HOMEOTOPY GROUPS

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Introduction. A principal goal in algebraic topology has been to classify and characterize spaces by means of topological invariants. One such is certainly the group $G(X)$ of homeomorphisms of a space X. And, for a large class of spaces X (including manifolds), the compact-open topology is a natural choice for $G(X)$, making it a topological transformation group on X. $G(X)$ is too large and complex for much direct study; however, homotopy invariants of $G(X)$ are *not* homotopy invariants of X. Thus, the *homeotopy groups of* X are defined to be the homotopy groups of $G(X)$. The $\mathscr{H}_k(X) = \pi_k[G(X)]$ are topological invariants of X which are shown (§2) not to be invariant even under isotopy, yet the powerful machinery of homotopy theory is available for their study.

In (2) some of the few published results concerning the component group $\mathscr{H}_0(X) = \pi_0[G(X)]$ are recounted. This group is then shown to distinguish members of some pairs of homotopic spaces.

In (3) the topological group $G(X)$ is given the structure of a fiber bundle over X, with the isotropy group ${}_xG(X)$ at $x \in X$ as fiber, for a class of homogeneous spaces X. This structure defines a topologically invariant, exact sequence, $\mathscr{H}_*(X)$.

In (4), $\mathscr{H}_*(X)$ is shown to be defined for manifolds, and this definition is extended to manifolds with boundary. Relations are then derived among the homeotopy groups of the set of manifolds got by deletion of finite point sets from a compact manifold.

Perhaps the most important result is given in (5): $\mathscr{H}_0(X,x) = \pi_0[{}_xG(X)]$ is represented there as a group of automorphisms of the sequence $\mathscr{H}_*(X)$. This representation generalizes, makes neater and constructively defines the well-known actions of $\pi_1(X,x)$ on the $\pi_k(X,x)$. Further, if X is a group with arc-component X_0 of the identity, e, then the well-known actions of $\pi_0(X) = X/X_0$ on $\pi_k(X_0,e)$ are always contained in this representation of $\mathscr{H}_0(X_0,e)$.

In (6) applications and examples are given.

This work is elementary and not exhaustive; so many unanswered questions lie close to the surface that I have made no attempt to list them.

1. **Definitions and preliminaries.** Let X be a topological space, and let $G(X)$

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be the group of homeomorphisms of X onto X. Arens [1; 2] has shown that, if X is locally compact, locally connected and Hausdorff, then the compact-open topology is the smallest topology on $G(X)$ in which $G(X)$ is a topological transformation group of X (see [3]: in the present work topological groups are required to be Hausdorff). The latter property is basic in the work below; therefore, only those base spaces X are considered which are locally compact, locally connected and Hausdorff; and $G(X)$ will always be given the compact-open topology. Such spaces X will be termed *admissible* to remind the reader of this assumption.

Now let $G_0(X)$ denote the arc-component of $1 \in G(X)$; $G_0(X)$ is a normal subgroup of $G(X)$. The equivalence relation defined by $G_0(X)$ will be called *homeotopy*. Let $I = [0, 1] \subset R$ (R will denote the real numbers); for $f_0, f_1 \in G(X)$, redefine f_0 to be homeotopic to f_1 iff there exists a map (i.e., continuous function) h: $X \times I \to X$ such that $h(x,i) = f_i(x)$ and, for each fixed $t \in I$, the function $h_t \in G(X)$, where h defines $h_t : X \to X$: $h_t(x) = h(x,t)$. Thus homeotopy is stronger than both homotopy and isotopy in demanding that h_t be a homeomorphism onto. It is shown in [4] that the two definitions of homeotopy are equivalent.

For $x \in X$, the *isotropy group* (or *group of stability*) at x will be denoted ${}_xG(X) = \{g \in G(X)\colon g(x) = x\}$. The arc-component of $1 \in {}_xG(X)$ will be denoted ${}_xG_0(X)$ (each subgroup has the relative topology). When confusion will not result, the groups $G(X)$, $G_0(X)$, ${}_xG(X)$ and ${}_xG_0(X)$ may be denoted simply G, G_0, ${}_xG$ and ${}_xG_0$, respectively.

The ith *homeotopy group of* X is defined to be $\mathscr{H}_i(X) = \pi_i(G)$. The group of arc-components of G, $\pi_0(G) = \mathscr{H}_0(X)$, will also be denoted simply by $\mathscr{H}(X)$. Homeotopy troups are thus topological invariants (up to isomorphism) associated with spaces of a certain class. The (quotient) topology of the $\mathscr{H}_i(X)$ is not considered here; however, it is discrete in some cases [5].

Let F be an admissible subspace of X, and define $G(X,F) = \{g \in G(X) : g \mid F \in G(F)\}$; denote the arc-component of $1 \in G(X,F)$ by $G_0(X,F)$. The ith *homeotopy group of the pair* (X,F) is defined to be $\mathscr{H}_i(X,F) = \pi_i[G(X,F)]$. Again, $\mathscr{H}_i(X,F)$ is a topological invariant of the pair (X,F). The group $\mathscr{H}_0(X,F)$ will also be denoted simply $\mathscr{H}(X,F)$.

Some symbols which will be used are: S^n = the n-sphere; H_g = a closed orientable surface of genus g; Z = the integers; Z_n = the integers mod n; S_n = the symmetric group on n marks.

