EXTREME EIGENVALUES OF N-DIMENSIONAL CONVOLUTION OPERATORS

BY
HAROLD WIDOM

PART I

1.1. Let k be a function belonging to L_1 of Euclidean N-space E_N and satisfying $k(-x) = \bar{k}(x)$; let Ω be a subset of E_N of finite but positive measure; let $t > 0$ be a parameter. The integral operator on $L_2(t\Omega)$ with kernel $k(x-y)$ is self-adjoint and completely continuous, so its spectrum is a discrete set of eigenvalues whose only possible limit point is zero. Denote the positive eigenvalues, if any, by

$$\mu_{1,t} \geqq \mu_{2,t} \geqq \cdots.$$

We are interested in the behavior of $\mu_{m,t}$ (for fixed m) as $t \to \infty$.

Denote the Fourier transform of k by K,

$$K(\xi) = \int e^{i\xi \cdot x} k(x) dx.$$

Set $M = \max K(\xi)$ and assume $M > 0$. Then there are positive eigenvalues for sufficiently large t, each is less than M, and $\mu_{m,t} \to M$ for each m. The proof of this is simple and is given in Part V. However in many cases much more can be said.

Assume $K(\xi) = M$ for exactly one value of ξ, which we may take to be $\xi = 0$, and assume that as $\xi \to 0$ we have

$$K(\xi) = M - |\xi|^\alpha \Phi(\xi/|\xi|) + o(|\xi|^\alpha) \tag{1}$$

where $\alpha > 0$ and Φ is a positive function defined on the unit sphere which is bounded and bounded away from zero. (In the simplest cases α will be 2 and Φ will be a positive definite quadratic form.) We shall associate with the triple (α,Φ,Ω) a certain unbounded self-adjoint positive definite operator on $L_2(\Omega)$, the inverse of a completely continuous operator, such that if its eigenvalues be denoted by $\lambda_1 \leqq \lambda_2 \leqq \cdots$ then for each m

$$\mu_{m,t} = M - \lambda_m t^{-\alpha} + o(t^{-\alpha}), \qquad t \to \infty.$$

To give the idea of how this is done let us consider, instead of the operator with kernel $k(x-y)$ on $t\Omega$, the operator with kernel $t^N k(t(x-y))$ on Ω. It is just

Received by the editors February 26, 1962.

a matter of changing variables; the eigenvalues are still $\mu_{m,t}$. Now we have a sequence of operators on the same space $L_2(\Omega)$ and we can also consider limiting properties of the eigenfunctions. The Fourier transform of $t^N k(tx)$ is $K(\xi/t)$ and for this reason we shall denote the operator by $\mathbf{A}_{K(\xi/t)}$. The corresponding bilinear form is

$$(2) \qquad (\mathbf{A}_{K(\xi/t)} f, g) = \int_\Omega \int_\Omega t^N k(t(x-y)) f(y)\, \bar{g}(x) dy dx, \qquad f, g \in L_2(\Omega).$$

To $\mathbf{A}_{K(\xi/t)}$ there corresponds in a natural way an operator $\hat{\mathbf{A}}_{K(\xi/t)}$ on $\hat{L}_2(\Omega)$, the space of Fourier transforms of L_2 functions vanishing almost everywhere outside Ω. The operator is determined by the bilinear form

$$(3) \qquad (\hat{\mathbf{A}}_{K(\xi/t)} F, G) = \int K(\xi/t) F(\xi) \bar{G}(\xi)\, d\xi, \qquad F, G \in \hat{L}_2(\Omega).$$

If

$$\hat{f}(\xi) = (2\pi)^{-N/2} \int e^{i\xi \cdot x} f(x) dx,$$

$$\hat{g}(\xi) = (2\pi)^{-N/2} \int e^{i\xi \cdot x} g(x) dx,$$

then we obtain from (2), (3), and Parseval's identity

$$(\mathbf{A}_{K(\xi/t)} f, g) = (\hat{\mathbf{A}}_{K(\xi/t)} \hat{f}, \hat{g}).$$

Of course the operator $\hat{\mathbf{A}}_{K(\xi/t)}$ is just multiplication by $K(\xi/t)$, which results in a function of $\hat{L}_2(E_N)$, followed by projection onto the subspace $\hat{L}_2(\Omega)$ of $\hat{L}_2(E_N)$.

For any bounded measurable function K we may similarly define the operator $\hat{\mathbf{A}}_K$ on $\hat{L}_2(\Omega)$ as multiplication by K followed by projection onto $\hat{L}_2(\Omega)$. The corresponding operator $\mathbf{A}_K$ on $L_2(\Omega)$ is then *defined* by the bilinear form

$$(\mathbf{A}_K f, g) = (\hat{\mathbf{A}}_K \hat{f}, \hat{g}), \qquad f, g \in L_2(\Omega).$$

Now

$$\mathbf{A}_{t^\alpha[M - K(\xi/t)]} = t^\alpha [M \cdot \mathbf{I} - \mathbf{A}_{K(\xi/t)}],$$

where $\mathbf{I}$ is the identity operator. Therefore the numbers $t^\alpha(M - \mu_{m,t})$ are the eigenvalues of $\mathbf{A}_{t^\alpha[M-K(\xi/t)]}$ and the eigenfunctions of $\mathbf{A}_{K(\xi/t)}$ are the eigenfunctions of $\mathbf{A}_{t^\alpha[M-K(\xi/t)]}$. What happens when $t \to \infty$? It follows from (1) that

$$t^\alpha[M - K(\xi/t)] \to |\xi|^\alpha \Phi(\xi/|\xi|)$$

for each ξ. There is therefore reason to hope that the eigenvalues and eigenfunctions of $\mathbf{A}_{t^\alpha[M-K(\xi/t)]}$ approach those of $\mathbf{A}_{|\xi|^\alpha \Phi(\xi/|\xi|)}$. This is what we shall show.

Now that the problem is stated this way one is naturally led to a more general question. If $J_t \to J$ pointwise, do the eigenvalues and eigenfunctions of $\mathbf{A}_{J_t}$ converge to those of $\mathbf{A}_J$? We shall show that this is true under certain conditions, which will be easily seen not to be superfluous.

The observant reader will have noticed that the J in which we are mainly interested, namely $J(\xi) = |\xi|^{\alpha}\Phi(\xi/|\xi|)$, is certainly not bounded, so that we have not defined $\mathbf{A}_J$. Part of our task will be to define $\mathbf{A}_J$ for a general non-negative function J, and to ensure the validity of the theorem we must do this correctly. For example if $J(\xi) = |\xi|^2$ and Ω is smooth $\mathbf{A}_J$ will be the negative of the Laplacian on the functions on Ω with zero boundary values.

1.2. Given a function $\phi \in L_1(-\pi, \pi)$ with Fourier coefficients c_j, the matrix $(c_{j-k})_{j,k=1,\cdots,t}$ is called the tth truncated Toeplitz matrix associated with ϕ. Similarly let $\phi(\xi) = (\xi^1, \cdots, \xi^N)$ be a function integrable on the N-torus with Fourier coefficients c_j ($j \in \Lambda$, the lattice points of E_N). Let Ω be a bounded set in E_N. Then we may consider the "N-dimensional Toeplitz matrix" $(c_{j-k})_{j,k\in\Omega}$. This is the operator which sends the vector $\{x_j\}_{j\in\Lambda\cap\Omega}$ into the vector

$$\left\{\sum_{k\in\Lambda\cap\Omega} c_{j-k}x_k\right\}, \qquad j \in \Lambda \cap \Omega.$$

If, as we shall assume, ϕ is real, then the matrix has only real eigenvalues. Denote the positive eigenvalues, if any, of the matrix associated with the set $t\Omega$ by

$$\mu_{1,t} \geqq \mu_{2,t} \geqq \cdots.$$

We are interested in the behavior of $\mu_{m,t}$ for fixed m as $t \to \infty$. Just as in the case of integral equations discussed in §1.1 we have $\mu_{m,t} \to \|\phi\|_\infty$ and we are interested in the next approximation.

Again it is convenient to consider not truncations, associated with $t\Omega$, of a single infinite matrix, but rather a family of matrices associated with the single set Ω. Let us set $\Omega_t = \Omega \cap t^{-1}\Lambda$. We consider the operator which sends the vector $\{u_p\}_{p\in\Omega_t}$ into the vector

$$\left\{\sum_{q\in\Omega_t} c_{t(p-q)}\, u_q\right\}, \qquad p \in \Omega_t.$$

The positive eigenvalues of this operator are the $\mu_{m,t}$. To show the connection with the situation discussed in §1.1 (not only the analogy, which is clear), we generalize. Let us assume that with each value of $t > 0$ we have a real-valued function $\phi_t(\xi) = \phi_t(\xi^1, \cdots, \xi^N)$ which is of period $2\pi t$ in each of its variables and integrable over the cube $|\xi^i| \leqq \pi t$. Denote by $L_2(\Omega_t)$ the space of vectors $u = \{u_p\}_{p\in\Omega_t}$ with norm

$$\|u\| = t^{-N/2}\{\sum |u|^2\}^{1/2}.$$

We set

$$c_{t,p} = (2\pi t)^{-N} \int_{|\xi^i|\leqq \pi t} e^{-i\xi\cdot p}\phi_t(\xi)d\xi$$

and define the operator $\mathbf{T}_t$ on $L_2(\Omega_t)$ by

$$(\mathbf{T}_t u)_p = \sum_{q\in\Omega_t} c_{t,p-q}u_q, \qquad p \in \Omega_t.$$

