s} x$ then $y \prec_{S_2} x$, this is also impossible. Combining (1), (2) and (3) it follows that $V = U \cup [\xi_\eta]$ solves $Q_1(\xi)$.

**Lemma II.4.** If $x \in D^k - \hat{A}^k$ then $Q_k(\xi)$ is solvable and if $V$ solves it then $\text{dom} V = T_k(\xi) - V$.

**Proof.** W.L.G. $k = 1$. Denote $J = Q_1(\xi) \cap D^1$. Let $\eta$ be a point where $x^1$ receives its minimum in $J$. We show that $Q_1(\eta) \cap \hat{Q}_1(\eta) = \emptyset$ for all $i \neq j$. First, since $\xi \notin \hat{A}^1$, $\hat{A}^1 \cap Q_1(\xi) = \emptyset$. So we have $\hat{Q}_1(\eta) \subset A^1 \cap Q_1(\eta) \subset \hat{A}^1 \cap Q_1(\xi) = \emptyset$. Next, since $x^1$ receives its minimum in $J$ at $\eta$, $Q_1(\eta) \cap D^1 = \{\eta\}$, $\eta \notin \hat{A}^1$ therefore $\eta \in Q_1(\eta)$-core. We have $\hat{Q}_1(\eta) \cap \hat{Q}_1(\eta) = Q_1(\eta) \cap D^1 \cap Q_1(\eta) = \{\eta\}$, so $\hat{Q}_1(\eta) \cap \hat{Q}_1(\eta) = \emptyset$. By II.1 $Q_1(\eta)$ is solvable. Let $U$ be a solution of $Q_1(\eta)$; by II.3 $U \cup [\xi_\eta]$ solves $Q_1(\xi)$. Now let $V$ be a solution of $Q_1(\xi)$. $\xi \in Q_1(\xi)$-core so $\xi \in V$. $\text{dom} \xi = T_1(\xi) - Q_1(\xi)$ therefore $\text{dom} V = (Q_1(\xi) - V) \cup (T_1(\xi) - Q_1(\xi)) = T_1(\xi) - V$.

**Lemma II.5.** Let $S_k = \{i, j\}$. If $F^k$ has $c_j$-shape and $\mu \in F^k \cap \hat{A}^k$ then there is a solution $V$ of $Q_1(\mu)$ such that $V \cap F^k = \{\mu\}$.

**Proof.** W.L.G. $i = 2$ and $j = 3$. We denote the ends of $F^1$ by $\alpha$ and $\beta$ such that $\alpha^2 > \beta^2$. Since $F^1$ has $c_j$-shape we have (1) $\hat{A}^2 \cap R_1(\alpha) = \emptyset$ and (2) $[\beta \alpha] \subset \hat{A}^3$. From (1) it follows that (3) $\hat{Q}_2(\mu) = \emptyset$. We show (4) $[\beta \mu] \subset \hat{Q}_3(\mu)$. Let $x \in [\beta \mu]$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
1963] SOLUTIONS TO COOPERATIVE GAMES WITHOUT SIDE PAYMENTS 287

\[ x^1 = \mu^1, \quad x^2 < \mu^2 \quad \text{and} \quad x^3 > \mu^3. \]

By (2) \( x \in \tilde{A}^3 \), therefore there is a \( y \in A^3 \) such that
\[ y \ni s_y x. \]

For small \( t > 0 \), \( z = ty + (1 - t)x \) satisfy \( z^1 > x^1, \quad z^2 > x^2 \) and \( z^3 > \mu^3 \),

so \( z \in \tilde{Q}_2^3 (\mu) \) and \( x \in \tilde{Q}_2^3 (\mu) \). For the relative position of \( Q_2^3 (\mu) \) and \( Q_2^3 (\mu) \) we have

the following possibilities: (a) \( Q_2^3 (\mu) \cap \tilde{Q}_2^3 (\mu) = \emptyset \) or (b) \( Q_2^3 (\mu) \cap \tilde{Q}_2^3 (\mu) \neq \emptyset \). If (a) holds then by (3) \( Q_2^3 (\mu) \cap \tilde{Q}_2^3 (\mu) = \emptyset \) for all \( i \neq j \). By II.1 \( Q_2 (\mu) - \text{dom} \ Q_2 (\mu) \) is a solution of \( Q_2 (\mu) \) and since \( \mu \in Q_2 (\mu) \)-core, \( (Q_2 (\mu) - \text{dom} \ Q_2 (\mu)) \cap F^1 = \{\mu\} \).

If (b) then by I.2 there is an interior point \( \zeta \) of \( Q_2 (\mu) \cap \tilde{Q}_2^3 (\mu) \). \( \zeta^1 > \mu^1, \quad \zeta^2 < \mu^2 \)

and \( \zeta^3 > \mu^3 \). \( \zeta \in D^2 - \tilde{A}^2 \), therefore by II.4 \( Q_2 (\zeta) \) is solvable. If \( U \) is a solution of \( Q_2 (\zeta) \) then, by II.3, \( \{\zeta, \mu\} \cup U \) solves \( Q_2 (\mu) \). Since \( \{\zeta, \mu\} \cup U \cap F^1 = \{\mu\} \) this completes the proof.