2. **Some examples.** In (2.1) there is a list of homeotopy groups, $\mathscr{H}(X)$, which either have been described or else can be easily inferred from published results. It should be noted here that, if X is a compact, connected manifold, then homeomorphisms f, $g \in G(X)$ are homeotopic iff isotopic. Fisher [6] has given a brief history of these results, plus a large bibliography; the reader may find details there. Examples of $\mathscr{H}_i(X)$, $i \geqq 1$, are given in (6).

2.1. $\mathscr{H}(R^0) \cong \mathscr{H}$ (half open interval) $\cong 1$. For $n = 1, 2$ or 3, $\mathscr{H}(R^n) \cong \mathscr{H}(S^n)$ $\cong \mathscr{H}$ (closed n-cell) $\cong Z_2$. For $n = 2, 3, \cdots$, $\mathscr{H}$ (n discrete points) $\cong$ $\mathscr{H}$(n-thistle) $\cong S_n$. For $g > 0$, $\mathscr{H}(H_g) = \mathscr{O}[\pi_1(H_g)]$, the group of outer automorphisms of the fundamental group of H_g (R. Baer [8]). H. Kneser [9] has shown that, for all n, $\mathscr{H}$ (closed n-cell) $\cong \mathscr{H}(S^{n-1})$. H. Gluck [13] reports that $\mathscr{H}(S^2 \times S^1) \cong (Z_2)^3$; i.e., the direct product $Z_2 \times Z_2 \times Z_2$.

By these examples one may see that homeotopy groups are not homotopy invariants. In fact, $\mathscr{H}(X)$ is not an isotopy invariant: an isotopy relation (due to S.-T. Hu) between $(-1,1) = X$ and $(-1,1] = Y$ is given by $f: X \to Y: x \to x$, and g: $Y \to X$: $x \to x/2$. The isotopies of fg and gf with the respective identities are both given by $h(x,t) = (1 + t)x$ for $t \in I$.

2.2. Let $r \geqq 3$ be a given integer and consider two linear graphs, X_r and Y_r, where X_r is obtained by joining the centroid v_0 of a regular polygon P_r to its vertices $v_1, \cdots, v_r$ by edges $v_0 v_i$ $(i = 1, \cdots, r)$; and Y_r is obtained by attaching r triangles $x_i y_i z_i$ $(i = 1, \cdots, r)$ at the end points $x_1, \cdots, x_r$ of an r-thistle with center x_0. Hu shows in [7] that X_r and Y_r are of the same homotopy type.

It is easy to see that $\mathscr{H}(X_3) = S_4$; and if $r > 3$, then $\mathscr{H}(X_r) = D_r$, the dihedral group of order $2r$. But $\mathscr{H}(Y_r)$ is the normal product of $(Z_2)^r$ by S_r, where S_r corresponds to permutations of the limbs of the r-thistle and each subgroup Z_2 corresponds to the homeotopy group of one of the triangles. Therefore, for all r, X_r and Y_r may be distinguished by their homeotopy groups.

3. **Homogeneous spaces.** Let the coset space $Y/Z = X$ be admissible; then G is transitive on X; i.e., X is *homogeneous*, and there is a one-to-one map of $G/{}_xG$ onto X, for arbitrary but fixed $x \in X$. If G is a fiber bundle over X with projection p: $g \to g(x)$, then there is a local cross-section at x of ${}_xG$ in G; hence p is an open map and $G/{}_xG$ is homeomorphic to X. (3.1) characterizes such spaces; it is shown in (4) that this class includes manifolds.

LEMMA 3.1. *Let X be an admissible coset space; say, X is homeomorphic to Y/Z; further, let Z have a local cross-section in Y. Then G is a principal fiber bundle over X with fiber and group ${}_xG$, for arbitrary $x \in X$.*

Proof. Let $Z_0 = \bigcap yZy^{-1}$: $y \in Y$. It is shown in [1, Theorem 2] that the inclusion of Y/Z_0 in G (defined by the action of Y/Z_0 on Y/Z) is continuous. Standard application of the bundle structure theorem [3] yields the conclusion.

DEFINITION 3.2. When G is a principal bundle over an arc-connected space X, then (with the notation of Hu [10, p. 152]) the following homotopy sequence is exact, for arbitrary $x \in X$:

$$\cdots \xrightarrow{p_*} \pi_{n+1}(X,x) \xrightarrow{d_*} \pi_n({}_xG,1) \xrightarrow{i_*} \pi_n(G,1) \xrightarrow{p_*} \pi_n(X,x) \rightarrow \cdots \xrightarrow{i_*} \pi_0(G,1).$$

Furthermore, G_0 is transitive on X; thus this sequence may be extended so that $\cdots \xrightarrow{i_*} \pi\ (G,1) \xrightarrow{p_*} 0$ is exact. In the case at hand, the sequence is a topological invariant of X. The notation introduced in (1) emphasizes this invariance: the *homeotopy sequence of* X, $\mathscr{H}_*(X)$, is defined to be

$$\cdots \xrightarrow{p_*} \pi_{n+1}(X,x) \xrightarrow{d_*} \mathscr{H}_n(X,x) \xrightarrow{i_*} \mathscr{H}_n(X) \xrightarrow{p_*} \pi_n(X,x) \rightarrow \cdots$$

$$\cdots \xrightarrow{p_*} \pi_1(X,x) \xrightarrow{d_*} \mathscr{H}(X,x) \xrightarrow{i_*} \mathscr{H}(X) \xrightarrow{p_*} 0.$$

