ON CERTAIN REPRESENTATIONS OF THE MEASURE ALGEBRA OF A LOCALLY COMPACT ABELIAN GROUP

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Throughout this paper let G be a locally compact abelian group. Furthermore, let \( \mathcal{B} \), resp. \( \mathcal{C}_0 \), resp. \( \mathcal{C}_\infty \), always be the Banach spaces of all bounded complex Borel functions, resp. bounded continuous functions, resp. continuous functions, which vanish at \( \infty \). Then we have \( \mathcal{C}_\infty \subset \mathcal{C}_0 \subset \mathcal{B} \), \( \mathcal{C}_\infty \mathcal{C}_0 \) iff G is compact and \( \mathcal{C}_0 = \mathcal{B} \) iff G is discrete. Every continuous complex linear functional on \( \mathcal{C}_\infty \) defines a bounded Radon measure on G and thus the set \( \mathcal{M} \) of all bounded Radon measures is the dual space of \( \mathcal{C}_\infty \). For the value of a measure \( \mu \in \mathcal{M} \) for a certain function \( f \in \mathcal{C}_\infty \), we use the following notations:

\[
\mu(f) = \int f \, d\mu = \int f(x) \, d\mu(x).
\]

The norm in \( \mathcal{M} \) is then defined as \( |\mu| = \sup_{|f| \leq 1} |\mu(f)| \) with \( |f| = \sup_G |f(x)| \). \( \mathcal{M} \) is a Banach algebra under convolution: For \( \mu, \nu \in \mathcal{M} \), the product \( \mu \ast \nu \) is defined as

\[
\mu \ast \nu(f) = \int f(x + y) \, d\mu(x) \, d\nu(y).
\]

For a complex function \( f \) on G let \( f^* \) be defined as \( f^*(x) = f(-x) \). Then we can define for \( \mu \in \mathcal{M} \): \( \mu^*(f) = \overline{\mu(f^*)} \). The mapping \( \mu \mapsto \mu^* \) is an isometric involution on \( \mathcal{M} \). Let a be a point in G and \( \varepsilon_a \) the corresponding point measure, defined as \( \varepsilon_a(f) = f(a) \). We have \( \varepsilon_0^* = \varepsilon_{-a} \). For \( a = 0 \) the measure \( \varepsilon_0 = \varepsilon \) is the identity in \( \mathcal{M} \).

Let \( \Omega^1 \) be the convolution algebra of all Haar integrable complex functions on G. If we identify \( g \in \Omega^1 \) with the Radon measure \( \lambda_g(f) = \int f(x) g(x) \, dx \), we may consider \( \Omega^1 \) as a norm closed and symmetric ideal in \( \mathcal{M} \). Here "symmetric" means, that with \( \lambda_y \in \Omega^1 \) always \( \lambda_y^* \subset \Omega^1 \). In fact, we have even \( \lambda_y^* = \lambda_{-y} \).

Every positive \( \mu \in \mathcal{M} \) defines a regular Borel measure on G, so that all bounded Borel functions are integrable with respect to \( \mu \). So we can uniquely extend every \( \mu \in \mathcal{M} \) to a bounded linear functional on \( \mathcal{B} \). For all definitions, results, etc. here mentioned, also for further questions, see, e.g., [1; 2] or [3].

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1. For $f \in \mathcal{B}$, $\mu \in \mathcal{M}$ we define 

$$(T_\mu f)(x) = \mu \ast \varepsilon_x(f) = \int_G f(x + t)d\mu(t).$$

Then $T_\mu f$ is a bounded complex function on $G$ and we have 

$$|T_\mu f| = \sup_{G} |T_\mu f(x)| \leq |\mu \ast \varepsilon_x| |f| = |\mu| |f|,$$

$$|\mu| = \sup_{|f| \leq 1} |\mu(f)| = \sup_{|f| \leq 1} |T_\mu f(0)| \leq \sup_{G} |T_\mu f| = |T_\mu|,$$

hence $|\mu| = |T_\mu|$. We have

(1) $T_\mu$ maps $\mathcal{C}_\infty$ into $\mathcal{C}_\infty$, $\mathcal{C}_0$ into $\mathcal{C}_0$ and $\mathcal{B}$ into $\mathcal{B}$.