**Definition II.6.** The pair \( \{A^i, A^j\} \) satisfies condition \( M \) if:

(1) \( \{A^i, A^j\} \) intersects maximally,

(2) \( F^k \cap a^i \cap a^j \neq A^k \) where \( \{k\} = N - \{i, j\} \).

We now formulate the induction hypothesis:

**II.7.** every 3-P.C.G. \( G \) for which \( m(G) \leq l - 1 \) is solvable. Let \( G \) be 3-P.C.G. for which \( m(G) = l \). We have to prove that \( G \) is solvable. We distinguish between the following possibilities:

**II.8.** there is at least one pair that satisfies condition \( M \).

**II.9.** there is no pair that satisfies condition \( M \).

**III. Second part:** case II.8. W.L.G. \( \{A^2, A^3\} \) satisfies condition \( M \). The ends of \( F^1 \) will be denoted by \( \alpha \) and \( \beta \) such that \( \alpha^2 \geq \beta^2 \). \( F^1 \cap a^2 \cap a^3 \neq A^1 \) therefore at least one of the ends is in \( a^2 \cap a^3 - \tilde{A}^1 \). We shall prove that \( G \) is solvable when:

**III.1.** \( \alpha \in a^2 \cap a^3 - \tilde{A}^1 \). The proof when \( \beta \in a^2 \cap a^3 - \tilde{A}^1 \) is similar to that in case (III.1). We shall distinguish three cases according to the three possible shapes of \( F^1 \) in case (III.1).

**III.a.** \( F^1 \) has a-shape. By (III.1) and II.4 \( Q_1 (\alpha) \) is solvable and if \( V \) solves it then (1) \( \text{dom} V \ni T_1 (\alpha) - V \). Since \( \alpha \in F^1 \) \( \tilde{A}^2 \cap \tilde{A}^3 \cap R_1 (\alpha) = \emptyset \). By I.19 \( \tilde{R}_1^2 (\alpha) \cap \tilde{R}_1^3 (\alpha) = \emptyset \), so \( \{R_1^2 (\alpha), R_1^3 (\alpha)\} \) does not intersect maximally. From I.20 it follows now that \( m(R_1 (\alpha)) \leq l - 1 \). By II.7 \( R_1 (\alpha) \) is solvable. If \( Q_1 (\alpha) \) and \( R_1 (\alpha) \) are independent then from I.22 and (1) it follows that \( A \) has a solution. If \( Q_1 (\alpha) \) and \( R_1 (\alpha) \) are not independent then if \( V \) solves \( Q_1 (\alpha) \) and \( W \) solves \( R_1 (\alpha) \) either (2) \( \text{dom} V \cap W \neq \emptyset \) or (3) \( \text{dom} W \cap V \neq \emptyset \). From III.a and I.17 it follows that (4) \( \text{dom} R_1 (\alpha) \cap Q_1 (\alpha) = \emptyset \). By (4) we have that (3) is impossible. By (III.1) \( \alpha \in A - \text{dom} A \) therefore (5) \( \alpha \in W \cap V \). From (2), (5) and I.12 it follows that there is a \( z \neq \alpha \) in \( V \cap A^1 \). By I.15 and due to III.a, we may assume that every \( y \in A^1 \) satisfies \( y^2 \leq \alpha^2 \).

Let \( \zeta \) be a point where \( x^3 \) receives its maximum(2) in \( V \cap A^1 \). If \( u \in V \cap A^1 \) then \( \zeta^2 = u^2 \) and \( \zeta^3 \geq u^3 \) and therefore (6) \( \text{dom} s, \zeta \ni \text{dom} s, u \). \( \alpha \in D^3 - \tilde{A}^3 \) therefore, by II.4, \( Q_3 (\alpha) \) is solvable. If \( U \) solves \( Q_3 (\alpha) \) then by (4) we have that (7) \( \text{dom} U \cap V = \emptyset \). We remark that (8) \( Q_3 (\alpha) \cap A^3 \subseteq \{x: \ x^2 = \alpha^2 \} \). Let \( x \) receive its maximum in \( U \cap A^3 \) at the point \( \eta \). We define: \( v = (\eta^1, e - \eta^1, -\zeta^3, \zeta^3) \). By I.12 and (6) we have that (9) \( R_1 (x) - \text{dom} V = R_1 (\alpha) \)

