ISLANDS AND PENINSULAS ON ARBITRARY RIEMANN SURFACES(1)

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1. In the present paper the nonintegrated form of the second fundamental theorem is proved for arbitrary Riemann surfaces and a general test is given for regular exhaustability.

In his theory of covering surfaces L. Ahlfors [1; 2] gave the second fundamental theorem for simply connected Riemann surfaces. The defect relation was generalized to parabolic Riemann surfaces by K. Noshiro [5], and the main inequality of the theory was extended to certain plane regions by K. Kunugui [4] and Y. Tumura [11]. Using this extension and Klein's fundamental domain, J. Tamura [10] generalized the second fundamental theorem to an important class of Riemann surfaces and to functions with at most two Picard values. Specifically, he obtained the interesting result that the Ahlfors condition $S(r)\,(1-r)\to\infty$ for regular exhaustability can be carried over to the disk of uniformization and also gives the functions whose defect sum does not exceed 2.

To study a general class of functions on arbitrary Riemann surfaces we shall separate these two properties. For regular exhaustability alone the Tamura condition can be sharpened by endowing the domain surface W with a conformal metric $d\rho$ with compact sets β_ρ of points at distance ρ from a fixed point. This metric can easily be formed on an arbitrary W. The condition

$$\text{(A)}\qquad \limsup_{\rho\to R}\left(S(\rho)\int_\rho^R \frac{d\rho}{l(\rho)}\right) = \infty,$$

where $R=\sup\rho$ and $l(\rho)$ is the length of β_ρ, then suffices for $\liminf\ (L(\rho)/S(\rho))=0$.

For the second fundamental theorem we replace the customary process by three steps: first we remove the peninsulas separated by cross-cuts, then those separated by cycles, and finally all islands. This leads to the main inequality (in No. 14) for an arbitrary Riemann surface. For the number P of Picard values we then obtain the bound

$$\text{(B)}\qquad P \leqq 2+\limsup_{\rho\to R}\frac{e(\rho)}{S(\rho)},$$

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where e is the Euler characteristic. The Tamura functions with $P \leqq 2$ are those for which S grows more rapidly than e.

The essence of bound (B) is that it is sharp. The testing of the sharpness is possible because the quantities involved can actually be computed. This is the main advantage of dealing with the surface itself rather than with its universal covering surface.

For further illumination we will also estimate $\limsup(e/S)$ directly, without invoking the first fundamental theorem, and arrive at the same bound for the number of Picard values.

§1. The Ahlfors theory

2. For our purposes it will suffice to consider covering surfaces of the extended w-plane W_0 and the stereographic metric

$$ds = \frac{|dw|}{1 + |w|^2}. \tag{1}$$

The area in this metric of W_0 is π.

Let W be an arbitrary open Riemann surface, represented as a covering surface of W_0. We consider a compact bordered subsurface $\Omega \subset W$ and, decomposing Ω in the usual manner into sheets, define the area A of Ω in metric (1) as the sum of the areas of the sheets. The mean sheet number S of Ω over W_0 is, by definition, $S = A/\pi$. The length L of the boundary of Ω is defined in the same fashion, as the sum of the lengths of the boundaries of the sheets constituting Ω.

3. On W_0 we choose a region Δ and denote by $S(\Delta)$ the area of the subset of Ω covering Δ, divided by the area of Δ. Should Ω have no relative boundary above W_0, we would clearly have $S = S(\Delta)$ for any Δ. The greater L, the more deficient can the coverage of Δ by Ω be. This is the meaning of Ahlfors' [1] well-known

COVERING THEOREM. *For every Δ there exists a constant k, independent of W and Ω, such that*

$$|S - S(\Delta)| \leqq kL. \tag{2}$$

4. The subset of Ω above Δ consists of two kinds of components: islands D_i that have no relative boundary above Δ, and peninsulas D_p possessing such relative boundary. The mean sheet number $n(\Delta)$ of $\bigcup D_i$ is, by definition, the area of $\bigcup D_i$ divided by the area of Δ. The mean sheet number $\mu(\Delta)$ of $\bigcup D_p$ is defined analogously. Obviously $S(\Delta) = n(\Delta) + \mu(\Delta)$, and the covering theorem (2) gives what could be called Ahlfors' [1]

FIRST FUNDAMENTAL THEOREM. *For every Ω and Δ,*

$$n(\Delta) + \mu(\Delta) = S + O(L). \tag{3}$$

The analogy with the Nevanlinna-type first fundamental theorem [8] is clear: $n(\Delta)$ and $\mu(\Delta)$ correspond to the (nonintegrated) counting function $N(\sigma,a)$ and proximity function $m(\sigma,a)$, respectively.