If the arc-connected space X is simply connected, then $\mathscr{H}(X,x) \cong \mathscr{H}(X)$. More generally, $i_*: \mathscr{H}(X,x) \cong \mathscr{H}(X)$ iff every loop at $x \in X$ can be lifted to a loop at $1 \in G$. This is always true when X is a group:

LEMMA 3.3. *If X is an admissible topological group, then G is a product bundle over X.*

Proof. The map $L: X \rightarrow G: x \rightarrow L_x$, the left translation of X by x, is continuous, by [1, Theorem 2]. Since $p \circ L$ is the identity map on X, L is a global cross-section of the bundle G with group and fiber ${}_eG$, where e is the identity of X.

Now define $m: X \times {}_eG \rightarrow G: (x, g) \rightarrow L_x \cdot g$; m is continuous since G is a topological group. The inverse map $m^{-1}: G \rightarrow X \times {}_eG: g \rightarrow (p(g), L^{-1}_{p(g)} \cdot g)$ is also continuous; m is a homeomorphism. Since X is homogeneous, the choice of e as base point for the isotropy group was arbitrary.

REMARK 3.31. When X is an admissible topological group, and $i \geqq 1$, $\mathscr{H}_i(X) \cong \mathscr{H}_i(X, x) + \pi_i(X, x)$, where the direct sum notation is used since all the groups are abelian. Further, if X is arc-connected, $i_*: \mathscr{H}(X, x) \cong \mathscr{H}(X)$.

THEOREM 3.4. *Let $q: X \rightarrow Y$ be a principal fiber bundle projection, where X is an admissible arc-connected space and Y is simply connected; and let $p: G(X) \rightarrow X: g \rightarrow g(x)$ be a bundle projection. Then i_*: $\mathscr{H}(X, x) \cong \mathscr{H}(X)$.*

Proof. Let $y = q(x)$, $F_y = q^{-1}(y)$, F be the fiber of $q: X \rightarrow Y$, and choose an admissible homeomorphism of F with F_y so that x corresponds to e, the identity of F. To simplify notation, rename the points of F_y by this homeomorphism; then $F_y = \{(y,f): f \in F\}$ and the operations of F on F_y appear as left multiplications: $f_1: F_y \rightarrow F_y: (y,f_2) \rightarrow (y,f_1 f_2)$. Let G, G_0, ${}_xG$ stand for $G(X)$, $G_0(X)$, ${}_xG(X)$.

Let A be a loop at $x \in X$, and let A^* be a covering arc, beginning at $g \in {}_xG$ and ending at 1, so that $p \circ A^* = A$. Then $q \circ p \circ A^* = q \circ A = A'$ is a loop at $y \in Y$. A' is homotopic to the point y; a covering homotopy of A in X followed by a covering homotopy of A^* in G yields a new arc (or path) B^* from g to 1. Its projection $p \circ B^* = B$ is a loop at $x \in X$ which lies entirely in the fiber F_y: let $B(t) = (y, B'(t))$, so B' is a loop at $e \in F$.

Now F is Hausdorff by hypothesis; the projections of local product representations in X show that F is admissible. Further, Steenrod [3, p. 40] shows that the map θ, which sends $f \in F$ into the right translation of the bundle X by f^{-1}, is a faithful representation of F as a topological transformation group on X. By [1, Theorem 2], $\theta: F \to G$ is continuous. Let $C^* = \theta \circ B'$ and consider the arc $D^*: I \to G: t \to B^*(t) \cdot C^*(t)$; D^* begins at g and ends at 1. But $p \circ D^*(t) = [B^*(t) \circ C^*(t)](y, e) = B^*(t)(y, B'(t)^{-1}) = (y, B'(t)B'(t)^{-1}) = (y, e)$ for all $t \in I$; so $D^*(I)$ lies in the fiber ${}_xG$ above x, and $g \in {}_xG_0$. Define $E^*: I \to G: t \to A^*(t) \cdot D^*(t)^{-1}$; E^* is a loop at $1 \in G$, and $p \circ E^* = p \circ (A^* \cdot D^{*-1}) = p \circ A^* = A$. Thus E^* is a loop at 1 covering A, p_* is an epimorphism at $\mathscr{H}_1(X)$, and i_* is an isomorphism.

REMARK 3.41. Since either F or Y in (3.4) may be taken to be a single point, this theorem is one possible generalization of the cases where X is simply connected or a group. Another is the fact that, if $i_*: \mathscr{H}(X_i, x_i) \cong \mathscr{H}(X_i)$ for $i = 1, \cdots, n$, then $i_*: \mathscr{H}(X_1 \times \cdots \times X_n, (x_1, \cdots, x_n)) \cong \mathscr{H}(X_1 \times \cdots \times X_n)$. The proof is easy.