(2) Observe that a solution of a compact set is compact. see [3, Theorem 3].
\[-\text{dom}_{s_1} \zeta = Q_3(\alpha) \cup \{x: x \in R_1(\alpha), x^3 \geq \zeta^3\}.\] By (8) we have that (10) \(Q_2(\alpha)\) is solvable. Let \(U = Q_2(\alpha) - \text{dom}_{s_1} \eta = \{x: x \in Q_2(\alpha), x^1 \geq \eta^1\}\). Combining (1), (9) and (10) we have (11) \(A - \text{dom}(U \cup V) = \{x: x \in A, x^{s_2} \geq v^{s_2}\}\). If \(v \notin A\) then from I.4, (11), (9) and (7) it follows that \(U \cup V\) solves \(A\). Suppose now that \(v \in A\). We define \(v_1 = (x^1, e - x^1 - \zeta^1, \zeta^2).\) By I.16 we have that (12) \(\emptyset = A^2 \cap Q_2\) and (13) \(Q_1(\alpha) \cup Q_3(\alpha) \subset R_2(\alpha)\) therefore by (12) and (13) we have (13) \(\text{dom} Q_2(\alpha) \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset, v \in D^2 - A^2\) therefore \(Q_2(\alpha)\) is solvable. If \(U_1\) solves \(Q_2(\alpha)\) then by (11) and (13) \(V \cup U \cup U_1\) is a solution of \(A\).

III.b. \(F^1\) has \(b\)-shape. Due to III.b, we have that \(R_1(\alpha) \cap (\hat{A}^2 \cup \hat{A}^3) = \emptyset\) and therefore (1) \(R_2(\alpha) = R_1(\alpha) = \emptyset\). From (1) it follows that (2) \(R_1(\alpha) \cap R_1(\alpha) = \emptyset\) for all \(i \neq j\). If \(x \in R_1(\alpha)\) then \(x^2 + x^3 \leq x^2 + x^3\) therefore (3) \([\alpha \beta] \cap \text{dom}_{s_1} R_1(\alpha) = \emptyset\). From (1) and (3) we have that (4) \(R_1(\alpha) - \text{dom} R_1(\alpha) \supset [\alpha \beta]\). We also have that (5) \(T_1(\alpha) - \text{dom} R_1(\alpha) = T_1(\alpha) - \text{dom} [\alpha \beta] = \{x: x \in T_1(\alpha), x^1 \leq \alpha^2, x^3 \leq \beta^3\}\). Define \(\mu = (e - x^2 - \beta^3, x^2, \beta^3).\)

III.b.1. \(\mu \notin A\). By I.4 we have that (6) \(\{x: x^{s_1} \leq \mu^{s_1}\} \cap A = \emptyset\). By (2) and II.1 the \(R_1(\alpha)\)-core is the solution of \(R_1(\alpha)\). By (4), (5) and (6) we have that \(R_1(\alpha) - \text{dom} R_1(\alpha)\) solves \(A\).

III.b.2. \(\mu \in A\). \(\mu \in D^1 - \hat{A}^1\) therefore \(Q_1(\mu)\) is solvable. We remark that (7) \(Q_1(\mu) \cap A^1 = \{\mu\}\). We distinguish several subcases of II.b.2.

III.b.2.1. There is a solution \(V_1\) of \(Q_1(\mu)\) such that \(V_1 \cap A^1 = \emptyset\). We have that (8) \(\text{dom} V_1 \cap R_1(\alpha) = \emptyset\). From (5) and (8) it follows that \(V_1 \cup (R_1(\alpha) - \text{dom} R_1(\alpha))\) is a solution of \(G\).

III.b.2.2. There is a solution \(V_2\) of \(Q_1(\mu)\) such that \(V_2 \cap A^1 = \{\mu\}\). In this case: \(R_1(\alpha) - \text{dom} V_2 = Q_3(\alpha) \cup Q_2(\beta)\). \(\alpha \in D^3 - \hat{A}^3\) and \(\beta \in D^2 - A^2\) so \(Q_3(\alpha)\) and \(Q_2(\beta)\) are solvable and if \(U\) solves \(Q_3(\alpha)\) and \(W\) solves \(Q_2(\beta)\) then, by I.13. \(\alpha \in U\) and \(\beta \in W\). Since \(Q_3(\alpha) \cap R_2(\beta)\) and \(Q_2(\beta) \subset R_3(\alpha)\) it follows from I.12 that \(\text{dom} U \cap W = \text{dom} W \cap U = \emptyset\). From these results and (5) it follows that \(U \cup W \cup V\) is a solution of \(G\).

III.b.2.3. There is a solution \(V_3\) of \(Q_1(\mu)\) such that \(V_3 \cap A^1 = \{\mu\}\) \(\neq \emptyset\). Let \(x^3\) receive its maximum in \(V_3 \cap A^1\) at the point \(\zeta^1, \zeta^2 = \mu^2\) and \(\zeta^3 > \mu^3\). By II.3 \([\alpha \beta] \cup V_3 = V_3\) solves \(Q_1(\alpha)\). We define \(v = (x^1, e - x^1 - \zeta^1, \zeta^2)\). \(R_1(\alpha) - \text{dom} V_3 = Q_3(\alpha) \cup \{x: x \in A, x^{s_3} \geq v^{s_3}\}\). Let \(U\) solve \(Q_3(\alpha)\). \(\text{dom} U \cap (Q_1(\alpha) \cup \{x: x \in A, x^{s_3} \geq v^{s_3}\}) = \emptyset\). If \(v \notin A\) then \(V_3 \cup U\) solves \(G\). If \(v \in A\) then \(v \in D^2 - A^2\). By I.16 \(Q_2(\alpha) \cap \emptyset = \emptyset\). \(Q_1(\alpha) \cup Q_3(\alpha) \subset R_1(\alpha)\) therefore by I.12 \(\text{dom} Q_2(\alpha) \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset\). If \(W\) solves \(Q_2(\alpha)\) we have that \(V_3 \cup U \cup W\) is a solution of \(G\).