5. In a triangulation of Ω we denote by V, E, and F the numbers of interior vertices, edges, and faces, and by e the Euler characteristic $-V+E-F$. If some interior vertices and edges are removed, the resulting subregions Ω_j have Euler characteristics $e(\Omega_j)$ in the original triangulation, with

$$e = \sum e(\Omega_j) - \bar{V} + \bar{E}. \tag{4}$$

Here $\bar{V}$ and $\bar{E}$ are the numbers of remaining vertices and edges.

We consider the case where these vertices and edges form only (disjoint) cross-cuts γ, with end points on the border of Ω, and cycles σ inside Ω. The contribution to $-\bar{V}+\bar{E}$ from every γ is 1 and from every σ is 0:

$$e = \sum e(\Omega_j) + n(\gamma), \tag{4$'$}$$

where $n(\gamma)$ is the number of cross-cuts.

On setting $e^+ = \max(e,0)$, and on denoting by $N_1(\Omega_j)$ the number of simply connected regions Ω_j we obtain the following equivalent formulation of (4)′:

$$e = \sum e^+(\Omega_j) + n(\gamma) - N_1(\Omega_j). \tag{5}$$

6. To evaluate $n(\gamma) - N_1(\Omega_j)$ we first consider the influence of crosscuts γ. A cross-cut divides Ω into at most two Ω_j, and these are simply connected if and only if Ω is simply connected. We infer that in any subdivision by only cross-cuts the number of resulting subregions exceeds $n(\gamma)$ at most by one. We shall consider separately the cases $e \geqq 0$ and $e = -1$.

If $e \geqq 0$, the number $N_1(\Omega_j)$ of simply connected subregions does not exceed $n(\gamma)$. We now assume, and this gives us more generality than we shall make use of, that every cycle σ produces only multiply connected subregions. Then the introduction of cycles has no effect on $n(\gamma) - N_1(\Omega_j) \geqq 0$, and it follows that $e^+ \geqq \sum e^+(\Omega_j)$. If $e = -1$, all Ω_j are simply connected, $N_1 \leqq n(\gamma)+1$, and it follows from (5) that no $e^+(\Omega_j)$ can be positive. We conclude that again

$$e^+ \geqq \sum e^+(\Omega_j).$$

LEMMA. *If a compact bordered surface Ω is subdivided by $n(\gamma)$ cross-cuts, and subsequently by cycles that do not create simply connected subregions, then the Euler characteristics $e(\Omega_j)$ of the resulting subregions Ω_j satisfy the inequality*

$$e^+ \geqq \sum e^+(\Omega_j), \tag{6}$$

where e is the Euler characteristic of Ω.

7. Let Ω be a complete covering surface with N sheets of a compact subregion $\Omega_0 \subset W_0$. Let V_0, E_0, F_0 be the numbers of vertices, edges, and faces in a triangulation of Ω_0, and denote by V, E, F the corresponding numbers for Ω when the triangulation of Ω_0 is lifted to Ω. Clearly E_0 and F_0 remain unchanged on each sheet of Ω, hence $E = NE_0$, $F = NF_0$. The number V is reduced from NV_0 by the sum $\sum b$ of the orders of branch points of Ω. This is the content of the Hurwitz formula

$$e = Ne_0 + \sum b, \tag{6$'$}$$

where e_0 is the Euler characteristic of Ω_0. A simpler version reads: $e^+ \geqq Ne_0$.

If Ω does not completely cover Ω_0, an increasingly long relative boundary of Ω can cut down e^+ with an increasing amount. This is the essence of the following far-reaching extension [1] of the Hurwitz formula:

AHLFORS' INEQUALITY. *For any region Ω_0 of the plane there exists a constant k, independent of the covering surface Ω of Ω_0, such that*

$$e^+ \geqq e_0 S - kL. \tag{7}$$

Here S is the mean sheet number of Ω above Ω_0 and L is the length of the relative boundary of Ω above Ω_0.