THEOREM 3.5. *Let X be an admissible coset space, say $X = Y/Z$; and let Z be an arc-connected subgroup having a local cross-section in Y. Then i_*: $\mathscr{H}(X,x) \cong \mathscr{H}(X)$, for $x \in X$.*

Proof. As before, it will suffice to choose x to be the image of Z in X. Let A be a loop at x representing $\phi \in \pi_1(X,x)$: there exists a lift A^* of A into Y which begins at 1 and ends at $z \in Z$. Let B^* be an arc in Z from z to 1, and define C^*: $I \to Y$ to be the homotopy product of A^* followed by B^*. By [1, Theorem 2], C^* may be considered a loop at 1 in $G(X)$, and its projection represents $\phi \in p_*[\mathscr{H}_1(X)]$. Theorem 3.5 is proved.

The reader should recall, for (3.1) and (3.5), the standard fact that, if Z is a closed subgroup of the Lie group Y, there always exists a local cross-section.

4. **Manifolds.** A *manifold* is a topological space in which each point has a neighborhood homeomorphic to some fixed R^n; the manifolds discussed below are further restricted to be connected and Hausdorff. A *manifold with boundary* is a connected Hausdorff space in which each point has an open neighborhood whose closure is homeomorphic to the closed unit ball B in a fixed euclidean space E^n; n is the dimension, $\dim(X)$, of X. Every manifold is a manifold with boundary. If X is a manifold with boundary, denote by $\mathrm{Core}(X)$ the subset of points of X which have neighborhoods homeomorphic to the interior of $B \subset E^n$; $\mathrm{Core}(X)$ is a manifold. Let $\mathrm{Rim}(X) = X - \mathrm{Core}(X)$.

LEMMA 4.1. *If X is a manifold with boundary and $x \in \mathrm{Core}(X)$, then G is a fiber bundle over* $\mathrm{Core}(X)$ *with projection $p: g \to g(x)$.*

Proof. It will suffice to exhibit a local cross-section for ${}_xG$ in G. Choose a neighborhood N of x and a homeomorphism ϕ: $N \to R^n$ of N with R^n, so that $\phi(x) = 0$. For $y \in N$, the translation homeomorphism $\phi^*(y)$: $R^n \to R^n$: $z \to z + \phi(y)$ furnishes a homeomorphism $\Phi(y) = \phi^{-1} \circ \phi^*(y) \circ \phi \in G(N)$ which carries x to y. $\Phi(y)$ is extensible, by the identity map on $X - N$, to a map $\Phi^*(y) \in G(X)$. Further, $\Phi^*: N \to G(X)$ is a cross-section over N. In the last two assertions, continuity is easily established for points not on the frontier of N. At the latter points the following remark may be used: If $y_i \in R^n$ and $\alpha \in S^{n-1}$, the sphere at infinity, so that $y_i \to \alpha$, and if $\| z_i - y_i \|$ is bounded in the usual norm, then $z_i \to \alpha$.

LEMMA 4.11. *If X is a manifold with boundary and $x \in \mathrm{Core}(X)$, then the injection j: $(\mathrm{Core}(X),x) \to (X,x)$ induces an isomorphism $j_* : \pi_k(\mathrm{Core}(X),x) \cong \pi_k(X,x)$ for all k.*

Proof. Since the image of a map f: $(S^k,s) \to (X,x)$ is compact, and $\mathrm{Rim}(X)$ is closed; $F = f(S^k) \cap \mathrm{Rim}(X)$ is also compact. Hence F may be covered by a finite number of open sets E_i such that each $E_i = \mathrm{Rim}(X) \cap D_i$, where $D_i \subset X$ is open, D_i^- is homeomorphic to $B \subset E^n$, and $x \notin D_i$. Let $F_i = F - \bigcup_{i \neq j} E_j \subset E_i$; F_i is closed and $F = \bigcup F_i$. Using the normality of X, one may construct shrinking homotopies H_i: $D_i^- \times I \to D_i^-$, each of which carries F_i into $\mathrm{Core}(X)$, and which may be extended by the identity map to all of X. The composition of these homotopies yields a homotopy of the range of f with a subset of $\mathrm{Core}(X)$; thus j_* is an epimorphism. That j_* is a monomorphism may be shown similarly. This lemma is doubtlessly known; however, I do not have a reference for it.

DEFINITION 4.12. If X is a manifold with boundary and $x \in \mathrm{Core}(X)$, the homotopy sequence of the bundle $G(X)$ over $\mathrm{Core}(X)$ is exact (see (3.2)). By (4.11), the isomorphism j_* may be used to define the *homeotopy sequence of* X, which resembles that of (3.2) except that the homomorphisms p_* and d_* are replaced by $j_* p_*$ and $d_* j_*^{-1}$; thus the homeotopy sequence of a manifold with boundary is exact. It will be denoted $\mathscr{H}_*(X)$ and is an invariant of X (with the understanding that $x \in \mathrm{Core}(X)$).

LEMMA 4.2 (ARENS). *If X is a compact admissible space and $x \in X$, then $G(X,x)$ is topologically isomorphic to $G(X - x)$, and so $\mathscr{H}_i(X,x) \cong \mathscr{H}_i(X - x)$ for all i.*

This is shown by Theorems 1, 3 and 4 of [2], plus the remark that $X - x$ is compact iff x is open.