III.c. \(F^3\) has \(c_3\)-shape. As in III.a, we have that \(R_1(\alpha)\) is solvable, and if \(Q_3(\alpha)\) and \(R_1(\alpha)\) are independent then \(G\) is solvable. If \(Q_3(\alpha)\) and \(R_1(\alpha)\) are not independent and \(V\) solves \(Q_1(\alpha)\) and \(W R_1(\alpha)\) then either \(\text{dom} V \cap W \neq \emptyset\) or \(\text{dom} W \cap V \neq \emptyset\).

III.c.1. There exist \(V_0\) and \(W_0\) such that \(\text{dom} V_0 \cap W_0 \neq \emptyset\). \(\alpha \in W_0\) therefore there must be a \(z \neq \alpha\) in \(V_0 \cap A^1\). We have that either \(z^3 = \alpha^3\) or \(z^2 = \alpha^2\).
III.c.1.1. \( z^3 = x^3 \). In this case we have that (1) \( y \in A^1 \) implies \( y^3 \leq z^3 \). Let \( x^2 \) take its maximum in \( V_0 \cap A^1 \) at the point \( \zeta \). Define \( v = (x^1, x^2, e - x^1 - x^2) \). From (1) it follows that (2) \( R_1(\zeta) = \text{dom} \, V_0 = Q_2(x) \cup \{x : x \in A, \, x^5 \geq v^5\} \).

By II.5 there is a solution \( U \) of \( Q_2(x) \) such that \( U \cap F^1 = \{\zeta\} \); it follows that (3) \( \text{dom} \, U \cap Q_1(\zeta) = \emptyset \). If \( v \notin A \) then \( U \cup V_0 \) is a solution of \( G \). If \( v \in A \) then \( v \in D^3 - A^3 \), so \( Q_3(v) \cap A^3 = \emptyset \). It follows that (4) \( \text{dom} \, Q_3(v) \cap (Q_2(x) \cup Q_1(\zeta)) = \emptyset \).

We also have that (5) \( \text{dom} \, U \cap Q_3(v) = \emptyset \). Now if \( U_1 \) is a solution of \( Q_3(v) \) then, combining (2), (3), (4) and (5), we have that \( V_0 \cup U \cup U_1 \) is a solution of \( G \).

III.c.1.2. \( z^2 = x^2 \). We now have that \( y \in A^1 \) implies \( y^2 \leq x^2 \). We show that we may suppose: (**) there is no \( u \in A^3 \) such that \( u^2 = x^2 \) and \( u^1 > x^1 \). If (**) fails then \( F^2 \) has \( b \)-shape and \( \{A^1, A^3\} \) satisfies condition \( M \), so by III.b \( G \) is solvable.

We also notice that: (**) if \( \partial \in A, \partial^2 = x^2, \partial^3 \leq \beta^3 \) and \( J \) solves \( Q_1(\partial) \) then \( J \cup [\partial \theta \partial] \) solves \( Q_1(\zeta) \). Now if (**) holds and \( U \) is a solution of \( Q_3(x) \) then \( \text{dom} \, U \cap (Q_1(x) \cup Q_2(x)) = \emptyset \). Let \( V \) solve \( Q_1(x) \). We denote by \( \mu(V) \) the point where \( x^3 \) takes its maximum in \( A^1 \cap V \). The point where \( x^3 \) takes its maximum in \( A^1 \cap \{x : x^1 = \alpha^1\} \) is denoted by \( \zeta \). \( U \) denotes a fixed solution of \( Q_3(x) \). We remark that \( \alpha \in U \).

III.c.1.2.1. There is a solution \( V_1 \) of \( Q_1(x) \) such that \( \beta^3 < \mu^3(V_1) \). Define \( v = (x^1, e - x^1 - x^2, \mu^3(V_1)) \). We have that \( R_1(\alpha) - \text{dom} \, V_1 = Q_3(x) \cup \{x : x \in A, \, x^5 \geq v^5\} \). If \( v \notin A \) then \( U \cup V_1 \) is a solution of \( G \). If \( v \in A \) then \( v \in \{\alpha^2\} - A^2 \). Let \( V_2 \) solve \( Q_2(x) \). \( \text{dom} \, U \cap (Q_1(x) \cup Q_2(x)) = \emptyset \), so \( U \cup U_1 \cup U_2 \) is a solution of \( G \).

