# SOLUTIONS TO COOPERATIVE GAMES WITHOUT SIDE PAYMENTS

BY

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An extension of Von Neumann Morgenstern solution theory to cooperative games without side payments has been outlined in [1]. In this paper we revise some of the definitions given in [1] and prove that in the new theory every three-person constant sum game is solvable (see [1, Theorem 1]). Other results that were formulated in [1] had already been proved in [2]. [1; 2] are also necessary for a full understanding of the basic definitions of this paper.

1. **Basic definitions.** If $N$ is a set with $n$ members, we denote by $E^N$ the $n$-dimensional euclidean space the coordinates of whose points are indexed by the members of $N$. Subsets of $N$ will be denoted by $S$. If $x \in E^N$ and $i \in N$, $x^i$ will denote the coordinate of $x$ corresponding to $i$; $x^S$ will denote the set $\{x^i : i \in S\}$. The superscript $N$ will be omitted, thus we write $x$ instead of $x^N$. We write $x^S \geqq y^S$ if $x^i \geqq y^i$ for all $i \in S$; similarly for $>$ and $=$. $\emptyset$ denotes the empty set.

DEFINITION 1.1. An *n-person characteristic function* is a pair $(N,v)$ where $N$ is a set with $n$ members, and $v$ is a function that carries each $S \subset N$ into a set $v(S) \subset E^N$ so that

(1) $v(S)$ is closed,

(2) $v(S)$ is convex,

(3) $v(\emptyset) = E^N$,

(4) if $x \in v(S)$ and $x^S \geqq y^S$ then $y \in v(S)$.

DEFINITION 1.2. An *n-person game* is a triad $(N, v, H)$, where $(N, v)$ is an $n$-person characteristic function and $H$ is a convex compact subset of $v(N)$.

We notice that this definition is not identical with that given in [1; 2]. In the first place $v$ is not assumed to be superadditive, i.e., the condition: $v(S_1 \cup S_2) \supset v(S_1) \cap v(S_2)$ for every pair of disjoint coalitions $S_1$ and $S_2$ is dropped. Secondly $H$ need not be a polyhedron.

2. **Solutions.** Let $G = (N, v, H)$ be an $n$-person game.

DEFINITION 2.1. Let $x, y \in E^N$, $S \neq \emptyset$. *$x$ dominates $y$ via $S$*, written $x \succ_S y$, if $x \in v(S)$ and $x^S > y^S$.

DEFINITION 2.2. *$x$ dominates $y$*, written $x \succ y$, if there is an $S$ such that $x \succ_S y$.

For $x \in E^N$ the following sets are defined: $\operatorname{dom}_S x = \{y : x \succ_S y\}$ and $\operatorname{dom} x = \{y : x \succ y\}$. Let $K \subset E^N$. We define $\operatorname{dom}_S K = \bigcup_{x \in K} \operatorname{dom}_S x$ and $\operatorname{dom} K = \bigcup_{x \in K} \operatorname{dom} x$.

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DEFINITION 2.3. $V$ is *K-stable* if $V = K - \operatorname{dom} V$.

DEFINITION 2.4. The *K-core* is the set $K - \operatorname{dom} K$.

We use the following abbreviation: P. S. O.—the proof, which is straightforward, will be omitted.

PROPOSITION 2.5. *Every K-stable set contains the K-core. P.S.O.*

PROPOSITION 2.6. *If for each* $x \in K \cap \operatorname{dom} K$ *there is a* $y \in K - \operatorname{dom} K$ *such that* $y \succ x$ *then the K-core is the only K-stable set. P.S.O.*

We denote: $v^i = \sup_{x \in v(\{i\})} x^i$.

DEFINITION 2.7. $x$ is *individually rational* if $x^i \geqq v^i$ for all $i \in N$.

DEFINITION 2.8. $x$ is *group rational* if there is no $y \in H$ such that $y > x$.

We denote: $\bar{A} = \{x : x \in H,\ x \text{ is individually rational}\}$ and $A = \{x : x \in \bar{A},\ x \text{ is group rational}\}$.

PROPOSITION 2.9. *K is* $\bar{A}$*-stable if and only if it is A-stable.*

**Proof.** Let $K$ be $\bar{A}$-stable. We show firstly that (1) $\bar{A} - A \subset \operatorname{dom} K$. If $x \in \bar{A} - A$ then there is a $y_0 \in \bar{A}$ such that $y_0 > x$. Define $f(y) = \min_{i \in N} (y^i - x^i)$. Since $f$ is continuous and $\bar{A}$ is compact $f$ receives its maximum in $\bar{A}$ at a point $z$, which must be in $A$. By 1.2 $z \in v(N)$. $f(z) \geqq f(y_0)$; therefore $z > x$. We have that $z \succ_N x$ and if $w \succ z$ then $w \succ x$. If $z \in K$ then $x \in \operatorname{dom} K$. If $z \in \operatorname{dom} K$ then there is a $w_0 \in K$ such that $w_0 \succ z$ and therefore $w_0 \succ x$, so $x \in \operatorname{dom} K$. From (1) it follows that $\bar{A} - \operatorname{dom} K = A - \operatorname{dom} K$ and therefore $K$ is $A$-stable. Now, let $K$ be $A$-stable. If $x \in \bar{A} - A$ we define $z$ as before and we see that $z \in A \subset K \cup \operatorname{dom} K$ implies that $x \in \operatorname{dom} K$. We conclude that (1) holds and therefore $K = A - \operatorname{dom} K = \bar{A} - \operatorname{dom} K$, i.e., $K$ is $\bar{A}$ stable.

DEFINITION 2.10. A *solution* of $G$ is an $A$-stable set.

If $G$ has a solution we say that $G$ is solvable.

THEOREM 2.11. *Every two person game has a unique solution, consisting of all of A. P.S.O.*

DEFINITION 2.12. $G$ is *constant-sum if* $H$ is contained in a plane

$$\sum_{i \in N} x^i = e.$$

3. **Three-person constant sum games.**

I. *Auxiliary lemmas.* We use the following abbreviations: 3-P.C.G. — three-person constant sum game, W. L. G.—without loss of generality.

Let $G = (N, v, H)$ be 3-P.C.G. We denote the members of $N$ by the first three positive integers and set $S_i = N - \{i\}$ for $i = 1, 2, 3$. Let $x \in H$. We denote: $\sum_{i=1}^{3} x^i = e$ and $L = \{y : \sum_{i=1}^{3} y^i = e\}$. We have that $\bar{A} = \{x : x \in H,\ x^i \geqq v^i,\ i = 1, 2, 3\} = A$. So $A$ is a convex compact subset of $L$. Domination between points of $A$ is possible only via the $S_i$, i.e., if $x, y \in A$ and $x \succ_S y$ then $S$ is one of the $S_i$. For a subset $B$ of $L$ and $i \in N$ the following sets are defined: $B^i = B \cap v(S_i)$, $\mathring{B}^i = B^i \cap \mathrm{dom}_{S_i} B^i$ and $b^i = B^i - \mathring{B}^i$.

LEMMA I.1. *If $B$ is convex then $\mathring{B}^i$ is convex.*

**Proof.** $B^i = B \cap v(S_i)$ is convex. If $x_1, x_2 \in \mathring{B}^i$ then there are $y_1, y_2 \in B^i$ such that $y_j \succ_{S_i} x_j$ for $j = 1,2$. If $0 < t < 1$, $x = t x_1 + (1-t) x_2$ and $y = t y_1 + (1-t) y_2$ then $x, y \in B^i$ and $y \succ_{S_i} x$, so $x \in \mathring{B}^i$.

We remark that $A^i$ is convex and compact, $\mathring{A}^i$ is convex and $a^i$ is compact.

Let $x \in L$ and $\varepsilon > 0$. The set $\{y : y \in L, \sum_{i=1}^{3} (y^i - x^i)^2 < \varepsilon^2\}$ is denoted by $S(x, \varepsilon)$. $x$ is an *interior point* of a subset $B$ of $L$ if there is an $\varepsilon > 0$ such that $S(x, \varepsilon) \subset B$.

LEMMA I.2. *If $B \subset L$ is convex and $K = \mathring{B}^i \cap \mathring{B}^j \neq \emptyset$, $i \neq j$, then $K$ contains an interior point.*

**Proof.** W.L.G. $i = 1$ and $j = 2$. We show firstly that $K \neq \emptyset$ implies that $B$ contains an interior point. If $B$ has no interior points then there are points $x_1$ and $x_2$ such that every $y \in B$ can be written as $y = t x_1 + (1-t) x_2$, $-\infty < t < \infty$. Let $x \in K$. $x = t_0 x_1 + (1 - t_0) x_2$. There are $y_l = t_l x_1 + (1 - t_l) x_2$, $y_l \succ_{S_1} x$ for $l = 1,2$. We have $y_1^2 > x^2$ and $y_1^3 > x^3$, i.e., $t_1 x_1^2 + (1 - t_1) x_2^2 > t_0 x_1^2 + (1 - t_0) x_2^2$ and $t_1 x_1^3 + (1 - t_1) x_2^3 > t_0 x_1^3 + (1 - t_0) x_2^3$. So $(t_1 - t_0)(x_1^2 - x_2^2) > 0$ and $(t_1 - t_0)(x_1^3 - x_2^3) > 0$. Therefore $\mathrm{sgn}(x_1^2 - x_2^2) = \mathrm{sgn}(x_1^3 - x_2^3)$. In the same way $y_2 \succ_{S_2} x$ implies that $\mathrm{sgn}\,(x_1^1 - x_2^1) = \mathrm{sgn}(x_1^3 - x_2^3)$. So the three differences $x_1^k - x_2^k$ have the same sign, which is impossible since $\sum_{k=1}^{3} x_1^k = \sum_{k=1}^{3} x_2^k$. Now, let $z$ be an interior point of $B$ and $y \in K$. For small positive $t$ the points $tz + (1-t)y$ are interior points of $K$.

