ON CERTAIN REPRESENTATIONS OF THE MEASURE ALGEBRA OF A LOCALLY COMPACT ABELIAN GROUP

BY
HORST LEPTIN

Throughout this paper let G be a locally compact abelian group. Furthermore, let $\mathscr{B}$, resp. $\mathscr{C}_0$, resp. $\mathscr{C}_\infty$, always be the Banach spaces of all bounded complex Borel functions, resp. bounded continuous functions, resp. continuous functions, which vanish at ∞. Then we have $\mathscr{C}_\infty \subset \mathscr{C}_0 \subset \mathscr{B}$, $\mathscr{C}_\infty = \mathscr{C}_0$ iff G is compact and $\mathscr{C}_0 = \mathscr{B}$ iff G is discrete. Every continuous complex linear functional on $\mathscr{C}_\infty$ defines a bounded Radon measure on G and thus the set $\mathfrak{M}$ of all bounded Radon measures is the dual space of $\mathscr{C}_\infty$. For the value of a measure $\mu \in \mathfrak{M}$ for a certain function $f \in \mathscr{C}_\infty$ we use the following notations:

$$\mu(f) = \int f d\mu = \int f(x) d\mu(x).$$

The norm in $\mathfrak{M}$ is then defined as $|\mu| = \sup_{|f| \leq 1} |\mu(f)|$ with $|f| = \sup_G |f(x)|$. $\mathfrak{M}$ is a Banach algebra under convolution: For $\mu, \nu \in \mathfrak{M}$, the product $\mu * \nu$ is defined as

$$\mu * \nu(f) = \int f(x+y) d\mu(x) d\nu(y).$$

For a complex function f on G let f^* be defined as $f^*(x) = \overline{f(-x)}$. Then we can define for $\mu \in \mathfrak{M}$: $\mu^*(f) = \overline{\mu(f^*)}$. The mapping $\mu \to \mu^*$ is an isomeric involution on $\mathfrak{M}$. Let a be a point in G and ε_a the corresponding point measure, defined as $\varepsilon_a(f) = f(a)$. We have $\varepsilon_a^* = \varepsilon_{-a}$. For $a = 0$ the measure $\varepsilon_0 = \varepsilon$ is the identity in $\mathfrak{M}$. Let $\mathfrak{L}^1$ be the convolution algebra of all Haar integrable complex functions on G. If we identify $g \in \mathfrak{L}^1$ with the Radon measure $\lambda_g(f) = \int f(x)g(x)dx$, we may consider $\mathfrak{L}^1$ as a norm closed and symmetric ideal in $\mathfrak{M}$. Here "symmetric" means, that with $\lambda_g \in \mathfrak{L}^1$ always $\lambda_g^* \subset \mathfrak{L}^1$. In fact, we have even $\lambda_g^* = \lambda_{g^*}$.

Every positive $\mu \in \mathfrak{M}$ defines a regular Borel measure on G, so that all bounded Borel functions are integrable with respect to μ. So we can uniquely extend every $\mu \in \mathfrak{M}$ to a bounded linear functional on $\mathscr{B}$. For all definitions, results, etc. here mentioned, also for further questions, see, e.g., [1; 2] or [3].

Received by the editors March 14, 1962.

1. For $f \in \mathscr{B}$, $\mu \in \mathfrak{M}$ we define

$$(T_\mu f)(x) = \mu * \varepsilon_x(f) = \int_G f(x+t)d\mu(t).$$

Then $T_\mu f$ is a bounded complex function on G and we have

$$|T_\mu f| = \sup_G |T_\mu f(x)| \leqq |\mu * \varepsilon_x| \; |f| = |\mu| \cdot |f|,$$

$$|\mu| = \sup_{|f| \leqq 1} |\mu(f)| = \sup |T_\mu f(0)| \leqq \sup |T_\mu f| = |T_\mu|,$$

hence $|\mu| = |T_\mu|$. We have

(1) *T_μ maps $\mathscr{C}_\infty$ into $\mathscr{C}_\infty$, $\mathscr{C}_0$ into $\mathscr{C}_0$ and $\mathscr{B}$ into $\mathscr{B}$.*