THEOREM 4.21. *Let X be a compact manifold with boundary, $\dim(X) > 1$, and let F be a finite subset of $\mathrm{Core}(X)$. Then $G(X,F)$ is topologically isomorphic to $G(X - F)$, and thus $\mathscr{H}_i(X,F) \cong \mathscr{H}_i(X - F)$, for all i.*

Proof. Let $F = \bigcup_{i=1}^n \{a_i\}$. Define a map $m : G(X,F) \to G(X - F)$: $g \to g|(X - F)$. The proof is divided into three parts showing that m respectively is a continuous monomorphism, onto, and has a continuous inverse.

(i) Trivially, m is an algebraic homomorphism. But if $f|(X - F) = g|(X - F)$, then $fg^{-1}|(X - F)$ is the identity map on $X - F$; then fg^{-1} fixes elements of nets converging to a_i; thus $f = g$ and m is one-to-one. If the set S is compact (or open) in $X - F$, then S is compact (open) in X; thus, the inverse image of a subbasic set $(K,W) = \{g \in G(X - F)\colon g(K) \subset W\}$ is a subbasic set $[K,W] = \{g \in G(X)\colon g(K) \subset W\}$.

(ii) That m is onto follows from de Groot [12, Satz III].

(iii) Let K be compact, W be open, and $K \subset W \subset X$, so that $[K,W]$ is a neighborhood of $1 \in G(X,F)$. By part (ii), $G(X,F)$ and $G(X-F)$ may be regarded as the same set with two topologies on it; to show that m is an open map, it will suffice that $m[K,W]$ is a neighborhood of 1 in $G(X-F)$.

Choose a neighborhood N_i of each a_i such that: the N_i are disjoint, if $a_i \in W$ then $N_i \subset W$, and for each i there exists a homeomorphism ϕ_i of N_i onto R^q, $\phi_i(a_i) =$ the origin. Give R^q the usual norm and define $T_i = \phi_i^{-1}\{t \in R^q\colon \|t\| = 1\}$ and $V_i = \phi_i^{-1}\{t \in R^q\colon \|t\| < 1\}$. Consider the open set $Q \subset G(X - F)$:

$$Q = (K - \bigcup_{i=1}^{n} V_i, W - F) \cap [\bigcap_{i=1}^{n} (T_i,N_i)].$$

By construction, $1 \in Q$. Let $h \in Q$ and $x \in K$; if $x \in K - \bigcup_i V_i$ then $h(x) \in W$. Now notice that $X - T_i$ has exactly two components, as does $X - h(T_i) = h(V_i) \cup h(V_i^{-\prime})$, where $V_i^{-\prime}$ denotes the complement of V_i^{-}. The Jordan-Brouwer theorem may be used to show that one of these components lies entirely inside N_i (which $h(T_i)$ separates). But, by (4.2) it may be assumed $n > 1$, and so there exists $j \neq i$ such that $T_j \subset V_i^{-\prime}$. Hence, if $h(V_i^{-\prime}) \subset N_i$, then $h(T_j) \subset N_j \cap N_i = \emptyset$, a contradiction. Thus $h(V_i) \subset N_i$ and, if $x \in K \cap (\bigcup_i V_i)$ then $h(x) \in W$. Therefore $h \in (K,W)$; so $Q \subset m[K,W]$ and m^{-1} is continuous.

REMARK 4.22. The preceding theorem is used, without proof, in [5].

REMARK 4.3. Let F,X be admissible, with $F \subset X$, and define $\kappa_*\colon \mathscr{H}(X,F) \to \mathscr{H}(X) \times \mathscr{H}(F)\colon fG_0(X,F) \to [fG_0(X),(f|F)G_0(F)]$. It is easy to see κ_* is a well-defined homomorphism. In general, however, κ_* is neither epic nor monic.

THEOREM 4.31. *Let X be a compact manifold with boundary,* $\dim(X) > 1$, *and let $F = \{a_i\}$ be a finite subset of* Core(X). *Define $X_j = X - \bigcup_{i=1}^{j}\{a_i\}$; and, for each $j = 0, \cdots, n-1$, let $i_*\colon \mathscr{H}(X_j, a_{j+1}) \cong \mathscr{H}(X_j)$. Then $\mathscr{H}(X - F) \cong \mathscr{H}(X) \times S_n$.*

Proof. By (4.21) and (4.3) it will suffice to show κ_* is an isomorphism.

Let $[fG_0(X),\alpha] \in \mathscr{H}(X) \times S_n$. Let $1 \leqq i < j \leqq n$; it is easy to construct $g_{ij} \in G_0(X)$ which interchanges a_i with a_j and fixes each a_k, $k \neq i,j$. Represent $\alpha \in S_n$ as a product of transpositions, and form the corresponding product $g \in G_0(X)$ of the g_{ij}; then $g(a_i) = a_{\alpha(i)}$ for each i. A similar construction deforms f to $h \in fG_0(X)$ so that $h(a_i) = a_i$ for each i. But $\kappa_*[(gh)G_0(X,F)] = [fG_0(X),\alpha]$, so κ_* is onto.

Now regard each $G(X_j) \subset G(X)$, by (4.21). Let f fix each a_i; then, for each $j \leqq n$, $f \in G(X_{j-1},a_j) \cap G_0(X_{j-1})$ implies $f \in G_0(X_j)$ since each injection i_* is an isomorphism. By induction, therefore, if f fixes each a_i and $f \in G_0(X)$ then

$f \in G_0(X_n)$. Thus, if $[fG_0,\alpha] = [G_0,1] \in \mathscr{H}(X) \times S_n$, then $f \in G_0(X_n) = G_0(X,F)$, and κ_* is a monomorphism.