(If $\phi_t(\xi) = \phi(\xi/t)$ then $\mathbf{T}_t$ is the operator described above.) The associated bilinear form is

$$(\mathbf{T}_t u, v) = t^{-N} \sum_{p,q} c_{t,p-q} u_q \bar{v}_p.$$

This can be written

$$(4) \quad (\mathbf{T}_t u, v) = \int_{|\xi^i| \leqq \pi t} \phi_t(\xi) \{(2\pi t^2)^{-N/2} \sum e^{i\xi \cdot p} u_p\} \ \{(2\pi t^2)^{-N/2} \sum e^{-i\xi \cdot q} \bar{v}_q\} \, d\xi.$$

Now suppose that as $t \to \infty$ the functions ϕ_t converge to a function ϕ and that u and v are restrictions to Ω_t of functions f and g defined on all of Ω. Then the limiting form of the right side of (4) is

$$\int \phi(\xi) \hat{f}(\xi) [\hat{g}(\xi)]^{-} \, d\xi,$$

the integration being extended over all of E_N. (The bar following the bracket in the above integral denotes complex conjugation.) Thus at least in some formal sense the operators $\mathbf{T}_t$ will converge to the operator $\mathbf{A}_\phi$ discussed in §1.1. We may therefore hope that the eigenvalues and eigenvectors of $\mathbf{T}_t$ will converge to the corresponding eigenvalues and eigenfunctions of $\mathbf{A}_\phi$. We shall show that this is the case under certain conditions. This will be enough to show that, with a smoothness condition imposed on Ω, the following holds for N dimensional Toeplitz matrices: Let ϕ and $\mu_{m,t}$ be as in the first paragraph of §1.2. Assume ϕ assumes its maximum M at $\xi = 0$, that the essential supremum of ϕ outside any neighborhood of $\xi = 0$ is smaller than M, and that as $\xi \to 0$

$$\phi(\xi) = M - |\xi|^\alpha \Phi(\xi/|\xi|) + o(|\xi|^\alpha)$$

where $\alpha > 0$ and Φ is bounded and bounded away from zero on the unit sphere. Then for fixed m as $t \to \infty$

$$\mu_{m,t} = M - \lambda_m t^{-\alpha} + o(t^{-\alpha}),$$

where $\lambda_1 \leqq \lambda_2 \leqq \cdots$ are the eigenvalues of $\mathbf{A}_{|\xi|^\alpha \Phi(\xi/|\xi|)}$.

1.3. An outline of the paper is as follows. In Part II we discuss the operators $\mathbf{A}_J$ and prove the perturbation theorem described in §1.1. In Part III we consider a technical question which is of interest for two reasons. It will allow us to replace (1) by the more general condition

$$K(\xi) = M - |\xi|^\alpha \Phi(\xi/|\xi|) L(|\xi|) + o(|\xi|^\alpha L(|\xi|))$$

where L is slowly varying near $|\xi| = 0$. More important, it is needed in Part IV, our discussion of Toeplitz matrices. The content of Part V was already mentioned.

1.4. A few words are in order concerning the history of the problems we consider here. For convenience we shall mean by "the integral equation theorem"

the theorem described in §1.1 whose conclusion is $\mu_{m,t} = M - \lambda_m t^{-\alpha} + o(t^{-\alpha})$ and perhaps also the related result concerning the eigenfunctions, which we have not explicitly stated; "the Toeplitz matrix theorem" is the corresponding theorem described in §1.2. In all previous work Ω was an interval on the line and $\Phi \equiv 1$. First the Toeplitz matrix theorem was proved in the case $\alpha = 2$ by Kac, Murdock, and Szegö [3]. (See also [1].) The integral equation theorem was proved by us in the case $\alpha = 2$ [8] and then for $\alpha \leqq 2$ but with k a probability density [9]. Parter proved the Toeplitz matrix theorem for $\alpha = 4$ [4] and then for α any even integer [5]. Finally we proved the integral equation theorem for general α [10] and Parter proved the corresponding Toeplitz matrix theorem [6].

In view of the papers [10] and [6] and the present paper it is clear, as was not clear in the early stages of development of the problem, that the Toeplitz matrix theorem and the integral equation theorem are not simply analogues of each other. The Toeplitz matrix theorem involves more.

Part II

2.1. We shall assume as stated in §1.1 that Ω is a subset of E_N of finite but positive measure. $L_2(\Omega)$ denotes the subspace of $L_2(E_N)$ consisting of the functions which vanish almost everywhere on Ω', the complement of Ω. $\hat{L}_2(\Omega)$ is the space of Fourier transforms of functions in $L_2(\Omega)$.

Let us be given a non-negative function $J(\xi)$, not identically zero. We denote by $\hat{\mathscr{H}}$ the set of $F \in \hat{L}_2(\Omega)$ for which

$$|||F|||^2 = \int [1 + J(\xi)]\,|F(\xi)|^2\,d\xi < \infty.$$

hen $\hat{\mathscr{H}}$ is a Hilbert space with inner product

$$\langle F,G\rangle = \int [1 + J(\xi)]\,F(\xi)\bar{G}(\xi)\,d\xi.$$

Given $F \in \hat{L}_2(\Omega)$ the functional $G \to (F,G)$ is bounded on $\hat{\mathscr{H}}$ since

$$|(F,G)| \leqq \|F\|\ \|G\| \leqq \|F\|\ |||G|||.$$

It follows that there is a unique element $\hat{\mathbf{B}}F \in \hat{\mathscr{H}}$ for which

$$(F,G) = \langle \hat{\mathbf{B}}F, G\rangle, \qquad G \in \hat{\mathscr{H}}. \tag{5}$$

Furthermore $|||\hat{\mathbf{B}}F||| \leqq \|F\|$, so certainly $\|\hat{\mathbf{B}}F\| \leqq \|F\|$. We have defined $\hat{\mathbf{B}}F$ for all $F \in \hat{L}_2(\Omega)$ but we shall consider only its restriction to $\hat{\mathscr{H}}_0$, the $\|\ \|$ closure of $\hat{\mathscr{H}}$ in $\hat{L}_2(\Omega)$. Thus $\hat{\mathbf{B}}$ is a bounded operator on $\hat{\mathscr{H}}_0$ of norm at most 1. It is also self-adjoint, since we have from (5)

$$(F,\hat{\mathbf{B}}G) = \langle \hat{\mathbf{B}}F, \hat{\mathbf{B}}G\rangle = (\hat{\mathbf{B}}F, G), \qquad F,G \in \hat{\mathscr{H}},$$

and $\hat{\mathscr{H}}$ is dense in $\hat{\mathscr{H}}_0$.

Define $\hat{\mathbf{A}} = \hat{\mathbf{B}}^{-1} - \mathbf{I}$. Then $\hat{\mathbf{A}}$ is a self-adjoint operator with domain $\hat{\mathscr{D}}$ = range $(\hat{\mathbf{A}})$ and from (5) we deduce

$$\int J(\xi)F(\xi)\bar{G}(\xi)d\xi = \int (\hat{\mathbf{A}}F)(\xi)\bar{G}(\xi)d\xi, \qquad F \in \hat{\mathscr{D}},\ G \in \hat{\mathscr{H}}. \tag{6}$$

Finally the spaces $\mathscr{H}$, $\mathscr{H}_0$, $\mathscr{D}$ and the operator $\mathbf{A}$ are defined by inverse Fourier transformation. Thus f belongs to the domain $\mathscr{D}$ of $\mathbf{A}$ if $\hat{f} \in \hat{\mathscr{D}}$ and we define $(\mathbf{A}f)\hat{} = \hat{\mathbf{A}}\hat{f}$. (A circumflex over a function denotes the Fourier transform with normalizing factor $(2\pi)^{-N/2}$.)

In most cases of interest we have $\mathscr{H}_0 = L_2(\Omega)$. For example if J is at most of polynomial growth the C^∞ functions with compact support and vanishing outside Ω all belong to $\mathscr{H}$, and these are dense in $L_2(\Omega)$ if the boundary of Ω has measure zero. However if Ω is a nowhere dense set of positive measure and, say, $J(\xi) = |\xi|^{N+1}$, then $\mathscr{H}$ consists only of the function identically zero, so certainly $\mathscr{H}_0 \neq L_2(\Omega)$. We shall not exclude this possibility.

Let us give the simplest examples of the operators $\mathbf{A}$. If $n = 1$, $\Omega = (-1,1)$, $J = \xi^2$, then $f \in \mathscr{H}$ means that f is (equivalent to) a function vanishing outside $(-1,1)$, absolutely continuous on $(-\infty,\infty)$ and with $f' \in L_2$. $f \in \mathscr{D}$ means that also f' is absolutely continuous on $(-1,1)$ and $f'' \in L_2(-1,1)$. For $f \in \mathscr{D}$, $Af = f''$. Notice that if $\Omega_1 = (-1,0) \cup (0,1)$ then the spaces $\mathscr{H}$ and $\mathscr{D}$ and the operator $\mathbf{A}$ are exactly the same for Ω_1 as for Ω since Ω_1 and Ω differ only by a set of measure zero. Thus it would not be accurate to say, for a general open set Ω, that the functions of $\mathscr{H}$ must vanish on the boundary of Ω. It is true if every neighborhood of every boundary point of Ω meets the complement of Ω in a set of positive measure. With this restriction imposed on an open set Ω in E_N it is not hard to see that the operator corresponding to $J = |\xi|^{2k}$ $(k = 1,2,\cdots)$ is the kth power of the negative of the Laplacian with Dirichlet boundary conditions.