Because the measures with compact carrier are dense in $\mathfrak{M}$ and $|\mu| = |T_\mu|$ we need prove (1) only for positive μ with compact carriers. If then f has compact carrier, $T_\mu f$ also has compact carrier, if f is continuous, it is uniformly continuous on every compact subset of G. From these facts it follows easily that $T_\mu \mathscr{C}_\infty \subset \mathscr{C}_\infty$ and $T_\mu \mathscr{C}_0 \subset \mathscr{C}_0$. For every positive Radon measure and every directed set(1) of continuous real functions f we have $\sup \mu(f) = \mu(\sup f)$ and therefore $\sup T_\mu f = T_\mu(\sup f)$ if $\sup f \in \mathscr{B}$. Therefore $T_\mu f$ is lower semi-continuous, if f is lower semi-continuous. Furthermore, we have $T_\mu(\lim f_n) = \lim T_\mu f_n$ if $\{f_n\}$ is a monotonic sequence in $\mathscr{B}$. It follows that $T_\mu \mathscr{B} \subset \mathscr{B}$.

Obviously the T_μ commute with the translations $f \to f_a$, where $f_a(x) = f(x+a)$. The following proposition yields a measurefree description of the measure algebra:

(2) *The mapping $\mu \to T_\mu$ is an isometric *-isomorphism from $\mathfrak{M}$ onto the algebra $\mathscr{A}$ of all bounded operators of $\mathscr{C}_\infty$, which commute with all translations $f \to f_a$ of $\mathscr{C}_\infty$.*

(Here the operator S^* for a bounded operator S of $\mathscr{C}_\infty$ is defined as $S^*f = (Sf^*)^*$.)

Proof. Clearly $\mu \to T_\mu$ is a linear and norm preserving mapping of $\mathfrak{M}$ into $\mathscr{A}$. For the product we have

$$T_{\mu * \nu} f(x) = \int f(x+t)d\mu * \nu(t) = \int f(x+u+v)d\mu(u)d\nu(v)$$

$$= \int T_\mu f(x+v)\, d\nu(v) = T_\nu T_\mu f(x) = T_\mu T_\nu f(x),$$

i.e., $T_{\mu * \nu} = T_\mu T_\nu$. Also

$$T_{\mu^*} f = \mu^* * \varepsilon_x(f) = (\mu * \varepsilon_x)^*(f) = (A\mu * \varepsilon_x)(f^*) = T_\mu(f^*)^* = T_\mu^*(f).$$

Let $A \in \mathscr{A}$. Then $\mu(f) = (Af)(0)$ defines a measure $\mu \in \mathfrak{M}$ and because A permutes with the translations, we have

(1) This means that, for each pair of functions f and g in the set, there exists a function h in the set such that $f \leqq h$ and $g \leqq h$ in the natural (pointwise) order.

$$T_\mu f(x) = \mu * \varepsilon_{-x}(f) = (Af_x)(0) = (Af)_x(0) = (Af)(x),$$

i.e., $T_\mu f = Af$ and consequently $T_\mu = A$.

2. A satisfactory description of all maximal ideals of the algebra $\mathfrak{M}$ is still an unsolved problem and so any step forward in this direction may be of some interest. In the case of the algebra $\mathfrak{L}^1$ we know, that all maximal ideals or equivalently all homomorphisms of $\mathfrak{L}^1$ onto the complex number field $\mathbf{C}$ are given by means of the formula $\Phi_\chi(g) = \int \overline{\chi(x)} g(x)dx$, where χ is a continuous character of G. The same formula gives a homomorphism of $\mathfrak{M}$ in $\mathbf{C}$, if we replace gdx by a general measure μ:

$$\Phi_\chi(\mu) = \int \overline{\chi(x)} d\mu(x) = \mu(\bar{\chi}).$$

In this way, as is well known, we may identify the maximal ideal space of $\mathfrak{L}^1$ with the open complement of the hull of $\mathfrak{L}^1$ in the maximal ideal space of $\mathfrak{M}$. Furthermore, it is well known(2) that these homomorphisms are just those given by minimal invariant subspaces of $\mathscr{B}$, invariant under the representation $\mu \to T_\mu$ of $\mathfrak{M}$ in $\mathscr{B}$: Every minimal invariant subspace of $\mathscr{B}$ is of the form $\{a\chi\}$, $a \in \mathbf{C}$, χ a continuous character, and each such function space is a minimal invariant subspace of $\mathscr{B}$. Now the question arises, if it is possible to obtain other homomorphisms of $\mathfrak{M}$ via minimal invariant subspaces in other reasonable representation spaces of $\mathfrak{M}$. In this paper we will show that in the cases of the factor spaces $\mathscr{D} = \mathscr{B}/\mathscr{C}_0$ and $\mathscr{E} = \mathscr{B}/\mathscr{C}_\infty$ the answer is negative. More explicitly we shall prove the following