COROLLARY 4.32. *Let X be a compact, simply connected, triangulable manifold of dimension > 2, and let $F = \{a_i\}$ be a set of n distinct points of X. Then $\mathscr{H}(X - F) \cong \mathscr{H}(X) \times S_n$.*

Proof. Each a_i may be chosen in the interior of a simplex of dimension > 2, and homotopies may be simplicially approximated in the 2-skeleton of the triangulation. Thus, the space got by deleting a finite number of points from X is simply connected. The result now follows from (4.31).

THEOREM 4.4. *Let $X = Y - F$, where Y is a compact manifold with boundary and F is a finite subset of* Core(Y), *and let $x \in$* Core(X). *Then $G_0(X,x)$ is topologically isomorphic to $G_0(X - x)$; consequently $\mathscr{H}_i(X,x) \cong \mathscr{H}_i(X - x)$ for all $i \geqq 1$.*

Proof. In case $\dim(X) = 1$, the component of x in X is a circle or a line segment; the theorem is true by (4.2) or by inspection.

When $\dim(X) > 1$, the isomorphism $m: G(Y,F) \cong G(X)$ maps the isotropy group at x isomorphically onto the isotropy group at x, by (4.21). Hence, m has a restriction mapping the group $G_0(Y,F \cup \{x\})$ onto $G_0(X,x)$. A second use of (4.21) shows that $G_0(Y,F \cup \{x\})$ is topologically isomorphic to $G_0(Y - [F \cup \{x\}]) = G_0(X - x)$.

5. **Automorphisms of $\mathscr{H}_*(X)$.** A homeomorphism g of an arbitrary couple (X,x), $x \in X$, induces an automorphism, g_*, of $\pi_k(X,x)$ for each k: if f represents $\phi \in \pi_k(X,x)$ then $g_*(\phi)$ is the homotopy class of $g \circ f$. Hence, if X is admissible, then ${}_xG$ has a representation, $\alpha: g \to g_*$, as a group of automorphisms of $\pi_k(X,x)$, for each k. Furthermore, if $g \in {}_xG_0$ then an arc from g to 1 in ${}_xG$ provides a homotopy of $g \circ f$ with f, so g_* is the identity automorphism. Thus α induces a representation, $\alpha_*: \mathscr{H}(X,x) \to \mathscr{A}[\pi_k(X,x)]$, of $\mathscr{H}(X,x)$ as a group of automorphisms of $\pi_k(X,x)$, for each k.

If $g \in {}_xG$ then $K_g: G \to G$: $h \to ghg^{-1}$ defines homeomorphisms of the couples $({}_xG,1)$ and $(G,1)$; these homeomorphisms may be used, just as above, to define representations, $\beta_*: \mathscr{H}(X,x) \to \mathscr{A}[\mathscr{H}_k(X,x)]$ and $\gamma_*: \mathscr{H}(X,x) \to \mathscr{A}[\mathscr{H}_k(X)]$, of $\mathscr{H}(X,x)$ in the automorphism groups of the homeotopy groups. When G is a bundle over X, so that the homeotopy sequence is defined, these automorphisms fit together in the natural way.

THEOREM 5.1. *Let G be a bundle over the arc-connected admissible space X with projection p. Then α_*, β_* and γ_* together furnish a representation r of $\mathscr{H}(X,x)$ in the group of automorphisms of $\mathscr{H}_*(X)$.*

Proof. Let g represent $\eta \in \mathscr{H}(X,x)$, and let $a_* = \alpha_*(\eta), b_* = \beta_*(\eta)$ and

$c_* = \gamma_*(\eta)$. The group property of the representation will be clear; it need only be shown that the following diagram is commutative:

$$\begin{array}{ccccccccc} \cdots \to & \pi_{k+1}(X,x) & \xrightarrow{d_*} & \mathscr{H}_k(X,x) & \xrightarrow{i_*} & \mathscr{H}_k(X) & \xrightarrow{p_*} & \pi_k(X,x) & \to \cdots \\ & \downarrow a_* & & \downarrow b_* & & \downarrow c_* & & \downarrow a_* & \\ \cdots \to & \pi_{k+1}(X,x) & \xrightarrow{d_*} & \mathscr{H}_k(X,x) & \xrightarrow{i_*} & \mathscr{H}_k(X) & \xrightarrow{p_*} & \pi_k(X,x) & \to \cdots. \end{array}$$

The conventions of Hu [10] will be followed in notation. For the first square, let $q:(G,{}_xG,1)\to(X,x)$ be defined by p; note $q \circ K_g = g \circ q$, since $g^{-1}\in {}_xG$. Let $\phi\in\pi_{k+1}(X,x)$, and let $[f_1]\in\pi_{k+1}(G,{}_xG,1)$ be such that $q_*[f_1]=\phi$, where $[f_1]$ denotes the homotopy class of $f_1:(I^{n+1},I^n,J^n)\to(G,{}_xG,1)$. Then $b_*d_*(\phi) = [K_g \circ (f_1 \mid I^n)] = [(K_g \circ f_1)\mid I^n]$; and $d_*a_*(\phi) = [f_2 \mid I^n]$, where

$$[f_2]\in\pi_{k+1}(G,{}_xG,1)$$

is such that $q_*[f_2] = g_*(\phi)$. But $q_*[K_g \circ f_1] = [q\circ K_g \circ f_1] = [g\circ q\circ f_1] = g_*q_*[f_1] = g_*(\phi)$, so the first square commutes.