LEMMA I.3. *If $x \in O = \bigcap_{i=1}^{3} \mathring{A}^i$ then $x$ is an interior point of $O$.*

**Proof.** There are $y_j \succ_{S_j} x$ for $j = 1,2,3$. We have: $y_1^2 > x^2, y_1^3 > x^3, y_2^1 > x^1$, $y_2^3 > x^3, y_3^1 > x^1$ and $y_3^2 > x^2$. There exist $0 < t_k < 1$ such that $z_k = t_k y_1 + (1 - t_k) y_k$ satisfy $z_k^1 = x^1$, $k = 2,3$. Since $z_2^3 > x^3$ and $z_3^3 < x^3$ there is a $0 < t_1 < 1$ such that $x = t_1 z_2 + (1 - t_1) z_3$. So $x$ is an interior point of the convex hull of $\{y_1, y_2, y_3\}$ and therefore of $A$. But if $x \in O$ is an interior point of $A$ then $x$ is also an interior point of $O$.

LEMMA I.4. *If $B \subset L$ is convex, $x_1, x_2 \in B$, $x_1^k = x_2^k$, $x_1^i < x_2^i$, $x_1^j > x_2^j$ and $y$ satisfies $y^i = x_2^i$, $y^j = x_1^j$ and $y^k = x_1^i + x_1^k - x_2^i$ then: $y \notin B$ if and only if $B \cap \{z : z^{S_k} \geqq y^{S_k}\} = \emptyset$. P.S.O.*

LEMMA I.5. *If $x \in \mathring{A}^i$ then there is a $y \in a^i$ such that $y \succ_{S_i} x$ and for every $\varepsilon > 0$, $S(y, \varepsilon) \cap \mathring{A}^i \neq \emptyset$.*

**Proof.** Define $f(z) = \min_{j \in S_i} (z^j - x^j)$. $f$ receives its maximum in $A^i$ at a point

$y$ which must be in $a^i$. Since $x \in \mathring{A}^i f(y) > 0$ and therefore $y \succ_{S_1} x$. If $0 \leqq t < 1$ then $ty + (1-t)x$ is in $\mathring{A}^i$, therefore for every $\varepsilon > 0$ $S(y, \varepsilon) \cap \mathring{A}^i \neq \emptyset$.

If $x, y \in L$ then the set $\{z : z = tx + (1-t)y, 0 \leqq t \leqq 1\}$ is denoted by $[x\,y]$ and is called an *interval*. $x$ and $y$ are called the *ends* of $[xy]$ $\cdot$ $(xy] = [xy] - \{x\}$ $\cdot [xy) = [xy] - \{y\} \cdot (xy) = [xy] - (\{x\} \cup \{y\})$. For $i = 1,2,3$ the following sets are defined: $D^i = A^j \cap A^k$, where $S_i = \{j,k\}$, and $F^i = \{x : x \in D^i, x^i \geqq y^i$ for every $y \in D^i\}\cdot$ $D^i$ is convex and compact. $F^i$ is an interval.

LEMMA I.6. *Let $S_k = \{i,j\}$. If $x^i$ receives its maximum in $F^k$ at a point $\alpha$, then: $\alpha \notin a^j$ if and only if $\mathring{A}^j \supset A^i$.*

**Proof.** W.L.G. $i = 2$ and $j = 3$. If $\mathring{A}^3 \supset A^2$ then $\alpha \in F^1 \subset D^1 \subset A^2 \subset \mathring{A}^3$. If $\alpha \notin a^3$ then there is an $x \in A^3$ such that $x \succ_{S_3} \alpha$. There is an $\varepsilon > 0$ such that $U = S(\alpha, \varepsilon) \cap A \subset \mathring{A}^3$. Now we show that if $y \in A^2$ then $y^1 \leqq \alpha^1$. If there is a $z \in A^2$ such that $z^1 > \alpha^1$ then for a small positive $tu = tz + (1-t)\alpha$ satisfies $u^1 > \alpha^1$ and $u \in U$. So we have $u \in D^1$ and $u^1 > \alpha^1$ which is impossible. Next we show that $y \in A^2$ implies that $y^2 \leqq \alpha^2$. Suppose that there is a $z \in A^2$ such that $z^2 > \alpha^2$. If $z^1 = \alpha^1$ then for a small positive $tu = tz + (1-t)\alpha$ satisfies $u \in U$, $u^1 = \alpha^1$ and $u^2 > \alpha^2$. So we have that $u \in F^1$ and $u^2 > \alpha^2$ which is impossible. If $z^1 < \alpha^1$ then there is a $0 < t < 1$ such that $w = tz + (1-t)x$ satisfies $w^1 = \alpha^1$ and $w^2 > \alpha^2$ $\cdot \alpha^{S_2} \geqq w^{S_2}$ therefore $w \in A^2$, but this is impossible as we have already shown. We have shown that every $y \in A^2$ satisfies $y^{S_3} \leqq \alpha^{S_3}$. Since $\alpha \in \mathrm{dom}_{S_3} x$ we have $A^2 \subset A \cap \mathrm{dom}_{S_3} x \subset \mathring{A}^3$.

The sets $\{A^1, A^2\}$, $\{A^2, A^3\}$ and $\{A^3, A^1\}$ will be called *pairs*.

DEFINITION I.7. The pair $\{A^i, A^j\}$ *intersects maximally* if:

(1) $\mathring{A}^i \cap \mathring{A}^j \neq \emptyset$.

(2) $a^i \cap a^j \neq \emptyset$.

The number of pairs that intersect maximally will be denoted by $m(G)$.

LEMMA I.8. *Let $i \neq j$ and $\mathring{A}^i \cap \mathring{A}^j \neq \emptyset$. $\mathring{A}^i \not\supset A^j$ and $\mathring{A}^j \not\supset A^i$ if and only if $\{A^i, A^j\}$ intersects maximally.*

**Proof.** W.L.G. $i = 2$ and $j = 3$. If $\mathring{A}^3 \supset A^2$ or $\mathring{A}^2 \supset A^3$ then $a^2 \cap a^3 = \emptyset$ and therefore $\{A^2, A^3\}$ does not intersect maximally. Now suppose that $\mathring{A}^2 \not\supset A^3$ and $\mathring{A}^3 \not\supset A^2$. Let $x^2$ and $x^3$ receive their maxima in $F^1$ at the points $\alpha$ and $\beta$ respectively. By I.6: $A^2 \not\subset \mathring{A}^3$ implies that $\alpha \in a^3$ and $\mathring{A}^2 \not\supset A^3$ implies that $\beta \in a^2$. We have (1) that $F^1 \subset a^2 \cup a^3$ and $F^1 \cap a^k \neq \emptyset$, $k = 2,3$. Since $F^1$ is connected and $F^1 \cap a^3$ and $F^1 \cap a^2$ are closed we must have $(F^1 \cap a^2) \cap (F^1 \cap a^3) = F^1 \cap a^2 \cap a^3 \neq \emptyset$.

From the proof of I.8 we can conclude that: (I.9) if $\{A^i, A^j\}$ intersects maximally then $F^k \cap a^i \cap a^j \neq \emptyset$ where $\{k\} = N - \{i,j\}$.

(1) Otherwise there is $x \in F^1 \cap \mathring{A}^2 \cap \mathring{A}^3$. Let $z \succ_{S_2} x$. For small $t > 0$, $u = tz + (1-t)x$ satisfy $u^1 > x^1$ and $u \in D^1$, which is impossible.

Lemma I.10. *If* $i \neq j, x, y \in a^i$, $x \neq y$ *and* $x^j = y^j$ *then every* $z \in A^i$ *satisfies* $z^j \leqq x^j$ *and* $\{u : u \in A^i, u^j = x^j\} \subset a^i$. *P.S.O.*

Lemma I.11. *Let* $S_k = \{i,j\}$. *If* $\{A^i, A^j\}$ *intersects maximally and* $x^i$ *and* $x^j$ *take their maxima in* $F^k$ *at the points* $\alpha$ *and* $\beta$ *respectively then* $F^k = [\alpha\beta]$ *and one of the following possibilities holds:*

| | | | |
|---|---|---|---|
| (a) | $\alpha = \beta$, | $\alpha \in a^i \cap a^j$ | |
| (b) | $\alpha \neq \beta$, | $[\alpha\beta] \subset a^i \cap a^j$, | |
| ($c_j$) | $\alpha \neq \beta$, | $(\alpha\beta] \subset a^i \cap \mathring{A}^j$, | $\alpha \in a^i \cap a^j$, |
| ($c_i$) | $\alpha \neq \beta$, | $[\alpha\beta) \subset a^j \cap \mathring{A}^i$, | $\beta \in a^i \cap a^j$. |

**Proof.** W.L.G. $i = 2$ and $j = 3$. We saw in the proof of I.8 that $\alpha \in F^1 \cap a^3$ and $\beta \in F^1 \cap a^2$. If $\alpha = \beta$ then (a) holds. If $\alpha \neq \beta$ we have the following possibilities for the relative positions of $a^3$ and $F^1$:

(1) There is no $x \neq \alpha$ in $a^3 \cap F^1$, i.e., $(\alpha\beta] \subset \mathring{A}^3$.