§2. The second fundamental theorem

8. We resume the notations W, W_0, Ω, S of No. 2 and let Δ_ν, $\nu = 1, \cdots, q$, $q \geqq 3$, be disjoint simply connected subregions of W_0. We remove from Ω all peninsulas D_p above all Δ_ν's and denote the components of the remaining part of Ω by Ω'. From each Ω' we remove all islands D_i above all Δ_ν's and denote the components of the remaining part of Ω' by $\bar{\Omega}$. Since only cycles are removed from Ω' in decomposing it into $\{D_i\}$ and $\{\bar{\Omega}\}$, we have by (4)

$$\sum e(D_i) = \sum e(\Omega') - \sum e(\bar{\Omega}) = N_1(\bar{\Omega}) + \sum e(\Omega') - \sum e^+(\bar{\Omega}), \tag{8}$$

where $N_1(\bar{\Omega})$ is the number of simply connected $\bar{\Omega}$.

9. We shall first estimate $N_1(\bar{\Omega})$. To this end we decompose the process of removing the D_p from Ω into two steps. First we cut Ω along those cross-cuts γ that lie above the boundaries of the Δ_ν and remove from Ω those peninsulas D_p that have thus become separated. We denote these peninsulas by $D_{p\gamma}$. The remaining part of Ω consists of components Ω_γ, say. Second, we cut each Ω_γ along the cycles σ that lie on the boundaries of the remaining D_p and above the boundaries of the Δ_ν. On removing these D_p, which we shall denote by $D_{p\sigma}$, the regions Ω' remain.

For the number $N_1(\Omega_\gamma)$ of simply connected Ω_γ we have

$$N_1(\bar{\Omega}) \leqq N_1(\Omega_\gamma). \tag{9}$$

In fact, if an Ω_γ is multiply connected, it may give rise to a simply connected Ω' when a cut is made along a σ. But since the number of Δ_ν's > 1, the subsequent removal of islands D_i must make the resulting $\bar{\Omega}$ multiply connected. We conclude that every simply connected $\bar{\Omega}$ is a simply connected Ω_γ. Inequality (9) follows.

10. We proceed to estimate $\sum e(\Omega')$ in (8). Since only cycles σ are used in dividing Ω_γ into $D_{p\sigma}$ and Ω', we have by (4)$'$

$$\sum e(\Omega_\gamma) = \sum e(D_{p\sigma}) + \sum e(\Omega').$$

Every $D_{p\sigma}$ is a peninsula and was separated from Ω_γ by a cycle σ; we infer that $D_{p\sigma}$ cannot be simply connected, $e(D_{p\sigma}) \geqq 0$, and

$$\sum e(\Omega') \leqq \sum e(\Omega_\gamma). \tag{10}$$

11. From (9) and (10) one obtains

$$N_1(\bar{\Omega}) + \sum e(\Omega') \leqq \sum e^+(\Omega_\gamma).$$

In the subdivision of Ω into $\{\Omega_\gamma\}$ and $\{D_{p\gamma}\}$ only cross-cuts γ were used. Lemma 6 applies and (6) gives

$$\sum e^+(\Omega_\gamma) \leqq \sum e^+(\Omega_\gamma) + \sum e^+(D_{p\gamma}) \leqq e^+(\Omega).$$

We have arrived from (8) to

$$\sum e(D_i) \leqq e^+(\Omega) - \sum e^+(\bar{\Omega}). \tag{11}$$

12. To estimate $e^+(\bar{\Omega})$ we apply (7). The Euler characteristic of $\bar{W}_0 = W_0 - \bigcup_1^q \Delta_\nu$ is $q-2$. If the mean sheet number of $\bigcup \bar{\Omega}$ above $\bar{W}_0$ is denoted by $S(\bar{W}_0)$, one obtains

$$\sum e^+(\bar{\Omega}) \geqq (q-2)\,S(\bar{W}_0) + O(L_0),$$

where L_0 is the length of the relative boundary of Ω above $\bar{W}_0$. Clearly $O(L_0)$ can be replaced by $O(L)$, where L is the length of the relative boundary of Ω above W_0. By (2), $S(\bar{W}_0)$ differs by $O(L)$ from the mean sheet number S of Ω above W_0 and we conclude that