Trivially, $i_*b_* = c_*i_*$; the second square is commutative. In the third square, $a_*p_*[f] = [g\circ p \circ f] = [p\circ K_g \circ f] = p_*c_*[f]$, and (5.1) is proved.

As usual, (5.1) and its proof are also valid, with minor changes of notation, for manifolds with boundary.

Remark 5.11. These automorphisms are generally nontrivial; e.g., an orientation-reversing element of $\mathscr{H}(S^n,x)$ is not identity at $\pi_n(S^n,x)$. A more interesting example is given by J. Nielsen [11]: if H_g is a sphere with $g>0$ handles, $x\in H_g$, then each automorphism of $\pi_1(H_g,x)$ is induced by some member of ${}_xG$.

Remark 5.12. If, in (5.1), X is the arc-component of the identity, e, in some group Y; then, for each $y\in Y$, the conjugation, K_y: $X\to X$: $x\to yxy^{-1}$, is in ${}_eG(X)$. Hence, the representation of $\mathscr{H}(X,e)$ contains the automorphisms induced by $\pi_0(Y)$, for every such Y. In the present context these automorphisms are viewed as properties of X itself.

Lemma 5.2. *Let G be a bundle over the arc-connected admissible space X with projection p, and let $\phi\in\pi_1(X,x)$. Then $\alpha_*d_*(\phi)$ is the standard automorphism of the $\pi_k(X,x)$ induced by translations around a loop, A, representing ϕ.*

Proof. The standard automorphism is described in [10, pp. 125–129]: a representative f of $\theta\in\pi_k(X,x)$ is chosen; a partial homotopy of $f\mid\partial I^n$ (which is a constant map) along A is constructed; the existence of an extension of the homotopy to all of I^n is shown; and finally, the result is shown to be unique up to homotopy. But let A^* be a lift of A running from g to 1 in G; then $d_*(\phi) = g({}_xG_0)$ and $\alpha_*d_*(\phi) = g_*\in\mathscr{A}[\pi_k(X,x)]$: the action of the homeotopy, $A^*(1-t)$, on $[f]\in\pi_k(X,x)$ by composition is a realization of the standard translation homotopy of $[f]$ along A, and the final stage is $g_*[f]$.

REMARK 5.21. The proof of (5.2) shows, more generally, that, if A is an arc from y to x in X and A^* lifts A, $A^*(1)=1$ in G; then the standard translation of $\pi_k(X,x)$ along A is the isomorphism

$$(A^*(0))_*: \pi_k(X,x) \cong \pi_k(X,y): [f] \to [A^*(0) \circ f].$$

COROLLARY 5.22. *With the assumptions of* (5.2), *the subgroup* $p_*\mathscr{H}_1(X) \subset \pi_1(X,x)$ *acts* (*in the standard way*) *trivially on each* $\pi_k(X,x)$.

This is true since $\phi \in p_*\mathscr{H}_1(X)$ has a lift represented by a loop at 1 in G.

REMARK 5.23. If, in (5.2), it is true that $i_*: \mathscr{H}(X,x) \cong \mathscr{H}(X)$, so that $p_*\mathscr{H}_1(X) = \pi_1(X,x)$, then X is n-simple for all n. This, together with (3.3) and (3.5), yields well-known results; however, the n-simplicity of the space X in (3.4) may be new.

REMARK 5.24. The preceding corollary shows that $p_*\mathscr{H}_1(X)$ is central in $\pi_1(X,x)$, since the standard action of $\pi_1(X,x)$ on itself is just the group of inner automorphisms. The real projective spaces, P^n, of even dimension $n \geqq 2$, are known to be not n-simple; yet $\pi_1(P^n,p) = Z_2$; thus $p_*\mathscr{H}_1(X)$ is not generally the center of the fundamental group.

EXAMPLE 5.25. Let X be the topological product of a circle with the space got from a 2-sphere by deleting three points. $\pi_1(X,x) \cong F_2 \times Z$, where F_2 is the free group on two generators; the center of this group is Z. The rotations of the circle induce rotations of the product which define loops in $G(X)$. These are lifts of loops representing elements of the subgroup of $\pi_1(X,x)$ corresponding to $\pi_1(S^1)$; i.e., elements of the center of $\pi_1(X,x)$. Since $p_*\mathscr{H}_1(X)$ is central, $p_*\mathscr{H}_1(X) = Z$.

Hence, $p_*\mathscr{H}_1(X)$ is in general a nontrivial subgroup of $\pi_1(X,x)$; of course, this subgroup is a topological invariant of the pair (X,x). It might be likened to the subgroup of orientation-preserving loops in the fundamental group of a nonorientable manifold: in its invariance and in that it is defined topologically.

6. **Some results.** It follows from (4.31) that: $\mathscr{H}(S^n) \cong \mathscr{H}(R^n)$ for $n \geqq 1$; $\mathscr{H}(S^{n-1} \times R) \cong \mathscr{H}(S^n) \times Z_2$ for $n \geqq 2$; more particularly, $\mathscr{H}(S^1 \times R) \cong \mathscr{H}(S^2 \times R) \cong (Z_2)^2$.