(2) There is an $x \neq \alpha$ in $a^3 \cap F^1$, and, therefore, by I.10, $F^1 \subset a^3$. And similarly for $a^2$ and $F^1$:

(3) There is no $y \neq \beta$ in $a^2 \cap F^1$, i.e., $[\alpha\beta) \subset \mathring{A}^2$.

(4) There is a $y \neq \beta$ in $a^2 \cap F^1$ and therefore $F^1 \subset a^2$.

Since $F^1 \subset a^2 \cup a^3$ (1) and (3) cannot hold together. If (2) and (4) hold together then we have (b). If (1) and (4) hold together, then we have ($c_j$). If (2) and (3) hold together then we have ($c_i$). We say that $F^k$ has $a$-shape if (a) holds; similarly for (b), ($c_i$) and ($c_j$).

For $x \in A$ the following sets are defined: $Q_i(x) = \{y : y \in A, \ y^{S_i} \geqq x^{S_i}\}$, $T_i(x) = \{y : y \in A, y^i < x^i\}$ and $R_i(x) = A - T_i(x)$. We remark that:

(I.12) $\operatorname{dom} Q_i(x) \cap R_i(x) = \operatorname{dom}_{S_i} Q_i(x) \cap R_i(x)$,

(I.13) $x \notin \mathring{A}^i$ if and only if $x \in Q_i(x) - \operatorname{dom} Q_i(x)$,

(I.14) $x \notin \mathring{A}^i$ if and only if $Q_i(x) \cap \mathring{A}^i = \emptyset$.

Lemma I.15. *If* $x \in A - \mathring{A}^i$; $y \neq x$, $y \in A^i \cap Q_i(x)$ *then there is a* $j \in S_i$ *such that every* $z \in A^i$ *satisfies* $z^j \leqq x^j$. *P.S.O.*

Lemma I.16. *Let* $S_k = \{i,j\}$. *We denote the ends of* $F^k$ *by* $\alpha$ *and* $\beta$ *such that* $\alpha^i \geqq \beta^i$. *If* $\gamma \in A$ *satisfies* $\gamma^k = \alpha^k$ *and* $\gamma^i < \beta^i$ *then* $Q_i(\gamma) \cap A^i = \emptyset$. *P.S.O.*

Lemma I.17. *Let* $S_k = \{i, j\}$. *If* $\gamma \in F^k \cap a^i \cap a^j$ *and* $x \in R_k(\gamma) - F^k$ *then* $\operatorname{dom} x \cap Q_k(\gamma) = \emptyset$.

**Proof.** W.L.G. $i = 2$ and $j = 3$. We denote the ends of $F^1$ by $\alpha$ and $\beta$ such that $\alpha^2 \geqq \beta^2$. Let $x \in R_1(\gamma) - F^1$ and $y \in Q_1(\gamma) \cdot x^2 + x^3 \leqq \gamma^2 + \gamma^3 \leqq y^2 + y^3$ so $x \succ_{S_1} y$ is impossible. If $x^1 > \gamma^1$ and $x \succ_{S_2} y$ or $x \succ_{S_3} y$ then $x \succ_{S_2} \gamma$ or $x \succ_{S_3} \gamma$ respectively, which is impossible. If $x^1 = \gamma^1$ then either $x^2 < \beta^2$ or $x^3 < \alpha^3$. If $x^3 < \alpha^3$ then, by I.16, $x \notin A^3$. Since $y^3 \geqq \gamma^3 \geqq \alpha^3 > x^3$, if $x \succ y$ then $x \succ_{S_3} y$, but this is impossible. Similarly if $x^2 < \beta^2$ then $x \succ y$ is impossible.

LEMMA I.18. *If a* $B \subset A$ *is convex and compact then* $(N,v,B)$ *is 3-P.C.G.,* $B^i = A^i \cap B$ *and* $\mathring{B}^i \subset \mathring{A}^i \cap B$. *P.S.O.*

*If* $B \subset A$ *is convex and compact we say that* $B$ *is solvable or that* $B$ *has a solution if* $(N,v,B)$ *is solvable. We also write* $m(B)$ *instead of* $m((N,v,B))$.

LEMMA I.19. *If* $x \in D^k$ *and* $l \in S_k$ *then* $\mathring{R}^l_k(x) = \mathring{A}^l \cap R_k(x)$.

**Proof.** If $y \in \mathring{A}^l \cap R_k(x)$ then $y^k \geqq x^k$ and there is a $z \in A^l$ such that $z \succ_{S_1} y$. Since $z^k > y^k$ $z \in R_k(x)$. So we have $z \in R_k(x) \cap A^l = R^l_k(x)$ and therefore $y \in \mathring{R}^l_k(x)$. We have shown that $\mathring{R}^l_k(x) \supset \mathring{A}^l \cap R_k(x)$. By I.18 $\mathring{R}^l_k(x) \subset \mathring{A}^l \cap R_k(x)$, so $\mathring{R}^l_k(x) = \mathring{A}^l \cap R_k(x)$.

LEMMA I.20. *If* $x \in D^k - \mathring{A}^k$, $l \in S_k$ *and* $\{A^k, A^l\}$ *does not intersect maximally, then* $\{R^1_k(x), R^k_k(x)\}$ *does not intersect maximally.*

**Proof.** Since $\{A^k, A^i\}$ does not intersect maximally, by I.8 at least one of the the following possibilities holds: $\mathring{A}^k \cap \mathring{A}^l = \emptyset$, $\mathring{A}^l \supset A^k$ or $\mathring{A}^k \supset A^l$. If $\mathring{A}^k \cap \mathring{A}^l = \emptyset$ then $\mathring{R}^k_k(x) \cap \mathring{R}^l_k(x) \subset \mathring{A}^k \cap \mathring{A}^l \cap R_k(x) = \emptyset$. If $\mathring{A}^l \supset A^k$ then $\mathring{R}^l_k(x) = \mathring{A}^l \cap R_k(x) \supset A^k \cap R_k(x) = R^k_k(x)$. $\mathring{A}^k \supset A^l$ is impossible since $x \in A^l - \mathring{A}^k$.

DEFINITION I.21. Let $B_1, \cdots, B_l$ be convex compact subsets of $A$. $B_1, \cdots, B_l$ are called *independent* if there exist solutions $V_1, \cdots, V_l$, $V_i$ solution of $B_i$ respectively, such that dom $V_k \cap (\bigcup_{j=1}^l V_j) = \emptyset$ for $k = 1, \cdots, l$.

LEMMA I.22. *If* $B_1, \cdots, Bl$ *are independent then there exist solutions* $V_1, \cdots, V_l$, $V_i$ *solution of* $B_i$ *for* $i = 1, \cdots, l$, *such that* $\bigcup_{j=1}^l V_j$ *is* $\bigcup_{j=1}^l B_j$*-stable. P.S.O.*

In the following three subsections we shall prove:

THEOREM. *Every 3-P.C.G. G is solvable.*

The proof will be by induction on $m(G)$.

II. *First part*: $m(G) = 0$. In this subsection we show that every 3-P.C.G. $G$ for which $m(G) = 0$ is solvable. We also prove some additional auxiliary lemmas:

LEMMA II.1. *Let G be 3-P.C.G. If* $\mathring{A}^1 \cap \mathring{A}^2 = \mathring{A}^2 \cap \mathring{A}^3 = \mathring{A}^3 \cap \mathring{A}^1 = \emptyset$ *then the A-core is the solution of G.*

**Proof.** Denote $C = A - \text{dom} A$. If $x \in A - C$ then there is a $y \in A$ that dominates it. There is an $i$ such that $y \succ_{S_i} x$, i.e., $x \in \mathring{A}^i$. By I.5 there is a $z \in a^i$ such that $z \succ_{S_i} x$. If $z \notin C$ then $z \in \mathring{A}^l$ where $l \neq i$. There is an $\varepsilon > 0$ such that $S(z,\varepsilon) \cap A \subset \mathring{A}^l$. But $S(z,\varepsilon) \cap \mathring{A}^i \neq \emptyset$; therefore $\mathring{A}^i \cap \mathring{A}^l \neq \emptyset$ which is impossible. We have shown that for every $x \in A - C$ there is a $z \in C$ such that $z \succ x$. By 2.6 $C$ is the only $A$-stable set.

