TRANSFORMATION GROUPS ON COHOMOLOGY PROJECTIVE SPACES(1)

BY

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1. **Introduction.** Let G be a topological transformation group on a compact Hausdorff space Y and $F(G;Y)$ its fixed point set. The present paper is devoted to the study of the cohomology structure of $F(G;Y)$ in the following three cases:

(1) G is the group Z_2 of integers modulo 2 and Y has the mod 2 cohomology ring of the real projective n-space.

(2) G is the group Z_p of integers modulo p, where p is an *odd* prime, and Y has the mod p cohomology structure of the lens $(2n+1)$-space mod p.

(3) G is the circle group S^1 (the group of reals mod 1) and Y has the integral cohomology ring of the complex projective n-space.

For the sake of simplicity, we shall call Y a cohomology real projective n-space or a cohomology lens $(2n+1)$-space mod p or a cohomology complex projective n-space if its cohomology structure is that described in (1) or (2) or (3). (Formal definition of these notions will be given later.)

The study of the problem proposed above is motivated by two recent theorems obtained separately by P. A. Smith and C. T. Yang. In [6], Smith proved that if Z_2 acts effectively on the real projective n-space, then the fixed point set is either empty, or it has exactly two components C_1 and C_2, where each C_i is a *cohomology* real projective n_i-space, $i = 1, 2$, and $n_1 + n_2 = n - 1$. Later in an unpublished work, Yang proved that if S^1 acts differentiably on the complex projective n-space, then the fixed point set is nonempty, has at most $n+1$ components, say $C_1, \cdots, C_k$, $k \leqq n+1$, where each C_i is a cohomology complex projective n_i-space, $i = 1, 2, \cdots, k$, and $n_1 + n_2 + \cdots + n_k = n - k + 1$. Our main purpose is to show that, under the more general setting of (1) and (3), essentially the same conclusions obtained by Smith and Yang still hold true. We also include a study of case (2), which is the natural counter part of case (1) when p is odd.

All topological spaces considered in this paper are assumed to be compact Hausdorff. For such a space X, $H^*(X;L) = \sum_{n=0}^{\infty} H^n(X;L)$ will denote the Alexander-Spanier-Wallace cohomology ring with coefficient domain L. Let G

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be a transformation group on a space X. For each $x \in X$ the closed subgroup $G_x = \{g \in G \mid gx = x\}$ of G is called the *isotropic subgroup at* x and the subset $G(x) = \{gx \in X \mid g \in G\}$ of X is called an *orbit*. The action is said to be *free* (or G acts *freely* on X) if $G_x = e$ (the identity subgroup of G) for all $x \in X$. The set $X/G = \{G(x) \mid x \in X\}$ endowed with the usual quotient topology is called the *orbit space*. The map (the word map will always mean a continuous map in this paper) sending $x \in X$ to $G(x) \in X/G$ is called the *canonical projection* and will be denoted by $\pi : X \to X/G$. We call X a cohomology n-sphere over L if $H^*(X;L) = H^*(S^n; L)$, where S^n denotes the n-sphere. If $L = Z_p$, we also call X a cohomology n-sphere mod p. If $L = Z$ (the group of integers), we also call X an integral cohomology n-sphere. As usual, empty space is regarded as cohomology (-1)-sphere over L.

2. **Cohomology real projective spaces and cohomology lens spaces.** Throughout this section, the coefficient domain L for cohomology will be the field Z_p of characteristic $p \neq 0$. We adopt the convention that $H^*(X)$ shall mean $H^*(X;Z_p)$ for $p \neq 2$ and for $p = 2$ the coefficient domain Z_2 shall be indicated out explicitly. A space Y is called a cohomology real projective n-space if the ring $H^*(Y;Z_2)$ is given by

$$H^*(Y;Z_2) = Z_2[x]/(x^{n+1}), \quad \text{degree } x = 1, \tag{2.1}$$

where $Z_2[x]$ is the polynomial ring with coefficients in Z_2 and (x^{n+1}) is the ideal generated by x^{n+1}. Y is called a cohomology lens $(2n+1)$-space mod $p, p \neq 2$, if the ring $H^*(Y)$ is given by

$$H^*(Y) = \Lambda[a] \otimes Z_p[x]/(x^{n+1}), \quad \text{degree } a = 1, \text{ degree } x = 2, \tag{2.2}$$

where $\Lambda[a]$ is the exterior algebra generated by a over Z_p, $Z_p[x]$ is the polynomial ring with coefficients in Z_p and (x^{n+1}) is the ideal generated by x^{n+1}. In addition, we also require that $\beta(a) = x$, where $\beta : H^1(Y) \to H^2(Y)$ is the Bockstein homomorphism. The following two propositions are well known.

PROPOSITION 2.3. *If Z_2 acts freely on a cohomology n-sphere* mod 2 X, *then X/Z_2 is a cohomology real projective n-space.*

PROPOSITION 2.4. *If Z_p $(p \neq 2)$ acts freely on a cohomology* $(2n+1)$*-sphere* mod p X, *then X/Z_p is a cohomology lens* $(2n+1)$*-space* mod p.

Much more interesting is their converse. We have

THEOREM 2.5. *If Z_2 acts freely on a connected space X such that X/Z_2 is a cohomology real projective n-space, then X is a cohomology n-sphere* mod 2.

Proof. Using Smith's special cohomology [1, p. 41], we have the Smith exact sequence which in the case of free action of Z_2 takes the following form.

$$
\begin{aligned}
(1)\qquad 0 \to H^0(X/Z_2;Z_2) &\xrightarrow{\pi^*} H^0(X;Z_2) \to H^0(X/Z_2;Z_2) \xrightarrow{\delta} H^1(X/Z_2;Z_2) \to \cdots \\
&\to H^{k-1}(X/Z_2;Z_2) \xrightarrow{\delta} H^k(X/Z_2;Z_2) \xrightarrow{\pi^*} H^k(X;Z_2) \\
&\to H^k(X/Z_2;Z_2) \to \cdots,
\end{aligned}
$$

where $\pi^* : H^k(X/Z_2;Z_2) \to H^k(X;Z_2)$ is the homomorphism induced by the canonical projection $\pi : X \to X/Z_2$. Since X is connected, $H^0(X;Z_2) = Z_2$ and hence $\pi^* : H^0(X/Z_2;Z_2) \to H^0(X;Z_2)$ must be an isomorphism. This implies that $\pi^* : H^1(X/Z_2;Z_2) \to H^1(X;Z_2)$ is trivial. The ring structure imposed on $H^*(X/Z_2;Z_2)$ then implies that $\pi^* : H^k(X/Z_2;Z_2) \to H^k(X;Z_2)$ is trivial for all $k \geqq 1$. The theorem follows immediately from the exactness of (1).