THEOREM. *If the locally compact abelian group G is separable, then the factorspace $\mathscr{D}$ has no proper minimal invariant subspaces with respect to the representation induced by the representation $\mu \to T_\mu$ of the measure algebra $\mathfrak{M}(G)$ in $\mathscr{B}$. If G is not compact, then every minimal invariant subspace of $\mathscr{E}$ is generated by a function $f = \chi g$, product of a continuous character χ and a continuous bounded function g, having the property that for every $a \in G$ the function $\tilde{g}_a : x \to g(a+x) - g(x)$ vanishes at infinity, i.e., lies in $\mathscr{C}_\infty$. Conversely, every function of this form $f = \chi g$ generates an invariant minimal subspace modulo $\mathscr{C}_\infty$ and the homomorphism Φ, which is defined by this subspace, is identical with Φ_χ.*

Call a function $g \in \mathscr{B}$ slowly oscillating at infinity, if $g(a+x) - g(x)$ vanishes at infinity for every fixed a. Of course every $g \in \mathscr{C}_\infty$ oscillates slowly at infinity, ut there also exist continuous functions *not* in $\mathscr{C}_\infty$, which are slowly oscillating

(2) Recall the fact, that every continous irreducible representation of a complex commutative Banach-algebra with unit is one-dimensional because the factor-algebra modulo the kernel is a field. By Gelfand's and Mazur's theorem this factor-algebra is equal to $\mathbf{C}$. Particularly every proper minimal invariant subspace must have dimension one.

at infinity. For example, let G be the real line and h be any periodic differentiable function with continuous derivative. Then $g(x) = h(\log(1 + x^2))$ oscillates slowly at infinity and is in $\mathscr{C}_\infty$ if and only if h is zero.

3. Let $\mathscr{A}$ be any T-invariant subspace of $\mathscr{B}$, T the representation $\mu \to T_\mu$ of $\mathfrak{M}$ in $\mathscr{B}$. If the function $f \in \mathscr{B}$ generates modulo $\mathscr{A}$ a minimal invariant subspace(2), there exists a homomorphism Φ of $\mathfrak{M}$ in $\mathbf{C}$, such that

$$T_\mu f \equiv \Phi(\mu) f \pmod{\mathscr{A}}$$

or more explicitly:

(3) $$(T_\mu f)(x) = \Phi(\mu) f(x) + H(\mu, x)$$

where $H(\mu, x)$ is a function in $\mathscr{A}$ for every $\mu \in \mathfrak{M}$. For $\mu = \varepsilon_y$ we have $(T_{\varepsilon_y} f)(x) = f(x + y)$ and $\Phi(\varepsilon_y) = \chi(y)$ is a character of G. We write

$$H(\varepsilon_y, x) = H(y, x).$$

Then we get from (3):

(4) $$f(x + y) = \chi(x) f(y) + H(x, y).$$

Now let $\mathscr{A}$ be always in $\mathscr{C}_0$, then $H(x, y)$ is continuous in y for every fixed x and from (4) follows immediately:

(5) *If f is continuous in a single point, then it is continuous everywhere.*

THEOREM 1. *Let G be separable. Let f be a bounded Borel-function on G, which satisfies identity* (4) *with a* (*not necessarily continuous*) *character χ. If then the function $H(x, y)$ is partially continuous in y, then f is continuous.*

Proof. We choose a sequence of integrable positive continuous functions u_i with $\int u_i dx = 1$ and $\lim_{i \to \infty} \int h(x) u_i(x) dx = h(0)$ for every bounded measurable function h, which is continuous at 0. Let in general $f_1 * f_2$ denote the convolution of the functions f_1 and f_2:

$$(f_1 * f_2)(x) = \int_G f_1(x - t) f_2(t) dt.$$

For every continuous function h on G, we then have $\lim_{i \to \infty} (h * u_i)(x) = h(x)$. We now write $f_i(y) = (f * u_i)(y)$ and $H_i(x, y) = (H(x, \cdot) * u_i)(y)$. The functions f_i are all continuous and bounded and $\lim H_i(x, y) = H(x, y)$, because $H(x, \cdot)$ is a continuous function of y. Let us assume that the bounded sequence $\{f_i(0)\}$ converges (otherwise we take a suitable subsequence). It follows from (4) for $y = 0$ that $\{f_i(x)\}$ then converges for every x. But then the bounded function $\tilde{f}(x) = \lim_{i \to \infty} f_i(x)$ is again measurable and satisfies the same identity (4). For the difference $d = f - \tilde{f}$ we then have