III.c.1.2.2. Every solution \( V \) of \( Q_1(x) \) satisfies \( \mu^3(V) \leq \beta^3 \) and there is a solution \( V_1 \) of \( Q_1(x) \) such that \( \mu^3(V_1) = \beta^3 \). \( \beta \in A^2 \cap D^2 \) therefore \( Q_2(\beta) \) is solvable. If \( U_1 \) solves \( Q_2(\beta) \) then \( U_1 \cap A^2 = \{\beta\} \) and \( \text{dom} \, U_1 \cap (Q_1(\mu(V_1)) \cup Q_3(x)) = \emptyset \). Let \( U_2 \) be a solution of \( Q_1(\mu(V_1)) \). From (**) it follows that \( \text{dom} \, U_2 \cap A^1 = \{\mu(V_1)\} \). So we have that \( U \cup U_1 \cup U_2 \) is a solution of \( G \).

III.c.1.2.3. If \( V \) solves \( Q_1(x) \) then \( \mu^3(V) < \beta^3 \).

III.c.1.2.3.1. \( \zeta^3 \geq \beta^3 \). Define \( v = (e - x^2 - \chi^3, x^2, \beta^3) \). By I.17 \( Q_1(x) \cap \text{dom} \, R_1(\alpha) = \text{dom} \, [x(\beta)] \cap Q_1(x) \), so \( \text{dom} \, R_1(\alpha) \cap \{x : x \in A, \, x^5 \geq v^5\} = \emptyset \). If \( v \notin A \) and \( U_1 \) solves \( Q_2(\beta) \) then \( \text{dom} \, U_2 \cap [x(\beta)] \cup U_1 \) solves \( G \). If \( v \in A \) let \( U_2 \) be a solution of \( Q_1(v) \). By (**) and the definition of \( U_2 \cup U_2 \cup \emptyset = \emptyset \). So in this case \( U \cup U_1 \cup U_2 \cup [x(\beta)] \) solves \( G \).

III.c.1.2.3.2. \( \zeta^3 < \beta^3 \). Define \( v = (e - x^2 - \chi^3, x^2, \chi^3) \). By II.5 there is a solution \( U_1 \) of \( Q_2(\chi) \) such that \( U_1 \cap F^1 = \{\chi\} \). If \( v \notin A \) then \( U \cup U_1 \cup [x \chi] \) is a solution of \( G \). If \( v \in A - A^1 \) and \( U_2 \) solves \( Q_1(v) \) then \( U \cup U_1 \cup U_2 \cup [x \chi] \) is a solution of \( G \). If \( v \in A^1 \) then \( U_1 \cup U_2 \cup U \) is a solution of \( G \).

III.c.2. If \( V \) solves \( Q_1(x) \) and \( W \) solves \( R_1(x) \) then \( \text{dom} \, W \cap V \neq \emptyset \) and \( \text{dom} \, V \cap W = \emptyset \). By I.17 we have that \( \text{dom} \, (R_1(x) - [\beta x]) \cap Q_1(x) = \emptyset \). By I.19 \( R_1(x) = \hat{A}^3 \cap R_1(x) \). \( [\beta x] \subseteq \hat{A}^3 \) therefore \( [\beta x] \subseteq \hat{A}^3 \). So we conclude that

\[
(1) \quad \text{dom} \, (R_1(x) - \text{dom} \, R_1(x)) \cap Q_1(x) = \emptyset.
\]
Since $F^1$ has $c_3$-shape we have also that $\hat{A}^2 \cap R_4(\alpha) = \emptyset$. If $\{A^1, A^3\}$ does not intersect maximally then, by I.20, $\{R_1^1(\alpha), R_3^1(\alpha)\}$ does not intersect maximally and we have that $m(R_1(\alpha)) = 0$. By II.2 the $R_1(\alpha)$-core is the solution of $R_1(\alpha)$ which, by (1), contradicts III.c.2. So $\{A^1, A^3\}$ intersects maximally. The $x^2$ coordinate of $F^2$ will be denoted by $d$.

III.c.2.1. $d < a^2$. We denote the ends of $F^2$ by $\gamma$ and $\delta$ such that $\delta^1 \geq \gamma^1$. We remark that $\delta \in D^2 - A^2$. So $Q_2(\delta)$ is solvable and if $U$ solves it then $dom U \supseteq T_2(\delta) \supseteq U$ and $dom U \cap R_2(\delta) = \emptyset$. Denote $\nu = (\alpha^1, d, e - d - \alpha^1)$ and $P = \{x: x \in A, x^3 \geq \nu^3\}$. We remark that $\hat{P}^i \cap \hat{P}^j = \emptyset$ for all $i \neq j$ and that $(P - dom P) \cap ((\alpha\beta] \cup [\gamma\delta)) = \emptyset$. So $P - dom P$ solves $P$ and

$$\text{dom}(P - dom P) \cap (Q_2(\delta) \cup Q_1(\alpha)) = \emptyset.$$ Summing we have that $U_1 = U \cup (P - dom P)$ solves $R_1(\alpha)$ and that $dom U_1 \cap Q_1(\alpha) = \emptyset$. Since this result contradicts III.c.2 $d < a^2$ is impossible.

III.c.2.2. $d = a^2$. If $F^2$ has $a$-shape or $b$-shape then $\{A^1, A^3\}$ satisfies condition $M$ and by III.a or III.b $G$ is solvable. It remains only to complete the proof when $F^2$ has $c_3$ or $c_1$-shape.