THEOREM 2.6. *If Z_p $(p \neq 2)$ acts freely on a connected space X such that X/Z_p is a cohomology lens $(2n+1)$-space* mod p, *then X is a cohomology $(2n+1)$-sphere* mod p.

We need some preliminary considerations. The action of Z_p on X induces an action of Z_p on $H^*(X)$. More precisely, let T be a generator of Z_p; then the induced homomorphism $T^* : H^*(X) \to H^*(X)$ satisfies $T^{*p} = 1$ (the identity homomorphism) and hence defines an action of Z_p on $H^*(X)$. We can therefore talk about the functor $H^*(Z_p;H^*(X))$, the cohomology of the group Z_p with coefficients in $H^*(X)$. Let $\tau^* = 1 - T^*$, $\sigma^* = \sum_{i=0}^{p-1} T^{*i}$ and $H^*(X)^0 = \ker \tau^*$, we have [2]

$$
(2.7)\qquad H^s(Z_p;H^*(X)) = \begin{cases} H^*(X)^0 & \text{if } s = 0, \\ \ker \tau^*/\operatorname{Im} \sigma^* & \text{if } s = 2k, k > 0, \\ \ker \sigma^*/\operatorname{Im} \tau^* & \text{if } s = 2k+1,\ k \geqq 0. \end{cases}
$$

Moreover, $H^*(Z_p;Z_p) = \sum_{s=0}^{\infty} H^s(Z_p;Z_p)$ has a ring structure which can be described as

$$
(2.8)\qquad H^*(Z_p;Z_p) = \Lambda[a] \otimes Z_p[x], \text{ degree } a = 1, \text{ degree } x = 2,
$$

where $\Lambda[a]$ and $Z_p[x]$ are the same as in (2.2). Furthermore, we also have $\beta(a) = x$, where $\beta : H^1(Z_p;Z_p) \to H^2(Z_p;Z_p)$ is the Bockstein homomorphism. The following two lemmas can be easily established.

LEMMA 2.9. *If $\sigma^* = 0$ and $H^k(X)^0 = 0$, then $H^k(X) = 0$.*

LEMMA 2.10. *If $\sigma^* = 0$ and $\dim H^k(X)^0 = 1$, then $\dim H^k(X) \leqq p - 1$.*

Proof of (2.6). We shall only give a proof for the case $n > 0$. Since Z_p acts freely on X, there is the Leray-Cartan spectral sequence [1] (E_r) whose E_2-term is given by $E_2^{s,t} = H^s(Z_p;H^t(X))$ and whose E_∞-term is associated with $H^*(X/Z_p)$. As $E_2^{s,t} = 0$ when either s or t is negative, we have the exact sequence for low dimensions [2]

$$
(1)\qquad 0 \to E_2^{1,0} \xrightarrow{\phi_1} H^1(X/Z_p) \xrightarrow{\pi^*} E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \xrightarrow{\phi_2} H^2(X/Z_p).
$$

Since X is connected, $E_2^{*,0} = \sum_{s=0}^{\infty} E_2^{s,0} = H^*(Z_p; Z_p)$. In particular, $E_2^{1,0} = Z_2$ and ϕ_1 must be an isomorphism. Let $a \in E_2^{1,0}$ be a generator, then $a' = \phi_1(a)$ is a generator of $H^1(X/Z_p)$. By (2.8), $x = \beta(a)$ is a generator of $E_2^{2,0}$ and we have $\phi_2(x) = \phi_2 \circ \beta(a) = \beta \circ \phi_1(a) = \beta(a') \neq 0$ by (2.2). Hence ϕ_2 must also be an isomorphism. As a and x generate $E_2^{*,0}$, it follows that

(2) $$\phi_s : E_2^{s,0} \to H^s(X/Z_p)$$

is an isomorphism for all $1 \leqq s \leqq 2n + 1$. Since $\pi^* \circ \phi_1(a) = 0$, we have $\pi^* \circ \phi_2(x) = \pi^* \circ \beta \circ \phi_1(a) = \beta \circ \pi^* \circ \phi_1(a) = 0$. But $\phi_1(a)$ and $\phi_2(x)$ generate the ring $H^*(X/Z_p)$ according to (2.2); it follows that

(3) $$\pi^* : H^s(X/Z_p) \to H^s(X)$$

is trivial for all $s \geqq 1$. Notice that (2) implies that $E_r^{s,0}$ has no cobounding elements for all $r \geqq 2$ and $1 \leqq s \leqq 2n + 1$. In particular we have

(4) $$d_{s+1} : E_{s+1}^{0,s} \to E_{s+1}^{s+1,0} \text{ is trivial for all } 1 \leqq s \leqq 2n.$$

To see the consequence of (3), consider the Smith exact sequences. Following the notations of [1, p. 41], these are

(5) $$\cdots \to H^s(X/Z_p) \xrightarrow{\pi^*} H^s(X) \to H^s(\tau) \to H^{s+1}(X/Z_p) \to \cdots,$$

(6) $$\cdots \to H^s(\tau) \to H^s(X) \xrightarrow{\mu} H^s(X/Z_p) \to H^{s+1}(\tau) \to \cdots.$$

It is known [1] that $\pi^* \circ \mu = \sigma^*$; hence (3) implies that

(7) $$\sigma^* : H^s(X) \to H^s(X) \text{ is trivial for all } s \geqq 1.$$

Now we proceed to prove by induction that $H^s(X) = 0$ for all $1 \leqq s \leqq 2n$. By exactness of (1), we have $E_2^{0,1} = H^1(X)^0 = 0$; hence $H^1(X) = 0$ by (7) and Lemma 2.9. Suppose it has been shown that $H^i(X) = 0$ for all $1 \leqq i < s \leqq 2n$. It is easily seen that this implies that the differentials $d_r : E_r^{0,s} \to E_r^{r,s-r+1}$ are trivial for all $2 \leqq r < s + 1$ are $r > s + 1$. This together with (4) gives $E_2^{0,s} = E_\infty^{0,s}$. By (2), we have $E_\infty^{s,0} = \operatorname{Im} \phi_s = H^s(S/Z_p)$. As $\dim H^s(X/Z_p) = \sum_{i=0} E_\infty^{s-i,i}$, we have $E_\infty^{0,s} = 0$. Thus we obtain $E^{0,s} = H^s(X)^0 = 0$. Applying Lemma 2.9 again, we obtain $H^s(X) = 0$.