$$d(x + y) = \chi(x) d(y).$$

But then either $d = 0$, i.e., $f = \tilde{f}$, or $\chi(x) = d(x)/d(0)$ is measurable, hence continuous, and also d is continuous. Now $\tilde{f}$ as a limit of a sequence of continuous functions, according to a well-known theorem, (see e.g. [4, p. 164]), is continuous in at least one point. It follows that $f = \tilde{f} + d$ is also continuous in at least one point and so is continuous everywhere. This proves Theorem 1.

COROLLARY. *The factorspace $\mathcal{B}/\mathcal{C}_0$ has no minimal invariant subspaces.*

We mention that according to a theorem of Rudin [2, p. 230] the kernel of the factor representation of $\mathfrak{M}$ in $\mathcal{D}$ is equal to $\mathfrak{L}^1$.

THEOREM 2. *If in* (4) $H(x,\cdot)$ *is in* $\mathcal{C}_\infty$ *for every* x *and* f *is not in* $\mathcal{C}_\infty$, *then* χ *is continuous and the function* $g = \bar{\chi}f$ *is a continuous function, which oscillates slowly at infinity.*

Proof. From Theorem 1 it follows that f is continuous. Now assume that f does not vanish at infinity. Then there exists a sequence $\{y_n\}$ in G, which converges to infinity and for which $\lim_{n\to\infty} f(y_n) = \alpha$ exists and is not 0. From (4) and $\lim H(x, y_n) = 0$ then follows $\lim_{n\to\infty} f(x + y_n) = \chi(x)\alpha$. This shows that χ must be measurable and hence continuous. Of course $g = \bar{\chi}f$ is also continuous and satisfies the relation

$$g(x + y) - g(y) = \bar{\chi}(x + y)H(x, y) \in \mathcal{C}_\infty$$

for every fixed x.

The next theorem holds for general locally compact abelian groups.

THEOREM 3. *A subspace* $(\tilde{f})$ *of* $\mathcal{B}/\mathcal{C}_\infty$ *is invariant under the representation* T *of* $\mathfrak{M}$ *in* $\mathcal{B}/\mathcal{C}_\infty$ *if and only if the function* f, *representative of* $\tilde{f} = f + \mathcal{C}_\infty$, *is of the form* $f = \chi g$, *where* χ *is a continuous character of* G *and* g *is a bounded continous function, which oscillates slowly at infinity. The homomorphism* Φ *of* $\mathfrak{M}$, *corresponding to* $(\tilde{f})$, *is identical with the homomorphism* Φ_χ, *corresponding to the character* χ.

We remark that in the separable case the "only if" part of this theorem is an immediate consequence of Theorem 2. For the general case we need a lemma:

(6) *A function* f *on* G *converges to zero for* x *converging to infinity, if and only if* $\lim_{n\to\infty} f(x_n) = 0$ *for every sequence* $\{x_n\} \subset G$ *with* $\lim_{n\to\infty} x_n = \infty$.

We must show only that the condition is sufficient. This is obvious, if G is countable at infinity, i.e., compactly generated. Now let G_0 be a compactly generated open subgroup of G and $\varepsilon > 0$. Let $\{x_n\} \subset G$ be such that $x_n - x_m \notin G_0$ for $n \neq m$. Then obviously $\lim_{n\to\infty} x_n = \infty$. It follows that $|f(x)| < \varepsilon$ on the residue-classes of G_0 with at most a finite number of exceptions, say $X_i = x_i + G_0$, $i = 1, \cdots, n$. Because X_i is homeomorphic to G_0, and is countable at infinity, there exists a compact subset $K_i \subset X_i$ with $|f(x)| < \varepsilon$ for $x \notin K_i$. Then $K = \bigcup_{i=1}^n K_i$ is again compact and for all $x \notin K$ we have $|f(x)| < \varepsilon$, which shows that $\lim_{x\to\infty} f(x) = 0$.