Because the measures with compact carrier are dense in $\mathcal{M}$ and $|\mu| = |T_\mu|$ we need prove (1) only for positive $\mu$ with compact carrier. If then $f$ has compact carrier, $T_\mu f$ also has compact carrier, if $f$ is continuous, it is uniformly continuous on every compact subset of $G$. From these facts it follows easily that $T_\mu \mathcal{C}_\infty \subseteq \mathcal{C}_\infty$ and $T_\mu \mathcal{C}_0 \subseteq \mathcal{C}_0$. For every positive Radon measure and every directed set $\{f\}$ of continuous real functions $f$ we have $\sup \mu(f) = \mu(\sup f)$ and therefore $\sup T_\mu f = T_\mu (\sup f)$ if $\sup f \in \mathcal{B}$. Therefore $T_\mu f$ is lower semi-continuous, if $f$ is lower semi-continuous. Furthermore, we have $T_\mu (\lim f_n) = \lim T_\mu f_n$ if $\{f_n\}$ is a monotonic sequence in $\mathcal{B}$. It follows that $T_\mu \mathcal{B} \subseteq \mathcal{B}$.

Obviously the $T_\mu$ commute with the translations $f \rightarrow f_a$, where $f_a(x) = f(x + a)$. The following proposition yields a measurefree description of the measure algebra:

(2) The mapping $\mu \rightarrow T_\mu$ is an isometric $*$-isomorphism from $\mathcal{M}$ onto the algebra $\mathcal{A}$ of all bounded operators of $\mathcal{C}_\infty$, which commute with all translations $f \rightarrow f_a$ of $\mathcal{C}_\infty$.

(Here the operator $S^*$ for a bounded operator $S$ of $\mathcal{C}_\infty$ is defined as $S^*f = (S^f)^*$.)

**Proof.** Clearly $\mu \rightarrow T_\mu$ is a linear and norm preserving mapping of $\mathcal{M}$ into $\mathcal{A}$. For the product we have 

$$T_\mu^* f(x) = \int f(x + t)d\mu^* v(t) = \int f(x + u + v)d\mu(u)dv(v)$$

$$= \int T_\mu f(x + v)dv(v) = T_{\mu^*} f(x) = T_\mu T_{\mu^*} f(x),$$

i.e., $T_{\mu^*} = T_\mu T_{\mu^*}$. Also

$$T_{\mu^*} f = \mu^* \ast \varepsilon_x(f) = (\mu \ast \varepsilon_x)^*(f) = (A\mu \ast \varepsilon_x)^*(f^*) = T_\mu(f^*) = T_\mu^*(f).$$

Let $A \in \mathcal{A}$. Then $\mu(f) = (Af)(0)$ defines a measure $\mu \in \mathcal{M}$ and because $A$ permutes with the translations, we have

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(1) This means that, for each pair of functions $f$ and $g$ in the set, there exists a function $h$ in the set such that $f \leq h$ and $g \leq h$ in the natural (pointwise) order.
\[ T_\mu f(x) = \mu \ast \varepsilon_x(f) = (Af_\mu)(0) = (Af)_\mu(0) = (Af)(x), \]
i.e., \( T_\mu f = Af \) and consequently \( T_\mu = A. \)

2. A satisfactory description of all maximal ideals of the algebra \( \mathfrak{M} \) is still an unsolved problem and so any step forward in this direction may be of some interest. In the case of the algebra \( \mathcal{L}^1 \) we know, that all maximal ideals or equivalently all homomorphisms of \( \mathcal{L}^1 \) onto the complex number field \( \mathbb{C} \) are given by means of the formula \( \Phi_\chi(g) = \int \overline{\chi(x)} g(x) dx \), where \( \chi \) is a continuous character of \( G \). The same formula gives a homomorphism of \( \mathfrak{M} \) in \( \mathbb{C} \), if we replace \( gdx \) by a general measure \( \mu \):

\[
\Phi_\chi(\mu) = \int \overline{\chi(x)} d\mu(x) = \mu(\overline{\chi}).
\]

In this way, as is well known, we may identify the maximal ideal space of \( \mathcal{L}^1 \) with the open complement of the hull of \( \mathcal{L}^1 \) in the maximal ideal space of \( \mathfrak{M} \). Furthermore, it is well known\(^{(2)}\) that these homomorphisms are just those given by minimal invariant subspaces of \( \mathcal{B} \), invariant under the representation \( \mu \rightarrow T_\mu \) of \( \mathfrak{M} \) in \( \mathcal{B} \): Every minimal invariant subspace of \( \mathcal{B} \) is of the form \( \{a\chi\} \), \( a \in \mathbb{C} \), \( \chi \) a continuous character, and each such function space is a minimal invariant subspace of \( \mathcal{B} \). Now the question arises, if it is possible to obtain other homomorphisms of \( \mathfrak{M} \) via minimal invariant subspaces in other reasonable representation spaces of \( \mathfrak{M} \). In this paper we will show that in the cases of the factor spaces \( \mathcal{D} = \mathcal{B}/\mathbb{C}_0 \) and \( \mathcal{E} = \mathcal{B}/\mathbb{C}_\infty \) the answer is negative. More explicitly we shall prove the following