$$\sum e^+(\bar{\Omega}) \geqq (q-2)S + O(L). \tag{12}$$

13. The first term $\sum e(D_i)$ in (11) is evaluated by Hurwitz' formula (6)$'$. The Euler characteristic of Δ_ν is -1, the total number of sheets in the union of all islands D_i covering Δ_ν is denoted by $n(\Delta_\nu)$, and the sum of orders of branch points in this union, by $b(\Delta_\nu)$. Then

$$\sum e(D_i) = -\sum n(\Delta_\nu) + \sum b(\Delta_\nu). \tag{13}$$

By the first fundamental theorem (3), $n(\Delta_\nu)$ can be expressed in terms of S and the mean sheet number $\mu(\Delta_\nu)$ of the union of all peninsulas D_ρ covering Δ_ν. We obtain an alternate form of (13):

$$\Sigma e(D_i) = \Sigma\mu(\Delta_\nu) - qS + \Sigma b(\Delta_\nu) + O(L). \tag{13$'$}$$

14. It remains to substitute (12) and (13)$'$ into (11). In analogy with other notations let e^+ stand for $e^+(\Omega)$. We have established the

SECOND FUNDAMENTAL THEOREM. *On an arbitrary Riemann surface W,*

$$\Sigma\mu(\Delta_\nu) < 2S - \Sigma b(\Delta_\nu) + e^+ + O(L). \tag{14}$$

An equivalent formulation is obtained by using (3):

$$(q - 2)\, S < \Sigma n(\Delta_\nu) - \Sigma b(\Delta_\nu) + e^+ + O(L). \tag{15}$$

The analogy with the Nevanlinna-type second fundamental theorem [7] is again clear. Our present result applies to arbitrary Riemann surfaces (cf. No. 16).

§3. Meromorphic functions

15. It will be assumed henceforth that the covering surface we have discussed is the image under a meromorphic function w of an open Riemann surface. Changing our notations slightly, we shall denote the latter by W, and the former by W_w.

Significant conclusions can be drawn from the second fundamental theorem (14) only if $L \subset W_w$ is negligibly small compared to S. Our immediate task will be to give a precise formulation to this property and to find a sufficient condition for w to possess it.

16. On an arbitrarily given open Riemann surface W let

$$d\rho = \lambda(z)\,|dz| \tag{16}$$

be a conformally invariant metric. Here $\lambda(z) \geqq 0$ is continuous in each parametric disk $|z| < 1$, and

$$\lambda(z_2) = \lambda(z_1)\left|\frac{dz_1}{dz_2}\right| \tag{17}$$

under change of parameter. The distance $\rho(z,\zeta)$ between two points z, ζ is defined as $\inf \int_\alpha d\rho$ over all rectifiable arcs α from z to ζ. We assume that, for a fixed ζ,

$$\lim_{z\to\beta} \rho(z,\zeta) = R = \text{const.} \leqq \infty \tag{18}$$

for every approach of z to the ideal boundary β of W. Then the regions W_ρ bounded by

$$\beta_\rho = \{z \,|\, \rho(z,\zeta) = \rho\}, \tag{19}$$

$0 < \rho < R$, exhaust W as $\rho \to R$.

There is a metric with these characteristics and with $R = \infty$ on every W. For instance a conformal mapping of the universal covering surface W^∞ of W onto the disk $|x| < 1$ gives the hyperbolic metric $d\rho = |dx(z)|/(1 - |x(z)|^2)$ with the desired properties. The degenerate case where W^∞ is conformally equivalent to the plane $x \neq \infty$ only occurs if W is simply or doubly connected and of parabolic type. But then the capacity metric directly on W can be used [8]. We conclude that our metric $d\rho$ can always be formed.

In case W is the interior of a compact bordered Riemann surface, then also a metric $d\rho$ with $R < \infty$ can easily be found and is perhaps the more natural choice. For this reason we shall cover both cases $R \leqq \infty$.