III.c.2.2.1. $\gamma^1 < \alpha^1$. In this case $F^2$ has $c_3$-shape. Let $\eta$ satisfy $\eta^2 = \alpha^2$ and $\alpha^1 > \eta^1 > \gamma$. $A^1 \supseteq [\alpha\gamma] \cup U_1$ solves $Q_1(\alpha)$. Let $\xi \in (\alpha\beta] \cap A^1$, $U_3$ be a solution of $Q_2(\xi)$ and $U_4$ a solution of $Q_3(\alpha)$. $U_5 = U_3 \cup U_4 \cup [\alpha\xi]$ is a solution of $R_1(\alpha)$. But $dom U_2 \cap U_5 = \emptyset$ contradicting III.c.2, so III.c.2.2.1 is impossible.

III.c.2.2.2. $\gamma^1 = \alpha^1$. In this case $F^2$ has $c_1$-shape. By II.5 there exist solutions $U_1$ of $Q_3(\alpha)$ and $U_2$ of $Q_2(\alpha)$ such that $U_1 \cap F^2 = \{\alpha\} = U_2 \cap F^1$. $U = U_1 \cup U_2$ is a solution of $R_1(\alpha)$ but $dom U_1 \cap Q_1(\alpha) = \emptyset$ contradicting III.c.2.

III.c.2.2.3. $\gamma^1 > \alpha^1$. In this case $F^2$ has $c^2$-shape and $\gamma \in a^1 \cap a^3 - A^2$. Let $U$ solve $Q_3(\gamma)$. $U_1 = U \cup (Q_3(\alpha) - dom Q_3(\alpha))$ solves $R_1(\alpha)$ but $dom U_1 \cap Q_1(\alpha) = \emptyset$ which is impossible.

III.c.2.3. $d > a^2$. In this case we show that $\hat{A}^1 \cap \hat{A}^3 \cap R_4(\alpha) = \emptyset$. It follows that $\{R_3^1(\alpha), R_3^3(\alpha)\}$ does not intersect maximally which as we have already seen. Suppose that $\hat{A}^1 \cap \hat{A}^3 \cap R_4(\alpha) \neq \emptyset$. Let $x \in \hat{A}^1 \cap \hat{A}^3 \cap R_4(\alpha)$.

Since $x^1 \geq \alpha^1$. Since $x^1 \geq \alpha^1$. Let $z \in a^1 \cap a^3 \cap F^2$ and $y$ be an interior point of $\hat{A}^1 \cap \hat{A}^3$. There is a $u \in [yz] \cap A^1 \cap \hat{A}^3$ such that $u^2 > \alpha^2$. So there is a $w \in [ux]$ for which $w^2 = \alpha^2$. $w \in \hat{A}^1 \cap \hat{A}^3$. If $w^1 \geq \alpha^1$ then $\alpha \in \hat{A}^3$ and if $w^3 \geq \alpha^3$ then $\alpha \in \hat{A}^1$.

Since both cases are impossible we must have $\emptyset = \hat{A}^1 \cap \hat{A}^3 \cap R_4(\alpha)$.

IV. Third part: case II.9.

IV.1. $m(G) \leq 2$. W.L.G. $\{A^2, A^3\}$ does not intersect maximally.

IV.1.1. $\hat{A}^2 \cap \hat{A}^3 = \emptyset$. We shall show that $m(G) = 0$. If $m(G) > 0$ then, W.L.G., $\{A^1, A^3\}$ intersects maximally. Let $z \in a^1 \cap a^3 \cap F^2$ and $y$ be an interior point of $\hat{A}^1 \cap \hat{A}^3$. If $z \in \hat{A}^2$ then we have $\hat{A}^2 \cap \hat{A}^3 \neq \emptyset$ which is impossible. If

(1) See I.11.
(2) If $x^2 \geq x^2$ then $\alpha S_3 \geq aS_3$ and $x \in \hat{A}^3$ imply $\alpha \in \hat{A}^3$ which is untrue.
z $ \notin \hat{A}^2$ then \( \{A^1, A^3\} \) satisfies condition \( M \), which is again impossible. Therefore \( \{A^1, A^3\} \) does not intersect maximally.

IV.1.2. \( \hat{A}^1 \supset A^2 \). If \( \{A^1, A^3\} \) intersects maximally then there is a \( z \in a^1 \cap a^3 \cap F^2 \). \( a^3 \cap \hat{A}^1 = \emptyset \); therefore \( z \notin \hat{A}^2 \). But since we have \( a^1 \cap a^3 \cap F^2 \subset \hat{A}^2 \), \( \{A^1, A^3\} \) cannot intersect maximally.

IV.1.2.1. \( \hat{A}^1 \cap \hat{A}^3 = \emptyset \). \( \hat{A}^2 \cap \hat{A}^1 \subset \hat{A}^1 \cap A^2 \subset \hat{A}^1 \cap \hat{A}^3 = \emptyset \), so \( m(G) = 0 \).