For other J or Ω the situation is more complicated. If $\Omega = (-1,1)$ and $J = |\xi|^\alpha$ $(\alpha > 0)$ the Green's function for $\mathbf{A}$ can be found explicitly. It is the function $K(x,y)$ of [10]. (See [10, Lemma 3].)

2.2. LEMMA 1. *Let $F_n \in \hat{\mathscr{H}}$ satisfy*

$$\|F_n\| = 1,\ \limsup_{n\to\infty} \int J|F_n|^2 d\xi < \liminf_{|\xi|\to\infty} J(\xi).$$

Then there is a subsequence $F_{n'}$ and an $F \in \hat{\mathscr{H}}$ with $F \neq 0$ so that $F_{n'} \to F$ weakly (in $\hat{L}_2(\Omega)$). If $\liminf J(\xi) = \infty$ the convergence is strong.

Proof. Since $\|F_n\| = 1$ there is a subsequence $F_{n'}$ weakly convergent to an $F \in \hat{L}_2(\Omega)$. Set

$$L = \liminf J(\xi), \qquad L' = \limsup \int J|F_n|^2 d\xi,$$

and assume first that $L < \infty$. Choose R and n_0 so that $n > n_0$ and $|\xi| \geqq R$ imply

$$\int J|F_n|^2 d\xi \leqq \frac{L + 2L'}{3}, \quad J(\xi) \geqq \frac{2L + L'}{3}.$$

Then

$$\frac{L + 2L'}{3} \geqq \int_{|\xi| \geqq R} J|F_n|^2 d\xi \geqq \frac{2L + L'}{3} \int_{|\xi| \geqq R} |F_n|^2 d\xi,$$

and so since $\|F_n\| = 1$,

$$\int_{|\xi| \leqq R} |F_n|^2 d\xi \geqq \frac{L - L'}{2L + L'}. \tag{7}$$

Now if $F_n = \hat{f}_n$ and $F = \hat{f}$ we have $\|f_n\| = 1$ and $f_{n'} \to f$ weakly. These imply, since Ω has finite measure, that $F_{n'} \to F$ pointwise and boundedly. Therefore from (7)

$$\int_{|\xi| \leqq R} |F|^2 d\xi \geqq \frac{L - L'}{2L + L'}$$

and so $F \neq 0$. That $F \in \hat{\mathscr{H}}$ follows from Fatou's lemma and the facts that $F_{n'} \to F$ pointwise and $\int J|F_n|^2 d\xi = O(1)$.

For $L = \infty$ we modify the argument as follows. Given $\varepsilon > 0$, choose R and n_0 so that $n > n_0$ and $|\xi| \geqq R$ imply

$$\int J|F_n|^2 d\xi \leqq L' + \varepsilon, \quad J(\xi) \geqq \varepsilon^{-1} L' + 1.$$

We obtain, analogously to (7),

$$\int_{|\xi| \leqq R} |F_n|^2 d\xi \geqq 1 - \varepsilon.$$

It follows that $\|F\|^2 \geqq 1 - \varepsilon$ and, since $\varepsilon > 0$ is arbitrary, that $\|F\| \geqq 1$. This, with weak convergence, implies strong convergence. Again $F \in \hat{\mathscr{H}}$ by Fatou's lemma.

THEOREM I. *The part of the spectrum of* $\mathbf{A}$ *to the left of* $L = \liminf J(\xi)$ *consists of eigenvalues, all positive and of finite multiplicity, whose only possible limit point is* L.

Proof. If λ is in the spectrum of $\mathbf{A}$ there is a sequence $F_m \in \hat{\mathscr{D}}$ so that $\|F_m\| = 1$ and

$$\|\hat{\mathbf{A}} F_m - \lambda F_m\| \to 0. \tag{8}$$

From (6), $\int J|F_m|^2 d\xi = \lambda + o(1)$. If $\lambda < L$ we can apply Lemma 1 to obtain a subsequence $F_{m'}$ converging weakly to $F \in \hat{\mathscr{H}}$, $F \neq 0$. It follows from (8) that for any $G \in \hat{\mathscr{D}}$, $\lambda(F, G) = \lim \lambda(F_{m'}, G) = \lim(\hat{\mathbf{A}}F_{m'}, G) = \lim(F_{m'}, \hat{\mathbf{A}}G) = (F, \hat{\mathbf{A}}G)$. Since $\hat{\mathbf{A}}$ is self-adjoint this implies $F \in \hat{\mathscr{D}}$ and $\hat{\mathbf{A}}F = \lambda F$. Thus λ is an eigenvalue. Also, F is a nonzero entire function and so vanishes only on a set of measure zero, and J is not identically zero. Thus $\lambda = \int J|F|^2 d\xi > 0$.

To prove the remaining two statements of the theorem, note that if either were false we could find a sequence $\lambda_n \to \lambda < L$ and an orthonormal sequence F_n with $\hat{\mathbf{A}}F_n = \lambda_n F_n$. But F_n is weakly convergent to zero and this contradicts Lemma 1.

COROLLARY. *If $L = \infty$ then* $\mathbf{A}$ *is the inverse of a positive definite completely continuous operator on* $\mathscr{H}_0$.

In case $L = \infty$ the eigenfunctions of $\mathbf{A}$ span $\mathscr{H}_0$, and it is possible to give a variational characterization of the eigenvalues. We denote the sequence of eigenvalues by $\lambda_1 \leqq \lambda_2 \leqq \cdots$. As we have seen, in pathological cases it is possible that $\mathscr{H}_0$ is not infinite dimensional. If $\mathscr{H}_0$ is d-dimensional ($d < \infty$) we shall define $\lambda_{d+1} = \lambda_{d+2} = \cdots = \infty$. In the theorem below, the infimum of the empty set is considered to be ∞.

THEOREM II. *For each n we have*

$$\lambda_n = \sup \inf \int J(\xi)|F(\xi)|^2 d\xi$$

where the supremum is taken over all sets of $n-1$ functions $G_1, \cdots, G_{n-1} \in \hat{L}_2(\Omega)$ and the infimum over all $F \in \hat{L}_2(\Omega)$ satisfying

$$\|F\| = 1, (F, G_m) = 0 \qquad (1 \leqq m < n). \tag{9}$$

Proof. We first consider the case $\lambda_n = \infty$, that is $d < n$. Choose $G_1, \cdots, G_{n-1}$ to contain a basis for $\mathscr{H}_0$. Either there is no nonzero F orthogonal to the G_m, in which case the infimum is ∞, or any such F does not belong to $\hat{\mathscr{H}}_0$, therefore not to $\hat{\mathscr{H}}$, so again the infimum is ∞.

We may assume therefore that $\lambda_n < \infty$. Let $\{F_m\}$ be an orthonormal basis of $\hat{\mathscr{H}}_0$ consisting of eigenfunctions of $\hat{\mathbf{A}}$ so that $\hat{\mathbf{A}}F_m = \lambda_m F_m$. For any set $G_1, \cdots, G_{n-1}$ we can find a linear combination

$$F = \sum_{m=1}^{n} \alpha_m F_m$$

satisfying (9). Then

$$\int J(\xi)|F(\xi)|^2 d\xi = \sum_{m=1}^{n} \lambda_m |\alpha_m|^2 \leqq \lambda .$$

It follows that $\sup \inf \leqq \lambda_n$. To prove the opposite inequality it suffices to show that

$$F \in \hat{\mathscr{H}}, \|F\| = 1, (F, F_m) = 0 \quad (1 \leqq m < n)$$

imply $\int J|F|^2 d\xi \geqq \lambda_n$. Now $\{F_m\}$ is an orthonormal basis for $\hat{\mathscr{H}}_0$; but also $\{(1+\lambda_m)^{-1/2} F_m\}$ is an orthonormal basis for $\hat{\mathscr{H}}$ with inner product $\langle\ ,\ \rangle$. Therefore

$$|||F|||^2 = \sum_{m=1}^{\infty} (1+\lambda_m)^{-1} |\langle F, F_m\rangle|^2 = \sum_{m=1}^{\infty} (1+\lambda_m) |(F, F_m)|^2$$

$$= \sum_{m=n}^{\infty} (1+\lambda_m) |(F, F_m)|^2 \geqq 1 + \lambda_n,$$

so $\int J|F|^2 d\xi \geqq \lambda_n$.

2.3. In this section we state and prove the perturbation theorem. We shall consider a family of functions $J_t, t > 0$, in addition to J. We denote by $\mathscr{H}$, $\mathscr{H}_0$, $\mathscr{D}$, $\mathbf{A}$ the spaces and operator corresponding to J and by $\mathscr{H}_0$, $\mathscr{H}_{0t}$, $\mathscr{D}_t$, $\mathbf{A}_t$ those corresponding to J_t. The spectral families corresponding to $\mathbf{A}$ [resp. $\mathbf{A}_t$] are $\mathbf{E}(\lambda)$ [resp. $\mathbf{E}_t(\lambda)$]. The conclusion of the theorem will be that $\mathbf{E}_t(\lambda) \to \mathbf{E}(\lambda)$ in the strong, or uniform, topology on operators on $\mathscr{H}_0$. Now it will be a consequence of hypothesis (ii) of the theorem that $\mathscr{H}_0 \subset \mathscr{H}_{0t}$ for all t, but it may happen that $\mathscr{H}_0 \neq \mathscr{H}_{0t}$. We must therefore make clear what these types of convergence mean. Strong convergence means that for any $f \in \mathscr{H}_0$

$$\|\mathbf{E}_t(\lambda) f - \mathbf{E}(\lambda) f\| \to 0.$$

Uniform convergence means that there are constant $v_t \to 0$ such that

$$\|\mathbf{E}_t(\lambda) f - \mathbf{E}(\lambda) f\| \leqq v_t \|f\|$$

for all $f \in \mathscr{H}_0$.