**Theorem.** If the locally compact abelian group \( G \) is separable, then the factorspace \( \mathcal{D} \) has no proper minimal invariant subspaces with respect to the representation induced by the representation \( \mu \rightarrow T_\mu \) of the measure algebra \( \mathfrak{M}(G) \) in \( \mathcal{B} \). If \( G \) is not compact, then every minimal invariant subspace of \( \mathcal{E} \) is generated by a function \( f = \chi g \), product of a continuous character \( \chi \) and a continuous bounded function \( g \), having the property that for every \( a \in G \) the function \( \tilde{g}_a : x \rightarrow g(a + x) - g(x) \) vanishes at infinity, i.e., lies in \( \mathbb{C}_\infty \). Conversely, every function of this form \( f = \chi g \) generates an invariant minimal subspace modulo \( \mathbb{C}_\infty \) and the homomorphism \( \Phi \), which is defined by this subspace, is identical with \( \Phi_\chi. \)

Call a function \( g \in \mathcal{B} \) slowly oscillating at infinity, if \( g(a + x) - g(x) \) vanishes at infinity for every fixed \( a \). Of course every \( g \in \mathbb{C}_\infty \) oscillates slowly at infinity, but there also exist continuous functions not in \( \mathbb{C}_\infty \), which are slowly oscillating

\(^{(2)}\) Recall the fact, that every continuous irreducible representation of a complex commutative Banach-algebra with unit is one-dimensional because the factor-algebra modulo the kernel is a field. By Gelfand's and Mazur's theorem this factor-algebra is equal to \( \mathbb{C} \). Particularly every proper minimal invariant subspace must have dimension one.
at infinity. For example, let \( G \) be the real line and \( h \) be any periodic differentiable function with continuous derivative. Then \( g(x) = h(\log(1 + x^2)) \) oscillates slowly at infinity and is in \( C_\infty \) if and only if \( h \) is zero.

3. Let \( \mathcal{A} \) be any \( T \)-invariant subspace of \( \mathcal{B} \), \( T \) the representation \( \mu \rightarrow T_\mu \) of \( \mathcal{M} \) in \( \mathcal{B} \). If the function \( f \in \mathcal{B} \) generates modulo \( \mathcal{A} \) a minimal invariant subspace\(^2\), there exists a homomorphism \( \Phi \) of \( \mathcal{M} \) in \( C \), such that

\[
T_\mu f \equiv \Phi(\mu)f \pmod{\mathcal{A}}
\]

or more explicitly:

\[
(T_\mu f)(x) = \Phi(\mu)f(x) + H(\mu, x)
\]

where \( H(\mu, x) \) is a function in \( \mathcal{A} \) for every \( \mu \in \mathcal{M} \). For \( \mu = \varepsilon_x \) we have \((T_{\varepsilon_x} f)(x) = f(x + y)\) and \( \Phi(\varepsilon_x) = \chi(y) \) is a character of \( G \). We write

\[
H(\varepsilon_x, x) = H(y, x).
\]

Then we get from (3):

\[
f(x + y) = \chi(x)f(y) + H(x, y).
\]

Now let \( \mathcal{A} \) be always in \( C_\infty \), then \( H(x, y) \) is continuous in \( y \) for every fixed \( x \) and from (4) follows immediately:

\[
\text{If } f \text{ is continuous in a single point, then it is continuous everywhere.}
\]

**Theorem 1.** Let \( G \) be separable. Let \( f \) be a bounded Borel-function on \( G \), which satisfies identity (4) with a (not necessarily continuous) character \( \chi \). If then the function \( H(x, y) \) is partially continuous in \( y \), then \( f \) is continuous.