IV.1.2.2. \( \hat{A}^3 \supset A^1 \). We denote \( C = A - \text{dom} \ A \). If \( x \in A - C \) then there is a \( y \in A \) such that \( y \preceq_{s_1} x \). So there is a \( z \in a^3 \) such that \( z \prec x \). \( a^2 \cap (A^1 \cup A^2) = \emptyset \) therefore \( z \in C \). By 2.6 \( C \) is a solution of \( G \).

IV.1.2.3. \( \hat{A}^1 \supset A^3 \). We have \( \hat{A}^1 \supset A^3 \supset \hat{A}^3 \supset A^2 \), so \( m(G) = 0 \).

IV.1.3. \( \hat{A}^2 \supset A^3 \). The proof in this case parallels that in IV.1.2.

IV.2. \( m(G) = 3 \). We denote \( F_k = [a_k b_k] \) and \( D = \bigcup_{k=1}^3 F_k \).

**Lemma IV.2.1.** Under the assumptions of IV.2 we can find \( i \) and \( k \) such that \( S_k = \{i, j\} \) and:

1. \( \alpha^i_k \geq \beta^i_k \)
2. \( \alpha^i_k \in a^i \cap a^3 \)
3. \( \alpha^i_1 \) takes its maximum in \( \{x : x^i = \alpha^i_1\} \cap D \) at a point \( \theta \in a^k \) such that every \( y \in A^i \) that satisfies \( y^i = \theta^i \) and \( y^i > \theta^i \) is in \( A^1 \), or
4. \( \alpha^i_1 \) takes its maximum in \( \{x : x^i = \alpha^i_1\} \cap D \) at a point \( \rho \in a^k \cap \hat{A}^1 \) and \( F^k \) has \( c_k \)-shape.

**Proof.** \( F^1 = [\alpha_1 \beta_1] \). W.L.G. \( \alpha^3_1 \geq \beta^3_1 \). We also suppose that \( \alpha \in a^2 \cap a^3 \). If \( \alpha \notin a^2 \cap a^3 \) then the proof is not altered much. We now consider \( F^3 \).

IV.2.1.1. Every \( y \in F^3 \) satisfies \( y^2 < \alpha^2_1 \). Let \( \theta \) be the point where \( x^3 \) takes its maximum in \( \{x : x^2 = \alpha^2_1\} \cap D \). \( \theta \neq \alpha_1 \). We show that \( \theta \in A^2 \). If \( F^3 = [\alpha_3 \beta_3] \) and \( \alpha^3_1 \geq \beta^3_1 \) then \( \alpha^3_3 > \theta^3 \) since \( \theta \notin F^3 \). If \( \alpha^3_1 > \theta^1 \) then \( \alpha^3_3 \preceq_{s_3} \theta \), and if \( \theta^1 \geq \alpha^3_1 \) then there is a \( u \in [\alpha_3, \alpha_3] \) such that \( u \preceq_{s_3} \theta \). Now if \( \theta \in A^3 \) then \( F^3 \) has \( c_1 \)-shape and \( \theta \in a^1 \) and if \( \theta \in \hat{A}^3 \) then it follows from I.3 that \( \theta \in a^1 \). So in this case we can choose \( k = 1 \) and \( i = 2 \).

IV.2.1.2. There is a \( y \in F^3 \) such that \( y^2 = \alpha^2_1 \). Let \( \theta = y \). If \( F^3 \) has \( a \)-shape then \( \theta \in a^1 \) and there is no \( u \in A^1 \) that satisfies \( u^3 = \theta^3 \) and \( u^2 > \theta^2 \). So we can choose \( k = 1 \) and \( i = 2 \). If \( F^3 \) has \( b \)-shape then \( \beta^2_3 > \alpha^3_2 \). If \( F^3 \) has \( b \)-shape and \( y = \beta_3 \) then there is no \( u \in A^1 \) such that \( u^3 = \theta^3 \), and \( u^2 > \theta^2 \) and we can choose \( k = 1 \) and \( i = 2 \).

IV.2.1.3. Every \( y \in F^3 \) satisfies \( y^2 > \alpha^2_1 \). If \( F^3 \) has \( a \), \( b \) or \( c_1 \)-shape then we have that \( \beta^2_3 \geq \alpha^3_2 \), \( \beta_3 \in a^2 \cap a^1 \) and every \( x \in F^1 \) satisfies \( x^2 < \beta^2_3 \). By IV.2.1.1. we can choose \( k = 3 \) and \( i = 2 \). Now suppose that \( F^3 \) has \( c_2 \)-shape. Let \( x^1 \) take its maxi-
mom in $D \cap \{x: x^2 = a_3^2\}$ at the point $p \in A^1 \cap A^2$. We shall show that $\rho \in A^1 \cap A^2$. Let

\[ \rho^2 = a_3^2 < \beta_3^2, \rho^1 > a_3^1 \]  

therefore $\rho^1 < a_3^1 = \beta_3^1$. We have also that $a_3^1 > \rho^1$ therefore there is a $u \in [a_3^1,a_3]$ such that $u < a_3^1 \rho$. It follows from I.3 that $\rho \in a^3$. Summing we have: $a_3^1 \geq \beta_3^1, a_3 \in a^2 \cap a^1$ and $x^1$ takes its maximum in $D \cap \{x: x^2 = a_3^2\}$ at a point $p \in a^1 \cap A^2$. So we can take $k = 3$ and $i = 1$.