Now let f generate an invariant subspace in $\mathscr{B}/\mathscr{C}_\infty$. From (4) the following identity follows:

$$\chi(x)f(y) + H(x,y) = \chi(y)f(x) + H(y,x). \tag{7}$$

Now choose a sequence $\{y_i\} \subset G$ with $\lim_{i\to\infty} y_i = \infty$ in such a way that $\lim f(y_i) = \alpha$ and $\lim \chi(y_i) = \beta \neq 0$ exists. In (7) set $y = y_i$ and take the limit. Because $\lim H(x,y_i) = 0$ for every x, we get

$$\chi(x)\cdot\alpha = \beta\cdot f(x) + h(x) \tag{8}$$

with $h(x) = \lim_{i\to\infty} H(y_i, x)$. This function $h(x)$ is measurable and as a limit of continuous functions is continuous in at least one point [4, p. 164]. If f vanishes at infinity, we have $\alpha = 0$ and $f = -\beta^{-1}h$; consequently f is continuous in at least one point and hence must be continuous everywhere. Therefore in this case $f \in \mathscr{C}_\infty$. If f does not vanish at infinity we may choose $\alpha \neq 0$. Then (8) shows that χ must be measurable, hence continuous, and again $f = \beta^{-1}(\alpha\chi - h)$ is continuous. Then $g = \bar{\chi}f$ oscillates slowly at infinity and $f = \chi g$ has the form which the theorem requires.

To prove the sufficiency of the condition in Theorem 3, let g be a bounded continuous function slowly oscillating at infinity. Then for every sequence $\{y_n\}$ tending to infinity,

$$h_n(x) = g(x + y_n) - g(y_n)$$

defines a sequence of uniformly bounded continuous functions with $\lim_{n\to\infty} h_n(x) = 0$ for every x. Hence we have by Lebesgue's theorem for every bounded Radon measure μ: $\lim \int h_n(x)d\mu(x) = 0$ which means

$$(T_\mu g)(y_n) - \mu(1)g(y_n) \to 0.$$

Therefore $T_\mu g - \mu(1)g$ is in $\mathscr{C}_\infty$, which means, that g generates a minimal invariant subspace $(\tilde{g})$ in $\mathscr{B}/\mathscr{C}_\infty$ with

$$T_\mu \tilde{g} = \mu(1)\tilde{g} = \hat{\mu}(\chi_0)\tilde{g}, \qquad \chi_0 \text{ the unit-character.}$$

If χ is a continuous character, then

$$\begin{aligned}(T_\mu \chi g)(x) &= \int \chi(x+y)g(x+y)d\mu(y) = \chi(x)\int g(x+y)d(\mu\cdot\bar{\chi})(y) \\ &\equiv \chi(x)(\mu\cdot\bar{\chi})^\wedge(\chi_0)g(x) \pmod{\mathscr{C}_\infty} = \hat{\mu}(\bar{\chi})\chi g(x) \pmod{\mathscr{C}_\infty},\end{aligned}$$

i.e., we have $T_\mu(\chi g)^\sim = \hat{\mu}(\bar{\chi})(\chi g)^\sim$, where $(\chi g)^\sim = \chi g + \mathscr{C}_\infty$. Now a reference to (6) completes the proof of Theorem 3.

Finally we remark that Lemma (6) is trivial, if G is separable; thus in this case the last paragraph already proves the theorem.

References

1. E. Hewitt, *A survey of abstract harmonic analysis*, Some aspects of analysis and probability; surveys in applied mathematics, Wiley, New York, 1958.

2. W. Rudin, *Measure algebras on abelian groups*, Bull. Amer. Math. Soc. **65** (1959), 227–247.

3. E. Hewitt and S. Kakutani, *A class of multiplicative linear functionals on the measure algebra of a locally compact abelian group*, Illinois J. Math. **4** (1960), 553–574.

4. G. Aumann, *Reelle Funktionen. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete*, Bd. 68, Springer, Berlin, 1954.

Universität Hamburg,
Hamburg, Germany
Tulane University
New Orleans, Louisiana

ERRATA TO VOLUME 104

Joseph A. Wolf. *Homogeneous manifolds of zero curvature*, pp. 462–469.

Page 462, line 13 of §2. Delete the sentence "M^n_s is complete if it is homogeneous." For if U is a nonzero totally isotropic linear subspace of R^n_s, then one can check that $R^n_s - U^\perp$ is homogeneous but not complete.

Add the hypothesis that M^n_s is complete in Theorem 1 (page 466) and in Theorem 2 (page 467).