Next we take up the case $s = 2n + 1$. As before, $H^i(X) = 0$ for all $1 \leqq i \leqq 2n$ implies that $d_r : E_r^{0,2n+1} \to E_r^{r,2n+2-r}$ is trivial for all $2 \leqq r < 2n + 2$ and $r > 2n + 2$. Hence we have $E_2^{0,2n+1} = E_{2n+2}^{0,2n+1}$ and $E_{2n+3}^{0,2n+1} = E_\infty^{0,2n+1} = 0$, where the last equation holds because $H^{2n+1}(X/Z_p) = E_\infty^{2n+1,0}$ by (2). Similarly, we have $E_2^{n+2,0} = E_{2n+2}^{2n+2,0}$ and $E_{2n+3}^{2n+2,0} = E_\infty^{2n+2,0} = 0$. (Since $H^{2n+2}(X/Z_p) = 0$.) But by definition, $E_{2n+3}^{2n+2,0} = E_{2n+2}^{2n+2,0}/\operatorname{Im}(E_{2n+2}^{0,2n+1} \xrightarrow{d_{2n+2}} E_{2n+2}^{2n+2,0})$ and $E_{2n+3}^{0,2\ +1} = \ker(E_{2n+2}^{0,2n+1} \xrightarrow{d_{2n+2}} E_{2n+2}^{2n+2,0})$. It follows that $d_{2n+2}: E_{2n+2}^{0,2n+1} \to E_{2n+2}^{2n+2,}$ is an somorphism. Thus we obtain $E_2^{0,2n+1} = E_{2n+2}^{0,2n+1} \cong E_{2n+2}^{2n+2,0} = E_2^{2n+2,0} = Z_p$. By (7), (2.7) and Lemma 2.10, we obtain $\dim H^{2n+1}(X/Z_p) \leqq p-1$.

For $s > 2n+1$, let us consider the exact sequences (5) and (6). It is not hard to see [1] that the composition map $H^s(X) \to H^s(\tau)$ and $H^s(\tau) \to H^s(X)$ is τ^*. (5) and (6) then imply that $\tau^* : H^s(X) \to H^s(X)$ is an epimorphism for all $s > 2n+1$. It follows that $H^s(X) = \tau^* H^s(X) = \cdots = \tau^{*\,p-1} H^s(X) = \sigma^* H^s(X) = 0$ by (7).

Finally let us return to $H^{2n+1}(X/Z_p)$. As we have now shown that $\dim H^*(X) < \infty$, the Euler-characteristic formula of E. E. Floyd [1, p. 40]

$$\sum_{s=0}^{\infty} (-1)^s \dim H\,(X) = p \sum_{s=0}^{\infty} (-1)^s \dim H^s(X/Z_p)$$

can be applied. In our case, this reduces to

$$1 - \dim H^{2n+1}(X) = 0$$

or $H^{2n+1}(X) = Z_p$. This completes the proof of (2.6).

3. **Cohomology covering spaces and lifting of actions.** It is a well-known fact that the real projective n-space admits the n-sphere as its two-folded covering space. The purpose of this section is to give a construction which among other things will insure that every cohomology real projective n-sphere admits a cohomology n-sphere mod 2 as its two-folded covering space. Throughout this section, cohomology always has Z_p as coefficient group with no distinction between p being even or odd, Y is a fixed *connected* space and $a \in H^1(Y)$ is a fixed *nonzero* element.

Let $f: Y \times Y \to Z_p$ be a 1-cocycle representing a, then there exists an open covering $\mathscr{V}$ of Y such that

$$(3.1) \qquad f(y_0, y_2) = f(y_0, y_1) + f(y_1, y_2) \text{ whenever } y_0, y_1, y_2 \in V \in \mathscr{V}.$$

By a *$\mathscr{V}$-chain* we mean a finite sequence $(y_i)_{i=0}^n$ of points of Y such that $\{y_{i-1}, y_i\} \in V_i \in \mathscr{V}$ for all $i = 1, 2, \cdots, n$. Let $b \in Y$ be a fixed base point. By a *$\mathscr{V}$-chain with base point b* we mean a $\mathscr{V}$-chain $(y_i)_{i=0}^n$ such that $y_0 = b$. The set of all $\mathscr{V}$-chains with base point b is denoted by χ. $(y_i)_{i=0}^n, (y'_j)_{j=0}^m \in \chi$ are said to be *equivalent* if

$$(3.2) \qquad y_n = y'_m$$

and

$$\sum_{i=1}^{n} f(y_{i-1}, y_i) = \sum_{j=1}^{m} f(y'_{j-1}, y'_j).$$

The quotient set under this equivalence relation is denoted by X and the equivalence class of $(y_i)_{i=0}^n \in \chi$ is denoted by $[y_i]_{i=0}^n$. The function $\pi : X \to Y$ given by $\pi([y_i]_{i=0}^n) = y_n$ is clearly well defined.

Now we topologize X as follows. Let $x = [y_i]_{i=0}^n \in X$ and $\mathscr{B}(y_n)$ be a base of neighborhoods of y_n such that every $B(y_n) \in \mathscr{B}(y_n)$ is contained in some $V \in \mathscr{V}$. To each $B(y_n) \in \mathscr{B}(y_n)$, define

$$B^*(x) = \Big\{[y'_j]_{j=0}^m \in X \,\big|\, y'_m \in B(y_n) \text{ and} \tag{3.3}$$
$$\sum_{i=1}^n f(y_{i-1}, y_i) + f(y_n, y'_m) + \sum_{j=1}^m f(y'_j, y'_{j-1}) = 0\Big\}.$$

It is easily verified that a Hausdorff topology is defined on X with $\mathscr{B}(x) = \{B^*(x) \mid B(y_n) \in \mathscr{B}(y_n)\}$ as a base of neighborhoods at x and that $\pi: X \to Y$ is a continuous map. In fact, π maps every $B^*(x)$ homeomorphically onto $B(y_n)$ hence it is even a local homeomorphism. In particular, there exists an open covering $\mathscr{V}^*$ of X such that every $V^* \in \mathscr{V}^*$ is mapped homeomorphically onto some $V \in \mathscr{V}$ by π.