LEMMA II.2. *Let G be 3-P.C.G. If* $m(G) = 0$ *then the A-core is the solution of G.*

**Proof.** If $\mathring{A}^1 \cap \mathring{A}^2 = \mathring{A}^2 \cap \mathring{A}^3 = \mathring{A}^3 \cap \mathring{A}^1 = \emptyset$ then by II.1 the $A$-core is the solution of $G$. If it is not the case then, W.L.G., we assume that $\mathring{A}^2 \cap \mathring{A}^3 \neq \emptyset$. Since $\{A^2, A^3\}$ does not intersect maximally we have that either $\mathring{A}^3 \supset A^2$ or $\mathring{A}^2 \supset A^3$.

W.L.G. we suppose that $\mathring{A}^3 \supset A^2$. There are three possibilities for the relative position of $A^1$ and $A^3$: (a) $\mathring{A}^1 \cap \mathring{A}^3 = \emptyset$, (b) $\mathring{A}^3 \supset A^1$ or (c) $\mathring{A}^1 \supset A^3$. In each case we show that $C$, the $A$-core, is the solution of $G$. (a) $\mathring{A}^1 \cap \mathring{A}^3 = \emptyset$. If $x \in A - C$ then there is a $y \in A$ such that $y \succ_{S_i} x$ for some $i$. Since $\mathring{A}^3 \supset A^2$ we may assume that $i \in S_2$. There is a $z \in a^i$ such that $z \succ_{S_i} x$. If $i = 1$ then, since $\mathring{A}^1 \cap \mathring{A}^j = \emptyset$ for $j = 2, 3$, $z \in C$. If $i = 3$ then, since $a^3 \cap \mathring{A}^2 = \emptyset$ and $\mathring{A}^1 \cap \mathring{A}^3 = \emptyset$, $z \in C$. By 2.6 $C$ is the solution of $G$. (b) $\mathring{A}^3 \supset A^1$. If $x \in A - C$ then there is a $y \in A$ such that $y \succ_{S_3} x$. So there is a $z \in a^3$ such that $z \succ x$. Since $a^3 \cap (\mathring{A}^1 \cup \mathring{A}^2) = \emptyset$, $z \in C$. So $C$ is the solution of $G$. (c) $\mathring{A}^1 \supset A^3$. The proof in this case parallels that in case (b).

Let $G = (N, v, H)$ be 3-P.C.G.

Lemma II.3. *If $\xi \in D^k - \mathring{A}^k$, $\eta \in D^k \cap Q_k(\xi)$ and $U$ is a solution of $Q_k(\eta)$ then $V = U \cup [\xi\eta]$ is a solution of $Q_k(\xi)$.*

**Proof.** W.L.G. $k = 1$. $\eta^1 \leqq \xi^1$, $\eta^2 \geqq \xi^2$ and $\eta^3 \geqq \xi^3$; therefore $(\mathrm{dom}_{S_3}[\xi\eta] \cup \mathrm{dom}_{S_2}[\xi\eta]) \cap [\xi\eta] = \emptyset$. $\xi \cdots \mathring{A}^1$; therefore $Q_1(\xi) \cap \mathring{A}^1 = \emptyset$. So $\mathrm{dom}_{S_1}[\xi\eta] \cap [\xi\eta] = \emptyset$. Summing we have (1) $\mathrm{dom}[\xi\eta] \cap [\xi\eta] = \emptyset$. Now we show (2) $Q_1(\xi) - [\xi\eta] - Q_1(\eta) \subset \mathrm{dom}[\xi\eta]$. Let $x \in Q_1(\xi) - [\xi\eta] - Q_1(\eta)$. If $x^1 \geqq \eta^1$ then there is a $y \in [\xi\eta]$ such that $y^1 = x^1$. $y \neq x$ so we may assume that $y^2 > x^2$. Under this assumption we can find a $z \in [\xi\eta]$ with $z^{S_3} > x^{S_3}$, so $z \succ x$. If $x^1 < \eta^1$ then, since $x \notin Q_1(\eta)$, either $x^2 < \eta^2$ or $x^3 < \eta^3$ and therefore $x \in \mathrm{dom}_{S_3}\eta \cup \mathrm{dom}_{S_2}\eta$. We now prove (3) $\mathrm{dom}\, U \cap [\xi\eta] = \emptyset$ and $\mathrm{dom}[\xi\eta] \cap U = \emptyset$. Let $x \in [\xi\eta]$ and $y \in U$. $y^1 \leqq x^1$, $y^2 \geqq x^2$ and $y^3 \geqq x^3$ therefore $x \succ y$ is impossible and if $y \succ x$ then $y \succ_{S_1} x$, but, since $x \notin \mathring{A}^1$, this is also impossible. Combining (1), (2) and (3) it follows that $V = U \cup [\xi\eta]$ solves $Q_1(\xi)$.

Lemma II.4. If $\xi \in D^k - \mathring{A}^k$ *then $Q_k(\xi)$ is solvable and if $V$ solves it then* $\mathrm{dom}\, V \supset T_k(\xi) - V$.

**Proof.** W.L.G. $k = 1$. Denote $J = Q_1(\xi) \cap D^1$. Let $\eta$ be a point where $x^1$ receives its minimum in $J$. We show that $\mathring{Q}_1^i(\eta) \cap \mathring{Q}_1^j(\eta) = \emptyset$ for all $i \neq j$. First, since $\xi \notin \mathring{A}^1$, $\mathring{A}^1 \cap Q_1(\xi) = \emptyset$. So we have $\mathring{Q}_1^1(\eta) \subset \mathring{A}^1 \cap Q_1(\eta) \subset \mathring{A}^1 \cap Q_1(\xi) = \emptyset$. Next, since $x^1$ receives its minimum in $J$ at $\eta$ $Q_1(\eta) \cap D^1 = \{\eta\}$. $\eta \notin \mathring{A}^1$ therefore $\eta \in Q_1(\eta)$-core. We have $\mathring{Q}_1^2(\eta) \cap \mathring{Q}_1^3(\eta) \subset Q_1^2(\eta) \cap Q_1^3(\eta) = D^1 \cap Q_1(\eta) = \{\eta\}$, so $\mathring{Q}_1^2(\eta) \cap \mathring{Q}_1^3(\eta) = \emptyset$. By II.1 $Q_1(\eta)$ is solvable. Let $U$ be a solution of $Q_1(\eta)$; by II.3 $U \cup [\xi\eta]$ solves $Q_1(\xi)$. Now let $V$ be a solution of $Q_1(\xi)$. $\xi \in Q_1(\xi)$-core so $\xi \in V$. $\mathrm{dom}\,\xi \supset T_1(\xi) - Q_1(\xi)$ therefore $\mathrm{dom}\, V \supset (Q_1(\xi) - V) \cup (T_1(\xi) - Q_1(\xi)) = T_1(\xi) - V$.

Lemma II.5. *Let $S_k = \{i, j\}$. If $F^k$ has $c_j$-shape and $\mu \in F^k \cap A^k$ then there is a solution $V$ of $Q_i(\mu)$ such that $V \cap F^k = \{\mu\}$.*

**Proof.** W.L.G. $i = 2$ and $j = 3$. We denote the ends of $F^1$ by $\alpha$ and $\beta$ such that $\alpha^2 > \beta^2$. Since $F^1$ has $c_3$-shape we have (1) $\mathring{A}^2 \cap R_1(\alpha) = \emptyset$ and (2) $[\beta\alpha) \subset \mathring{A}^3$. From (1) it follows that (3) $\mathring{Q}_2^2(\mu) = \emptyset$. We show (4) $[\beta\mu) \subset \mathring{Q}_2^3(\mu)$. Let $x \in [\beta\mu)$.

$x^1 = \mu^1$, $x^2 < \mu^2$ and $x^3 > \mu^3$. By (2) $x \in \mathring{A}^3$, therefore there is *a* $y \in A^3$ such that $y \succ_{S_3} x$. For small $t > 0$ $z = ty + (1-t)x$ satisfy $z^1 > x^1$, $z^2 > x^2$ and $z^3 > \mu^3$, so $z \in Q_2^3(\mu)$ and $x \in \mathring{Q}_2^3(\mu)$. For the relative position of $Q_2^1(\mu)$ and $Q_2^3(\mu)$ we have the following possibilities: (a) $\mathring{Q}_2^1(\mu) \cap \mathring{Q}_2^3(\mu) = \emptyset$ or (b) $\mathring{Q}_2^1(\mu) \cap \mathring{Q}_2^3(\mu) \neq \emptyset$. If (a) holds then by (3) $\mathring{Q}_2^i(\mu) \cap \mathring{Q}_2^j(\mu) = \emptyset$ for all $i \neq j$. By II.1 $Q_2(\mu) - \operatorname{dom} Q_2(\mu)$ is a solution of $Q_2(\mu)$ and since $\mu \in Q_2(\mu)$-core, $(Q_2(\mu) - \operatorname{dom} Q_2(\mu)) \cap F^1 = \{\mu\}$. If (b) then by I.2 there is an interior point $\zeta$ of $\mathring{Q}_2^1(\mu) \cap \mathring{Q}_2^3(\mu)$. $\zeta^1 > \mu^1$, $\zeta^2 < \mu^2$ and $\zeta^3 > \mu^3$. $\zeta \in D^2 - A^2$, therefore by II.4 $Q_2(\zeta)$ is solvable. If $U$ is a solution of $Q_2(\zeta)$ then, by II.3, $[\zeta\mu] \cup U$ solves $Q_2(\mu)$. Since $([\zeta\mu] \cup U) \cap F^1 = \{\mu\}$ this completes the proof.