As before $L = \liminf J(\xi)$.

THEOREM III. *Let J_t be a family of non-negative functions satisfying the following conditions*:

(i) $\lim_{t\to\infty} J_t(\xi) = J(\xi)$ *for almost every* ξ,

(ii) *there are constants* $c_1, c_2 > 0$ *such that*

$$J_t(\xi) \leqq c_1 + c_2 J(\xi)$$

for all t *and* ξ,

(iii) *given* $\varepsilon > 0$ *there exist* R *and* t_0 *such that* $t > t_0$ *and* $|\xi| \geqq R$ *imply*

$$J_t(\xi) \geqq \begin{cases} L - \varepsilon & \text{if } L < \infty, \\ 1/\varepsilon & \text{if } L = \infty. \end{cases}$$

If $\lambda < L$ *is not an eigenvalue of* $\mathbf{A}$ *then* $\mathbf{E}_t(\lambda) \to \mathbf{E}(\lambda)$ *in the strong topology on operators on* $\mathscr{H}_0$. *If* $L = \infty$ *the convergence holds with respect to the uniform operator topology.*

We shall need a few lemmas, the hypotheses being those of the theorem. $\{t'\}$ will be a sequence of t's tending to ∞.

LEMMA 2. *Assume* $F_{t'} \in \hat{\mathscr{H}}_{t'}$ *satisfy*

$$\|F_{t'}\| = 1, \qquad \limsup_{t' \to \infty} \int J_{t'}(\xi)\,|F_{t'}(\xi)|^2 d\xi < L.$$

Then the conclusions of Lemma 1 *hold.*

The proof of this is almost identical with that of Lemma 1 and so is omitted.

LEMMA 3. *Assume* $F_{t'} \in \hat{\mathscr{H}}_{t'}$ *satisfy*

$$\int J_{t'}(\xi)\,|F_{t'}(\xi)|^2 d\xi = O(1), \quad F_{t'} \to F \text{ weakly}.$$

Then

$$\int J_{t'}(\xi) F_{t'}(\xi)\,\bar{G}(\xi) d\xi \to \int J(\xi) F(\xi)\,\bar{G}(\xi) d\xi, \qquad G \in \hat{\mathscr{H}}. \tag{10}$$

Proof. Since $\|J_{t'}^{1/2} F_{t'}\| = O(1)$ every subsequence of $\{J_{t'}^{1/2} F_{t'}\}$ has a subsequence which converges weakly. But $J_{t'}^{1/2} \to J^{1/2}$ pointwise and, as observed in the proof of Lemma 1, $F_{t'} \to F$ pointwise. It follows that $J_{t'}^{1/2} F_{t'} \to J^{1/2} F$ weakly. In addition $J_{t'}^{1/2} G \to J^{1/2} G$ in norm. (Here we have used (ii) and the dominated convergence theorem.) These two facts imply (10).

LEMMA 4. *Let I be a closed interval to the left of L and disjoint from the spectrum of* $\mathbf{A}$. *Then for sufficiently large t the spectrum of* $\mathbf{A}_t$ *is disjoint from I.*

Proof. It follows from (iii) that for sufficiently large t, I lies to the left of $L_t = \liminf J_t$. It follows from Theorem I that for such t any point λ_t belonging to I and the spectrum of $\mathbf{A}_t$ is an eigenvalue of $\mathbf{A}_t$. Assume there is a sequence of t's tending to ∞ for each of which there is such a λ_t. Let F_t be a normalized eigenfunction of $\hat{\mathbf{A}}_t$ corresponding to λ_t. By Lemma 2 we can find a subsequence $F_{t'}$ converging weakly to a nonzero $F \in \hat{\mathscr{H}}$. We may also assume $\lambda_{t'}$ converges to some number λ. If we use Lemma 3 with $G \in \hat{\mathscr{D}}$ we obtain

$$\lambda(F, G) = \lim \lambda_{t'}(F_{t'}, G) = (F, \hat{\mathbf{A}} G). \tag{11}$$

Since $\hat{\mathbf{A}}$ is self-adjoint this implies $F \in \hat{\mathscr{D}}$ and $\hat{\mathbf{A}} F = \lambda F$. This is a contradiction since $\lambda \in I$.

We now prove the theorem. Let C be a circle in the complex plane, described in the positive direction, with center $\lambda/2$ and radius λ. We have

$$\hat{\mathbf{E}}(\lambda) = \frac{1}{2\pi i} \int_C (\hat{\mathbf{A}} - z)^{-1} dz.$$

From Lemma 4, C is disjoint from the spectrum of $\hat{\mathbf{A}}_t$ for sufficiently large t, so we have

$$\hat{\mathbf{E}}_t(\lambda) = \frac{1}{2\pi i}\int_C (\hat{\mathbf{A}}_t - z)^{-1}dz.$$

On C the operators $(\hat{\mathbf{A}}_t - z)^{-1}$ are uniformly bounded for sufficiently large t since C is bounded away from the spectrum of $\hat{\mathbf{A}}_t$, also by Lemma 4. If we can show, for each $z \in C$, that $(\hat{\mathbf{A}}_t - z)^{-1} \to (\hat{\mathbf{A}} - z)^{-1}$ weakly, then we can conclude that $\hat{\mathbf{E}}_t(\lambda) \to \hat{\mathbf{E}}(\lambda)$ weakly, and so strongly.

Let $F \in \hat{\mathscr{H}}_0$. Since

$$\hat{\mathbf{A}}_t(\hat{\mathbf{A}}_t - z)^{-1}F = F + z(\hat{\mathbf{A}}_t - z)^{-1}F \tag{12}$$

we have

$$\int J_t|(\hat{\mathbf{A}}_t - z)^{-1}F|^2 d\xi = (\hat{\mathbf{A}}(\hat{\mathbf{A}}_t - z)^{-1}F, (\hat{\mathbf{A}}_t - z)^{-1}F) = O(1).$$

By Lemma 2 every sequence $\{t'\}$ has a subsequence $\{t''\}$ for which $(\hat{\mathbf{A}}_{t''} - z)^{-1}F$ converges weakly. Denote this weak limit by H. Then by Lemma 3 with $G \in \hat{\mathscr{D}}$

$$(\hat{\mathbf{A}}_{t''}(\hat{\mathbf{A}}_{t''} - z)^{-1}F, G) \to (H, \hat{\mathbf{A}}G). \tag{13}$$

Thus (see (12))

$$(H, \hat{\mathbf{A}}G) = (F, G) + z(H, G), \qquad G \in \hat{\mathscr{D}}.$$

It follows that $H \in \hat{\mathscr{D}}$ and $(\hat{\mathbf{A}} - z)H = F$, so $H = (\hat{\mathbf{A}} - z)^{-1}F$. We have shown that every sequence $\{t'\}$ has a subsequence $\{t''\}$ for which $(\hat{\mathbf{A}}_{t''} - z)^{-1}F \to (\hat{\mathbf{A}} - z)^{-1}F$ weakly. This implies $(\hat{\mathbf{A}}_t - z)^{-1}F \to (\hat{\mathbf{A}} - z)^{-1}F$ weakly.

It remains to show that if $L = \infty$ then $\hat{\mathbf{E}}_t(\lambda) \to \hat{\mathbf{E}}(\lambda)$ uniformly. If not, we could find $F_{t'} \in \hat{\mathscr{H}}$ with $\|F_{t'}\| = 1$ and a $\delta > 0$ so that

$$\|\hat{\mathbf{E}}_{t'}(\lambda)F_{t'} - \hat{\mathbf{E}}(\lambda)F_{t'}\| \geqq \delta. \tag{14}$$

By Lemma 2 there is a subsequence so that both $\hat{\mathbf{E}}_{t''}(\lambda)F_{t''}$ and $\hat{\mathbf{E}}(\lambda)F_{t''}$ converge strongly, say to G_1 and G_2 respectively. It follows from (14) that $\|G_1 - G_2\| \geqq \delta$. But the weak convergence of $\hat{\mathbf{E}}_t(\lambda)$ to $\hat{\mathbf{E}}(\lambda)$ implies $G_1 = G_2$, a contradiction.

It is now easy to obtain the integral equation theorem described in §1.1. Recall that we have a function $k \in L_1(E_N)$ which satisfies $k(-x) = \bar{k}(x)$ and we set $K(\xi) = \int e^{-i\xi\cdot x}k(x)dx$. We assume that $K(0) = M > 0$, that

$$\max_{|\xi'| \geqq \delta} K(\xi) < M$$

for each $\delta > 0$, and that as $\xi \to 0$

$$K(\xi) = M - |\xi|^\alpha \Phi(\xi/|\xi|) + o(|\xi|^\alpha) \tag{15}$$

where $\alpha > 0$ and Φ is bounded and bounded away from zero. The positive eigenvalues of $\mathbf{A}_{K(\xi/t)}$ are denoted by $\mu_{1,t} \geqq \mu_{2,t} \geqq \cdots$ and corresponding normalized eigenfunctions by $f_{1,t}, f_{2,t}, \cdots$. Denote the eigenvalues of $\mathbf{A}_J$ with $J(\xi) = |\xi|^\alpha \Phi(\xi/|\xi|)$ by $\lambda_1 \leqq \lambda_2 \leqq \cdots$ and corresponding normalized eigenfunctions by $f_1, f_2, \cdots$. (Recall the convention by which we may define certain λ_n to be ∞.)