LEMMA 3.4. *For each $y \in Y$, $\pi^{-1}(y)$ has exactly p points.*

Proof. It suffices to consider the case $y = b$. The function $\phi: \pi^{-1}(b) \to Z_p$ given by $\phi([y_i]_{i=0}^n) = \sum_{i=1}^n f(y_{i-1}, y_i)$, where $[y_i]_{i=0}^n \in \pi^{-1}(b)$, is clearly injective. Moreover, $\operatorname{Im}\phi \subset Z_p$ is a subgroup; hence we have either $\operatorname{Im}\phi = 0$ or $\operatorname{Im}\phi = Z_p$. If $\operatorname{Im}\phi = 0$, we define a 0-cochain $g: Y \to Z_p$ by $g(y) = \sum_{i=1}^n f(y_{i-1}, y_i)$ where $(y_i)_{i=0}^n$ is any $\mathscr{V}$-chain with base point b such that $y_n = y$. Such a $\mathscr{V}$-chain exists (since Y is connected) and g is well defined. Now if $y, y' \in V \in \mathscr{V}$, we have $g(y') - g(y) = f(y, y')$ since we can represent $g(y')$ by $\sum_{i=1}^n f(y_{i-1}, y_i) + f(y, y')$. But this means $f - \delta g$ has empty support, contradicting the assumption that $a \neq 0$.

Notice that (3.4) also implies that X is compact.

By a *$\mathscr{V}^*$-chain* we mean a finite sequence $(x_i)_{i=0}^n$ of points of X such that that $\{x_{i-1}, x_i\} \in V_i^* \in \mathscr{V}^*$ for all $i = 1, 2, \cdots, n$. A $\mathscr{V}^*$-chain $(x_i)_{i=0}^n$ is said to *cover* a $\mathscr{V}$-chain $(y_i)_{i=0}^n$ (or $(y_i)_{i=0}^n$ is *covered by* $(x_i)_{i=0}^n$) if $\pi(x_i) = y_i$ for all $i = 0, 1, \cdots, n$. The function $h: X \to Z_p$ given by

$$h([y_i]_{i=0}^n) = -\sum_{i=1}^n f(y_{i-1}, y_i) \tag{3.5}$$

is clearly well defined having the property that

$$h(x_0) - h(x_n) = \sum_{i=1}^n f(y_{i-1}, y_i) \tag{3.6}$$

whenever a $\mathscr{V}^*$-chain $(x_i)_{i=0}^n$ covers a $\mathscr{V}$-chain $(y_i)_{i=0}^n$. The following lemma is immediate.

LEMMA 3.7. (*Chain lifting property and monodromy property.*) *Given a $\mathscr{V}$-chain $(y_i)_{i=0}^n$ and a point $x \in \pi^{-1}(y_0)$, there exists a unique $\mathscr{V}^*$-chain $(x_i)_{i=0}^n$ covering $(y_i)_{i=0}^n$ with $x_0 = x$. Given two $\mathscr{V}$-chains $(y_i)_{i=0}^n$ and $(y'_j)_{j=0}^m$ with $y_0 = y'_0$ and $y_n = y'_m$, two $\mathscr{V}^*$-chains $(x_i)_{i=0}^n$ and $(x'_j)_{j=0}^m$ covering $(y_i)_{i=0}^n$ and $(y'_j)_{j=0}^m$ respectively with $x_0 = x'_0$, then $x_n = x'_m$ if and only if $\sum_{i=1} f(y_{i-1}, y_i) = \sum_{j=1} f(y'_{j-1}, y'_j)$.*

LEMMA 3.8. *There exists a free action of Z_p on X such that $X/Z_p = Y$ and π coincides with the canonical projection.*

Proof. Define a map S: $\pi^{-1}(b) \to \pi^{-1}(b)$ by $S(x) = \phi^{-1}(\phi(x) + 1)$, where $x \in \pi^{-1}(b)$ and $\phi : \pi^{-1}(b) \to Z_p$ is the function defined in Lemma 3.4. Extend S to a map $X \to X$ as follows. Let $x = [y_i]_{i=0}^n \in X$ be an arbitrary point of X. Choose a $\mathscr{V}^*$-chain $(x_i)_{i=0}^n$ covering $(y_i)_{i=0}^n$ with $x_0 = S([b])$, where $[b] \in \pi^{-1}(b)$ is the class of the $\mathscr{V}$-chain (b,b); then define $S(x) = x$. S is well defined in view of Lemma 3.8 and it is easily verified to be a periodic map on X of period p having no fixed point. Later we shall refer to S as the *deck-transformation.*

LEMMA 3.9. *The homomorphism $\pi^* : H^1(Y) \to H^1(X)$ takes a into* 0.

Proof. In fact, $\pi^*(a)$ contains the coboundary δh, where h is the 0-cochain defined by (3.5).

LEMMA 3.10. *X is connected.*

Proof. It is not hard to see that if X is not connected, then it has exactly p components $X_1, \cdots, X_p$ and every $\pi \mid X_i : X_i \to Y$ is a homeomorphism. It follows that $\pi^*(a) \neq 0$, contrary to (3.9).

Collecting (3.1) through (3.10), we have thus proved

THEOREM 3.11. *Let Y be a connected space and $a \in H^1(Y)$ a nonzero element; then there exists a space X and a free action of Z_p on X such that $Y = X/Z_p$ and $\pi^* : H^1(Y) \to H^1(X)$ maps a into zero.*

Because of the last property of π^*, the space X may be called a cohomology covering space of Y with respect to $a \in H^1(Y)$. This space can actually be characterized abstractly. We formulate this in the following way. By a cohomology covering space of Y with respect to a, we mean a principal bundle (X, Y, Z_p, π) such that $\pi^*(a) = 0$, where Y is a connected space and $a \in H^1(Y)$ is a preassigned nonzero element. We state without proof the following uniqueness theorem.

THEOREM 3.12. *Let (X, Y, Z_p, π) and (X', Y, Z_p, π') be two cohomology covering spaces of Y with respect to $a \in H^1(Y)$ and $g : Y \to Y$ a homeomorphism such that $g^*(a) = a$, where $g^* : H^1(Y) \to H^1(Y)$ is the homomorphism induced by g. Then there exists a homeomorphism $\tilde{g} : X \to X'$ such that $\pi' \circ \tilde{g} = g \circ \pi$. Moreover, let $x \in X$ and $x' \in X'$ be any two preassigned points such that $\pi'(x') = g \circ \pi(x)$; then $\tilde{g}$ can be chosen in such a way that $\tilde{g}(x) = x'$, and it is completely determined by this condition.*

Suppose that G is a transformation group on Y, a *bundle lifting* of G is an action of G on X such that each $g \in G$ acts on X as a bundle map $\tilde{g}$ [7] (i.e., it commutes with the deck-transformation), and $\pi \circ \tilde{g}(x) = g \circ \pi(x)$ for all $x \in X$. It follows that a bundle lifting of G defines an action of $Z_p \times G$ on X.