**Proof.** We choose a sequence of integrable positive continuous functions \( u_i \) with \( \int u_i dx = 1 \) and \( \lim_{i \to \infty} \int h(x)u_i(x)dx = h(0) \) for every bounded measurable function \( h \), which is continuous at 0. Let in general \( f_1 * f_2 \) denote the convolution of the functions \( f_1 \) and \( f_2 \):

\[
(f_1 * f_2)(x) = \int_G f_1(x - t)f_2(t)dt.
\]

For every continuous function \( h \) on \( G \), we then have \( \lim_{i \to \infty} (h * u_i)(x) = h(x) \). We now write \( f_j(y) = (f * u_j)(y) \) and \( H_j(x, y) = (H(x, \cdot) * u_j)(y) \). The functions \( f_j \) are all continuous and bounded and \( \lim_{i \to \infty} H_j(x, y) = H(x, y) \), because \( H(x, \cdot) \) is a continuous function of \( y \). Let us assume that the bounded sequence \( \{f_j(0)\} \) converges (otherwise we take a suitable subsequence). It follows from (4) for \( y = 0 \) that \( \{f_k(x)\} \) then converges for every \( x \). But then the bounded function \( f_j(x) \)

\[
\lim_{i \to \infty} f_j(x) = d \text{ is again measurable and satisfies the same identity (4).}
\]

For the difference \( d = f - \tilde{f} \) we then have

\[
d(x + y) = \chi(x)d(y).
\]

But then either \( d = 0 \), i.e., \( f = \tilde{f} \), or \( \chi(x) = d(x)/d(0) \) is measurable, hence con-
continuous, and also \( d \) is continuous. Now \( f \) as a limit of a sequence of continuous functions, according to a well-known theorem, (see e.g. [4, p. 164]), is continuous in at least one point. It follows that \( f = \hat{f} + d \) is also continuous in at least one point and so is continuous everywhere. This proves Theorem 1.

**Corollary.** The factorspace \( \mathcal{B}/\mathcal{C}_0 \) has no minimal invariant subspaces.

We mention that according to a theorem of Rudin [2, p. 230] the kernel of the factor representation of \( \mathcal{M} \) in \( \mathcal{D} \) is equal to \( \Omega^1 \).

**Theorem 2.** If in (4) \( H(x, \cdot) \) is in \( \mathcal{C}_\infty \) for every \( x \) and \( f \) is not in \( \mathcal{C}_\infty \), then \( \chi \) is continuous and the function \( g = \overline{\chi} f \) is a continuous function, which oscillates slowly at infinity.

**Proof.** From Theorem 1 it follows that \( f \) is continuous. Now assume that \( f \) does not vanish at infinity. Then there exists a sequence \( \{y_n\} \) in \( G \), which converges to infinity and for which \( \lim_{n \to \infty} f(y_n) = \alpha \) exists and is not 0. From (4) and \( \lim H(x, y_n) = 0 \) then follows \( \lim_{n \to \infty} f(x + y_n) = \chi(x) \alpha \). This shows that \( \chi \) must be measurable and hence continuous. Of course \( g = \overline{\chi} f \) is also continuous and satisfies the relation

\[
g(x + y) - g(y) = \overline{\chi}(x + y)H(x, y) \in \mathcal{C}_\infty
\]

for every fixed \( x \).

The next theorem holds for general locally compact abelian groups.

**Theorem 3.** A subspace \( (\mathcal{J}) \) of \( \mathcal{B}/\mathcal{C}_\infty \) is invariant under the representation \( T \) of \( \mathcal{M} \) in \( \mathcal{B}/\mathcal{C}_\infty \) if and only if the function \( f, \) representative of \( \mathcal{J} = f + \mathcal{C}_\infty \), is of the form \( f = \chi g \), where \( \chi \) is a continuous character of \( G \) and \( g \) is a bounded continuous function, which oscillates slowly at infinity. The homomorphism \( \Phi \) of \( \mathcal{M} \), corresponding to \( (\mathcal{J}) \), is identical with the homomorphism \( \Phi_\chi \), corresponding to the character \( \chi \).

We remark that in the separable case the “only if” part of this theorem is an immediate consequence of Theorem 2. For the general case we need a lemma:

(6) A function \( f \) on \( G \) converges to zero for \( x \) converging to infinity, if and only if \( \lim_{n \to \infty} f(x_n) = 0 \) for every sequence \( \{x_n\} \subset G \) with \( \lim_{n \to \infty} x_n = \infty \).