COROLLARY. *For each m we have*

$$\lim_{t\to\infty} t^\alpha(M - \mu_{m,t}) = \lambda_m. \tag{16}$$

If $\lambda_m < \infty$ then every sequence $\{t'\}$ has a subsequence $\{t''\}$ for which $f_{m,t''}$ converges strongly to a linear combination of those f_n for which $\lambda_m = \lambda_n$.

Proof. Apply Theorem III with $J(\xi) = |\xi|^\alpha \Phi(\xi/|\xi|)$ and $J_t(\xi) = t^\alpha[M - K(\xi/t)]$.

PART III

3.1. If we try to extend the corollary of Theorem III to the case when the hypothesis (15) is generalized to

$$K(\xi) = M - |\xi|^\alpha \Phi(\xi/|\xi|) L(|\xi|) + o(|\xi|^\alpha L(|\xi|)), \tag{17}$$

L being a non-negative slowly varying function near zero, we would take

$$J_t(\xi) = L(t^{-1})^{-1} t^\alpha [M - K(\xi/t)],\ J(\xi) = |\xi|^\alpha \Phi(\xi/|\xi|). \tag{18}$$

But we cannot apply the theorem since the domination condition (ii) may not hold. However we have,

THEOREM IV. *The conclusions of Theorem* III *hold if* (ii) *is replaced by* (ii′) *for each $G \in \hat{\mathscr{H}}$ there is a sequence $G_k \in \hat{\mathscr{H}}$ satisfying*

$$\text{(a)} \quad \int \sup_t J_t(\xi) |G_k(\xi)|^2 d\xi < \infty, \qquad \textit{each } k,$$

$$\text{(b)} \quad \int J(\xi) |G_k(\xi) - G(\xi)|^2 d\xi \to 0, \qquad k \to \infty.$$

Proof. Where in the proof of Theorem III was (ii) used? Only in the proof of Lemma 3. Now this lemma was used at two points of the proof of the theorem, namely at (11) and at (13). Going back to (11), we can conclude in our situation that

$$\lambda(F, G_k) = \int J F \bar{G}_k d\xi.$$

(Recall that $F \in \hat{\mathscr{H}}$.) But then from (b) we can conclude that

$$\lambda(F, G) = \int J F \bar{G}\, d\xi.$$

Similarly the argument at (13) can be taken care of.

In the following corollary we denote by $C_0^\infty(\Omega)$ the C^∞ functions with compact support contained in Ω.

COROLLARY. *Assume that $C_0^\infty(\Omega)$ is $|||\ |||$ dense in $\mathscr{H}$. Then the corollary to Theorem* III *holds with* (15) *replaced by* (17) *if* (16) *is replaced by*

$$\lim_{t\to\infty} L(t^{-1})^{-1}t^\alpha[M-\mu_{m,t}] = \lambda_m.$$

Proof. It is sufficient to show that with J_t and J given by (18) the condition (ii′) of Theorem IV holds. It is an elementary fact about slowly varying functions that for each $\varepsilon > 0$ there is a $\delta > 0$ and a $c_1 > 0$ so that

$$\frac{L(au)}{L(u)} \leqq c_1(a^{-\varepsilon} + a^\varepsilon) \qquad \text{if } (1+a)u \leqq \delta.$$

Thus

$$L(t^{-1})^{-1}L(|\xi|/t) \leqq c_2(|\xi|^{-\alpha/2} + |\xi|) \qquad \text{if } t^{-1}(1+|\xi|) \leqq \delta.$$

This shows

$$J_t(\xi) \leqq c_3(1+|\xi|^{\alpha+1}), \qquad 1+|\xi| \leqq \delta t.$$

But if $1+|\xi| \geqq \delta t$ then

$$J_t(\xi) \leqq L(t^{-1})^{-1}t^\alpha \max_\eta |K(\eta)| \leqq c_4 t^{\alpha+1}.$$

Therefore for all t and ξ

$$J_t(\xi) \leqq c_5(1+|\xi|^{\alpha+1}).$$

It follows that part (a) of condition (ii′) is satisfied for $G_k = \hat{g}_k$ where $g_k \in C_0^\infty(\Omega)$, and (b) is a consequence of this fact and our assumption.

3.2. Because of the assumption of the preceding corollary, and because we shall need it in our discussion of Toeplitz matrices, we now give a sufficient condition for $C_0^\infty(\Omega)$ to be $|||\ |||$ dense in $\mathscr{H}$.

We call Ω star-shaped with respect to the origin if for each $r < 1$ the closure of $r\Omega$ is contained in the interior of Ω. Ω is star-shaped if some translate is star-shaped with respect to the origin. Ω is locally star-shaped if every point of $\overline{\Omega}$ (the closure of Ω) has a neighborhood whose intersection with Ω is star-shaped.

THEOREM V. *Assume that Ω is locally star-shaped and that for some constants $\delta, M > 0$*

$$\frac{1+J(\xi)}{1+J(\eta)} \leqq M \qquad \text{if } \frac{|\xi|}{|\eta|} \leqq 1+\delta. \tag{19}$$

Then $C_0^\infty(\Omega)$ is $|||\ |||$ dense in $\mathscr{H}$.

Proof. It follows from (19) that $J(\xi) \leqq c(1+|\xi|^{M_1})$ with appropriate constants c, M_1. Thus $C_0^\infty(\Omega) \subset \mathscr{H}$.

Assume first that Ω is bounded and star-shaped; we may suppose it is star-shaped with respect to the origin. If $r < 1$,

$$\varepsilon_r = \operatorname{dist}(r\Omega, \Omega') > 0.$$

Let ϕ be a non-negative C^∞ function on E_N, supported on $|x| \leqq 1$, and with $\int \phi dx = 1$. Given $f \in \mathscr{H}$, set

$$f_r(x) = \varepsilon_r^{-N} \int \phi(\varepsilon_r^{-1}(x-y)) f(r^{-1}y) dy.$$

Then $f_r \in C_0^\infty(\Omega)$. If $\Phi(\xi) = \int e^{i\xi\cdot x}\phi(x)dx$ then

$$\hat{f}_r(\xi) = r^N \Phi(\varepsilon_r \xi) \hat{f}(r\xi)$$

and

$$\begin{aligned} |||f_r|||^2 &= \int [1 + J(\xi)] |\hat{f}_r(\xi)|^2 d\xi \\ &\leqq r^N \int [1 + J(r^{-1}\xi)] |\hat{f}(\xi)|^2 d\xi \leqq M r^N |||f|||^2 \end{aligned}$$

whenever $r > (1+\delta)^{-1}$. Thus we have

$$|||f_r||| = O(1), \quad \|f_r - f\| \to 0,$$

as $r \to 1$. These two statements imply the result. For by the Banach-Saks theorem (see [6, §38]) there is a sequence $r_n \to 1$ and an $f_1 \in \mathscr{H}$ such that

$$\left|\left|\left| \frac{1}{n} \sum_{k=1}^{n} f_{r_k} - f_1 \right|\right|\right| \to 0.$$

Then $f_1 = f$ and

$$\left\{ \frac{1}{n} \sum_{k=1}^{n} f_{r_k} \right\}$$

is the required sequence of functions in $C_0^\infty(\Omega)$.

Next assume Ω is bounded and locally star-shaped. Let $U_1, \cdots, U_n$ be bounded open sets covering $\bar{\Omega}$ such that each $\Omega \cap U_i$ is star-shaped. Find functions ϕ_i belonging to C^∞ so that each ϕ_i is supported in U_i and $\sum \phi_i(x) = 1$ on Ω. Let $f \in \mathscr{H}$ and consider a fixed ϕ_i. We have, using Schwarz's inequality,

$$\begin{aligned} |||\phi_i f|||^2 &= (2\pi)^N \int [1 + J(\xi)] d\xi \left| \int \hat{\phi}_i(\xi - \eta) \hat{f}(\eta) d\eta \right|^2 \\ &\leqq (2\pi)^N \int [1 + J(\xi)] d\xi \int |\hat{\phi}_i(\xi - \eta)| \, |\hat{f}(\eta)|^2 d\eta \int |\hat{\phi}_i(\eta)| \, d\eta. \end{aligned}$$

It follows from (19) that

$$1 + J(\xi) \leqq M_2 [1 + J(\xi - \eta)] \, [1 + J(\eta)]$$

for some M_2 and all ξ, η. Therefore

$$(20) \qquad |||\phi_i f|||^2 \leqq (2\pi)^N M_2 \int |\phi_i(\eta)|\, d\eta \int [1 + J(\xi)]\, |\hat{\phi}_i(\xi)|\, d\xi\, |||f|||^2.$$

This shows that $\phi_i f \in \mathscr{H}$. (The argument we have just given was suggested by a similar one used by Hörmander and Lions [2, Lemma 6].) Now since $\phi_i f$ vanished outside the star-shaped set $\Omega \cap U_i$ we can find a sequence $f_{ik} \in C_0^\infty(\Omega \cap U_i)$ so that $|||f_{ik} - \phi_i f||| \to 0$. But then

$$\left|\left|\left| \sum_i f_{ik} - f \right|\right|\right| \to 0$$

and $\{\sum_i f_{ik}\}$ is the required sequence.