THEOREM 3.13. *Let (X,Y,Z_p,π) be a cohomology covering space of Y with respect to $a \in H^1(Y)$ and G a finite transformation group on Y. Suppose that a is invariant under G (i.e., $g^*(a) = a$ for all $g \in G$) and that the fixed point set $F(G;Y) \neq \emptyset$. Then G has a bundle lifting.*

Proof. In view of (3.12), it suffices to consider a particular X. This can be obtained by a slight modification of the construction given in the beginning of this section. There exists an open covering $\mathscr{V}$ of Y, a 1-cocycle $f: Y \times Y \to Z_p$ representing a and to each $g \in G$ a 0-cochain $k_g : Y \to Z_p$ such that (i) $gV \in \mathscr{V}$ for all $g \in G$ and $V \in \mathscr{V}$, (ii) (3.1) holds true and (iii) $f(y,y') - f(g(y),g(y'))$ $= k_g(y) - kg(y')$ whenever $y,y' \in V \in \mathscr{V}$. Let χ be the set of all $\mathscr{V}$-chains with base point b, where b is chosen as a point in $F(G;Y)$. By (i), every $g \in G$ induces a function $g : \chi \to \chi$ defined as $g((y_i)_{i=0}^n) = (g(y_i))_{i=0}^n$. Define an equivalence relation in χ as (3.2). Condition (iii) insures that each g takes equivalent $\mathscr{V}$-chains into equivalent $\mathscr{V}$-chains and hence induces a map $\tilde{g} : X \to X$. The rest of the theorem is obvious.

4. Fixed point sets of actions of Z_p on cohomology real projective spaces or cohomology lens spaces.

With the machineries built up in the previous two sections, it is now easy to establish two of the main theorems of this paper.

THEOREM 4.1. *If Z_2 acts on a cohomology real projective n-space, then the fixed point set F is either empty or it has at most two components. If F has k components $C_1,\cdots,C_k$, $1 \leqq k \leqq 2$, then each C_i is a cohomology real projective n_i-space, $i = 1,\cdots,k$ and*

$$\sum_{i=1}^{k} n_i = n - k + 1.$$

Proof. Let Z_2 act on a cohomology real projective n-space Y, that is, an involution $T: Y \to Y$ and suppose that $F(Z_2;Y) = F \neq \emptyset$. According to (3.11), there is a cohomology covering space X with respect to a, where $a \in H^1(Y;Z_2)$ is a generator. As X is connected, we know that X is a cohomology n-sphere mod 2 by (2.5). Since $F \neq \emptyset$ and a is necessarily invariant under T^*, T can be lifted to an involution $\tilde{T}: X \to X$ which commutes with the deck-transformation S on X (see Lemma 3.8). $\tilde{T}$ and S together then define an action of $Z_2 \times Z_2$ on X. Consider the subgroups in $Z_2 \times Z_2$ generated by $(S,1)$ $(1,\tilde{T})$ and $(S,\tilde{T})$, their fixed point sets F_0, F_1 and F_2 respectively. By the well-known theorem of P. A. Smith [1], F_i is a cohomology n_i-sphere mod 2, $i = 0,1,2$. Moreover, by a theorem of P. A. Smith [6] and A. Borel [1, p. 175], we have the relation that $\sum_{i=0}^{2}(n_i + 1)$ $= n + 1$. Obviously, we have $n_0 = -1$ since S acts freely on X. Hence the equation reduces to $n_1 + n_2 = n - 1$. It is not hard to see that $\pi(F_1)$ and $\pi(F_2)$, if not empty, are precisely the components of $F(Z_2;Y)$. Now S acts on F_1 and F_2, freely of course; therefore by (2.3) $\pi(F_1)$ and $\pi(F_2)$ are cohomology real projective n_i-spaces, $i = 1, 2$. This completes the proof of (4.1).

In precisely the same way, one can prove

THEOREM 4.2. *If* Z_p $(p \neq 2)$ *acts on a cohomology lens* $(2n+1)$-*space* mod p, *then the fixed point set* F *is either empty or it has at most* p *components. If* F *has* k *components* $C_1, \cdots, C_k$, $1 \leqq k \leqq p$, *then each* C_i *is a cohomology lens* $(2n_i + 1)$-*space* mod p, $i = 1, 2, \cdots, k$ *and*

$$\sum_{i=1}^{k} n_i = n - k + 1.$$

5. **Cohomology complex projective spaces.** We now turn to actions of the circle group S^1. An action of S^1 on a space X is said to have *finite orbit type* [1] if the set $\{S^1_x \subset S^1 \mid x \in X\}$ is finite, i.e., if there is only a finite number of distinct isotropic subgroups. We shall need the notion of universal bundle and classifying space [1, p. 52]. The universal bundle for S^1 is the space $\bigcup_{n=1}^{\infty} S^{2n+1}$ and the classifying space for S^1 is the infinite dimensional complex projective space CP^{∞}. These spaces are not compact (in fact, not even locally compact) and hence do not fit into the cohomology we are using now. As usual, this complication can be avoided by confining ourselves to spaces X on which S^1 acts to have finite cohomology dimension over Z [1, p. 6] (notation: $\dim_Z X < \infty$). Then by taking a sufficiently large $N \gg \dim_Z X$, we may "regard" S^{2N+1} and CP^N (complex projective N-space) as the universal bundle E_{S^1} and the classifying space B_{S^1} for the group S^1 (see [1, p. 52] for detailed explanation). This convention shall be used from now on. Throughout the rest of this paper, cohomology will always have the group of integers Z as coefficient domain unless otherwise stated. By a cohomology complex projective n-space we mean a space Y whose integral cohomology ring $H^*(Y)$ is given by

(5.1) $$H^*(Y) = Z[x]/(x^{n+1}), \text{ degree } x = 2,$$

where $Z[x]$ is the polynomial ring over Z and (x^{n+1}) is the ideal generated by x^{n+1}.