DEFINITION II.6. The pair $\{A^i, A^j\}$ *satisfies condition M* if:

(1) $\{A^i, A^j\}$ intersects maximally,

(2) $F^k \cap a^i \cap a^j \not\subset \mathring{A}^k$ where $\{k\} = N - \{i,j\}$.

We now formulate the induction hypothesis:

II.7. *every* 3-*P.C.G. G for which* $m(G) \leqq l - 1$ *is solvable.* Let $G$ be 3-P.C.G. for which $m(G) = l$. We have to prove that $G$ is solvable. We distinguish between the following possibilities:

II.8. *there is at least one pair that satisfies condition M.*

II.9. *there is no pair that satisfies condition M.*

III. *Second part*: *case* II.8. W.L.G. $\{A^2, A^3\}$ satisfies condition $M$. The ends of $F^1$ will be denoted by $\alpha$ and $\beta$ such that $\alpha^2 \geqq \beta^2$. $F^1 \cap a^2 \cap a^3 \not\subset \mathring{A}^1$ therefore at least one of the ends is in $a^2 \cap a^3 - \mathring{A}^1$. We shall prove that $G$ is solvable when: (III.1) $\alpha \in a^2 \cap a^3 - \mathring{A}^1$. The proof when $\beta \in a^2 \cap a^3 - \mathring{A}^1$ is similar to that in case (III.1). We shall distinguish three cases according to the three possible shapes of $F^1$ in case (III.1).

III.a. $F^1$ *has a-shape.* By (III.1) and II.4 $Q_1(\alpha)$ is solvable and if $V$ solves it then (1) $\operatorname{dom} V \supset T_1(\alpha) - V$. Since $\alpha \in F^1$ $\mathring{A}^2 \cap \mathring{A}^3 \cap R_1(\alpha) = \emptyset$. By I.19 $\mathring{R}_1^2(\alpha) \cap \mathring{R}_1^3(\alpha) = \emptyset$, so $\{R_1^2(\alpha), R_1^3(\alpha)\}$ does not intersect maximally. From I.20 it follows now that $m(R_1(\alpha)) \leqq l - 1$. By II.7 $R_1(\alpha)$ is solvable. If $Q_1(\alpha)$ and $R_1(\alpha)$ are independent then from I.22 and (1) it follows that $A$ has a solution. If $Q_1(\alpha)$ and $R_1(\alpha)$ are not independent then if $V$ solves $Q_1(\alpha)$ and $W$ solves $R_1(\alpha)$ either (2) $\operatorname{dom} V \cap W \neq \emptyset$ or (3) $\operatorname{dom} W \cap V \neq \emptyset$. From III.a and I.17 it follows that (4) $\operatorname{dom} R_1(\alpha) \cap Q_1(\alpha) = \emptyset$. By (4) we have that (3) is impossible. By (III.1) $\alpha \in A - \operatorname{dom} A$ therefore (5) $\alpha \in W \cap V$. From (2), (5) and I.12 it follows that there is a $z \neq \alpha$ in $V \cap A^1$. By I.15 and due to III.a, we may assume that every $y \in A^1$ satisfies $y^2 \leqq \alpha^2$. Let $\zeta$ be a point where $x^3$ receives its maximum(2) in $V \cap A^1$. If $u \in V \cap A^1$ then $\zeta^2 = u^2$ and $\zeta^3 \geqq u^3$ and therefore (6) $\operatorname{dom}_{S_1} \zeta \supset \operatorname{dom}_{S_1} u$. $\alpha \in D^3 - \mathring{A}^3$ therefore, by II.4, $Q_3(\alpha)$ is solvable. If $U$ solves $Q_3(\alpha)$ then by (4) we have that (7) $\operatorname{dom} U \cap V = \emptyset$. We remark that (8) $Q_3(\alpha) \cap A^3 \subset \{x\colon x^2 = \alpha^2\}$. Let $x^1$ receive its maximum in $U \cap A^3$ at the point $\eta$. We define: $v = (\eta^1, e - \eta^1 - \zeta^3, \zeta^3)$. By I.12 and (6) we have that (9) $R_1(\alpha) - \operatorname{dom} V = R_1(\alpha)$

(2) Observe that a solution of a compact set is compact. see [3, Theorem 3].

$-\operatorname{dom}_{S_1}\zeta = Q_3(\alpha) \cup \{x: x \in R_1(\alpha), x^3 \geqq \zeta^3\}$. By (8) we have that (10) $Q_2(\alpha) - \operatorname{dom} U = Q_2(\alpha) - \operatorname{dom}_{S_3}\eta = \{x: x \in Q_2(\alpha), x^1 \geqq \eta^1\}$. Combining (1), (9) and (10) we have (11) $A - \operatorname{dom}(U \cup V) = \{x: x \in A, x^{S_2} \geqq v^{S_2}\}$. If $v \notin A$ then from I.4, (11), (9) and (7) it follows that $U \cup V$ solves $A$. Suppose now that $v \in A$. We define $v_1 = (\alpha^1, e - \alpha^1 - \zeta^3, \zeta^3)$. By I.16 we have that (12) $\emptyset = A^2 \cap Q_2(v_1) \supset A^2 \cap Q_2(v)$. $Q_1(\alpha) \cup Q_3(\alpha) \subset R_2(v)$ therefore by (12) and I.12 we have (13) $\operatorname{dom} Q_2(v) \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset$. $v \in D^2 - A^2$ therefore $Q_2(v)$ is solvable. If $U_1$ solves $Q_2(v)$ then by (11) and (13) $V \cup U \cup U_1$ is a solution of $A$.

III.b. *$F^1$ has b-shape.* Due to III.b, we have that $R_1(\alpha) \cap (\mathring{A}^2 \cup \mathring{A}^3) = \emptyset$ and therefore (1) $\mathring{R}_1^2(\alpha) = \mathring{R}_1^3(\alpha) = \emptyset$. From (1) it follows that (2) $\mathring{R}_1^i(\alpha) \cap \mathring{R}_1^j(\alpha) = \emptyset$ for all $i \neq j$. If $x \in R_1(\alpha)$ then $x^2 + x^3 \leqq \alpha^2 + \alpha^3$ therefore (3) $[\alpha\beta] \cap \operatorname{dom}_{S_1} R_1(\alpha) = \emptyset$. From (1) and (3) we have that (4) $R_1(\alpha) - \operatorname{dom} R_1(\alpha) \supset [\alpha\beta]$. We also have that (5) $T_1(\alpha) - \operatorname{dom} R_1(\alpha) = T_1(\alpha) - \operatorname{dom}\{\alpha, \beta\} = \{x: x \in T_1(\alpha), x^2 \geqq \alpha^2, x^3 \geqq \beta^3\}$. Define $\mu = (e - \alpha^2 - \beta^3, \alpha^2, \beta^3)$.

III.b.1. $\mu \notin A$. By I.4 we have that (6) $\{x: x^{S_1} \geqq \mu^{S_1}\} \cap A = \emptyset$. By (2) and II.1 the $R_1(\alpha)$-core is the solution of $R_1(\alpha)$. By (4), (5) and (6) we have that $R_1(\alpha) - \operatorname{dom} R_1(\alpha)$ solves $A$.

III.b.2. $\mu \in A$. $\mu \in D^1 - \mathring{A}^1$ therefore $Q_1(\mu)$ is solvable. We remark that (7) $Q_1(\mu) \cap A^1 \subset \{x: x^2 = \alpha^2\}$. We distinguish several subcases of II.b.2.

III.b.2.1. There is a solution $V_1$ of $Q_1(\mu)$ such that $V_1 \cap A^1 = \emptyset$. We have that (8) $\operatorname{dom} V_1 \cap R_1(\alpha) = \emptyset$. From (5) and (8) it follows that

$$V_1 \cup (R_1(\alpha) - \operatorname{dom} R_1(\alpha))$$

is a solution of $G$.

III.b.2.2. There is a solution $V_2$ of $Q_1(\mu)$ such that $V_2 \cap A^1 = \{\mu\}$. In this case: $R_1(\alpha) - \operatorname{dom} V_2 = Q_3(\alpha) \cup Q_2(\beta)$. $\alpha \in D^3 - \mathring{A}^3$ and $\beta \in D^2 - \mathring{A}^2$ so $Q_3(\alpha)$ and $Q_2(\beta)$ are solvable and if $U$ solves $Q_3(\alpha)$ and $W$ solves $Q_2(\beta)$ then, by I.13, $\alpha \in U$ and $\beta \in W$. Since $Q_3(\alpha) \subset R_2(\beta)$ and $Q_2(\beta) \subset R_3(\alpha)$ it follows from I.12 that

$$\operatorname{dom} U \cap W = \operatorname{dom} W \cap U = \emptyset.$$

From these results and (5) it follows that $U \cup W \cup V_2$ is a solution of $G$.