Finally we remove the restriction that Ω be bounded. Assume $f \in \mathscr{H}$ and again let ϕ be a non-negative C^∞ function on E_N, supported on $|x| \leqq 1$, with $\int \phi dx = 1$; in addition assume $\phi = 1$ on a neighborhood of the origin. It follows from (20) that

$$\begin{aligned} |||\phi(\varepsilon x) f(x)|||^2 &\leqq (2\pi)^N M_2 \int |\hat{\phi}(\eta)|\, d\eta \int [1 + J(\varepsilon\xi)]\, |\hat{\phi}(\xi)|\, d\xi\, |||f|||^2 \\ &\leqq M_3 \int (1 + |\varepsilon\xi|^{M_1})\, |\hat{\phi}(\xi)|\, d\xi\, |||f|||^2 \leqq M_4 |||f|||^2 \end{aligned}$$

if, say, $\varepsilon < 1$. By the Banach-Saks theorem once again there is a sequence $\varepsilon_n \to 0$ so that

$$\left|\left|\left| \frac{1}{n} \sum_{k=1}^{n} \phi(\varepsilon_k x) f(x) - f(x) \right|\right|\right| \to 0.$$

Now each of the functions

$$\frac{1}{n} \sum_{k=1}^{n} \phi(\varepsilon_k x) f(x)$$

is supported in a set of the form $\Omega \cap \{x : |x| \leqq R\}$. If we can show each of these sets is contained in a bounded locally star-shaped subset of Ω we shall be through. Each point x of $\bar{\Omega} \cap \{x : |x| \leqq R\}$ has a neighborhood U_x such that $\Omega \cap U_x$ is star-shaped, and we can take U_x to be bounded. Since $\bar{\Omega} \cap \{x : |x| \leqq R\}$ is compact we can find a finite set $U_{x_1}, \cdots, U_{x_k}$ which covers it. Then $\Omega \cap (U_{x_1} \cup \cdots \cup U_{x_k})$ is bounded, locally star-shaped, and contains $\Omega \cap \{x : |x| \leqq R\}$.

Part IV

4.1. In this section we shall prove a perturbation theorem similar to Theorem III but where the approximating operators are matrices. We assume that for each value of $t > 0$ we have a function $\phi_t(\xi) = \phi_t(\xi^1, \cdots, \xi^N)$ defined, non-negative, and integrable over the cube $|\xi^i| \leqq \pi t$. Ω is a bounded subset of E_N and $\Omega_t = \Omega \cap t^1\Lambda$ where Λ is the set of lattice points in E_N. $L_2(\Omega_t)$ is the space of vectors $\{u_p\}_{p \in \Omega_t}$ with norm

$$\| u \| = t^{-N/2} \left\{ \sum_{p \in \Omega_t} |u_p|^2 \right\}^{1/2}.$$

$\mathbf{T}_t$ is the operator on $L_2(\Omega_t)$ defined by

$$(\mathbf{T}_t u)_p = \sum_{q \in \Omega_t} c_{t,p-q} u_q$$

where

$$c_{t,p} = (2\pi t)^{-N} \int_{|\xi^i| \leqq \pi t} e^{-i\xi \cdot p} \phi_t(\xi) d\xi, \qquad p \in t^{-1}\Lambda.$$

We shall use a certain mapping $\mathbf{U}_t$ of $L_2(\Omega_t)$ into $L_2(E_N)$. It is given by

$$\mathbf{U}_t u(x) = (2\pi t)^{-N} \sum_{p \in \Omega_t} u_p \int_{|\xi^i| \leqq \pi t} e^{i\xi \cdot (p-x)} d\xi.$$

The Fourier transform of $\mathbf{U}_t u$ is

$$(\mathbf{U}_t u)^\wedge(\xi) = \begin{cases} (2\pi t^2)^{-N/2} \sum_{p \in \Omega_t} u_p e^{i\xi \cdot p} & |\xi^i| \leqq \pi t, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

From (21) and the definition of the norm in $L_2(\Omega_t)$ it is easy to verify that $\mathbf{U}_t$ is an isometry. Moreover we have

$$(\mathbf{T}_t u, v) = \int \phi_t(\xi)(\mathbf{U}_t u)^\wedge(\xi)(\mathbf{U}_t v)^{\wedge -}(\xi) d\xi, \tag{22}$$

where $^{\wedge -}$ denotes Fourier transformation followed by complex conjugation.

THEOREM VI. *Assume Ω is bounded and locally star-shaped, ϕ is a non-negative function for which there exist constants δ, $M > 0$ such that*

$$\frac{1 + \phi(\xi)}{1 + \phi(\eta)} \leqq M \qquad \text{if } \frac{|\xi|}{|\eta|} \leqq 1 + \delta.$$

We set $L = \liminf \phi(\xi)$.

Let ϕ_t be as described above and assume

(i) $\lim_{t \to \infty} \phi_t(\xi) = \phi(\xi)$ *for almost every* ξ,

(ii) *there are constants $c_1, c_2 > 0$ such that*

$$\phi_t(\xi) \leqq c_1(1 + |\xi|^{c_2}), \qquad |\xi^i| \leqq \pi t,$$

(iii) *given $\varepsilon > 0$ there exist R and t_0 such that $t > t_0$ and $|\xi| \geqq R$, $|\xi^i| \leqq \pi t$ imply*

$$\phi_t(\xi) \geqq \begin{cases} L - \varepsilon & \text{if } L < \infty, \\ 1/\varepsilon & \text{if } L = \infty. \end{cases}$$

Denote by $\mathbf{E}_t(\lambda), \mathbf{E}(\lambda)$ the spectral families of $\mathbf{T}_t$ and $\mathbf{A}(= \mathbf{A}_\phi)$ respectively. Then if $\lambda < L$ is not an eigenvalue of $\mathbf{A}$ we have $\mathbf{U}_t \mathbf{E}_t(\lambda) \mathbf{U}_t^* \to \mathbf{E}(\lambda)$ in the strong topology on the operators on $L_2(E_N)$. If in addition $L = \infty$ the convergence holds with respect to the uniform operator topology.

REMARK. Strictly speaking $\mathbf{E}(\lambda)$ is an operator on $L_2(\Omega)$, not $L_2(E_N)$. We have made the obvious identification.

4.2. The proof of the theorem requires a series of lemmas some of which are analogous to those used in the proof of Theorem III. For this reason not all the lemmas will be proved in detail. The spaces $\mathscr{H}$, $\mathscr{H}_0$, $\mathscr{D}$ will be those corresponding to ϕ.

LEMMA 5. *Assume $u_t \in L_2(\Omega_t)$ satisfy*

$$\|u_t\| = 1 \qquad \limsup_{t\to\infty} \int \phi_t |(\mathbf{U}_t u_t)^\wedge|^2 d\xi < \infty.$$

Then there is a subsequence $\{t'\}$ and an $F \in \hat{\mathscr{H}}$ with $F \neq 0$ such that $(\mathbf{U}_t u_t)^\wedge \to F$ weakly in $\hat{L}_2(E_N)$. If $L = \infty$ the convergence is strong.

Proof. We may assume that $(\mathbf{U}_t u_t)^\wedge$ converges weakly to $F \in \hat{L}_2(E_N)$. We must show that $F \neq 0$, that $F \in \hat{\mathscr{H}}$, and that $\|F\| = 1$ if $L = \infty$. It follows from (21) that

$$|(\mathbf{U}_t u_t)^\wedge(\xi)|^2 \leqq (2\pi t^2)^{-N} \sum_{p\in\Omega_t} 1 \sum_{p\in\Omega_t} |u_p|^2 = (2\pi t)^{-N} \sum_{p\in\Omega_t} 1.$$

The sum on the right represents the number of lattice points in $t\Omega$ and so is $O(t^N)$. Therefore the functions $(\mathbf{U}_t u_t)^\wedge$ are uniformly bounded. Similarly, using the notation $p = (p^1, \cdots, p^N)$,

$$\left| \frac{\partial}{\partial \xi^i} (\mathbf{U}_t u_t)^\wedge(\xi) \right| \leqq (2\pi t)^{-N} \sum_{p\in\Omega_t} |p^i|^2 = O(1),$$

so $(\mathbf{U}_t u_t)^\wedge$ are equicontinuous. It follows that $(\mathbf{U}_t u_t)^\wedge \to F$ boundedly and pointwise, and the method of Lemma 1 shows that $F \neq 0$, $\int \phi |F|^2 d\xi < \infty$, and $\|F\| = 1$ if $L = \infty$. It remains to show that $F \in \hat{\mathscr{H}}$, that is, if $\hat{f} = F$ then $f \in L_2(\Omega)$. If $g \in C_0^\infty(\Omega')$ then

$$\int e^{-i\xi\cdot p} \hat{g}(\xi) d\xi = 0, \qquad p \in \Omega_t$$

and so (see (21))

$$((\mathbf{U}_t u_t)^\wedge, \hat{g}) = 0.$$

It follows that f is orthogonal to g for all $g \in C_0^\infty(\Omega')$. But since Ω is locally star-shaped the boundary of Ω has measure zero, so that $C_0^\infty(\Omega')$ is dense in $L_2(\Omega')$. Therefore f is orthogonal to $L_2(\Omega')$ so $f \in L_2(\Omega)$.

LEMMA 6. *Let $u_t \in L_2(\Omega_t)$ satisfy*

$$\limsup \int (1 + \phi_t) |(\mathbf{U}_t u_t)^\wedge|^2 d\xi < \infty, \qquad (\mathbf{U}_t u_t)^\wedge \to F \text{ weakly}.$$

Then

$$\lim \int \phi_t (\mathbf{U}_t u_t)^\wedge (\hat{g})^- d\xi = \int \phi F (\hat{g})^- d\xi, \qquad g \in C_0^\infty(\Omega).$$

Proof. See the proof of Lemma 3.