17. Let $w(z)$ be a meromorphic function on an arbitrary open W. We denote by $L(\rho)$ the length, in the stereographic metric of the w-sphere, of the image under w of β_ρ. Similarly, $S(\rho)$ shall stand for the area of the image of W_ρ divided by π. To answer the question raised in No. 15 we shall study when the condition

$$\liminf_{\rho \to R} \frac{L(\rho)}{S(\rho)} = 0 \tag{20}$$

is satisfied.

On setting

$$w_\rho = \frac{dw}{dz} \Big/ \frac{d\rho}{|dz|} = w_z \lambda^{-1}$$

we have

$$L(\rho) = \int_{\beta_\rho} \frac{|w_\rho|}{1 + |w|^2} d\rho \tag{21}$$

and

$$S(\rho) = \frac{1}{\pi} \int_0^\rho d\rho \int_{\beta_\rho} \frac{|w_\rho|^2}{(1 + |w|^2)^2} d\rho. \tag{22}$$

We set $l(\rho) = \int_{\beta_\rho} d\rho$ and state:

THEOREM. *For* $0 < \rho < R$,

$$\frac{d\rho}{l(\rho)} \leqq \pi \frac{dS(\rho)}{L(\rho)^2}. \tag{23}$$

This is a direct consequence of Schwarz's inequality:

$$L(\rho)^2 \leqq \int_{\beta_\rho} \frac{|w_\rho|^2}{(1 + |w|^2)^2} d\rho \cdot \int_{\beta_\rho} d\rho = \pi \frac{dS(\rho)}{d\rho} \cdot l(\rho).$$

18. We are now ready to establish a criterion for (20). It is in the nature of the problem to exclude the degenerate case of a bounded $S(\rho)$.

THEOREM. *Let w be a meromorphic function on an arbitrary Riemann surface W. Then* $\liminf(L(\rho)/S(\rho)) = 0$ *if $S(\rho)$ increases so rapidly that*

$$\limsup_{\rho\to R}\left(S(\rho)\int_\rho^R \frac{d\rho}{l(\rho)}\right) = \infty. \tag{24}$$

Proof. Suppose the conclusion were not true: $\liminf(L/S) > 0$. Then there would exist constants $q > 0$ and $0 < \rho_0 < R$ such that $L(\rho) > qS(\rho)$ for $\rho_0 < \rho < R$. It would follow that

$$\int_\rho^R \frac{d\rho}{l(\rho)} \leqq \pi\int_\rho^R \frac{dS(\rho)}{L(\rho)^2} < \frac{\pi}{q^2}\int_\rho^R \frac{dS(\rho)}{S(\rho)^2} \leqq \frac{\pi}{q^2}\,\frac{1}{S(\rho)}.$$

Hence

$$S(\rho)\int_\rho^R \frac{d\rho}{l(\rho)} \leqq \frac{\pi}{q},$$

which contradicts (24).

19. To illustrate the meaning of condition (24), we consider some concrete cases of Riemann surfaces W of increasing generality. First let W be the finite or infinite disk $|z| < R \leqq \infty$. We choose the metric $d\rho = |dz|$, $\rho = r$, $l(\rho) = 2\pi r$, and find that $\int_\rho^R l(\rho)^{-1}\,d\rho$ diverges for $R = \infty$ and dominates $(R-\rho)/2\pi R$ for $R < \infty$.

COROLLARY 1. *The condition* $\liminf(L/S) = 0$ *is satisfied by all meromorphic functions in the plane and by those meromorphic functions in the disk $|z| < R$ for which*

$$\limsup\,(S(r)(R-r)) = \infty. \tag{25}$$

20. More generally, we consider Riemann surfaces W_p characterized by the property that the capacity function $p_\beta(z)$ (see e.g. [3; 8], or [9]) tends to a constant $k_\beta \leqq \infty$ for any approach of z to the ideal boundary β of W. Here the logarithmic singularity of $p_\beta(z)$ is taken in a fixed parametric disk. The capacity of β is defined as $c_\beta = e^{-k_\beta}$. It is known [3] that W_p is parabolic, $W_p \in O_G$, if and only if $k_\beta = \infty$. We choose the capacity metric [8; 9]

$$d\rho = \frac{1}{2\pi}\,|\operatorname{grad} p_\beta|\;|dz| \tag{26}$$

and set $\rho = p_\beta = k$ with $0 \leqq k < k_\beta$. The exclusion of values $k < 0$ is for convenience and has no bearing on our conclusions, which concern a boundary property. We can even allow several logarithmic singularities as in No. 26. We have $l(\rho) = (2\pi)^{-1}\int_{\beta_k} dp_\beta^* = 1$ and $\int_\rho^R l(\rho)^{-1}d\rho = k_\beta - k$.