III.b.2.3. There is a solution $V_3$ of $Q_1(\mu)$ such that $V_3 \cap A^1 - \{\mu\} \neq \emptyset$. Let $x^3$ receive its maximum in $V_3 \cap A^1$ at the point $\zeta \cdot \zeta^2 = \mu^2$ and $\zeta^3 > \mu^3$. By II.3 $[\mu\alpha] \cup V_3 = V_3'$ solves $Q_1(\alpha)$. We define $v = (\alpha^1, e - \alpha^1 - \zeta^3, \zeta^3)$. $R_1(\alpha) - \operatorname{dom} V_3' = Q_3(\alpha) \cup \{x: x \in A, x^{S_2} \geqq v^{S_2}\}$. Let $U$ solve $Q_3(\alpha)$. $\operatorname{dom} U \cap (Q_1(\alpha) \cup \{x: x \in A, x^{S_2} \geqq v^{S_2}\}) = \emptyset$. If $v \notin A$ then $V_3' \cup U$ solves $G$. If $v \in A$ then $v \in D^2 - A^2$. By I.16 $Q_2(v) \cap A^2 = \emptyset$. $Q_1(\alpha) \cup Q_3(\alpha) \subset R_2(v)$ therefore by I.12 $\operatorname{dom} Q_2(v) \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset$. If $W$ solves $Q_2(v)$ we have that $V_3' \cup U \cup W$ is a solution of $G$.

III.c. *$F^1$ has $c_3$-shape.* As in III.a, we have that $R_1(\alpha)$ is solvable and if $Q_1(\alpha)$ and $R_1(\alpha)$ are independent then $G$ is solvable. If $Q_1(\alpha)$ and $R_1(\alpha)$ are not independent and $V$ solves $Q_1(\alpha)$ and $W$ $R_1(\alpha)$ then either $\operatorname{dom} V \cap W \neq \emptyset$ or $\operatorname{dom} W \cap V \neq \emptyset$.

III.c.1. There exist $V_0$ and $W_0$ such that $\operatorname{dom} V_0 \cap W_0 \neq \emptyset$. $\alpha \in W_0$ therefore there must be a $z \neq \alpha$ in $V_0 \cap A^1$. We have that either $z^3 = \alpha^3$ or $z^2 = \alpha^2$.

III.c.1.1. $z^3 = \alpha^3$. In this case we have that (1) $y \in A^1$ implies $y^3 \leqq z^3$. Let $x^2$ take its maximum in $V_0 \cap A^1$ at the point $\zeta$. Define $v = (\alpha^1, \zeta^2, e - \alpha^1 - \zeta^2)$. From (1) it follows that (2) $R_1(\alpha) - \text{dom}\, V_0 = Q_2(\alpha) \cup \{x : x \in A,\ x^{S_3} \geqq v^{S_3}\}$. By II.5 there is a solution $U$ of $Q_2(\alpha)$ such that $U \cap F^1 = \{\alpha\}$; it follows that (3) $\text{dom}\, U \cap Q_1(\alpha) = \emptyset$. If $v \notin A$ then $U \cup V_0$ is a solution of $G$. If $v \in A$ then $v \in D^3 - A^3$, so $Q_3(v) \cap A^3 = \emptyset$. It follows that (4) $\text{dom}\, Q_3(v) \cap (Q_2(\alpha) \cup Q_1(\alpha)) = \emptyset$. We also have that (5) $\text{dom}\, U \cap Q_3(v) = \emptyset$. Now if $U_1$ is a solution of $Q_3(v)$ then, combining (2), (3), (4) and (5), we have that $V_0 \cup U \cup U_1$ is a solution of $G$.

III.c.1.2. $z^2 = \alpha^2$. We now have that $y \in A^1$ implies $y^2 \leqq \alpha^2$. We show that we may suppose: (*) there is no $u \in A^3$ such that $u^2 = \alpha^2$ and $u^1 > \alpha^1$. If (*) fails then $F^2$ has $b$-shape and $\{A^1, A^3\}$ satisfies condition $M$, so by III.b $G$ is solvable. We also notice that: (**) if $\vartheta \in A$, $\vartheta^2 = \alpha^2$, $\vartheta^3 \leqq \beta^3$ and $J$ solves $Q_1(\vartheta)$ then $J \cup [\alpha\vartheta]$ solves $Q_1(\alpha)$. Now if (*) holds and $U$ is a solution of $Q_3(\alpha)$ then $\text{dom}\, U \cap (Q_1(\alpha) \cup Q_2(\alpha)) = \emptyset$. Let $V$ solve $Q_1(\alpha)$. We denote by $\mu(V)$ the point where $x^3$ takes its maximum in $A^1 \cap V$. The point where $x^3$ takes its maximum in $A^1 \cap \{x : x^1 = \alpha^1\}$ is denoted by $\zeta$. $U$ denotes a fixed solution of $Q_3(\alpha)$. We remark that $\alpha \in U$.

III.c.1.2.1. There is a solution $V_1$ of $Q_1(\alpha)$ such that $\beta^3 < \mu^3(V_1)$. Define $v = (\alpha^1, e - \alpha^1 - \mu^3(V_1), \mu^3(V_1))$. We have that $R_1(\alpha) - \text{dom}\, V_1 = Q_3(\alpha) \cup \{x : x \in A,\ x^{S_2} \geqq v^{S_2}\}$. If $v \notin A$ then $U \cup V_1$ is a solution of $G$. If $v \in A$ then $v \in D^2 - A^2$. Let $U_1$ solve $Q_2(v)$. $\text{dom}\, U_1 \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset$, so $U \cup U_1 \cup V_1$ is a solution of $G$.

III.c.1.2.2. Every solution $V$ of $Q_1(\alpha)$ satisfies $\mu^3(V) \leqq \beta^3$ and there is a solution $V_1$ of $Q_1(\alpha)$ such that $\mu^3(V_1) = \beta^3$. $\beta \in a^2 \cap D^2$ therefore $Q_2(\beta)$ is solvable. If $U_1$ solves $Q_2(\beta)$ then $U_1 \cap A^2 = \{\beta\}$ and $\text{dom}\, U_1 \cap (Q_1(\mu(V_1)) \cup Q_3(\alpha)) = \emptyset$. Let $U_2$ be a solution of $Q_1(\mu(V_1))$. From (**) it follows that $U_2 \cap A^1 = \{\mu(V_1)\}$. So we have that $U \cup U_1 \cup U_2$ is a solution of $G$.

III.c.1.2.3. If $V$ solves $Q_1(\alpha)$ then $\mu^3(V) < \beta^3$.

III.c.1.2.3.1. $\zeta^3 \geqq \beta^3$. Define $v = (e - \alpha^2 - \beta^3, \alpha^2, \beta^3)$. By I.17 $Q_1(\alpha) \cap \text{dom}\, R_1(\alpha) = \text{dom}[\alpha\beta] \cap Q_1(\alpha)$, so $\text{dom}\, R_1(\alpha) \cap \{x : x \in A,\ x^{S_1} \geqq v^{S_1}\} = \emptyset$. If $v \notin A$ and $U_1$ solves $Q_2(\beta)$ then $U \cup [\alpha\beta] \cup U_1$ solves $G$. If $v \in A$ let $U_2$ be a solution of $Q_1(v)$. By (**) and the definition of $U_2$ $U_2 \cap A^1 = \emptyset$. So in this case $U \cup U_1 \cup U_2 \cup [\alpha\beta]$ solves $G$.

III.c.1.2.3.2. $\zeta^3 < \beta^3$. Define $v = (e - \alpha^2 - \zeta^3, \alpha^2, \zeta^3)$. By II.5 there is a solution $U_1$ of $Q_2(\zeta)$ such that $U_1 \cap F^1 = \{\zeta\}$. If $v \notin A$ then $U \cup U_1 \cup [\alpha\zeta]$ is a solution of $G$. If $v \in A - A^1$ and $U_2$ solves $Q_1(v)$ then $U \cup U_1 \cup U_2 \cup [\alpha\zeta]$ is a solution of $G$. If $v \in A^1$ then $U_1 \cup U_2 \cup U$ is a solution of $G$.

III.c.2. If $V$ solves $Q_1(\alpha)$ and $W$ $R_1(\alpha)$ then $\text{dom}\, W \cap V \neq \emptyset$ and $\text{dom}\, V \cap W = \emptyset$. By I.17 we have that $\text{dom}(R_1(\alpha) - [\beta\alpha)) \cap Q_1(\alpha) = \emptyset$. By I.19 $\mathring{R}_1^3(\alpha) = \mathring{A}^3 \cap R_1(\alpha)$. $[\beta\alpha) \subset \mathring{A}^3$ therefore $[\beta\alpha) \subset \mathring{R}_1^3(\alpha)$. So we conclude that

$$\text{dom}\,(R_1(\alpha) - \text{dom}\, R_1(\alpha)) \cap Q_1(\alpha) = \emptyset. \tag{1}$$

Since $F^1$ has $c_3$-shape we have also that $\mathring{A}^2 \cap R_1(\alpha) = \emptyset$. If $\{A^1, A^3\}$ does not intersect maximally then, by I.20, $\{R_1^1(\alpha), R_1^3(\alpha)\}$ does not intersect maximally and we have that $m(R_1(\alpha)) = 0$. By II.2 the $R_1(\alpha)$-core is the solution of $R_1(\alpha)$ which, by (1), contradicts III.c.2. So $\{A^1, A^3\}$ intersects maximally. The $x^2$ coordinate of $F^2$ will be denoted by $d$.