H. Kneser [9] has shown that the real line, R, is a strong deformation retract of $G_0(R)$. Thus the homeotopy sequence $\mathscr{H}_*(R)$ is

$$\cdots \to 0 \to \cdots \to 0 \to Z_2 \xrightarrow{i_*} Z_2 \to 0.$$

He then shows that S^1 is a strong deformation retract of $G_0(S^1)$; thus the sequence $\mathscr{H}_*(S^1)$ is

$$\cdots \to 0 \to \cdots \to 0 \to Z \xrightarrow{p_*} Z \xrightarrow{d_*} Z_2 \xrightarrow{i_*} Z_2 \to 0.$$

Further, he shows $G(R^2,0)$ is a strong deformation retract of $G(R^2)$, and that the group of isometries of the plane is a deformation retract of $G(R^2,0)$. Hence, S^1 is a deformation retract of $G_0(R^2)$, and $\mathscr{H}_*(R^2)$ is

$$\cdots \to 0 \to \cdots \to 0 \to Z \xrightarrow{i_*} Z \xrightarrow{p_*} 0 \xrightarrow{d_*} Z_2 \xrightarrow{i_*} Z_2 \to 0.$$

By (4.4), $i_*: \mathscr{H}_j(S^2,x) \cong \mathscr{H}_j(R^2) = 0$ for $j \geqq 2$ and $x \in S^2$, and $i_*: \mathscr{H}_1(S^2,x) \cong \mathscr{H}_1(R^2) \cong Z$. Therefore, $\mathscr{H}_j(S^2) \cong \pi_j(S^2)$ for $j > 3$. Kneser [ibid.] next shows that the special orthogonal group, $SO(3,R)$, is a deformation retract of $G_0(S^2)$. Since $\pi_1(SO[3,R]) = Z_2$, $\mathscr{H}_*(S^2)$ has the ending:

$$\cdots \to 0 \to Z \xrightarrow{d_*} Z \xrightarrow{i_*} Z_2 \xrightarrow{p_*} 0 \xrightarrow{d_*} Z_2 \xrightarrow{i_*} Z_2 \to 0.$$

Though much of the result for S^2 can be deduced from Kneser's retraction, the sequence has the advantage of exhibiting the homomorphisms explicitly, as well as relating the homeotopy groups of S^2 to those of R^2.

Now consider $S^1 \times R = R^2 - \{0\}$; this is a Lie group, and $\pi_j(S^1 \times R) \cong \pi_j(S^1) = 0$ for $j \geqq 2$. By (3.31), (4.31) and (4.4), i_*:

$$\mathscr{H}_j(S^1 \times R,x) \cong \mathscr{H}_j(S^1 \times R) \cong \mathscr{H}_j(R^2,0) = 0$$

for $x \in S^1 \times R$ and $j \geqq 2$; and $\mathscr{H}_*(S^1 \times R)$ is

$$\cdots \to 0 \to \cdots \to 0 \xrightarrow{i_*} Z \xrightarrow{p_*} Z \xrightarrow{d_*} Z_2^2 \xrightarrow{i_*} Z_2^2 \to 0.$$

In general, let X_k be the space got from S^2 by deleting k points, $k \geqq 3$. The plane covers X_k, so $\pi_j(X_k) = 0$ for $j \geqq 2$; additionally, $\pi_1(X_k)$ is a free group with trivial center; hence, by (5.24), $p_*: \mathscr{H}_1(X_k) \to 0$. These facts, together with (4.4), imply by induction: $i_*: \mathscr{H}_j(X_k,x) \cong \mathscr{H}_j(X_k) \cong \mathscr{H}_j(S^1 \times R,y) = 0$ for $j \geqq 1$, $x \in X_k$ and $y \in S^1 \times R$. In other words, $G_0(X_k)$ is j-connected for all j (note also that $G_0(X_k)$ is locally contractible [5]). By (4.31), $\mathscr{H}_*(X_3)$ is

$$\cdots \to 0 \to \cdots \to 0 \xrightarrow{p_*} F_2 \xrightarrow{d_*} \mathscr{H}(X_k,x) \xrightarrow{i_*} Z_2 \times S_3 \to 0,$$

where F_2 is the free group on two generators.

AUTOMORPHISMS OF $\mathscr{H}_*(X)$ 6.1. There are no automorphisms of $\mathscr{H}_*(R)$. In case $X = S^1, R^2$ or S^2, the automorphism group $\mathscr{A}[\mathscr{H}_*(X)] \cong Z_2 \cong \mathscr{H}(X,x)$ and the representation r of $\mathscr{H}(X,x)$ there is faithful. Thus $r\mathscr{H}_*(X,x)$ nontrivially contains the standard action of $\pi_1(X,x)$ or $\pi_0(X)$.

$\mathscr{A}[\mathscr{H}_*(S^1 \times R)] \cong Z_2^3$ and $\mathscr{H}(S^1 \times R,x) \cong Z_2^2$, yet $r\mathscr{H}(S^1 \times R,x) \cong Z_2$ is neither faithful nor onto. Again, however, the standard action is trivial.

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