PROPOSITION 5.2. *If* S^1 *acts freely on an integral cohomology* $(2n+1)$-*sphere* X *and if* $\dim_Z X < \infty$, *then* X/S^1 *is a cohomology complex projective* n-*space.*

Proof. There exists a spectral sequence (E_r) [1] whose E_2-term is given by $E_2^{s,t} = H^s(B_{S^1}; H^t(X))$ and whose E_∞-term is associated with $H^*(X/S^1)$ (up to certain dimension). The assertion follows readily from the Gysin sequence of (E_r) and the fact that $H^*(B_{S^1})$ is given by

(5.3) $$H^*(B_{S^1}) = Z[x]/(x^{N+1}), \text{ degree } x = 2, 2N+1 > \dim_Z X.$$

THEOREM 5.4. *If S^1 acts freely on X such that X/S is a cohomology complex projective n-space and* $\dim_Z X/S^1 < \infty$. *If moreover* $\pi^* : H^2(X/S^1) \to H^2(X)$ *is trivial, then X is an integral cohomology* $(2n+1)$*-sphere and* $\dim_Z X < \infty$.

Proof. We shall only prove the case $n > 0$. Since S^1 acts freely on X, $\pi : X \to X/S^1$ is a fiber map by a theorem of A. Gleason [4]. This implies easily that $\dim_Z X < \infty$. We can therefore consider the spectral sequence (E_r) of (5.2) where we have $E_2^{s,t} = H^s(B_{S^1}; H^t(X)) = H^s(B_{S^1}) \otimes H^t(X)$ ($H^*(B_{S^1})$ has no torsion). In particular, we have

(1) $$E_2^{s,t} = 0 \text{ whenever } s \text{ is odd.}$$

Consider the sequence

$$0 \to E_2^{1,0} \xrightarrow{\phi_1} H^1(X/S^1) \xrightarrow{\pi^*} E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \xrightarrow{\phi_2} H^2(X/S^1) \xrightarrow{\pi^*} E_2^{0,2}.$$

In general, the last place of this sequence is not exact but only satisfies $\operatorname{Im}\phi_2 \subset \ker \pi^*$ and $\ker\pi^*/\operatorname{Im}\phi_2 = E_\infty^{1,1}$. But in our case $E_\infty^{1,1} = 0$ by (1). Taking account of the fact that $H^1(X/S^1) = 0$, we have thus the following exact sequence:

(2) $$0 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \xrightarrow{\phi_2} H^2(X/S^1) \xrightarrow{\pi^*} E_2^{0,2}.$$

Now $\pi^* : H^2(X/S^1) \to E_2^{0,2} = H^2(X)$ is trivial by hypothesis; hence (2) reduces to an exact sequence $0 \to E_2^{0,1} \to Z \to^{\phi_2} Z \to 0$. This implies $\phi_2 : E_2^{2,0} \to H^2(X/S^1)$ is an isomorphism and $E_2^{0,1} = H^1(X) = 0$. As $E_2^{*,0} = \sum_{s=0}^{\infty} E_2^{s,0} = H^*(B_{S^1})$, by (5.3) and (5.1) one deduces that

(3) $$\phi_s : E_2^{s,0} \to H^s(X/S)$$

is an isomorphism for all $1 \leqq s \leqq 2n+1$. Just as in Theorem 2.6, this fact and $H^1(X) = 0$ enables one to prove inductively that $H^s(X) = 0$ for all $1 \leqq s \leqq 2n$. Again as in (2.6), one then uses this and proceeds to show that $E_2^{0,2n+1} = E_{2n+2}^{0,2n+1}$, that $E^{2n+2,0} = E_{2n+2}^{2n+2,0}$ and that $d_{2n+2} : E_{2n+2}^{0,2n+1} \to E_{2n+2}^{2n+2,0}$ is an isomorphism. Hence $H^{2n+1}(X) = E_2^{0,2n+1} \cong E_2^{2n+2,0} = Z$.

For higher dimensional groups, of course, no special cohomology theory is available here. Instead, we propose to prove by induction that $H^{2n+k}(X) = 0$ for all $k \geqq 2$. We have seen that $\theta_0 = d_{2n+2}$: $E_2^{0,2n+1} \to E_2^{2n+2,0}$ is an isomorphism. This can be described as follows. Let a be a generator of $H^2(B_{S^1})$ and 1 denote the generator of $H^0(B_{S^1})$ as well as that of $H^0(X)$. Consider $a^{n+1} \otimes 1 \in E_2^{2n+2,0}$, regard it as in $E_{2n+2}^{2n+2,0}$; then there exists uniquely an element $b \in H^{2n+1}(X)$ such that $1 \otimes b \in E_2^{0,2n+1}$, considered as in $E_{2n+2}^{0,2n+1}$, satisfies $d_{2n+2}(1 \otimes b) = a^{n+1} \otimes 1$, and θ_0 is entirely determined by the relation $\theta_0(1 \otimes b) = a^{n+1} \otimes 1$.

Let $Z(E_r^{s,t})$ be the cocycles of $E_r^{s,t}$, $\mu_r^{s,t} : Z(E_r^{s,t}) \to E_{r+1}^{s,t}$ the projection and $j_r^{s,t} : Z(E_r^{s,t}) \to E_r^{s,t}$ the inclusion. We agree that if we write $\mu_r^{s,t} : E_r^{s,t} \to E_{r+1}^{s,t}$, it is tacitly assumed that $Z(E_r^{s,t}) = E_r^{s,t}$. Similarly, if we write $j_r^{s,t} : E_{r+1}^{s,t} \to E_r^{s,t}$, it is tacitly assumed that $Z(E_r^{s-r,t+r-1}) = E_r^{s-r,t+r-1}$ and $E_{r+1}^{s,t}$ has been identified with $Z(E_r^{s,t})$. It is easily verified that we have the following diagram:

$$\begin{array}{ccccccccc}
E_2^{0,2n+2} & \xleftarrow{j_2^{0,2n+2}} & E_3^{0,2n+2} = E_{2n+3}^{0,2n+2} & = & E_{2n+4}^{0,2n+2} & = & E_\infty^{0,2n+2} = 0 & & \\
\downarrow d_2 & & & & & & & & \\
E_2^{2,2n+1} & \xrightarrow{\mu_2^{2,2n+1}} & E_3^{2,2n+1} = E_{2n+2}^{2,2n+1} & \xleftarrow{j_{2n+2}^{2,2n+1}} & E_{2n+3}^{2,2n+1} & = & E_\infty^{2,2n+1} = 0 & & \\
& & \downarrow d_{2n+2} & & & & & & \\
E_2^{2n+4,0} & = & E_{2n+2}^{2n+4,0}. & & & & & &
\end{array}$$