III.c.2.1. $d < \alpha^2$. We denote the ends of $F^2$ by $\gamma$ and $\delta$ such that $\delta^1 \geqq \gamma^1$. We remark that $\delta \in D^2 - A^2$. So $Q_2(\delta)$ is solvable and if $U$ solves it then dom $U \supset T_2(\delta) - U$ and dom $U \cap R_2(\delta) = \emptyset$. Denote $v = (\alpha^1, d, e - d - \alpha^1)$ and $P = \{x: x \in A,\ x^{S_3} \geqq v^{S_3}\}$. We remark that $\mathring{P}^i \cap \mathring{P}^j = \emptyset$ for all $i \neq j$ and that $(P - \operatorname{dom} P) \cap ((\alpha\beta] \cup [\gamma\delta)) = \emptyset$. So(3) $P - \operatorname{dom} P$ solves $P$ and

$$\operatorname{dom}(P - \operatorname{dom} P) \cap (Q_2(\delta) \cup Q_1(\alpha)) = \emptyset.$$

Summing we have that $U_1 = U \cup (P - \operatorname{dom} P)$ solves $R_1(\alpha)$ and that dom $U_1 \cap Q_1(\alpha) = \emptyset$. Since this result contradicts III.c.2 $d < \alpha^2$ is impossible.

III.c.2.2. $d = \alpha^2$. If $F^2$ has $a$-shape or $b$-shape then $\{A^1, A^3\}$ satisfies condition $M$ and by III.a or III.b $G$ is solvable. It remains only to complete the proof when $F^2$ has $c_3$ or $c_1$-shape.

III.c.2.2.1. $\gamma^1 < \alpha^1$. In this case $F^2$ has $c_3$-shape. Let $\eta$ satisfy $\eta^2 = \alpha^2$ and $\alpha^1 > \eta^1 > \max.(\gamma^1, e - \alpha^2 - \beta^3)$ and $U_1$ be a solution of $Q_1(\eta)$. $U_2 = [\alpha\eta] \cup U_1$ solves $Q_1(\alpha)$. Let $\zeta \in (\alpha\beta] \cap A^1$, $U_3$ be a solution of $Q_2(\zeta)$ and $U_4$ a solution of $Q_3(\alpha)$. $U_5 = U_3 \cup U_4 \cup [\alpha\zeta]$ is a solution of $R_1(\alpha)$. But dom $U_2 \cap U_5 \neq \emptyset$ contradicting III.c.2, so III.c.2.2.1 is impossible.

III.c.2.2.2. $\gamma^1 = \alpha^1$. In this case $F^2$ has $c_1$-shape. By II.5 there exist solutions $U_1$ of $Q_3(\alpha)$ and $U_2$ of $Q_2(\alpha)$ such that $U_1 \cap F^2 = \{\alpha\} = U_2 \cap F^1$. $U = U_1 \cup U_2$ is a solution of $R_1(\alpha)$ but(3) dom $U \cap Q_1(\alpha) = \emptyset$ contradicting III.c.2.

III.c.2.2.3. $\gamma^1 > \alpha^1$. In this case $F^2$ has $c_1$-shape and $\gamma \in a^1 \cap a^3 - A^2$. Let $U$ solve $Q_2(\gamma)$. $U_1 = U \cup (Q_3(\alpha) - \operatorname{dom} Q_3(\alpha))$ solves $R_1(\alpha)$ but dom $U_1 \cap Q_1(\alpha) = \emptyset$ which is impossible.

III.c.2.3. $d > \alpha^2$. In this case we show that $\mathring{A}^1 \cap \mathring{A}^3 \cap R_1(\alpha) = \emptyset$. It follows that $\{R_3^1(\alpha), R_3^3(\alpha)\}$ does not intersect maximally which is impossible as we have already seen. Suppose that $\mathring{A}^1 \cap \mathring{A}^3 \cap R_1(\alpha) \neq \emptyset$. Let $x \in \mathring{A}^1 \cap \mathring{A}^3 \cap R_1(\alpha)$. $x^1 \geqq \alpha^1$. Since(4) $\alpha \in a^3$ $x^2 < \alpha^2$. Let $z \in a^1 \cap a^3 \cap F^2$ and $y$ be an interior point of $\mathring{A}^1 \cap \mathring{A}^3$. There is a $u \in [yz] \cap \mathring{A}^1 \cap \mathring{A}^3$ such that $u^2 > \alpha^2$. So there is a $w \in [ux]$ for which $w^2 = \alpha^2$. $w \in \mathring{A}^1 \cap \mathring{A}^3$. If $w^1 \geqq \alpha^1$ then $\alpha \in \mathring{A}^3$ and if $w^3 \geqq \alpha^3$ then $\alpha \in \mathring{A}^1$. Since both cases are impossible we must have $\emptyset = \mathring{A}^1 \cap \mathring{A}^3 \cap R_1(\alpha)$.

IV. *Third part*: *case* II.9.

IV.1. $m(G) \leqq 2$. W.L.G. $\{A^2, A^3\}$ does not intersect maximally.

IV.1.1. $\mathring{A}^2 \cap \mathring{A}^3 = \emptyset$. We shall show that $m(G) = 0$. If $m(G) > 0$ then, W.L.G., $\{A^1, A^3\}$ intersects maximally. Let $z \in a^1 \cap a^3 \cap F^2$ and $y$ be an interior point of $\mathring{A}^1 \cap \mathring{A}^3$. $[yz) \subset \mathring{A}^3$. If $z \in \mathring{A}^2$ then we have $\mathring{A}^2 \cap \mathring{A}^3 \neq \emptyset$ which is impossible. If

(3) See I.17.

(4) If $x^2 \geqq x^2$ then $\alpha^{S_3} \geqq \alpha^{S_3}$ and $x \in \mathring{A}^3$ imply $\alpha \in \mathring{A}^3$ which is untrue.

$z \notin \mathring{A}^2$ then $\{A^1, A^3\}$ satisfies condition $M$, which is again impossible. Therefore $\{A^1, A^3\}$ does not intersect maximally.

IV.1.2. $\mathring{A}^3 \supset A^2$. If $\{A^1, A^3\}$ intersects maximally then there is a $z \in a^1 \cap a^3 \cap F^2$. $a^3 \cap \mathring{A}^2 = \emptyset$; therefore $z \notin \mathring{A}^2$. But since we have $a^1 \cap a^3 \cap F^2 \subset \mathring{A}^2$, $\{A^1, A^3\}$ cannot intersect maximally.

IV.1.2.1. $\mathring{A}^1 \cap \mathring{A}^3 = \emptyset$. $\mathring{A}^2 \cap \mathring{A}^1 \subset \mathring{A}^1 \cap A^2 \subset \mathring{A}^1 \cap \mathring{A}^3 = \emptyset$, so $m(G) = 0$.

IV.1.2.2. $\mathring{A}^3 \supset A^1$. We denote $C = A - \operatorname{dom} A$. If $x \in A - C$ then there is a $y \in A$ such that $y \succ_{S_3} x$. So there is a $z \in a^3$ such that $z \succ x$. $a^3 \cap (\mathring{A}^1 \cup \mathring{A}^2) = \emptyset$ therefore $z \in C$. By 2.6 $C$ is a solution of $G$.

IV.1.2.3. $\mathring{A}^1 \supset A^3$. We have $\mathring{A}^1 \supset A^3 \supset \mathring{A}^3 \supset A^2$, so $m(G) = 0$.

IV.1.3. $\mathring{A}^2 \supset A^3$. The proof in this case parallels that in IV.1.2.

IV.2. $m(G) = 3$. We denote $F^k = [\alpha_k \beta_k]$ and $D = \bigcap_{h=1}^{3} A^h$.

LEMMA IV.2.1. *Under the assumptions of* IV.2 *we can find $i$ and $k$ such that $S_k = \{i,j\}$ and*:

(1) $\alpha_k^i \geqq \beta_k^i$,

(2) $\alpha_k \in a^i \cap a^j$,

(3.a) *$x^j$ takes its maximum in $\{x : x^i = \alpha_k^i\} \cap D$ at a point $\vartheta \in a^k$ such that every $y \in A^k$ that satisfies $y^j = \vartheta^j$ and $y^i > \vartheta^i$ is in $\mathring{A}^i$, or*

(3.b) *$x^i$ takes its maximum in $\{x : x^j = \alpha_k^j\} \cap D$ at a point $\rho \in a^k \cap \mathring{A}^j$ and $F^k$ has $c_j$-shape.*

**Proof.** $F^1 = [\alpha_1 \beta_1]$. W.L.G. $\alpha_1^2 \geqq \beta_1^2$. We also suppose that $\alpha \in a^2 \cap a^3$. If $\alpha \notin a^2 \cap a^3$ then $\beta \in a^2 \cap a^3$ and the proof is not altered much. We now consider $F^3$.