COROLLARY 2. *The condition* $\liminf(L/S)=0$ *is satisfied by all meromorphic functions on a parabolic* W_p *and by those meromorphic functions on a hyperbolic* W_p *for which* $\limsup(S(k)(k_\beta - k)) = \infty$.

The first part of this corollary continues to hold on arbitrary parabolic W for they can always be endowed with a $d\rho$-metric with a divergent $\int^\infty l(\rho)^{-1}d\rho$.

21. Somewhat more generally, consider a Riemann surface W_s [8] with a $d\rho$-metric ds satisfying the additional condition $l(\rho) = 1$. On setting $\rho = \sigma$ we again have $\int_\rho^R l(\rho)^{-1}d\rho = \sigma_\beta - \sigma$.

COROLLARY 3. *All meromorphic functions on a* W_s *with* $\sigma_\beta = \infty$ *and those meromorphic functions on a* W_s *with* $\sigma_\beta < \infty$ *that satisfy the condition* $\limsup(S(\sigma)(\sigma_\beta - \sigma)) = \infty$ *have the property* $\liminf(L/S) = 0$.

22. In the trivial case of the plane or the punctured plane Corollary 2 applies and we exclude this case in the sequel. For all other Riemann surfaces W the universal covering surface W^∞ can be conformally mapped onto the disk $|x| < 1$ and W can be endowed with the invariant hyperbolic metric $d\rho = |dx(z)|/(1-|x(z)|^2)$. The surface W is represented by a fundamental region W_x bounded by circular arcs perpendicular to $|x| = 1$, identified, by pairs, by linear transformations. The level line β_ρ appears as the intersection $\beta_{\rho x}$ of a circle $|x| = r$ and W_x. Its Euclidian length is $l_e(\rho) = \int_{\beta\rho x} dx$ and its hyperbolic length is

$$l(\rho) = \int_{\beta\rho x} \frac{|dx|}{1-|x|^2} = \frac{1}{1-r^2}\, l_e(\rho).$$

We conclude that $d\rho/l(\rho) = |dx|/l_e(\rho)$. Consequently we can express our criterion (24) in Euclidian metric:

COROLLARY 4. *A meromorphic function on an arbitrary Riemann surface* W *has the property* $\liminf(L/S) = 0$ *if, in the uniformization into* $|x| < 1$,

$$\limsup_{r\to 1}\left(S(\rho(r))\int_r^1 \frac{dr}{l_e(r)}\right) = \infty. \tag{27}$$

For a parabolic surface it is known that $\int^1 l_e(r)^{-1}dr = \infty$ and we again have no restriction on $w(z)$.

§4. Consequences of the second fundamental theorem

23. We are now in a position to draw conclusions from the second fundamental theorem (14)–(15). First we obtain a bound for the number P of Picard values. To this end let each Δ_ν contain at least one Picard value. Then there can be no islands D_i above Δ_ν. We denote by $e(\rho)$ the Euler characteristic of W_ρ, set

$$\varepsilon = \limsup_{\rho\to R} \frac{e^+(\rho)}{S(\rho)}, \tag{28}$$

and obtain from (15) the following extension of Picard's theorem (cf. [8]):

THEOREM. *Let W be an arbitrary Riemann surface. The bound*

$$P \leqq 2 + \varepsilon \tag{29}$$

for the number P of Picard values is valid for every meromorphic function w on W with property (24) *or its analogues in Nos.* 19–22. *In particular, the bound holds for all w on a parabolic W.*

24. To arrive at the analogue of Picard-Borel's theorem let Δ_ν shrink to a point a_ν and denote by $n(a_\nu)$ the number of points, counted with their multiplicities, covering a_ν. Then (15) implies

$$(q-2)S < \sum n(a_\nu) + e^+ + O(L) \tag{30}$$

and one obtains:

THEOREM. *Under the assumptions of the preceding theorem, the number of Picard-Borel points a characterized by* $\limsup(n(a)/S) = 0$ *cannot exceed* $2 + \varepsilon$.