We now prove that $G$ is solvable in case IV.2. W.L.G. the results of IV.2.1 hold for $k = 1$ and $i = 2$.

IV.2.2. $(3.a)$ holds in IV.2.1. We remark that if $z \in A^1 \cap Q_1(\vartheta)$ then $z^3 = \vartheta^3$. If there is a $z \in A^1$ such that $z^3 = \vartheta^3$ and $z^2 > \vartheta^2$ then $F^3$ has $c_2$-shape. By Lemma II.5 there is a solution $V$ of $Q_1(\vartheta)$ such that $V \cap F^3 = \{\vartheta\}$. So we can always find a solution $V_1$ of $Q_1(\vartheta)$ such that $V_1 \cap A^1 = \{\vartheta\}$. Similar reasoning shows that there is always a solution $V_2$ of $Q_3(\alpha_1)$ such that $V_2 \cap A^3 = \{\alpha_1\}$.

IV.2.2.1. $\beta_3^1 > \beta_3^1$. We define $v = (a_3^1, e - a_3^1 - \vartheta^3, \beta_3^1)$. Suppose $v \in A$. We have that $Q_2(v) \cap A^2 = \emptyset$. So if $U$ is a solution of $Q_2(v)$ then $V_1 \cup V_2 \cup U \cup [\vartheta \alpha_1]$ solves $G$. If $v \notin A$ then $V_1 \cup V_2 \cup [\vartheta \alpha_1]$ solves $G$.

IV.2.2.2. $\beta_3^1 = \beta_3^1$. If $U$ is a solution of $Q_2(\beta_3^1)$ then $V_1 \cup V_2 \cup U$ is a solution of $G$.

IV.2.2.3. $\beta_3^1 < \beta_3^1$. Let $x^3$ take its maximum in $A^1 \cap \{x: x^1 = a_3^1\}$ at $\zeta$. We define $\mu = (e - a_3^1 - \beta_3^1, a_3^1, a_3^1)$. Suppose $\xi^3 \geq \beta_3^1$ and $\mu \in A$. In this case if $W$ is a solution of $Q_2(\beta_3^1)$ and $W_1$ is a solution of $Q_1(\mu)$ then $V_2 \cup W \cup W_1 \cup F^1$ solves $G$. If $\mu \notin A$ then $V_1 \cup W \cup F^1$ is a solution of $G$. If $\xi^3 < \beta_3^1$ then $F^1$ has $c_3$-shape. We define $\eta = (e - \xi^3 - a_3^1, a_3^1, a_3^1)$. By II.5 there is a solution $U$ of $Q_2(\xi^3)$ such that $U \cap F^1$. If $\eta \notin A$ then $U \cup V_2 \cup [\xi \alpha_1]$ solves $G$. If $\eta \in A - A^1$ and $U_1$ solves $Q_1(\eta)$ then $V_2 \cup U \cup U_1 \cup [\xi \alpha_1]$ is a solution for $G$. If $\eta \in A^1$ then $\eta = \vartheta$ and $V_1 \cup V_2 \cup U$ is a solution of $G$.

IV.2.3. $(3.b)$ holds in IV.2.1. So $F^1$ has $c_2$-shape. By II.5 there is a solution $V$ of $Q_2(\alpha_1)$ such that $V \cap F^1 = \{\alpha_1\}$. Next we show that there is a solution $V_1$ of $Q_1(\rho)$ such that $V_1 \cap A^1 = \{\rho\}$. If $Q_1(\rho) \cap A^1 = \{\rho\}$ this follows from the fact that $\rho$ belongs to every solution of $Q_1(\rho)$. If there is $x \in Q_1(\rho) \cap A^1, x \neq \rho$, then $x \in a^1$ and $(6)$ $x^2 = \rho^2$. Using I.10 and observing that $\rho \in A^3$ we see that $F^2$ has $C_3$-shape and $\rho \in F^2$. II.5 yields a desired $V_1$. Now define $v = (a_3^1, \rho^2, e - a_3^1 - \rho^2)$. Observe that $Q_3(v) \cap A^3 = \emptyset$. If $v \in A$ and $V_2$ is a solution of $Q_3(v)$ then $V \cap V_1 \cup V_2 \cup [\rho \alpha_1]$ is a solution of $G$. If $v \notin A$ then $V \cup V_1 \cup [\rho \alpha_1]$ solves $G$.

REFERENCES


THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL

(5) Suppose there is $z \in A^1 \cap Q_1(\vartheta)$ with $z^3 > \vartheta^3, a_3^1 \in A^1$ so there is $u \in A^1$ with $u^2 > a_3^1 \vartheta^2$. For small $t > 0, y = tu + (1 - t)z$ satisfy $y^3 > \vartheta^3$ and $y \in A^1$ which is impossible since $\vartheta \in a^1$.

(6) By an argument similar to that in footnote (5).