LEMMA 7. *Assume $g \in C_0^\infty(\Omega)$. Then for every integer $k > 0$,*

$$\lim_{t\to\infty} \int (1+|\xi|)^{2k} \, |\hat{g}(\xi) - (\mathbf{U}_t\mathbf{U}_t^* g)^\wedge(\xi)|^2 d\xi = 0.$$

Proof. We have

$$\begin{aligned}(\mathbf{U}_t^* g)_p &= (2\pi)^{-N} \int g(x)dx \int_{|\xi^i| \leqq \pi t} e^{-i\xi\cdot(p-x)} \, d\xi \\ &= (2\pi)^{-N/2} \int_{|\xi^i| \leqq \pi t} e^{-i\xi\cdot p}\hat{g}(\xi)d\xi\end{aligned} \tag{23}$$

and so for $|\xi^i| \leqq \pi t$ (see (21)),

$$\begin{aligned}(\mathbf{U}_t\mathbf{U}_t^* g)^\wedge(\xi) &= (2\pi t)^{-N} \sum_{p\in\Omega_t} e^{i\xi\cdot p} \int_{|\eta^i| \leqq \pi t} e^{-i\eta\cdot p}\hat{g}(\eta)d\eta \\ &= (2\pi t^2)^{-N/2} \sum_{p\in\Omega_t} e^{i\xi\cdot p}[g(p) + O(t^{-2k})]\end{aligned} \tag{24}$$

where the term $O(t^{-2k})$ is independent of ξ.

We denote by Γ_j the second difference operator acting on the jth coordinate. Thus for any v defined on $t^{-1}\Lambda$

$$(\Gamma_j v)(p) = 2v(p^1, \cdots, p^N) - v(p^1, \cdots, p^j + t^{-1}, \cdots, p^N) - v(p^1, \cdots, p^j - t^{-1}, \cdots, p^N).$$

Then a simple computation shows, using (24),

$$\begin{aligned}&\int_{|\xi^i| \leqq \pi t} |1 - e^{i\xi^j/t}|^{2k} |(\mathbf{U}_t\mathbf{U}_t^* g)^\wedge(\xi)|^2 d\xi \\ &\qquad = t^{-N} \sum_{p\in\Omega_t} \{g(p)(\Gamma_j^k \bar{g})(p) + O(t^{-2k})\}.\end{aligned}$$

Since $\Gamma_j^k \bar{g}(p) = O(t^{-2k})$ and the number of points in Ω_t is $O(t^N)$ we deduce

$$\int_{|\xi^i| \leqq \pi t} |1 - e^{i\xi^j/t}|^{2k} |(\mathbf{U}_t\mathbf{U}_t^* g)^\wedge(\xi)|^2 d\xi = O(t^{-2k}).$$

Since

$$|1 - e^{i\xi^j/t}| \geqq (\pi t)^{-1} |\xi^j|, \qquad |\xi^j| \leqq \pi t,$$

we have

$$\int_{|\xi^i| \leqq \pi t} |\xi^j|^{2k} |(\mathbf{U}_t\mathbf{U}_t^* g)^\wedge(\xi)|^2 d\xi = O(1).$$

Therefore

$$\int_{|\xi^t| \leqq nt} (1 + |\xi|)^{2k} |(\mathbf{U}_t \mathbf{U}_t^* g)^\wedge(\xi)|^2 d\xi = O(1). \tag{25}$$

Now if we go back to (24), say with $k = 1$, we see that $(\mathbf{U}_t \mathbf{U}_t^* g)^\wedge$ converges to $\hat{g}$ boundedly and pointwise. Let

$$S = \sup_t \|(\mathbf{U}_t \mathbf{U}_t^* g)^\wedge\|_\infty.$$

Then

$$\int_{|\xi| \geqq R} (1 + |\xi|)^{2k} |\hat{g} - (\mathbf{U}_t \mathbf{U}_t^* g)^\wedge|^2 d\xi$$

$$\leqq \frac{4S^2}{(1 + R)^2} \int (1 + |\xi|)^{2k+2} |\hat{g} - (\mathbf{U}_t \mathbf{U}_t^* g)^\wedge|^2 d\xi$$

and by (25) with k replaced by $k + 1$ this can be made less than $\varepsilon/2$ by choosing R sufficiently large. Then

$$\int_{|\xi| \leqq R} (1 + |\xi|)^{2k} |\hat{g} - (\mathbf{U}_t \mathbf{U}_t^* g)^\wedge|^2 d\xi$$

can be made less than $\varepsilon/2$ by choosing t sufficiently large, and the lemma is es tablished.

LEMMA 8. *Let I be a closed interval to the left of L and disjoint from the spectrum of* $\mathbf{A}$. *Then for sufficiently large t the spectrum of* $\mathbf{T}_t$ *is disjoint from I.*

Proof. We begin as in the proof of Lemma 4. We assume the lemma false, so there is a sequence $\{\lambda_{t'}\} \subset I$ such that each λ_t is an eigenvalue of $\mathbf{T}_{t'}$, say with normalized eigenvector $u_{t'}$, and such that $\lambda_{t'} \to \lambda \in I$. By Lemma 5 we may assume that $\mathbf{U}_{t'} u_{t'}$ converges weakly to $f \in \mathscr{H}$, $f \neq 0$. Assume $g \in C_0^\infty(\Omega)$. Then

$$\lambda(f, g) = \lim \lambda_{t'} (\mathbf{U}_{t'} u_{t'}, g) = \lim (\mathbf{T}_{t'} u_{t'}, \mathbf{U}_{t'}^* g).$$

By (22) this is

$$\lim \int \phi_{t'} (\mathbf{U}_{t'} u_{t'})^\wedge (\mathbf{U}_{t'} \mathbf{U}_{t'}^* g)^{\wedge -} d\xi$$

$$= \lim \left\{ \int \phi_{t'} (\mathbf{U}_{t'} u_{t'})^\wedge (\hat{g})^- d\xi + \int \phi_{t'} (\mathbf{U}_{t'} u_{t'}) [(\mathbf{U}_{t'} \mathbf{U}_{t'}^* g)^\wedge - \hat{g}]^- d\xi \right\}.$$

By Lemma 6 the first integral approaches

$$\int \phi \hat{f} (\hat{g})^- d\xi;$$

by the boundedness of

$$\int \phi_{t'} |(\mathbf{U}_{t'} u_{t'})^\wedge|^2 d\xi,$$

hypothesis (ii) of the theorem, and Lemma 7, the second integral approaches zero. Thus

$$\lambda(f,g) = \int \phi \hat{f}(\hat{g})^- \, d\xi.$$

This holds for all $g \in C_0^\infty(\Omega)$. But by Theorem V, $C_0^\infty(\Omega)$ is $|||\ |||$ dense in $\mathscr{H}$, so the identity can be extended to hold for all $g \in \mathscr{H}$. It follows that $f \in \mathscr{D}$ and $\mathbf{A}f = \lambda f$, a contradiction.

4.3. We now complete the proof of Theorem VI. We have, by Lemma 8, with an appropriate circle C,

$$\mathbf{E}(\lambda) = \frac{1}{2\pi i} \int_C (\mathbf{A} - z)^{-1} dz,$$

$$\mathbf{U}_t\mathbf{E}_t(\lambda)\mathbf{U}_t^* = \frac{1}{2\pi i} \int_C \mathbf{U}_t(\mathbf{T}_t - z)^{-1}\, \mathbf{U}_t^* dz,$$

and the operators $(\mathbf{T}_t - z)^{-1}$ are uniformly bounded on C. We first prove that for $f \in L_2(E_N)$

$$\mathbf{U}_t(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f \to (\mathbf{A} - z)^{-1}\, \mathbf{E}f$$

weakly, where $\mathbf{E} = \mathbf{E}(\infty)$ is the projection of $L_2(E_N)$ onto $L_2(\Omega)$. This will establish the weak, hence strong, convergence of the projections $\mathbf{U}_t\mathbf{E}_t(\lambda)\mathbf{U}_t^*$ to $\mathbf{E}(\lambda)$. We may assume that $\mathbf{U}_t(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f$ converges weakly to some $h \in L_2(\mathbf{E}^N)$.

Let $g \in C_0^\infty(\Omega')$. Then it follows from (23), which holds for any function $g \in L_2(E_N)$, that $\mathbf{U}_t^* g \to 0$ uniformly on Ω_t. Therefore

$$(h, g) = \lim (\mathbf{U}_t(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f, g) = \lim((\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f, \mathbf{U}_t^* g) = 0.$$

Since this holds for all $g \in C_0^\infty(\Omega')$ we obtain, as at the end of the proof of Lemma 5, that $h = \mathbf{E}h$.

Next let $g \in C_0^\infty(\Omega)$. Then

$$\begin{aligned} (\mathbf{U}_t\mathbf{U}_t^* f, g) &= (\mathbf{U}_t\mathbf{T}_t(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f, g) - z(\mathbf{U}_t(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f, g). \\ &= (\mathbf{T}_t(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f, \mathbf{U}_t^* g) - z(\mathbf{U}_t(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f, g). \end{aligned} \tag{26}$$

The left side converges to (f,g) by Lemma 7, and the second term on the right side to $-z(h,g)$. As for the first term on the right, it is, by (22),

$$\int \phi_t \hat{h}_t(\hat{g})^- d\xi + \int \phi_t \hat{h}_t[(\mathbf{U}_t\mathbf{U}_t^* g)\hat{} - \hat{g}]^- d\xi, \tag{27}$$

where we have set

$$h_t = \mathbf{U}_t(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f.$$

The first term on the right of (27) converges to

$$\int \phi \hat{h}(\hat{g})^{-} d\xi \tag{28}$$

by Lemma 6. As for the second term we have

$$\begin{aligned}\int \phi_t |\hat{h}_t|^2 d\xi &= (\mathbf{T}_t(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f, (\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f) \\ &= (\mathbf{U}_t^* f, (\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f) + z\|(\mathbf{T}_t - z)^{-1}\mathbf{U}_t^* f\|^2\end{aligned}$$

which is $O(1)$. Therefore by hypothesis (ii) and Lemma 7 the second term on the right of (27) approaches zero. It follows that (27), i.e., the first term on the right of (25), converges to (28). Therefore

$$(\mathbf{E}f,g) = (f,g) = \int \phi \hat{h}(\hat{g})^{-} d\xi - z(h,g), \qquad g \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is $|||\ |||$ dense in $\mathscr{H}$ this identity holds for all $g \in \mathscr{H}$. It follows that $h \in \mathscr{D}$ and $(\mathbf{A} - z)h = \mathbf{E}f$.