25. Relax further the condition on the deficient coverage of Δ by permitting $\limsup(n(\Delta)/S) > 0$. Set

$$\delta(\Delta) = 1 - \limsup \frac{n(\Delta)}{S} = \liminf \frac{\mu(\Delta)}{S}, \tag{31}$$

where the latter form is a consequence of the first fundamental theorem (3) in the case $\liminf(L/S) = 0$ under consideration. Formula (14) gives the following generalization of the classical defect relation to arbitrary Riemann surfaces W:

THEOREM. *Under the conditions of Theorem* 23, *the defect sum has the bound*

$$\sum \delta(\Delta_\nu) \leqq 2 + \varepsilon. \tag{32}$$

Here the sum is extended over any finite or infinite number of disjoint simply connected regions Δ_ν of the w-plane.

26. We proceed to show that the bound $2 + \varepsilon$ is sharp at least for even numbers. (For odd numbers we refer to a surface constructed by B. Rodin in his doctoral dissertation [6].)

THEOREM. *For any integer $n > 0$ there exists a Riemann surface W and a meromorphic function w on W such that*

$$P = 2 + \varepsilon = 2n.$$

Proof. Consider the n-sheeted Riemann surface W above the z-plane whose branch points are at $z_j = i(\pi/2 + j\pi)$, $j = 0, \pm 1, \pm 2, \cdots$, all of multiplicity n.

Choose on W the metric $d\rho = |dz|/2\pi n|z|$ and set $\rho = (2\pi n)^{-1}\log|z|$ (see No. 20). Then β_ρ is the n-sheeted circle $|z| = e^{2\pi h\rho}$ and $l(\rho) = (2\pi n)^{-1}\int_{\beta_\rho} d\arg|z| = 1$. We have $\int_\rho^\infty l(\rho)^{-1}d\rho = \infty$, and condition (24), hence also (20), is always satisfied. This is directly implied also by Corollary 2.

Consider on W the meromorphic function

$$w(z) = \sqrt[n]{\frac{e^z + i}{e^z - i}}\,. \tag{33}$$

Because $\liminf(L/S) = 0$, we find

$$\varepsilon = \limsup \frac{e^+}{S} = \lim\sup \frac{e^+}{n(\Delta) + \mu(\Delta)} \leqq \lim\sup \frac{e^+}{n(\Delta)},$$

where we choose for Δ a small disk about $w = 0$. Then $n(\Delta)$ is the number of zeros in w. Exhaust W by n-sheeted disks $W_m : |z| < 2\pi m$, $m = 1, 2, \cdots$, and indicate quantities referring to W_m by the subindex m. Then

$$\limsup \frac{e^+}{n(\Delta)} = \lim\sup \frac{e_m^+}{n_m(\Delta)}\,. \tag{34}$$

When bounded terms are disregarded, one finds from Hurwitz' formula that $e_m \sim 4m(n-1)$. The zeros of w are at $e^z = -i$, $z_j = i(-\pi/2 + j\cdot 2\pi)$, $j = 0, \pm 1, \pm 2, \cdots$, and therefore $n_m(\Delta) \sim 2m$. It follows that $\varepsilon \leqq 2(n-1)$. By (29) the number of Picard values cannot exceed $2 + \varepsilon \leqq 2n$. But $2n$ is exactly the number of Picard values, equidistantly distributed on $|w| = 1$. This proves the theorem.

27. The significance of ε is perhaps best illustrated by also estimating S directly, without invoking the first fundamental theorem, and by comparing the results. We shall do this for the slightly more general function

$$\eta(z) = w(z)^h = \left(\frac{e^z + i}{e^z - i}\right)^{h/n}, \tag{35}$$

which has $2n/h$ Picard values, h being an integral factor of $2n$ (cf. [8]).