IV.2.1.1. Every $y \in F^3$ satisfies $y^2 < \alpha_1^2$. Let $\vartheta$ be the point where $x^3$ takes its maximum in $\{x : x^2 = \alpha_1^2\} \cap D$. $\vartheta \neq \alpha_1$. We show that $\vartheta \in \mathring{A}^2$. If $F^3 = [\alpha_3 \beta_3]$ and $\alpha_3^1 \geqq \beta_3^1$ then $\alpha_3^3 > \vartheta^3$ since $\vartheta \notin F^3$. If $\alpha_3^1 > \vartheta^1$ then $\alpha_3 \succ_{S_2} \vartheta$, and if $\vartheta^1 \geqq \alpha_3^1$ then there is a $u \in [\alpha_1 \alpha_3]$ such that $u \succ_{S_2} \vartheta$. Now if $\vartheta \in a^3$ then $F^2$ has $c_1$-shape and $\vartheta \in a^1$ and if $\vartheta \in \mathring{A}^3$ then it follows from I.3 that $\vartheta \in a^1$. So in this case we can choose $k = 1$ and $i = 2$.

IV.2.1.2. There is a $y \in F^3$ such that $y^2 = \alpha_1^2$. Let $\vartheta = y$. If $F^3$ has $a$-shape then $\vartheta \in a^1$ and there is no $u \in A^1$ that satisfies $u^3 = \vartheta^3$ and $u^2 > \vartheta^2$. So we can choose $k = 1$ and $i = 2$. If $F^3$ has not $a$-shape then $\beta_3^2 > \alpha_3^2$. If $F^3$ has $b$-shape and $y = \beta_3$ then there is no $u \in A^1$ such that $u^3 = \vartheta^3$, and $u^2 > \vartheta^2$ and we can choose $k = 1$ and $i = 2$. If $y \neq \beta_3$ then we have that $\beta_3 \in a^2 \cap a^1$ and every $x \in F^1$ satisfies $x^2 < \beta_3^2$. By IV.2.1.1. we may take $k = 3$ and $i = 2$. If $F^3$ has $c_2$-shape then $\vartheta \in a^1$ and if $u \in A$, $u^3 = \vartheta^3$ and $u^2 > \vartheta^2$ then $u \in \mathring{A}^2$. So we can take $k = 1$ and $i = 2$. If $F^3$ has $c_1$-shape and $y = \beta_3$ we choose $k = 1$ and $i = 2$. If $y \neq \beta_3$ then we have that $\beta_3 \in a^2 \cap a^1$ and every $x \in F^1$ satisfies $x^2 < \beta_3^2$. By IV.2.1.1. we can choose $k = 3$ and $i = 2$.

IV.2.1.3. Every $y \in F^3$ satisfies $y^2 > \alpha_1^2$. If $F^3$ has $a$, $b$ or $c_1$-shape then we have that $\beta_3^2 \geqq \alpha_3^2$, $\beta_3 \in a^2 \cap a^1$ and every $x \in F^1$ satisfies $x^2 < \beta_3^2$. By IV.2.1.1, we can choose $k = 3$ and $i = 2$. Now suppose that $F^3$ has $c_2$-shape. Let $x^1$ take its maximum in $D \cap \{x: x^2 = \alpha_3^2\}$ at the point $\rho$. $\rho \neq \alpha_3$. We shall show that $\rho \in \mathring{A}^1 \cap \mathring{A}^2$. $\rho^2 = \alpha_3^2 < \beta_3^2$. $\rho^1 > \alpha_3^1$ therefore $\rho^3 < \alpha_3^3 = \beta_3^3$. So $\beta_3 \succ_{S_1} \rho$. We have also that $\alpha_1^1 > \rho^1$ therefore there is a $u \in [\alpha_1 \alpha_3]$ such that $u \succ_{S_2} \rho$. It follows from I.3 that $\rho \in a^3$. Summing we have: $\alpha_3^1 \geqq \beta_3^1$, $\alpha_3 \in a^2 \cap a^1$ and $x^1$ takes its maximum in $D \cap \{x: x^2 = \alpha_3^2\}$ at a point $\rho \in a^3 \cap \mathring{A}^2$. So we can take $k = 3$ and $i = 1$.

We now prove that $G$ is solvable in case IV.2. W.L.G. the results of IV.2.1 hold for $k = 1$ and $i = 2$.

IV.2.2. (3.a) holds in IV.2.1. We remark(5) that if $z \in A^1 \cap Q_1(\vartheta)$ then $z^3 = \vartheta^3$. If there is a $z \in A^1$ such that $z^3 = \vartheta^3$ and $z^2 > \vartheta^2$ then $F^3$ has $c_2$-shape. By Lemma II.5 there is a solution $V$ of $Q_1(\vartheta)$ such that $V \cap F^3 = \{\vartheta\}$. So we can always find a solution $V_1$ of $Q_1(\vartheta)$ such that $V_1 \cap A^1 = \{\vartheta\}$. Similar reasoning shows that there is always a solution $V_2$ of $Q_3(\alpha_1)$ such that $V_2 \cap A^3 = \{\alpha_1\}$.

IV.2.2.1. $\vartheta^3 > \beta_1^3$. We define $v = (\alpha_1^1, e - \alpha_1^1 - \vartheta^3, \vartheta^3)$. Suppose $v \in A$. We have that $Q_2(v) \cap A^2 = \emptyset$. So if $U$ is a solution of $Q_2(v)$ then $V_1 \cup V_2 \cup U \cup [\vartheta\alpha_1]$ solves $G$. If $v \notin A$ then $V_1 \cup V_2 \cup [\vartheta\alpha_1]$ solves $G$.

IV.2.2.2. $\vartheta^3 = \beta_1^3$. If $U$ is a solution of $Q_2(\beta_1)$ then $V_1 \cup V_2 \cup U$ is a solution of $G$.

IV.2.2.3. $\vartheta^3 < \beta_1^3$. Let $x^3$ take its maximum in $A^1 \cap \{x: x^1 = \alpha_1^1\}$ at $\zeta$. We define $\mu = (e - \alpha_1^2 - \beta_1^3, \alpha_1^2, \beta_1^3)$. Suppose $\zeta^3 \geqq \beta_1^3$ and $\mu \in A$. In this case if $W$ is a solution of $Q_2(\beta_1)$ and $W_1$ is a solution of $Q_1(\mu)$ then $V_2 \cup W \cup W_1 \cup F^1$ solves $G$. If $\mu \notin A$ then $V_2 \cup W \cup F^1$ is a solution of $G$. If $\zeta^3 < \beta_1^3$ then $F^1$ has $c_3$-shape. We define $\eta = (e - \zeta^3 - \alpha_1^2, \alpha_1^2, \zeta^3)$. By II.5 there is a solution $U$ of $Q_2(\zeta)$ such that $\{\zeta\} = U \cap F^1$. If $\eta \notin A$ then $U \cup V_2 \cup [\zeta\alpha_1]$ solves $G$. If $\eta \in A - A^1$ and $U_1$ solves $Q_1(\eta)$ then $V_2 \cup U \cup U_1 \cup [\zeta\alpha_1]$ is a solution for $G$. If $\eta \in A^1$ then $\eta = \vartheta$ and $V_1 \cup V_2 \cup U$ is a solution of $G$.

IV.2.3. (3.b) holds in IV.2.1. So $F^1$ has $c_3$-shape. By II.5 there is a solution $V$ of $Q_2(\alpha_1)$ such that $V \cap F^1 = \{\alpha_1\}$. Next we show that there is a solution $V_1$ of $Q_1(\rho)$ such that $V_1 \cap A^1 = \{\rho\}$. If $Q_1(\rho) \cap A^1 = \{\rho\}$ this follows from the fact that $\rho$ belongs to every solution of $Q_1(\rho)$. If there is $x \in Q_1(\rho) \cap A^1$, $x \neq \rho$, then $x \in a^1$ and(6) $x^2 = \rho^2$. Using I.10 and observing that $\rho \in \mathring{A}^3$ we see that $F^2$ has $C_3$-shape and $\rho \in F^2$. II.5 yields a desired $V_1$. Now define $v = (\alpha_1^1, \rho^2, e - \alpha^1 - \rho^2)$. Observe that $Q_3(v) \cap A^3 = \emptyset$. If $v \in A$ and $V_2$ is a solution of $Q_3(v)$ then $V \cup V_1 \cup V_2 \cup [\rho\alpha_1]$ is a solution of $G$. If $v \notin A$ then $V \cup V_1 \cup [\rho\alpha_1]$ solves $G$.

## References


1. R. J. Aumann and B. Peleg, *Von Neumann Morgenstern solutions to cooperative games without side payments*, Bull. Amer. Math. Soc. **66** (1960), 173–179.

2. R. J. Aumann, *The core of cooperative game without side payments*, Trans. Amer. Math. Soc. **98** (1961), 539–552.

3. D. B. Gillies, *Solutions to general non-zero-sum games*, Contributions to theory of games. IV, pp. 47–85, Princeton Univ. Press, Princeton, N. J., 1959.



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(5) Suppose there is $z \in A^1 \cap Q_1(\vartheta)$ with $z^3 > \vartheta^3$. $\alpha_1 \in \mathring{A}^1$ so there is $u \in A^1$ with $u^2 > \alpha_1^2 = \vartheta^2$. For small $t > 0$ $y = tu + (1-t)z$ satisfy $y^{S_1} > \vartheta^{S_1}$ and $y \in A^1$ which is impossible since $\vartheta \in a^1$.

(6) By an argument similar to that in footnote (5).