In this diagram, $\ker d_2 = \operatorname{Im} j_2^{0,2n+2} = 0$. Define $\theta_2 : E_2^{2,2n+1} \to E_2^{2n+4,0}$ by $\theta_2 = d_{2n+2}\, \mu_2^{2,2n+1}$. Since $\ker d_{2n+2} = \operatorname{Im} j_{2n+2}^{2,2n+1} = 0$, we have $\ker \theta_2 = \ker \mu_2^{2,2n+1} = \operatorname{Im} d_2$. In other words, $0 \to E_2^{0,2n+2} \to^{d_2} E_2^{2,2n+1} \to^{\theta_2} E_2^{2n+4,0}$ is exact. Now consider $a \otimes b \in E_2^{2,2n+1}$, we have $\theta_2(a \otimes b) = d_{2n+2}((a \otimes 1)(1 \otimes b)) = (a \otimes 1)\, d_{2n+2}(1 \otimes b)$ because $d_{2n+2}(a \otimes 1) = 0$. That is, we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 \to & E_2^{0,2n+2} & \xrightarrow{d_2} & E_2^{2,2n+1} & \xrightarrow{\theta_2} & E_2^{2n+4,0} & \\
& \gamma_a \otimes 1 & & \uparrow & & \uparrow \ \gamma_a \otimes 1 & \\
& & & E_2^{0,2n+1} & \xrightarrow{\theta_0} & E_2^{2n+2,0} & ,
\end{array}$$

where $\gamma_a : H^*(B_{S^1}) \to H^*(B_{S^1})$ is the multiplication by $a \in H^2(B_{S^1})$. Since γ_a and θ_0 are isomorphisms, θ_2 must be an isomorphism and hence the exactness of the upper row implies that $H^{2n+2}(X) = E_2^{0,2n+2} = 0$.

Suppose it has been proved that $H^{2n+i}(X) = 0$ for all $2 \leqq i < k$. In a similar manner as above, one can show that there exists a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & E_2^{0,2n+k} & \longrightarrow & E_2^{k,2n+1} & \xrightarrow{\theta_k} & E_2^{2n+k+2,0} \\
& & \gamma_a^{k/2} \otimes 1 & \uparrow & & \uparrow \ \gamma_a^{k/2} \otimes 1 \\
& & & E_2^{0,2n+1} & \xrightarrow{\theta_0} & E_2^{2n+2,0}
\end{array}$$

for k even and

$$\begin{array}{ccccccc}
E_2^{k-1,2n+1} & \xrightarrow{\theta_k} & E_2^{2n+k+1,0} & \longrightarrow & E_2^{0,2n+k} & \longrightarrow 0 \\
\gamma_a^{(k-1)/2} \otimes 1 \uparrow & & \uparrow \ \gamma_a^{(k-1)/2} \otimes 1 & & & \\
E_2^{0,2n+1} & \xrightarrow{\theta_0} & E_2^{2n+2,0} & & &
\end{array}$$

for k odd with exact upper rows. In either case, one concludes that $H^{2n+k}(X) = E_2^{0,2n+k} = 0$. The proof of (5.4) is completed.

6. **Lifting of an action in a principal bundle** (X, Y, S^1, π).

PROPOSITION 6.1. *Let Y be a space and a an element of $H^2(Y)$, then there exists a principal bundle (X, Y, S^1, π) such that $\pi^* : H^2(Y) \to H^2(X)$ maps a into zero.*

Proof. Represent Y as the inverse limit of an inverse system $\{Y_m, \phi_m^{m'}\}$ of triangulable spaces [3]. Let $\phi_m : Y \to Y_m$ be the projection. By the continuity property of cohomology, there exists an index m and $a_m \in H^2(Y_m)$ such that $\phi_m^*(a_m) = a$. Consider the principal bundle (S^{2N+1}, CP^N, S^1, p), where N is so chosen that $2N + 1 > \dim Y_m$. According to the standard obstruction theory [5], there exists a map $g : Y_m \to CP^N$ such that $g^*(x) = a_m$, where $x \in H^2(CP^N)$ is a generator. We have therefore a map $f : Y \to CP^N$ $(f = g \circ \phi_m)$ such that $f^*(x) = a$. The bundle induced by f has all the desired properties stated in (6.1).

Now suppose that an action of S^1 is given on the base space Y in a principal bundle (X, Y, S^1, π). The notion of bundle lifting is defined in the same way as in §3. More precisely, let $\beta : S^1 \times X \to X$ represent the action of the structural group S^1 on X and $\alpha : S^1 \times Y \to Y$ the given action of S^1 on Y. Then a bundle lifting $\tilde{\alpha}$ of α is a map $\tilde{\alpha} : S^1 \times X \to X$ which defines an action of S^1 on X and satisfying the conditions (i) $\pi \circ \tilde{\alpha}(g,x) = \alpha(g, \pi(x))$ and (ii) $\tilde{\alpha}(g_1, \beta(g_2, x)) = \beta(g_2, \tilde{\alpha}(g_1, x))$. The following result is essentially due to T. E. Stewart [8].

PROPOSITION 6.2. *Let (X, Y, S^1, π) be a principal bundle and $\alpha : S^1 \times Y \to Y$ an action of S^1 on Y. If $H^1(Y) = 0$, then α has a bundle lifting.*

Proof. Let R be the additive group of reals and $\varphi : R \to S^1$ the usual exponential map. The map $\alpha' : R \times Y \to Y$ given by $\alpha'(t, y) = \alpha(\varphi(t), y)$ defines an action of R on Y. Using the same argument employed in [8, Lemma 3.3], one deduces that α' has a bundle lifting $\alpha'' : R \times X \to X$. Now to "push" α'' down to S^1, all we have to do is to adjust it in such a way that it becomes periodic. Define a map $g : Y \to S^1$ by the condition that $\alpha''(0, x) = \beta(g(y), \alpha''(1, x))$ for all $x \in \pi^{-1}(y)$. Since $H^1(Y) = 0$, i.e., $\pi^1(Y) = 0$, where $\pi^1(Y)$ is the Bruschlinsky group [5] of Y, g is homotopic to zero. Hence g can be factored as $g = \varphi \circ h$ by a map $h : Y \to R$. Define $\hat{\alpha} : R \times X \to X$ as

$$\hat{\alpha}(t,x) = \beta(\varphi\ th(\pi(x))),\ \alpha''(t, x)).$$

It is easily verified that $\hat{\alpha}$ defines an action of R on X satisfying $\hat{\alpha}(0,x) = \hat{\alpha}(1,x)$. Hence the map $\tilde{\alpha} : S^1 \times X \to X$ given by

$$\tilde{\alpha}(\varphi(t), x) = \hat{\alpha}(t, x)$$

is well defined, it gives an action of S^1 on X which is a bundle lifting of α.