We have shown that $\mathbf{U}_t\mathbf{E}_t(\lambda)\mathbf{U}_t^* \to \mathbf{E}(\lambda)$ weakly. To show that if $L = \infty$ the convergence is uniform we use Lemmas 5 and 2 just as in the proof of the analogous fact in Theorem III we used Lemma 2. The details are left to the reader.

In the following corollary we shall assume that $\mathbf{T}_t$ come from expanding Toeplitz matrices as described in §1.2. Thus we are given a bounded non-negative function $\psi(\xi)$ on the N-torus. We assume that for each $\delta > 0$

$$\sup_{\delta \leqq |\xi^i| \leqq \pi} \psi(\xi) < M \qquad (M > 0)$$

and as $\xi \to 0$

$$\psi(\xi) = M - |\xi|^\alpha \Phi(\xi/|\xi|) + o(|\xi|^\alpha)$$

where $\alpha > 0$ and Φ is bounded and bounded away from zero. We set $\psi_t(\xi) = \psi(\xi/t)$ and $\phi(\xi) = |\xi|^\alpha \Phi(\xi/|\xi|)$. The positive eigenvalues of the operator $\mathbf{T}_t$ corresponding to ψ_t are denoted by $\mu_{1,t} \geqq \mu_{2,t} \geqq \cdots$, and a set of corresponding normalized eigenvectors is $u_{1,t}, u_{2,t}, \cdots$. The numbers and functions $\lambda_1 \leqq \lambda_2 \leqq \cdots$, $f_1, f_2, \cdots$ are the eigenvalues and corresponding normalized eigenfunctions $\mathbf{A}_\phi$.

COROLLARY. *Assume Ω is locally star-shaped. Then for each m we have*

$$\lim_{t \to \infty} t^\alpha (M - \mu_{m,t}) = \lambda_m.$$

Morever every sequence $\{t'\}$ has a subsequence $\{t''\}$ for which the functions $\mathbf{U}_{t''}u_{m,t''}$ converge strongly to a linear combination of those f_n for which $\lambda_n = \lambda_m$.

Proof. Apply Theorem VI with $\phi_t = t^\alpha[M - \psi_t]$.

PART V

We prove here the assertion made in §1.1. To repeat: *Assume* $k \in L_1(E_N)$ $k(-x) = \bar{k}(x)$. *Set*

$$K(\xi) = \int e^{i\xi\cdot x} k(x)dx$$

and assume $M = \max K(\xi) > 0$. *Let* Ω *have finite but positive measure and denote by* $\mu_{1,t} \geqq \mu_{2,t} \geqq \cdots$ *the positive eigenvalues of the integral operator on* $L_2(t\Omega)$ *with kernel* $k(x-y)$. *Then*

$$\lim_{t\to\infty} \mu_{n,t} = M$$

for each n.

Let B be a ball with center the origin. Then for almost every $x \in E_N$ we have as $t \to \infty$

$$\int_{t^{-1}B} |\chi_\Omega(x+y) - \chi_\Omega(x)|\, dy = o(t^{-N}),$$

where χ_Ω denotes the characteristic function of Ω. In particular for almost every $x \in \Omega$ we have, for every B,

$$\int_{t^{-1}B} |\chi_\Omega(x+y) - 1|\, dy = o(t^{-N}).$$

Since the eigenvalues $\mu_{n,t}$ are unchanged if we translate Ω, we may take $x = 0$. Thus for every B

$$\int_B |\chi_\Omega(t^{-1}y) - 1|\, dy = o(1).$$

It follows that for any $f \in L_2(E_N)$,

$$\lim_{t\to\infty} \int |f(y)\chi_{t\Omega}(y) - f(y)|^2 dy = 0. \tag{29}$$

We have

$$\mu_{n,t} = \inf \sup \int K(\xi)\, |F(\xi)|^2 d\xi$$

where the infimum is taken over all $G_1, \cdots, G_{n-1} \in \hat{L}_2(t\Omega)$ and the supremum over all $F \in \hat{L}_2(t\Omega)$ satisfying

$$\|F\| = 1 \quad (F, G_i) = 0 \quad (1 \leqq i < n).$$

Let $\varepsilon > 0$ and find disjoint sets $I_1, \cdots, I_n$ each of finite positive measure on each of which $K(\xi) \geqq M - \varepsilon$. Let $\mathbf{M}_t$ denote the matrix of inner products

$$(2\pi)^{-2N}\ (\chi_{I_i} * \hat{\chi}_{t\Omega}, \chi_{I_j} * \hat{\chi}_{t\Omega})$$

where the asterisk denotes convolution.

Since from (29)

$$(2\pi)^{-N}\chi_{I_i} * \hat{\chi}_{t\Omega} \to \chi_{I_i}$$

in L_2, $\mathbf{M}_t$ approaches the matrix

$$\mathbf{M} = \begin{bmatrix} |I_1| & & 0 \\ & \ddots & \\ 0 & & |I_n| \end{bmatrix}.$$

If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a vector such that $(\mathbf{M}\alpha, \alpha) = 1$ then of course

$$\sum |\alpha_i|^2 |I_i| = 1.$$

It follows that for sufficiently large t, say $t > t_1$, $(\mathbf{M}_t\alpha, \alpha) = 1$ implies

$$1 + \varepsilon \geqq \sum |\alpha_i|^2 |I_i| \geqq 1 - \varepsilon.$$

We can also find t_2 so that $t > t_2$ implies

$$\|(2\pi)^{-N} \chi_{I_i} * \hat{\chi}_{t\Omega} - \chi_{I_i}\| \leqq \varepsilon/n$$

for each i.

Assume $t \geqq \max(t_1, t_2)$. Given $G_1, \cdots, G_{n-1}$ find $\alpha_t = (\alpha_{1,t}, \cdots, \alpha_{n,t})$ so that

$$F_t(\xi) = (2\pi)^{-N} \sum \alpha_{i,t} \chi_{I_i} * \hat{\chi}_{t\Omega}$$

satisfies

$$\|F_t\| = 1, \quad (F_t, G_i) = 0 \quad (1 \leqq i < n).$$

The condition $\| F_t \| = 1$ is equivalent to $(\mathbf{M}_t\alpha_t, \alpha_t) = 1$, so

$$1 + \varepsilon \geqq \sum |\alpha_{i,t}|^2 \ |I_i| \geqq 1 - \varepsilon.$$

Also

$$\| F_t - (2\pi)^{-N} \sum \alpha_{i,t}\chi_{I_i} \| \leqq \sum |\alpha_{i,t}| \ \|(2\pi)^{-N}\chi_{I_i} * \hat{\chi}_{t\Omega} - \chi_{I_i}\| \leqq 2\,\varepsilon.$$

Therefore, for sufficiently small ε,

$$\begin{aligned} \int K|F_t|^2 d\xi &\geqq (M - \varepsilon) \int I_1 \cup \cdots \cup I_n |F_t|^2 d\xi \\ &\geqq (M - \varepsilon) \left\{ \int |\sum \alpha_{i,t}\chi_{I_i}|^2 d\xi - 6\varepsilon \right\} \\ &= (M - \varepsilon) \ \{ \sum |\alpha_{i,t}|^2 |I_i| - 6\varepsilon \} \\ &\geqq (M - \varepsilon) \ (1 - 7\varepsilon). \end{aligned}$$

Thus $\mu_{n,t} \geqq (M - \varepsilon)\ (1 - 7\varepsilon)$ for $t > \max(t_1, t_2)$ and the assertion is established.

References

1. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Univ. California Press, Berkeley, Calif., 1958.

2. L. Hörmander and J. L. Lions, *Sur la completion par rapport à une intégrale de Dirichlet*, Math. Scand. **2** (1956), 259–270.

3. M. Kac, W. L. Murdock and G. Szegö, *On the eigenvalues of certain Hermitian forms*, J. Rat. Mech. Anal. **2** (1953), 767–800.

4. S. V. Parter, *Extreme eigenvalues of Toeplitz forms and applications to elliptic difference equations*, Trans. Amer. Math. Soc. **99** (1961), 153–193.

5. ———, *On the extreme eigenvalues of truncated Toeplitz matrices*, Bull. Amer. Math. Soc. **67** (1961), 191–196.

6. ———, *On the extreme eigenvalues of Toeplitz matrices*, Trans. Amer. Math. Soc. **100** (1961), 263–276.

7. F. Riesz and B. Sz.-Nagy, *Functional analysis*, Unger, New York, 1955.

8. H. Widom, *On the eigenvalues of certain Hermitian operators*, Trans. Amer. Math. Soc. **88** (1958), 491–522.

9. ———, *Stable processes and integral equations*, Trans. Amer. Math. Soc. **98** (1961), 430–449.

10. ———, *Extreme eigenvalues of translation kernels*, Trans. Amer. Math. Soc. **100** (1961), 252–262.

Cornell University,
Ithaca, New York
University of Chicago,
Chicago, Illinois