Let $R_{mj} \subset W_m$ be the n-sheeted rectangle

$$|x| < 2\pi\sqrt{m^2 - j^2}\,,$$

$$(j-1)\cdot 2\pi < y < j\cdot 2\pi,$$

$j = 1, 2, \cdots, m-1$. If the contribution of R_{mj} to S_m is denoted by S_{mj}, then clearly

$$S_m > 2\sum_{j=1}^{m-1} S_{mj}. \tag{36}$$

Under the mapping $s = e^z$ the rectangle R_{mj} becomes an n-sheeted annulus with outer radius

$$R = \exp(2\pi\sqrt{m^2 - j^2}) \tag{37}$$

and inner radius R^{-1}. The function $t = (s + i)/(s - i)$ maps the annulus onto the n-sheeted complement of two Steiner circles encircling $t = 1$ and -1 respectively, symmetrically placed about the real and imaginary t-axes and intersecting the real axis at the images of $s = \pm iR, \pm iR^{-1}$, that is, at distances

$$t_1 = \frac{R + 1}{R - 1} \tag{38}$$

and t_1^{-1} from $t = 0$. The function $w = \sqrt[n]{t}$ maps the n-sheeted complement of the two Steiner disks onto the 1-sheeted complement of the $2n$ images of the disks, which appear as distorted disks encircling points $w = e^{i\phi_\nu}$, $\phi_\nu = \nu\pi/n$, $\nu = 1, \cdots, 2n$, and are located in the annulus

$$\sqrt[n]{t_1^{-1}} < |w| < \sqrt[n]{t_1}. \tag{39}$$

The function $\eta = w^h$ gives as the final image $\eta(R_{mj})$ of R_{mj} the h-sheeted complement of $2n/h$ distorted disks encircling points $\eta = e^{i\alpha_\nu}$, $\alpha_\nu = \nu h\pi/n$, $\nu = 1, \cdots, 2n/h$, and located in the annulus

$$r_1 = t_1^{-h/n} < |\eta| < t_1^{h/n} = r_1^{-1}. \tag{40}$$

By definition, the mean sheet number S_{mj} of the image of R_{mj} in the stereographic η-metric is the π^{-1}-fold area of $\eta(R_{mj})$. By omitting the annulus (40) we obtain

$$S_{mj} > \frac{h}{\pi}\left(\int_0^{2\pi}\int_0^{r_1}\frac{r\,d\phi\,dr}{(1+r^2)^2} + \int_0^{2\pi}\int_{r_1^{-1}}^{\infty}\frac{r\,d\phi\,dr}{(1+r^2)^2}\right)$$

$$= -h\left(\frac{1}{1 + r_1^2} - 1 - \frac{1}{1 + r_1^{-2}}\right) = \frac{2hr_1^2}{1 + r_1^2}.$$

On setting

$$\varepsilon_{mj} = \frac{2}{\exp(2\pi\sqrt{m^2 - j^2}) - 1} \tag{41}$$

one obtains $t_1 = 1 + \varepsilon_{mj}$ and

$$S_{mj} > \frac{2h(1 + \varepsilon_{mj})^{-2h/n}}{1 + (1 + \varepsilon_{mj})^{-2h/n}} > h(1 + \varepsilon_{mj})^{-2h/n}.$$

Here

$$\varepsilon_{mj} \leqq \frac{2}{\exp(2\pi\sqrt{2m-1}) - 1} = \varepsilon_m$$

and we find by (36) that

$$S_m > 2 \sum_{j-1}^{m-1} h(1 + \varepsilon_m)^{-2h/n} = 2(m - 1)\, h\,(1 + \varepsilon_m)^{-2h/n}.$$

For the Euler characteristic we have as before $e_m \sim 4m(n - 1)$. Hence

$$\limsup_{m\to\infty} \frac{e_m}{S_m} \leqq \limsup_{m\to\infty} \frac{2m(n-1)}{(m-1)\,h\,(1 + \varepsilon_m)^{-2h/n}} = \frac{2(n-1)}{h}.$$

In the special case $h = 1$ the value is $2(n - 1)$, in perfect agreement with our result in No. 26.

28. Whether or not there are functions on a given W with $P = 2 + \varepsilon$, or with $P = 0$ but $\sum\delta = 2 + \varepsilon$ are open questions. Further problems of possible interest in the theory of meromorphic functions on Riemann surfaces were listed at the end of [8].

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