We must show only that the condition is sufficient. This is obvious, if \( G \) is countable at infinity, i.e., compactly generated. Now let \( G_0 \) be a compactly generated open subgroup of \( G \) and \( \varepsilon > 0 \). Let \( \{x_n\} \subset G \) be such that \( x_n - x_m \not\in G_0 \) for \( n \neq m \). Then obviously \( \lim_{n \to \infty} x_n = \infty \). It follows that \( |f(x)| < \varepsilon \) on the residue-classes of \( G_0 \) with at most a finite number of exceptions, say \( X_i = x_i + G_0 \), \( i = 1, \ldots, n \). Because \( X_i \) is homeomorphic to \( G_0 \), and is countable at infinity, there exists a compact subset \( K_i \subset X_i \) with \( |f(x)| < \varepsilon \) for \( x \not\in K_i \). Then \( K = \bigcup_{i=1}^n K_i \) is again compact and for all \( x \notin K \) we have \( |f(x)| < \varepsilon \), which shows that \( \lim_{x \to \infty} f(x) = 0 \).
Now let \( f \) generate an invariant subspace in \( \mathcal{B} / \mathcal{C}_\infty \). From (4) the following identity follows:

\[
(7) \quad \chi(x)f(y) + H(x,y) = \chi(y)f(x) + H(y,x).
\]

Now choose a sequence \( \{y_i\} \subset G \) with \( \lim_{i \to \infty} y_i = \infty \) in such a way that \( \lim f(y_i) = \alpha \) and \( \lim \chi(y_i) = \beta \neq 0 \) exists. In (7) set \( y = y_i \) and take the limit. Because \( \lim H(x,y_i) = 0 \) for every \( x \), we get

\[
(8) \quad \chi(x) \cdot \alpha = \beta \cdot f(x) + h(x)
\]

with \( h(x) = \lim_{i \to \infty} H(y_i, x) \). This function \( h(x) \) is measurable and as a limit of continuous functions is continuous in at least one point [4, p. 164]. If \( f \) vanishes at infinity, we have \( \alpha = 0 \) and \( f = -\beta^{-1}h \); consequently \( f \) is continuous in at least one point and hence must be continuous everywhere. Therefore in this case \( f \in \mathcal{C}_\infty \).

If \( f \) does not vanish at infinity we may choose \( \alpha \neq 0 \). Then (8) shows that \( \chi \) must be measurable, hence continuous, and again \( f = \beta^{-1}(\alpha \chi - h) \) is continuous. Then \( g = \chi f \) oscillates slowly at infinity and \( f = \chi g \) has the form which the theorem requires.

To prove the sufficiency of the condition in Theorem 3, let \( g \) be a bounded continuous function slowly oscillating at infinity. Then for every sequence \( \{y_n\} \) tending to infinity,

\[
h_n(x) = g(x + y_n) - g(y_n)
\]

defines a sequence of uniformly bounded continuous functions with \( \lim_{n \to \infty} h_n(x) = 0 \) for every \( x \). Hence we have by Lebesgue's theorem for every bounded Radon measure \( \mu \): \( \lim \int h_n(x) d\mu(x) = 0 \) which means

\[
(T_{\mu}g)(y_n) - \mu(1)g(y_n) \to 0.
\]

Therefore \( T_{\mu}g - \mu(1)g \) is in \( \mathcal{C}_\infty \), which means, that \( g \) generates a minimal invariant subspace \( \langle g \rangle \) in \( \mathcal{B} / \mathcal{C}_\infty \) with

\[
T_{\mu}g = \mu(1)g = \mu(\chi_0)g,
\]

where \( \chi_0 \) the unit-character.

If \( \chi \) is a continuous character, then

\[
(T_{\mu}(\chi g))(x) = \int \chi(x + y)g(x + y) d\mu(y) = \chi(x) \int g(x + y) d\mu(\chi)(y)
\]

\[
\equiv \chi(x) (\mu \cdot \chi)^{-1} (\chi_0)g(x) \mod \mathcal{C}_\infty = \mu(\chi)\chi g(x) \mod \mathcal{C}_\infty,
\]

i.e., we have \( T_{\mu}(\chi g)^- = \mu(\chi)(\chi g)^- \), where \( (\chi g)^- = \chi g + \mathcal{C}_\infty \). Now a reference to (6) completes the proof of Theorem 3.

Finally we remark that Lemma (6) is trivial, if \( G \) is separable; thus in this case the last paragraph already proves the theorem.
REFERENCES


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Page 462, line 13 of §2. Delete the sentence “$M^n$ is complete if it is homogeneous.” For if $U$ is a nonzero totally isotropic linear subspace of $\mathbb{R}^n$, then one can check that $\mathbb{R}^n - U^\perp$ is homogeneous but not complete.

Add the hypothesis that $M^n$ is complete in Theorem 1 (page 466) and in Theorem 2 (page 467).