7. **Actions of S^1 on cohomology complex projective spaces.** We first prove a proposition that will be needed in the proof of the main theorem and it is also interesting by itself.

PROPOSITION 7.1. *Let S^1 act on a cohomology complex projective n-space Y and assume that* $\dim_Z Y < \infty$. *Then the fixed point set F is nonempty and has at most $n+1$ components.*

Proof. The proof is based on the following well-known device. Let Y_{S^1} be the orbit space of the diagonal action of S^1 on $Y \times E_{S^1}$, where E_{S^1} is taken as the sphere S^{2N+1} with $2N+1 > \dim_Z Y$. There are [1, p. 50] natural maps $\pi_1 : Y_{S^1} \to Y/S^1$ and $\pi_2 : Y_{S^1} \to B_{S^1}$ and π_2 is always a fibering with Y as fiber. Consider the spectral sequence (E_r) of π_2 with the rational field Q as coefficient domain. We have $E_2^{s,t} = H^s(B_{S^1}; H^t(Y;Q))$ and the E_∞-term is associated with $H^*(Y_{S^1};Q)$. Since $E_2^{s,t} = 0$ when either s or t is odd, it is easily seen that (E_r) is trivial. This enables one to compute readily that

$$(1) \qquad \dim H^k(Y_S^1;Q) = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ m+1 & \text{if } k = 2m,\ 0 \leqq m \leqq n, \\ n+1 & \text{if } k = 2m,\ n \leqq m \leqq N. \end{cases}$$

The map π_1 is in general not a fibering but for each $z = \pi(y) \in Y/S^1$ we have $\pi^{-1}(z) = S^{2N+1}/S^1y$, where S^1y is the isotropic subgroup at y. Now suppose that $F = \phi$. Then each S^1y is a finite group and the rational cohomology of S^{2N+1}/S^1y is trivial, i.e., $H^k(S^{2N+1}/S^1y;Q) = 0$ for all $1 \leqq k \leqq 2N$. It follows from the Vietoris mapping theorem that $\pi_1{}^* : H^k(Y_{S^1};Q) \to H^k(Y_{S^1};Q)$ is an isomorphism for all $0 \leqq k \leqq 2N$. In particular, take k even with $\dim_Z Y < k \leqq 2N$, we obtain from (1) that $H^k(Y/S^1;Q) \neq 0$. But this is a contradiction because $\dim_Q Y/S^1 \leqq \dim_Z Y/S^1 \leqq \dim_Z Y$ [1, p. 111].

Just as Y_{S^1}, we can form the space F_{S^1} which is simply $B_{S^1} \times F$. The inclusion $i : F \to Y$ induces a homomorphism $i^* : H^k(Y_{S^1};Q) \to H^k(F_{S^1};Q)$. It is known [1, p. 54] that i^* is an isomorphism for all $\dim_Q Y < k \leqq 2N$. Take $k = 2N$, by (1) and the Kunneth formula we obtain $\dim H^0(F;Q) \leqq n+1$.

We now present the last main theorem of this paper.

THEOREM 7.2. *Let S^1 act on a cohomology complex projective n-space Y. Suppose that* $\dim_Z Y < \infty$ *and that the action has finite orbit type. Then the fixed point set F is nonempty, it has at most $n+1$ components, say $C_1, \cdots, C_k$ $1 \leqq k \leqq n+1$, where each C_i is a cohomology complex projective n_i-space, $i = 1, 2, \cdots, k$, and*

$$\sum_{i=1}^{k} n_i = n - k + 1.$$

Proof. Let $a \in H^2(Y)$ be a generator. By (6.1), there exists a principal bundle (X, Y, S^1, π) such that $\pi^* : H^2(Y) \to H^2(X)$ is trivial and therefore X is an integral cohomology $(2n+1)$-sphere according to (5.4). Let $\alpha : S^1 \times Y \to Y$ denote the given action and $\beta : S^1 \times X \to X$ the action of the structural group. By (6.2), α

has a bundle lifting $\tilde{\alpha}: S^1 \times X \to X$. $\tilde{\alpha}$ and β together then define an action $\gamma: (S^1 \times S^1) \times X \to X$ of $S^1 \times S^1$ on X by $\gamma((g_1, g_2), \tilde{x}) = \alpha(g_1, \beta(g_2, x))$. We claim that γ has finite orbit type. Suppose that $(g_1, g_2) \in G_x \subset S^1 \times S^1$. Let $y = \pi(x)$, then $g_1 \in Gy$. If $Gy \neq S^1$, i.e., Gy is finite, it is easily seen that $G_x = Gy \times N$, where $N \subset Gy$ is a subgroup. Hence there is only a finite number of G_x of this type since α has finite orbit type. If $Gy = S^1$, i.e., $y \in F$, it is easily seen that G_x is then of the form $G_x = \{(g, g^{ky}) \mid g \in S^1\}$, where ky is some integer depending on y. Moreover, the function $y \to ky$ is continuous on F, hence it must be constant on each component of F. By (7.1), F has at most $n+1$ components, say $C_1, \cdots, C_k$, $1 \leqq k \leqq n+1$. To each C_i, $i = 1, 2, \cdots, k$. Let $k_i = ky$, $y \in C_i$ and $H_i = \{(g, g^k i) \mid g \in S^1\}$. Then G_x must be one of the H_i, $i = 1, 2, \cdots, k$. This proves the assertion that γ has finite orbit structure.

Now let F_i be the fixed point set of H_i. By a theorem of E. E. Floyd [1, p. 63], F_i is an integral cohomology m_i-sphere and by dimension parity, m_i must be odd, say $m_i = 2n_i + 1$, $i = 1, 2, \cdots, k$. It is easily seen that $\beta: S^1 \times F_i \to F_i$ is a free action of S^1 on F_i for which $F_i/S^1 = C_i$. By (5.2), C_i is a cohomology complex projective n_i-space, $i = 1, 2, \cdots, k$. Finally, by a theorem of A. Borel [1, p. 175], we have the relation

$$\sum_{i=1}^{k} (m_i - (-1)) = (2n + 1) - (-1),$$

that is,

$$\sum_{i=1}^{k} n_i = n - k + 1.$$

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